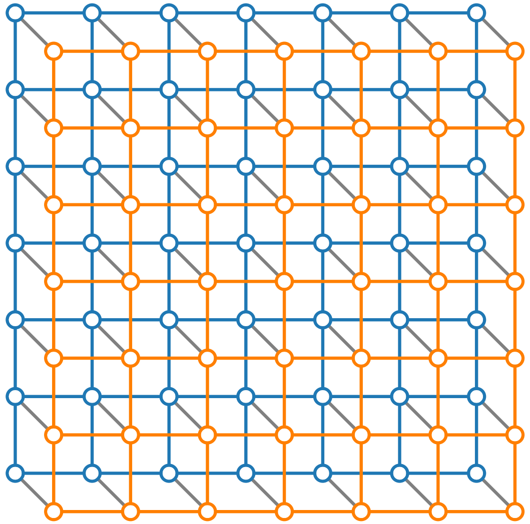
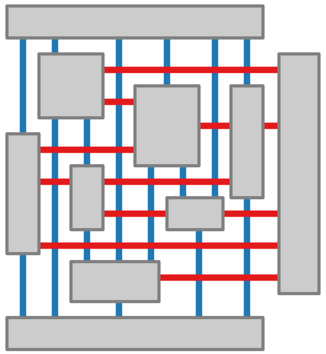
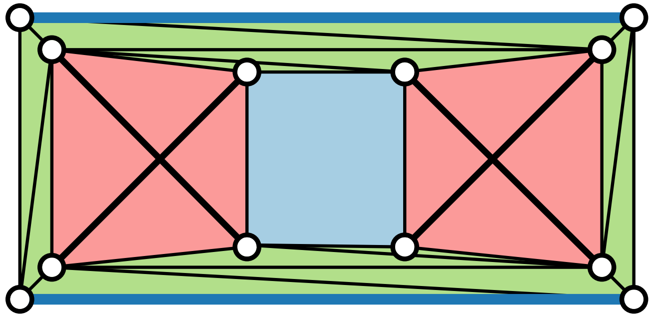
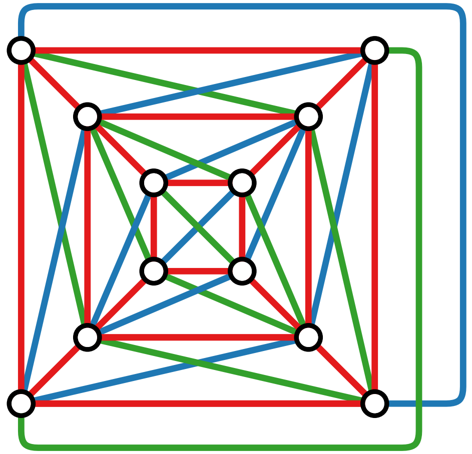
Visualization of Graphs





Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings





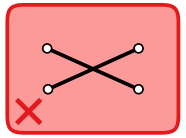
Part I:
Graph Classes and Drawing Styles

Alexander Wolff

Partially based on slides by Fabrizio Montecchini, Michalis Bekos, and Walter Didimo.

Planar Graphs

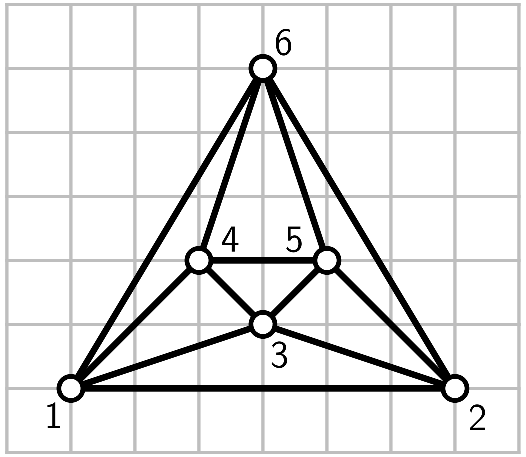
Planar graphs admit drawings in the plane without crossings.



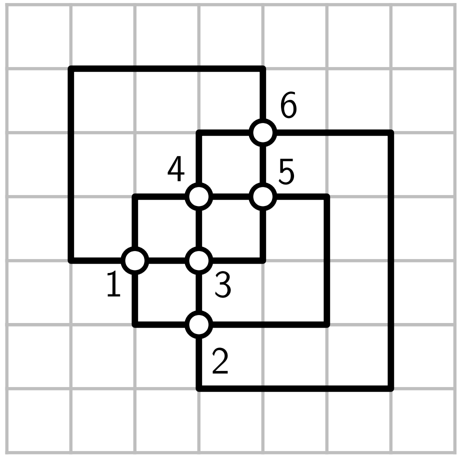
Plane graph is a planar graph with a plane embedding = rotation system.

Planarity is recognizable in linear time.

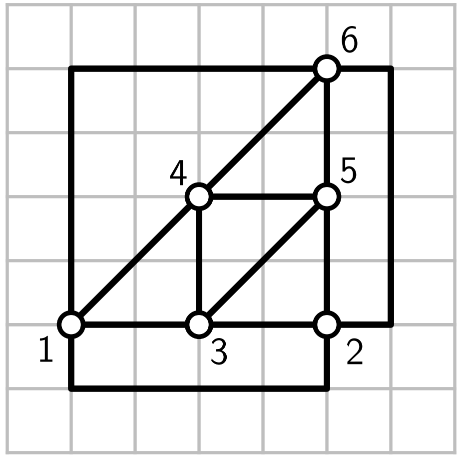
Different drawing styles...



