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Advanced Algorithms

Succinct Data Structures

Indexable Dictionaries and Trees

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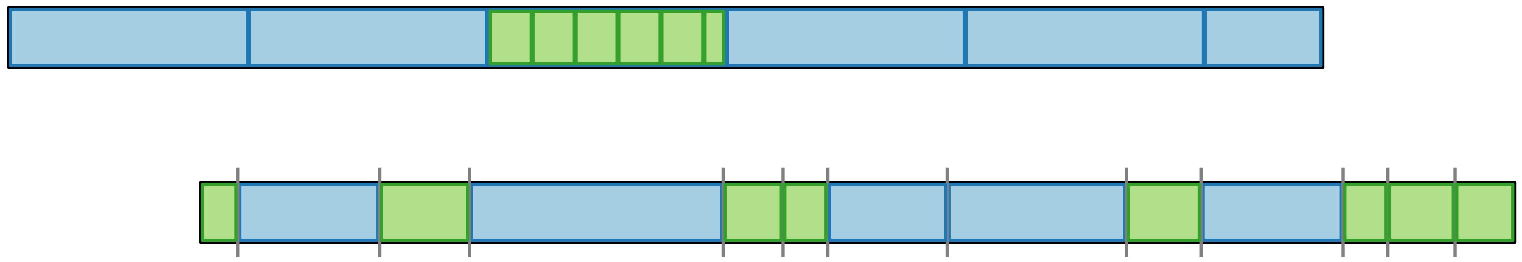
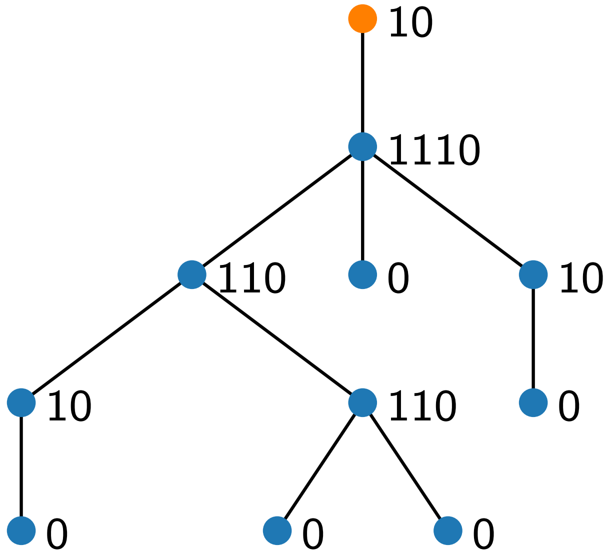




Data Structures – Informal Definition

A **data structure** is a concept to
- **store**,
- **organize**, and
- **manage** data.

As such, it is a collection of
- **data values**,
- their **relations**, and
- the **operations** that be can applied to the data.

$\Rightarrow$

- What do we represent?
- How much space is required?
- Dynamic or static?
- Which operations are defined?
- How fast are they?

Remarks.
- We look at data structures as a designer/implementer (and not necessarily as a user).
- To define a data structure and to implement it are two different tasks.

Succinct Data Structures

Goal.

- Use space "close" to information-theoretical minimum,
- but still support time-efficient operations.

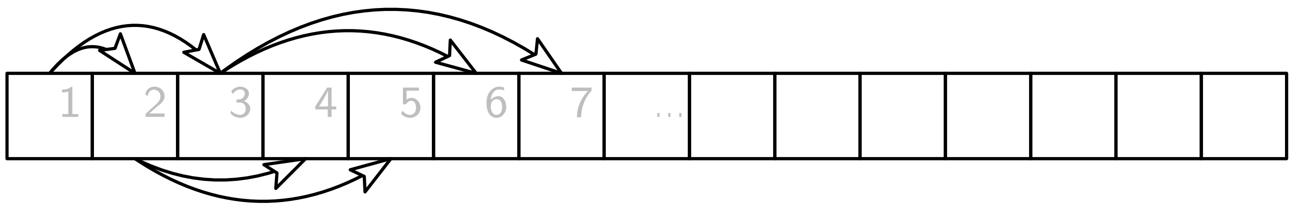
Let L be the information-theoretical lower bound to represent a class of objects.
Then a data structure, which still supports *time-efficient* operations, is called

- **implicit**, if it takes $L + O(1)$ bits of space;
- **succinct**, if it takes $L + o(L)$ bits of space;
- **compact**, if it takes $O(L)$ bits of space.

