Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

Würzburg, May 22, 2023

Prof. Dr. Alexander Wolff
M.Sc. Marie Diana Sieper

Homework Assignment #3
Algorithms for Geographic Information Systems (SS 2023)

Exercise 1 – Maximum-Likelihood Estimation

Consider a (possibly biased) coin with sides $\mathtt{H}$ and $\mathtt{T}$. The probability distribution for a single flip is $\Pr[\mathtt{H}] = p$ and $\Pr[\mathtt{T}] = 1 - p$, where $p \in [0, 1]$ is a parameter. We flip the coin n times, independently, and k of these flips result in $\mathtt{H}$. Derive the maximum-likelihood estimate for p; prove your result. **[4 points]**

Exercise 2 – Markov Students

Consider a student who is, at every time step, in one of the states in $Z = \{\mathtt{Study}, \mathtt{Party}, \mathtt{Sleep}\}$. The transition probabilities are as follows.

$$\Pr[X_{n+1} = \mathtt{Study} \mid X_n = \mathtt{Study}] = 0.3$$
$$\Pr[X_{n+1} = \mathtt{Party} \mid X_n = \mathtt{Study}] = 0.5$$
$$\Pr[X_{n+1} = \mathtt{Sleep} \mid X_n = \mathtt{Study}] = 0.2$$

$$\Pr[X_{n+1} = \mathtt{Study} \mid X_n = \mathtt{Party}] = 0.0$$
$$\Pr[X_{n+1} = \mathtt{Party} \mid X_n = \mathtt{Party}] = 0.5$$
$$\Pr[X_{n+1} = \mathtt{Sleep} \mid X_n = \mathtt{Party}] = 0.5$$

$$\Pr[X_{n+1} = \mathtt{Study} \mid X_n = \mathtt{Sleep}] = 0.3$$
$$\Pr[X_{n+1} = \mathtt{Party} \mid X_n = \mathtt{Sleep}] = 0.3$$
$$\Pr[X_{n+1} = \mathtt{Sleep} \mid X_n = \mathtt{Sleep}] = 0.4$$

a) Draw the state transition diagram, with a node for every state and arrows indicating the transition probabilities. **[2 points]**

b) You may notice that, independent of the initial distribution $\Pr[X_1]$, the probability distribution for later time steps always seems to converge to the same distribution as time goes on. (Try some.)

   This doesn't hold for all possible Markov chains, but it is in fact true for this particular Markov chain: independently of $\Pr[X_1]$, the probability distribution $\Pr[X_n]$ converges as n goes to infinity. This limit distribution is called a *stationary distribution*, defined as a probability distribution such that $\Pr[X_{n+1}] = \Pr[X_n]$ in a particular Markov chain. (It is called "stationary" because it "doesn't move.") Calculate the stationary distribution for the given Markov chain; give exact values. **[5 points]**

c) Find a Markov model that is not independent of its initial distribution. Draw its state diagram and give two initial distributions with different convergence properties. **[3 points]**

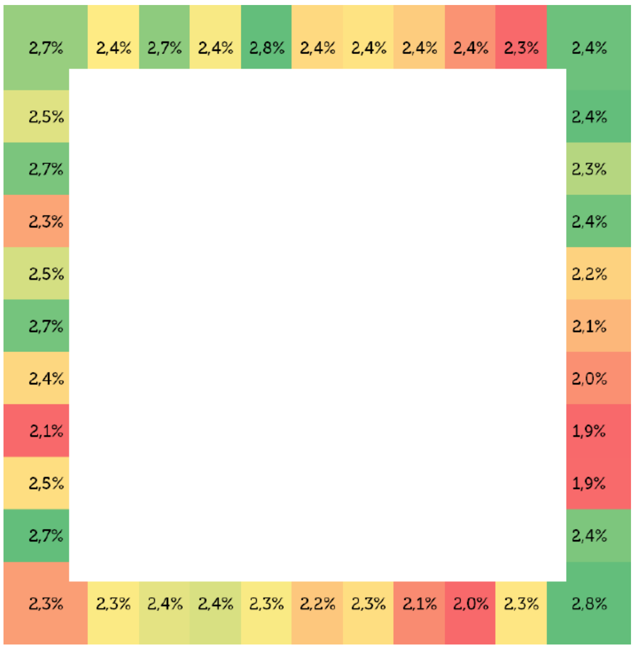
Exercise 3 – In Jail

Consider the well-known boardgame Monopoly, where the players travel along a loop of 40 game tiles (or possibly are 'in jail'). See Figure 1a (a) for a version of the game board. During the course of the game, the player can buy and trade tiles in order to later make money from 'rent' paid by other players that land on the tile. This means that any tiles that players are more likely to land on are more valuable.

We want to model the movement of a player along the board as a Markov chain, so that we can easily calculate a stationary distribution: as the game goes longer and (infinitely) longer, what is the probability of a player landing on each tile? This information is useful when making buying decisions in the game. See Figure 1b (b) for the stationary distribution of Dutch-rules Monopoly, which has been calculated in this way.

Model, as Markov chain, a basic version of how a player goes around the board during the game. What are the states and transition probabilities? Then find out about more advanced rules and incorporate some of them in your model. The exact rules vary from edition to edition, but certainly discuss: going to jail, getting out of jail, and chance cards. **[6 points]**

(a) An English version of the game board.



(b) Stationary distribution using the Dutch chance cards, colour-coded from red (low probability) to green (high probability). There is an additional 5.5% probability of being in jail.

FIGURE 1: Monopoly

Please hand in your solutions on WueCampus until 12:00 on Thuesday, June 1.