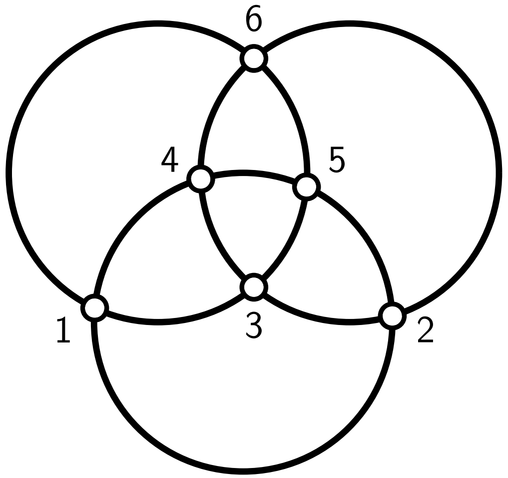
straight-line drawing



orthogonal drawing



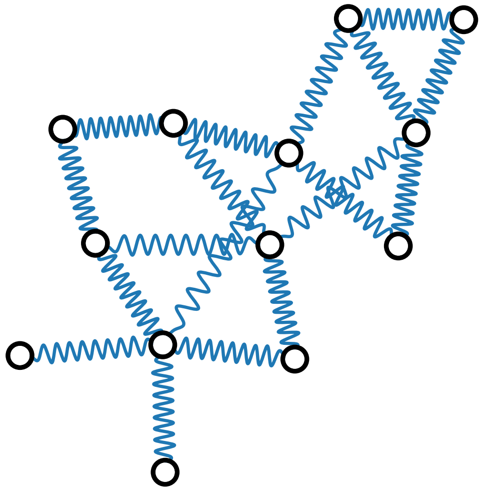
grid drawing with bends & 3 slopes



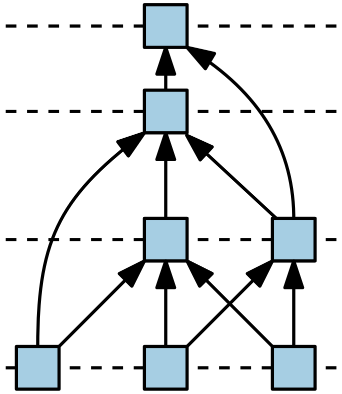
circular-arc drawing

And Non-Planar Graphs?

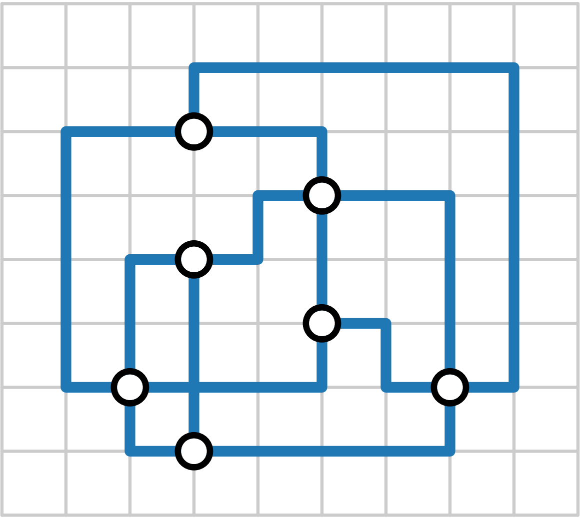
We have seen a few drawing styles:



force-directed drawing

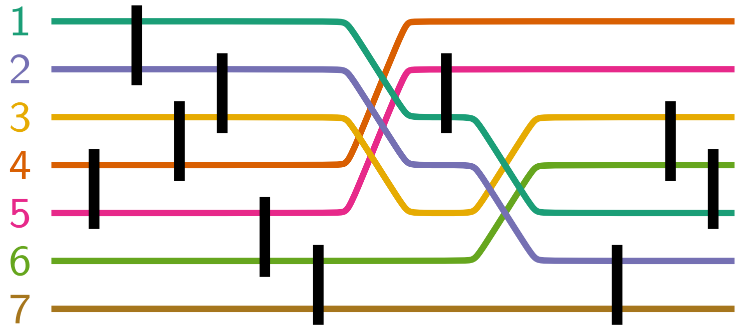


hierarchical drawing

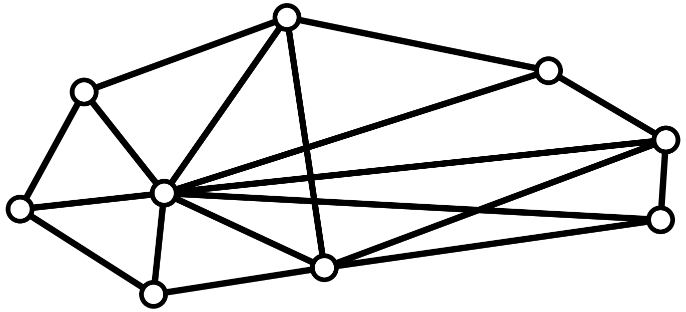


orthogonal layouts
(via planarization)

Maybe not all crossings are equally bad?



block crossings



Which crossings feel worse?

Eye-Tracking Experiment

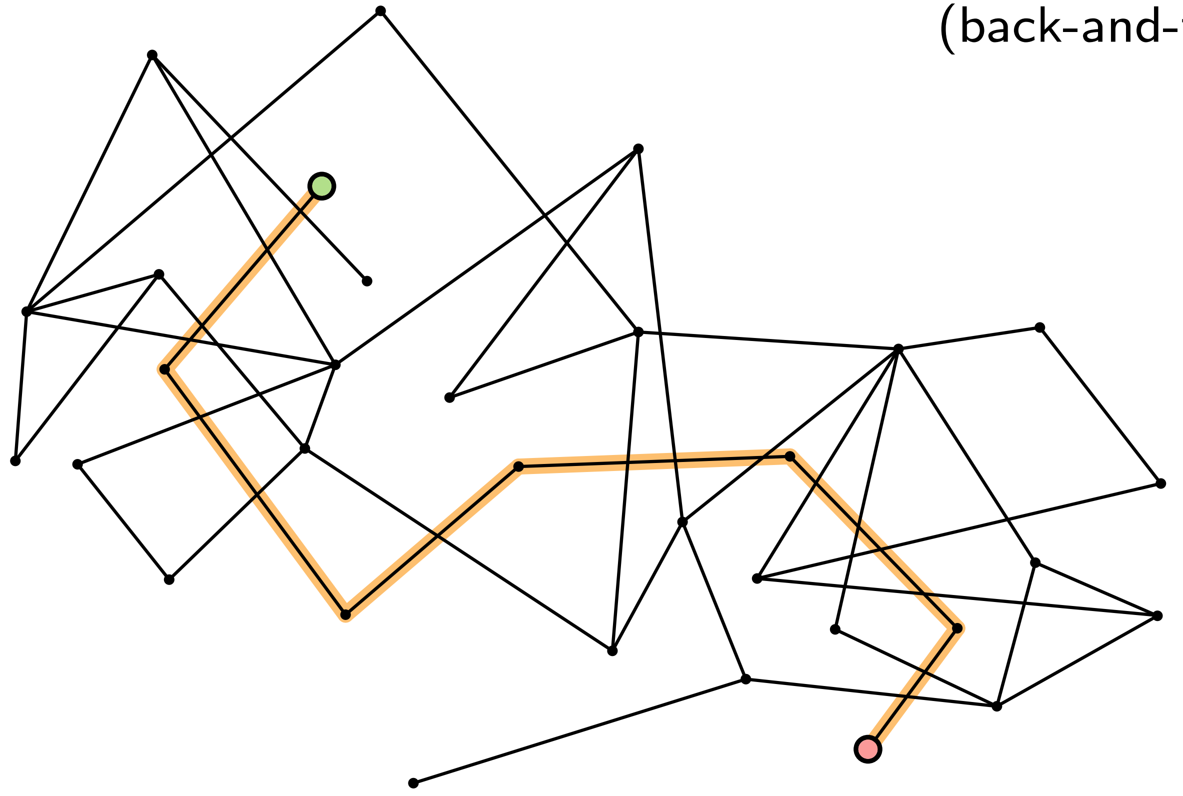
[Eades, Hong & Huang 2008]