Examples?

Examples for Implicit Data Structures

- **arrays** to represent lists
  - but why not linked lists?
- **1-dim arrays** to represent multi-dimensional arrays
- **sorted arrays** to represent sorted lists
  - but why not binary search trees?
- **arrays** to represent complete binary trees and heaps



$\texttt{leftChild}(i) = 2i$

$\texttt{rightChild}(i) = 2i + 1$

$\texttt{parent}(i) = \lfloor \frac{i}{2} \rfloor$

And unbalanced trees?

Succinct Indexable Dictionary

Represent a subset $S \subseteq \{1, 2, \ldots, n\}$ and support the following operations in $O(1)$ time:

- member(i) returns if $i \in S$
- rank(i) = number of elements in S that are less or equal to i
- select(j) = j-th element in S
- predecessor(i)
- successor(i)

How many different subsets of $\{1, 2, \ldots, n\}$ are there? 2^n

How many bits of space do we need to distinguish them?

$\log 2^n = n$ **bits**

our logarithms are all to basis 2, i.e., $\log_2$

Succinct Indexable Dictionary

Represent S with a bit vector b of length n where

$$b[i] = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

plus $o(n)$-space structures to answer in $O(1)$ time

- $\texttt{rank}(i) = \#$ 1s at or before position i
- $\texttt{select}(j) =$ position of j-th 1 bit

$\Rightarrow$ Exercise: Use these methods to answer $\texttt{predecessor}(i)$ and $\texttt{successor}(i)$ in $O(1)$ time.

$S = \{3, 4, 6, 8, 9, 14\}$ where $n = 15$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
b	0	0	1	1	0	1	0	1	1	0	0	0	0	1	0

$\texttt{select}(5) = 9$

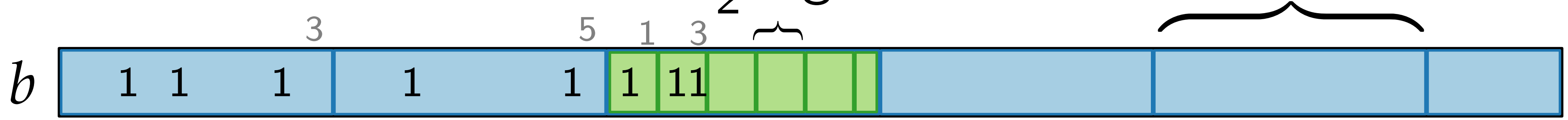
$\texttt{rank}(9) = 5 = \texttt{rank}(12)$

$\texttt{rank}(15) = 6$

$\texttt{member}(i)$ can trivially be answered in $O(1)$ time
(assuming that we can access any entry in constant time)

