Lehrstuhl für Informatik I
– Algorithmen und Komplexität –
Universität Würzburg

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Johannes Zink
Oksana Firman

Exercise Sheet #10

Graph Visualization (SS 2023)

Exercise 1 – Adding an edge with a minimum number of new crossings

Suppose you are given a planar drawing of a graph $G = (V, E)$ with corresponding combinatorial embedding $\mathcal{E}$. Let $u, v \in V$ be vertices such that $e = \{u, v\} \notin E$.

Devise a polynomial-time algorithm that adds e to the existing drawing causing as few crossings as possible, show its correctness and give its running time. **6 Points**

Hint: Use the dual graph of $\mathcal{E}$.

Exercise 2 – Fixed linear crossing number

In the lecture we mentioned the problem *fixed linear crossing number*, which is the crossing number for a fixed linear layout: For a graph $G = (V, E)$ with given vertex numbering $V = \{v_1, v_2, \dots, v_n\}$, each vertex v_i has position $(i, 0)$ and every edge is a semicircle. The only decision when drawing an edge is, therefore, whether it is drawn in the halfplane above or below the horizontal line at height 0. Given a graph G with numbered vertices and an integer k, it is NP-hard to decide whether a fixed linear layout with at most k crossings exists.

a) Devise an algorithm that decides for a given graph $G = (V, E)$ with vertex numbering $V = \{v_1, \dots, v_n\}$ in polynomial time whether a fixed linear layout with zero crossings exists. Argue why your algorithm is correct and give its runtime. **6 Points**

b) Show that, when restricting the input graphs to matchings (i.e. all vertices have degree 1), that the decision problem for the fixed linear crossing number problem is still NP-hard. **4 Points**

   Hint: Use the a reduction from the general fixed linear crossing number problem.

Exercise 3 – Sharp asymptotic lower bound on the crossing number

To show that the bound $\mathrm{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2} \in \Omega(\frac{m^3}{n^2})$ from the lecture is asymptotically tight, we need to find a graph G (or rather a family of graphs) with n vertices and m edges and a drawing style for G with $O(\frac{m^3}{n^2})$ crossings.

For $0 < k < n/2$, consider the graph G with vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{\{v_i, v_j\} \mid i < j' \leq i + k \text{ and } j = j' \mod n\}$. We draw the vertices in the order of their indices on a circle and edges as straight lines.

Compute the number of crossings first per edge and then for G in total to show that the asymptotic lower bound is sharp. **4 Points**

This assignment is due at the beginning of the next lecture, that is, on July 14 at 10:15 am. Please submit your solutions via WueCampus. The questions can be asked in the tutorial session on July 12 at 16:00 and the solutions will be discussed one week after that on July 19.