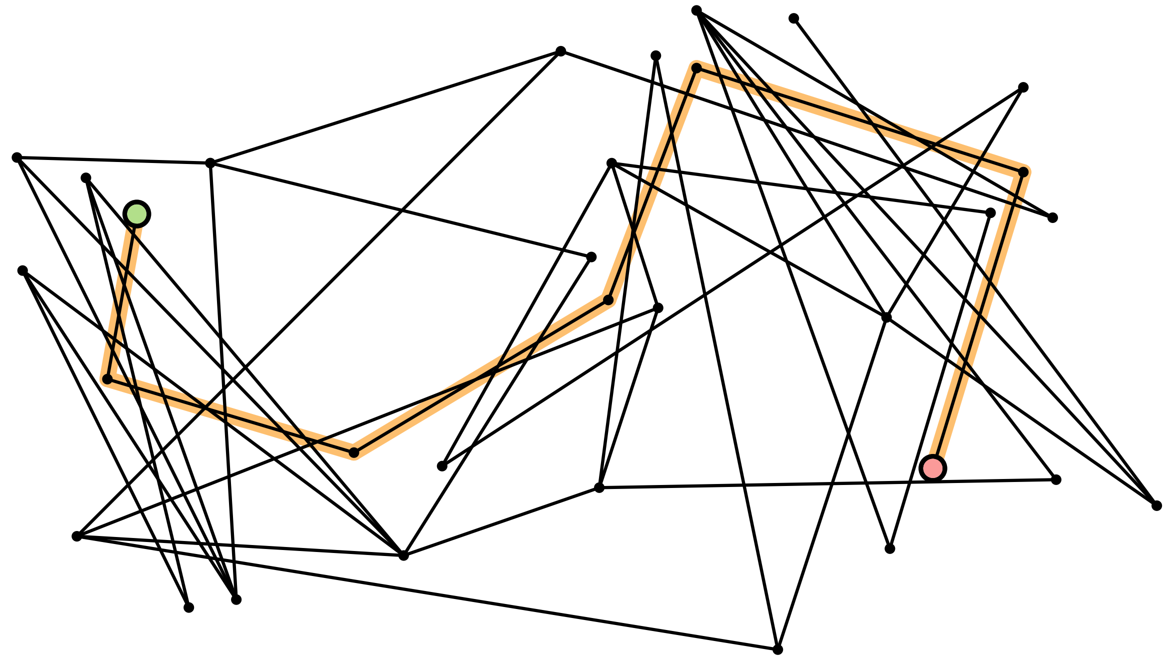
Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

Results:

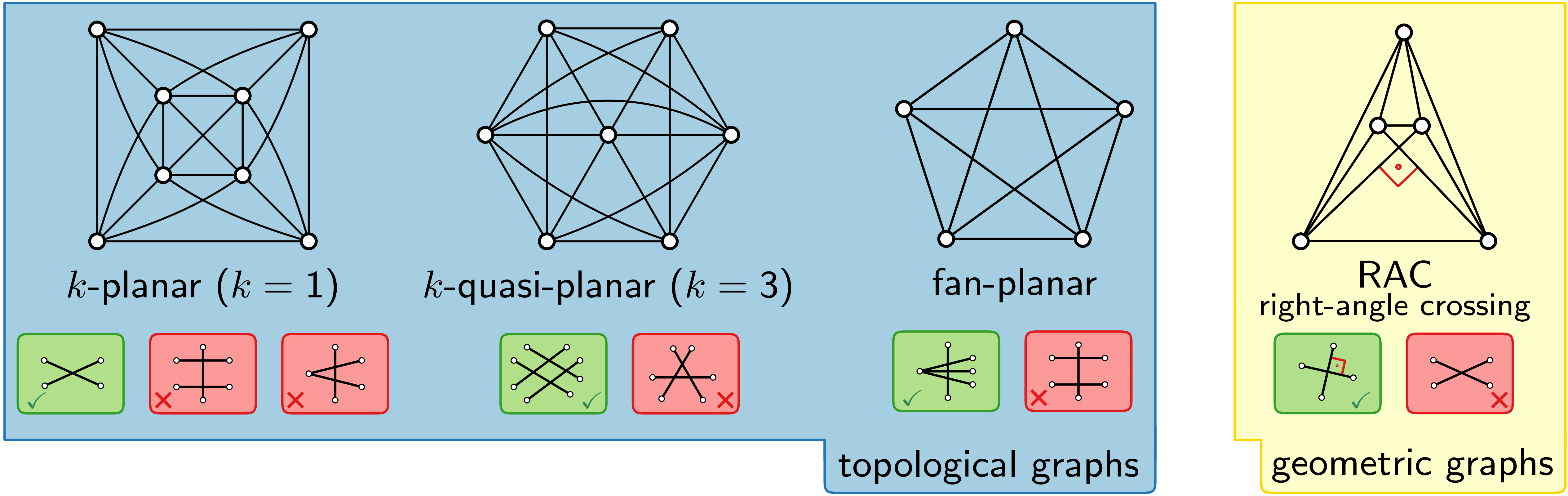
no crossings	eye movements smooth and fast
large crossing angles	eye movements smooth but slightly slower
small crossing angles	eye movements no longer smooth and very slow (back-and-forth movements at crossing points)





Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.