Rank in $o(n)$ Bits + $O(1)$ Time

$\log^2 n = (\log n)^2$

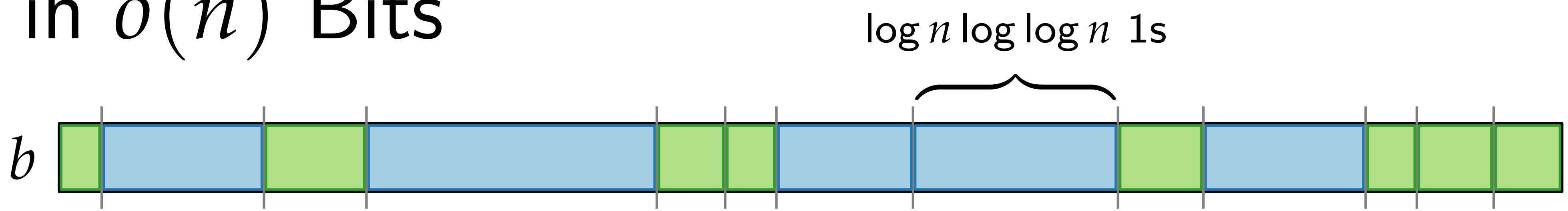


1. Split into $(\log^2 n)$-bit **chunks**
   and store cumulative rank: each needs $\leq \log n$ bits
   $$\Rightarrow O\left(\frac{n}{\log^2 n}\log n\right) = O\left(\frac{n}{\log n}\right) \subseteq o(n) \text{ bits}$$
2. Split **chunks** into $(\frac{1}{2}\log n)$-bit **subchunks**
   and store cumulative rank within **chunk**: each needs $\leq \log\log^2 n = 2\log\log n$ bits
   $$\Rightarrow O\left(\frac{n}{\log n}\log\log n\right) \subseteq o(n) \text{ bits}$$
3. Use **lookup table** for bitstrings of length $(\frac{1}{2}\log n)$: $2^{\frac{1}{2}\log n} = \sqrt{n}$ entries
   $$\Rightarrow O(\sqrt{n}\log n\log\log n) \subseteq o(n) \text{ bits}$$
4. $\texttt{rank}(i) = \texttt{rank}$ of **chunk**
   $+$ relative $\texttt{rank}$ of **subchunk** within **chunk**
   $+$ relative $\texttt{rank}$ of element i within **subchunk**

$\Rightarrow O(1)$ time

(assume read & write in $O(1)$ time)

Select in $o(n)$ Bits



1. Store indices of every $(\log n \log\log n)$-th 1 bit in array

$$\Rightarrow O\left(\frac{n}{\log n \log\log n} \log n\right) = O\left(\frac{n}{\log\log n}\right) \subseteq o(n) \text{ bits}$$

2. Within group of $(\log n \log\log n)$ 1 bits of length r bits:

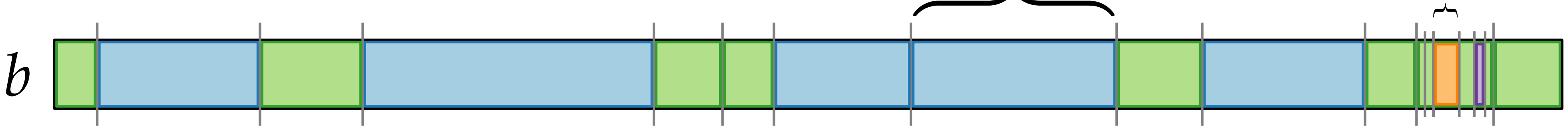
   if $r \geq (\log n \log\log n)^2$
   then store indices of 1 bits in group in array

$$\Rightarrow O\left(\frac{n}{(\log n \log\log n)^2} (\log n \log\log n) \log n\right) \subseteq O\left(\frac{n}{\log\log n}\right) \text{ bits}$$

   else problem is reduced to bitstrings of length $r < (\log n \log\log n)^2$

3. Repeat 1. and 2. on reduced bitstrings

Select in $o(n)$ Bits+ $O(1)$ Time



3. Repeat 1. and 2. on reduced bitstrings ($r < (\log n \log\log n)^2$):

   1' Store relative indices of every $(\log\log n)^2$-th 1 bit in array

   $$\Rightarrow O(\frac{n}{(\log\log n)^2} \log\log n) = O(\frac{n}{\log\log n}) \text{ bits}$$

   2' Within group of $(\log\log n)^2$ 1 bits of length r' bits:

   if $r' \geq (\log\log n)^4$
   then store relative indices of 1 bits in subgroup in array