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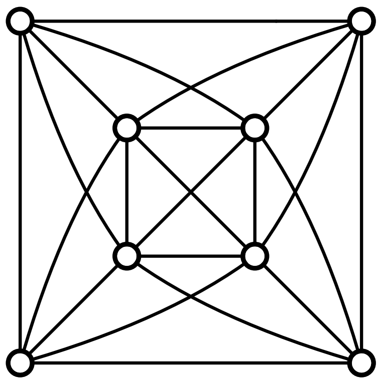
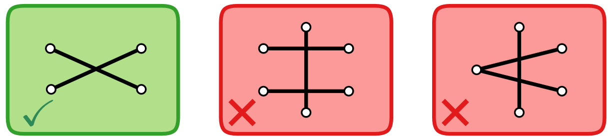
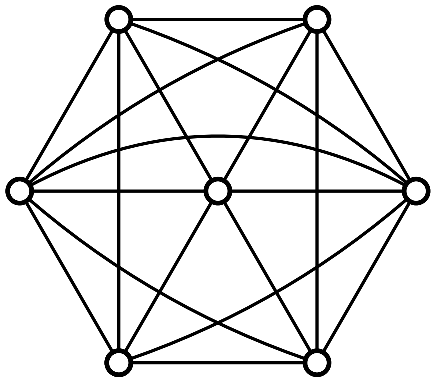
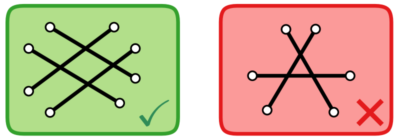
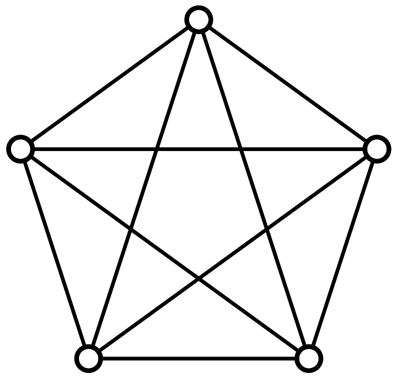
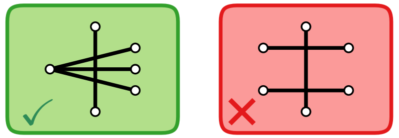



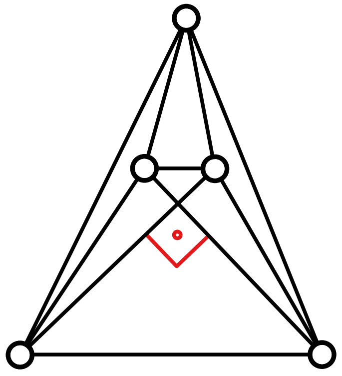
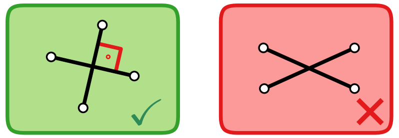
k-planar ($k = 1$)

k-quasi-planar ($k = 3$)

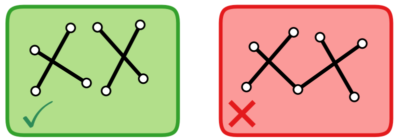



fan-planar

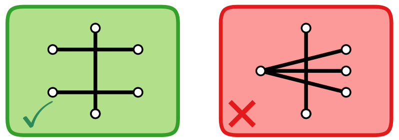



RAC
right-angle crossing

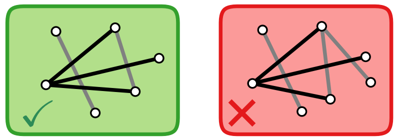
There are many more beyond-planar graph classes. . .



IC (independent crossing)



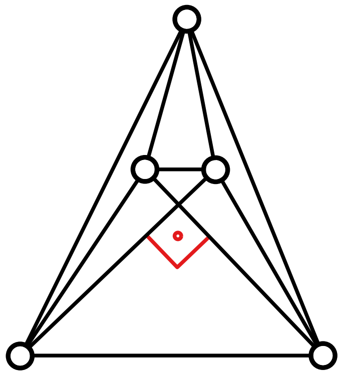
fan-crossing-free



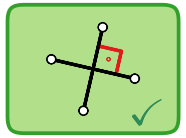
skewness-k ($k = 2$)

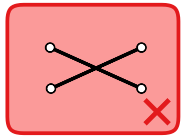
combinations, . . .

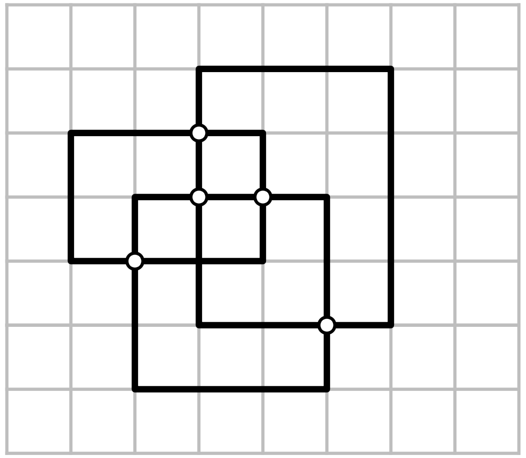
Drawing Styles for Crossings



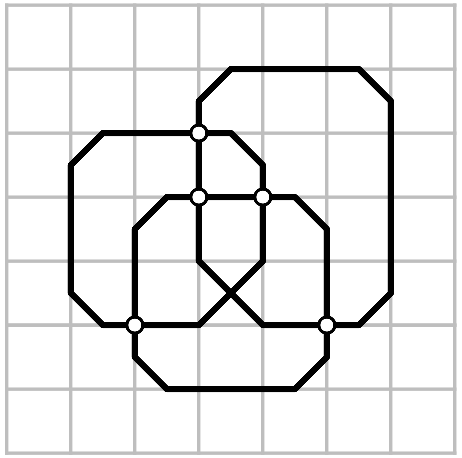
RAC
right-angle crossing



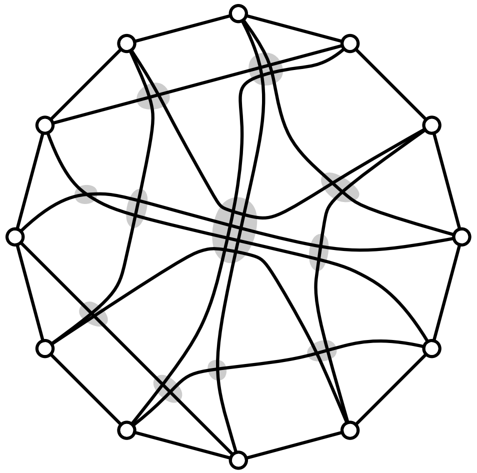




orthogonal

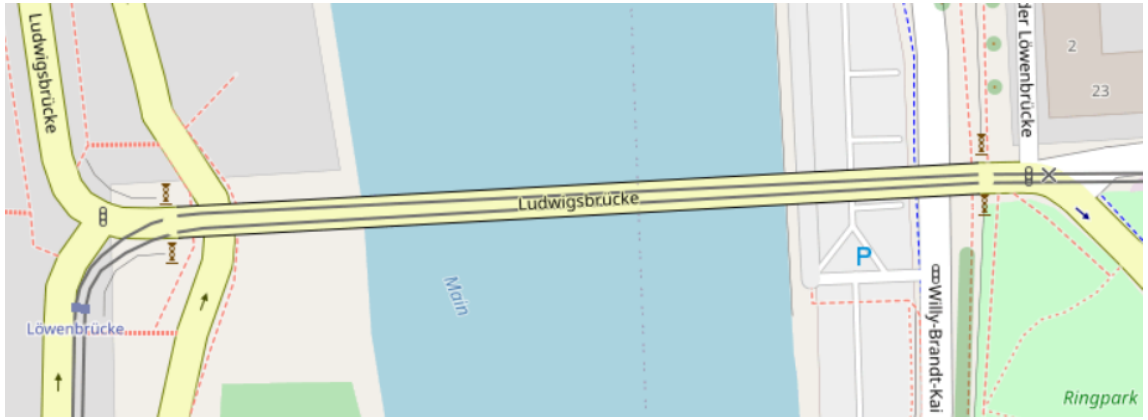


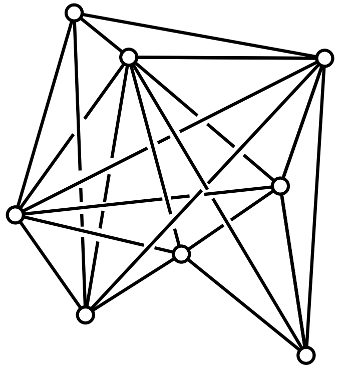
slanted orthogonal



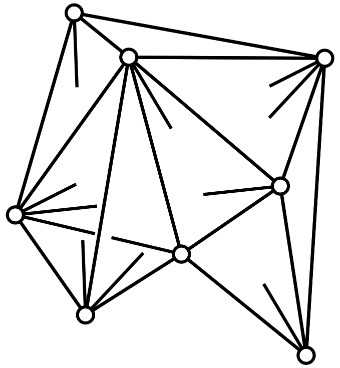
block / bundled crossings

circular layout: 28 invididual
vs. 12 bundle crossings

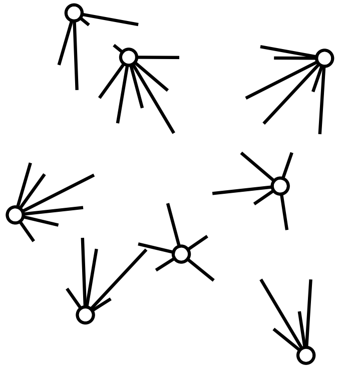




cased crossings

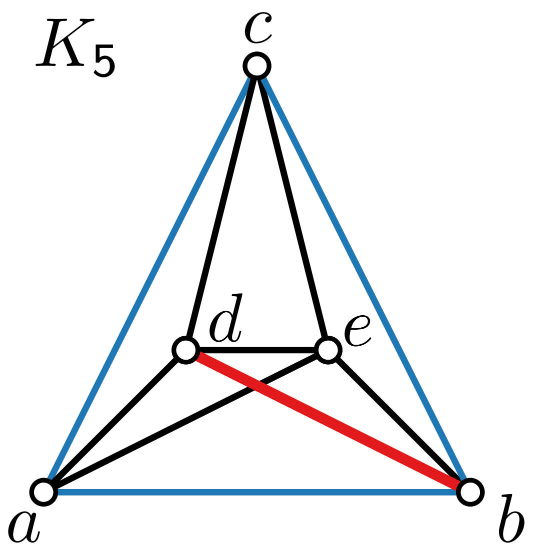


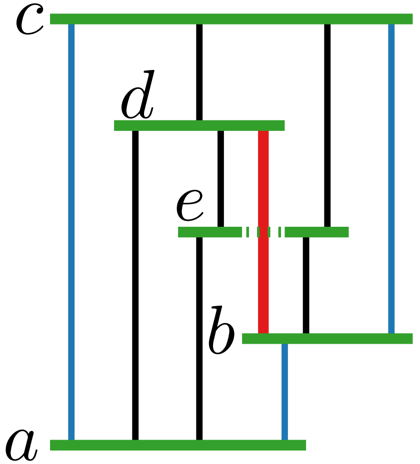
symmetric partial
edge drawing



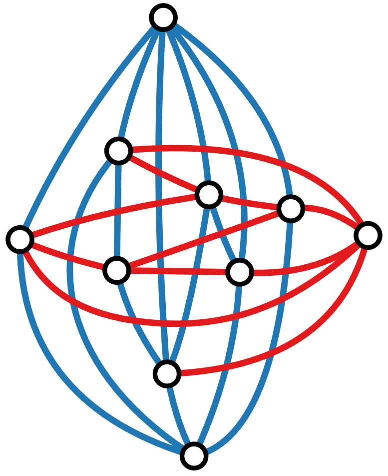
1/4-SHPED

Geometric Representations

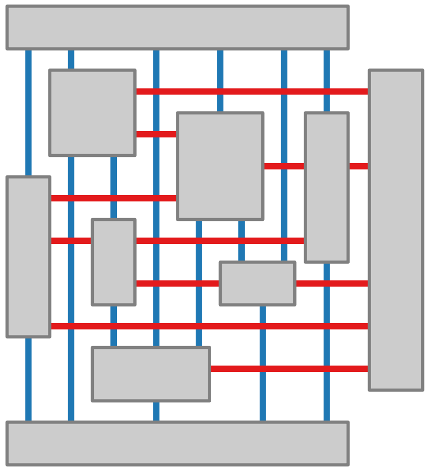




bar 1-visibility
representation (B1VR)



thickness-2
graph G



rectangle visibility
representation

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

- G has at most $6n - 20$ edges. [Bose et al. 1997]
- Recognition is NP-complete. [Shermer 1996]
- Recognition becomes polynomial if embedding is fixed. [Biedl et al. 2018]