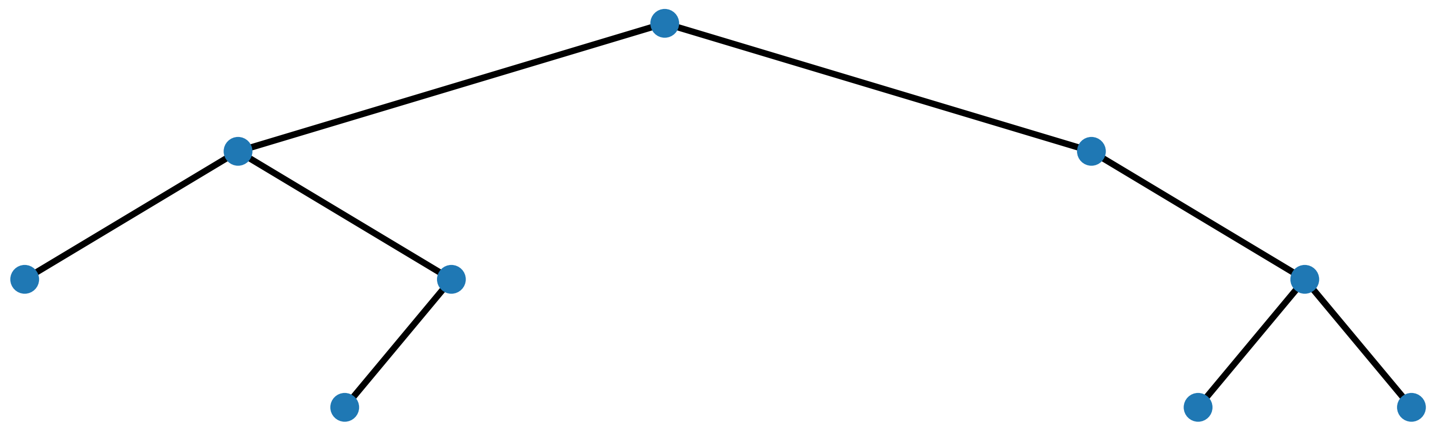
   $$\Rightarrow O(\frac{n}{(\log\log n)^4}(\log\log n)^2 \log\log n) = O(\frac{n}{\log\log n}) \text{ bits}$$

   else problem is reduced to bitstrings of length $r' < (\log\log n)^4$

4. Use lookup table for bitstrings of length $r' \leq (\log\log n)^4 \leq \frac{1}{2}\log n$

   $$\Rightarrow O(\underbrace{\sqrt{n}}_{\#\text{ bitstrings}}\ \underbrace{\log n}_{\text{query } j}\ \underbrace{\log\log n}_{\text{answer}}) = o(n) \text{ bits}$$

Succinct Representation of Binary Trees



C_n is the n-th *Catalan number* and $C_0 = 1$

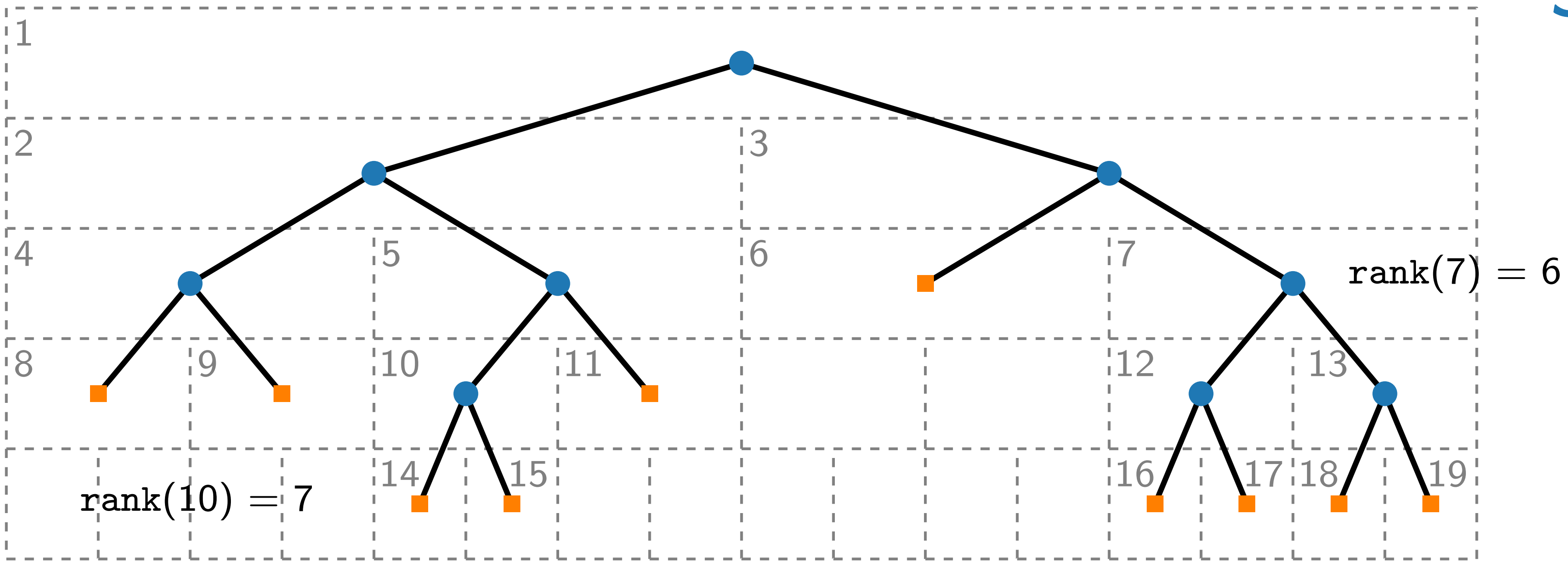
Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} = \frac{(2n)!}{(n+1)!\,n!}$

$$\log C_n = 2n + o(n) \text{ (by Stirling's approximation)}$$

$\Rightarrow$ We can use $2n + o(n)$ bits to represent binary trees.

Difficulty is when a binary tree is not full.

Succinct Representation of Binary Trees



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.

- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

Proof is exercise.

Idea.

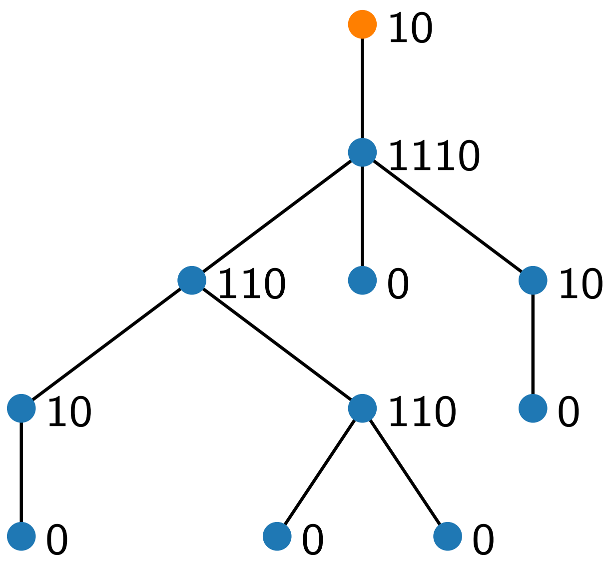
- Add external nodes to have out-degree 2
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.