Julius-Maximilians-
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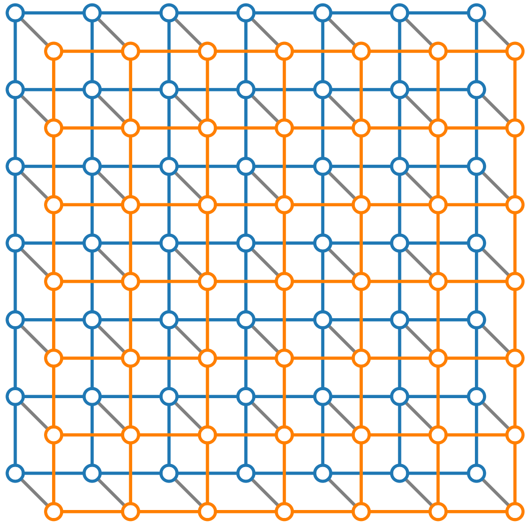
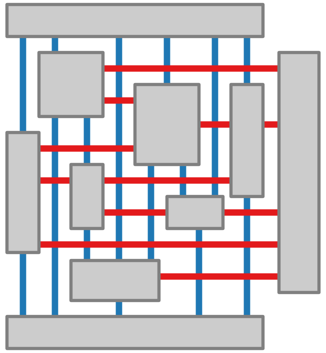
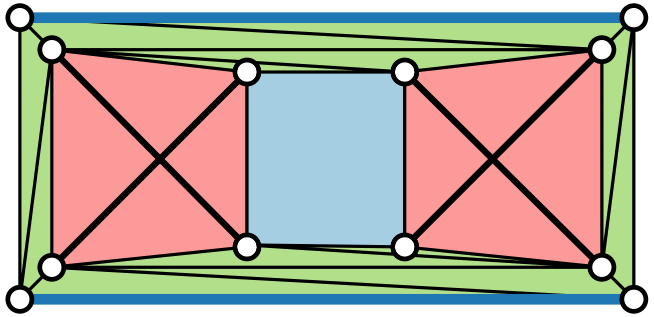
WÜRZBURG