- $\texttt{parent}(i) = \texttt{select}(\lfloor \frac{i}{2} \rfloor)$
- $\texttt{leftChild}(i) = 2\,\texttt{rank}(i)$
- $\texttt{rightChild}(i) = 2\,\texttt{rank}(i) + 1$
- $\texttt{rank}(i)$ is index for extra storing array

Succinct Representation of Trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



- unary decoding of outdegree
- gives LOUDS sequence

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

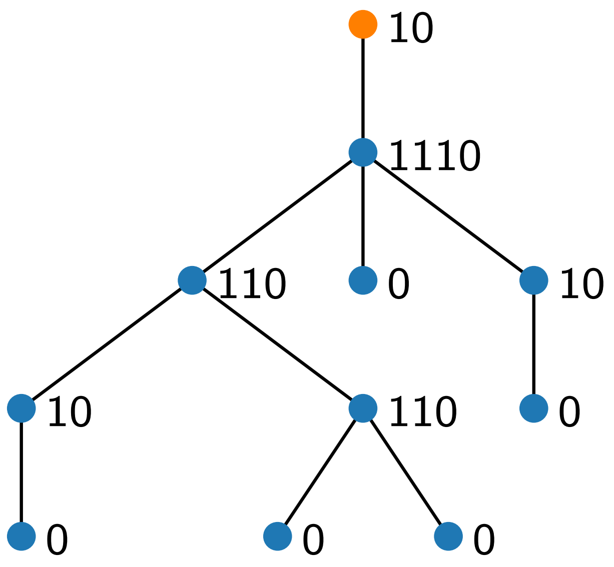
Size.

- each vertex (except root) is represented twice, namely with a 1 and with a 0
- $o(n)$ bits for `rank` and `select`

$\Rightarrow 2n + o(n)$ bits

Succinct Representation of Trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



- unary decoding of outdegree
- gives LOUDS sequence

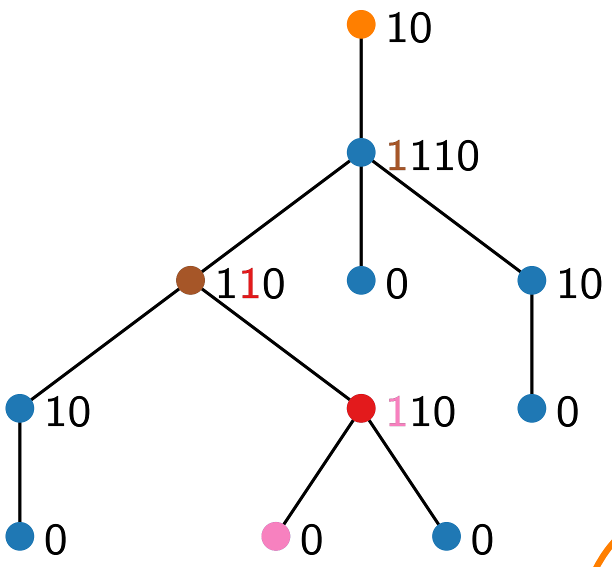
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

Operations.

- Let i be index of 1 in LOUDS sequence.
- $\texttt{rank}(i)$ is index for array storing vertex objects/values.

Succinct Representation of Trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



- unary decoding of outdegree
- gives LOUDS sequence

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

1	0	1	1	1	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0

execute select(j) on the 0s instead of the 1s

execute rank(i) on the 1s (as before)

- $\texttt{firstChild}(i) = \texttt{select}_0(\texttt{rank}_1(i)) + 1$

  $\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$
  $= \texttt{select}_0(6) + 1 = 14 + 1 = 15$

- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$
  $= \texttt{select}_1(2) = 3$

- $\texttt{nextSibling}(i) = i + 1$

Exercise: $\texttt{child}(i, j)$
with validity check

Discussion

- Succinct data structures are
  - space efficient
  - support fast operations

  but
  - are mostly static (dynamic at extra cost),
  - number of operations is limited,
  - complex $\rightarrow$ harder to implement

- Rank and select form basis for many succinct representations

Literature

Main reference:

- Lecture 17 of Advanced Data Structures (MIT, Fall'17) by Erik Demaine
- [Jac '89] "Space efficient Static Trees and Graphs"

Recommendations:

- Lecture 18 of Demaine's course on compact & succinct arrays & trees