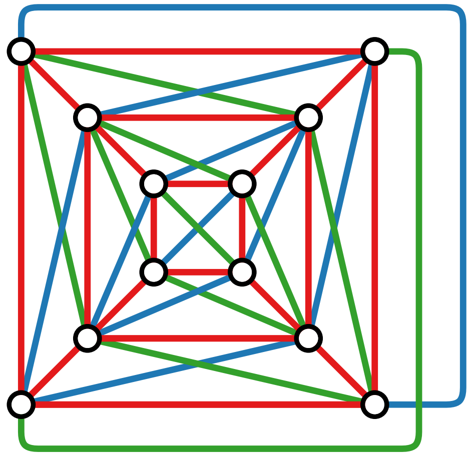
Visualization of Graphs





Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings

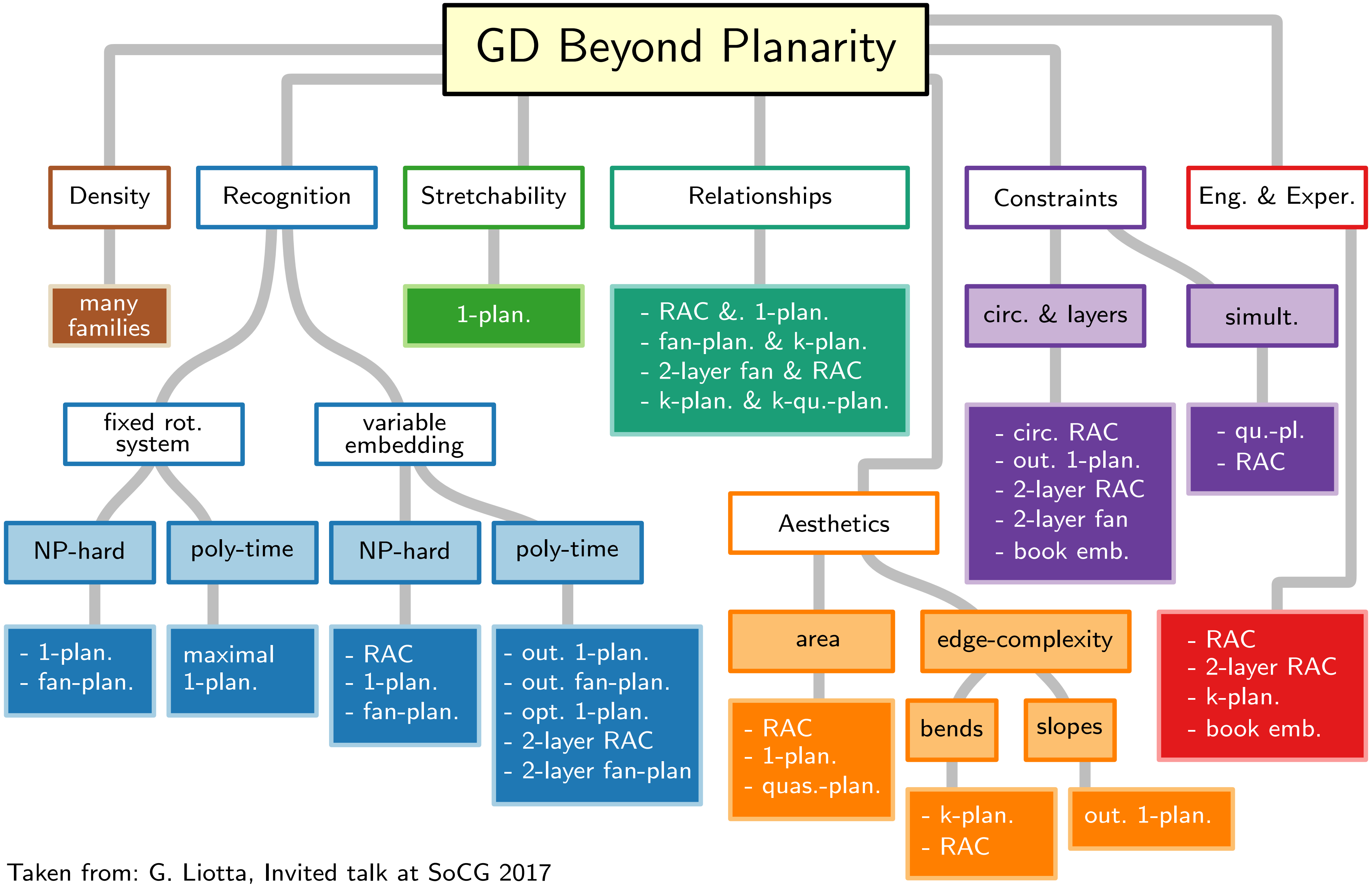




Part II:
Density & Relationships

Alexander Wolff

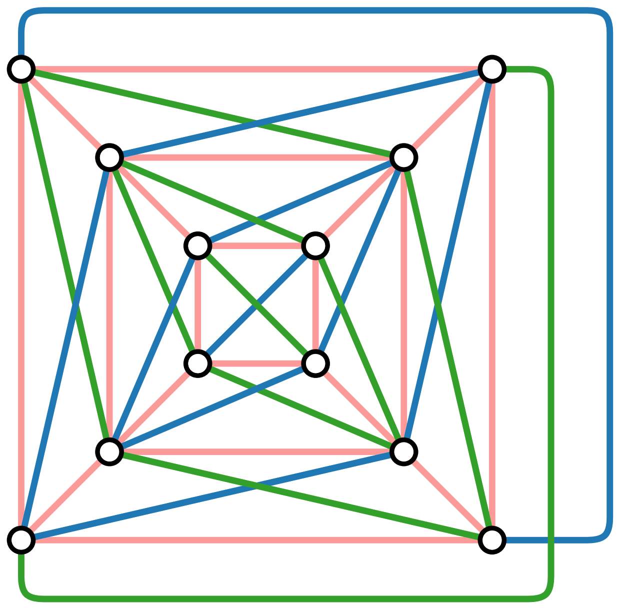
GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n-8$ edges, which is a tight bound.



Proof sketch.

- Let the red edges be those that do not cross.
- Each blue edge crosses a green edge.

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- Let the red edges be those that do not cross.
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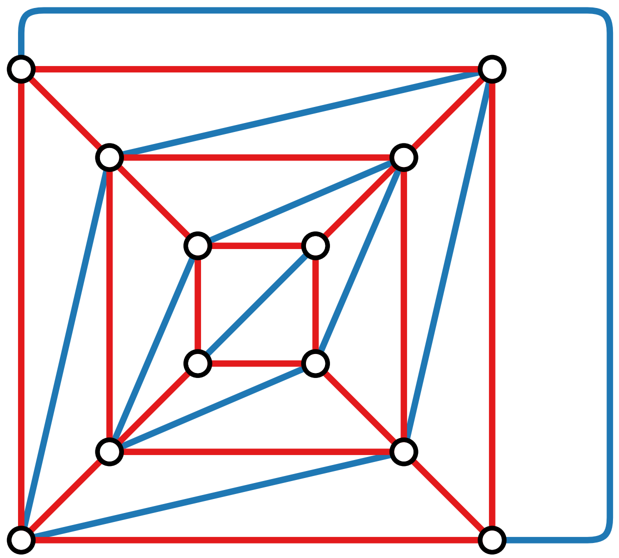
- This yields a red-blue plane graph G_{rb} with

  $m_{rb} \leq 3n - 6$



G_{rb}

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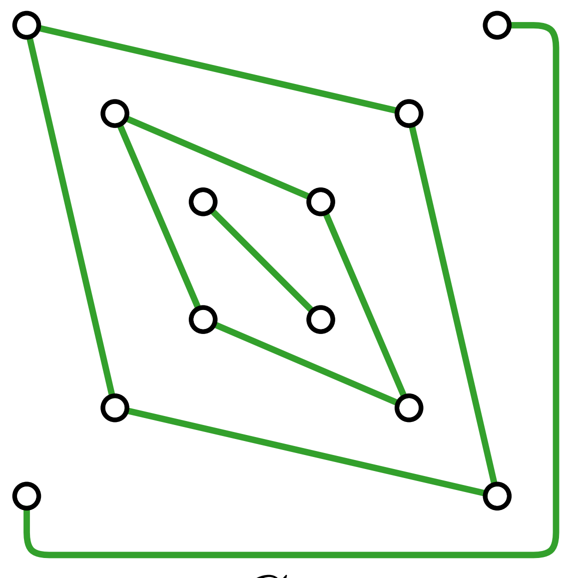

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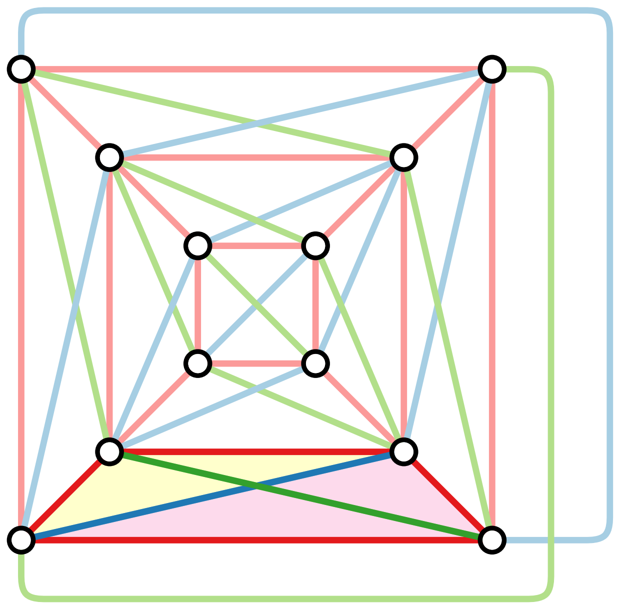

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- and a green plane graph G_g with

  $m_g \leq 3n - 6 \quad \Rightarrow \quad m \leq m_{rb} + m_g \leq 6n - 12$

Observe that each green edge joins two faces in G_{rb}.

$$m_g \leq f_{rb}/2 \leq (2n-4)/2 = n - 2$$

Density of 1-Planar Graphs

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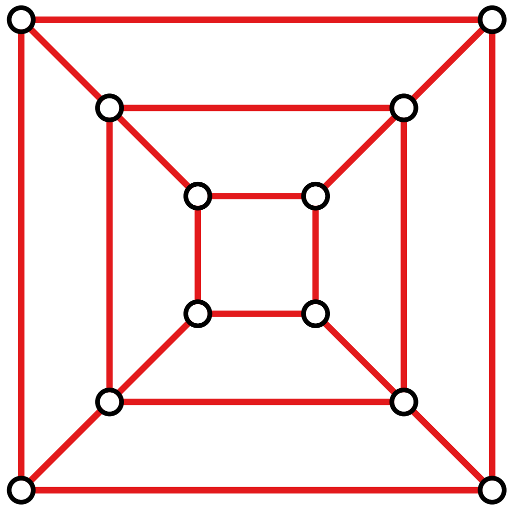
$$m_g \leq f_{rb}/2 \leq (2n-4)/2 = n - 2$$

$$\Rightarrow \quad m = m_{rb} + m_g \leq 3n - 6 + n - 2 = 4n - 8$$



Planar structure:
$2n - 4$ edges
$n - 2$ faces

Density of 1-Planar Graphs

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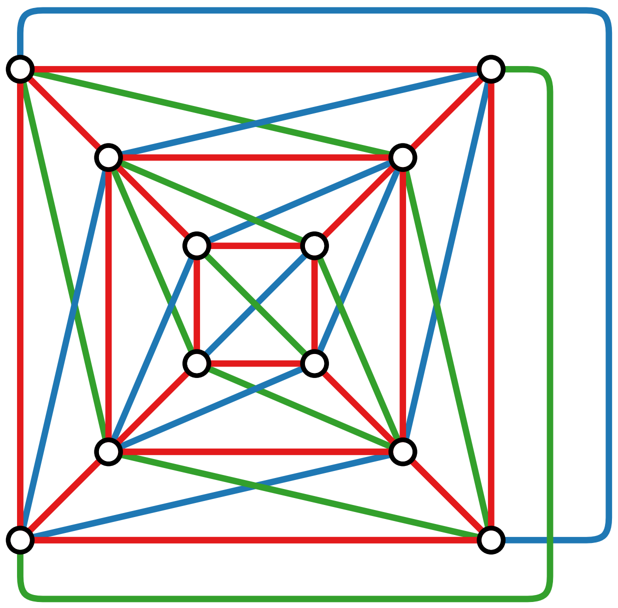

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$$m_g \leq f_{rb}/2 \leq (2n-4)/2 = n - 2$$

$$\Rightarrow \quad m = m_{rb} + m_g \leq 3n - 6 + n - 2 = 4n - 8$$

Planar structure: $2n - 4$ edges

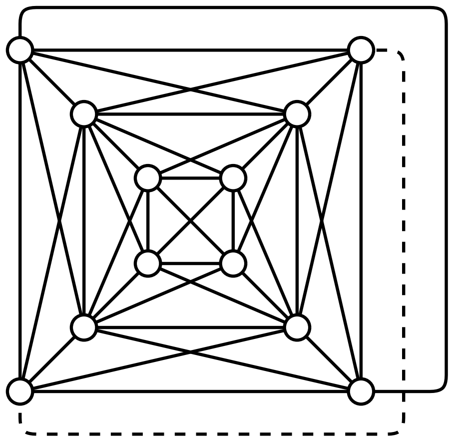
$n - 2$ faces

Edges per face: 2 edges

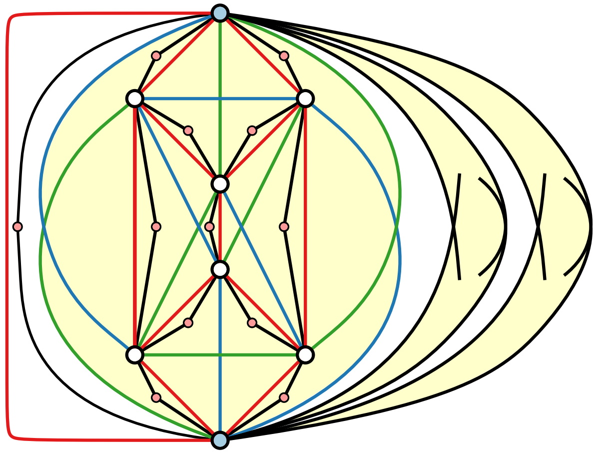
Total: $4n - 8$ edges

Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n-8$ edges, which is a tight bound.



A 1-planar graph with n vertices is called **optimal** if it has exactly $4n-8$ edges.
A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.



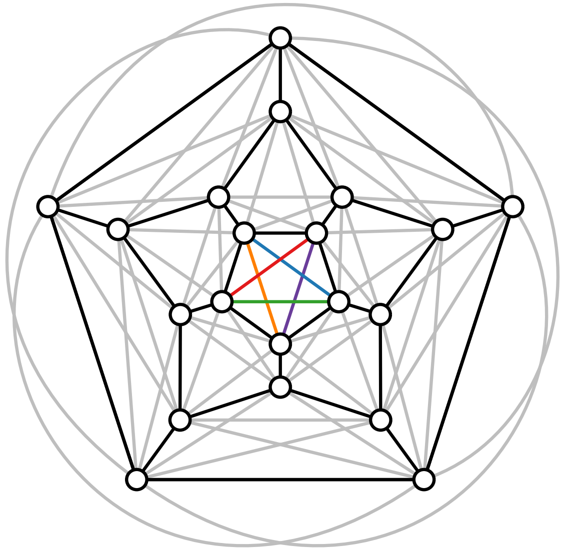
Theorem. [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with n vertices and $\underbrace{45/17n}_{\approx 2.65n} - O(1)$ edges.

Theorem. [Didimo 2013]
A 1-planar graph with n vertices that admits a **straight-line drawing** has at most $4n-9$ edges.

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2		[Pach and Tóth 1997]



optimal 2-planar

Planar structure:

$n - m + f = 2$

$m = c \cdot f$?

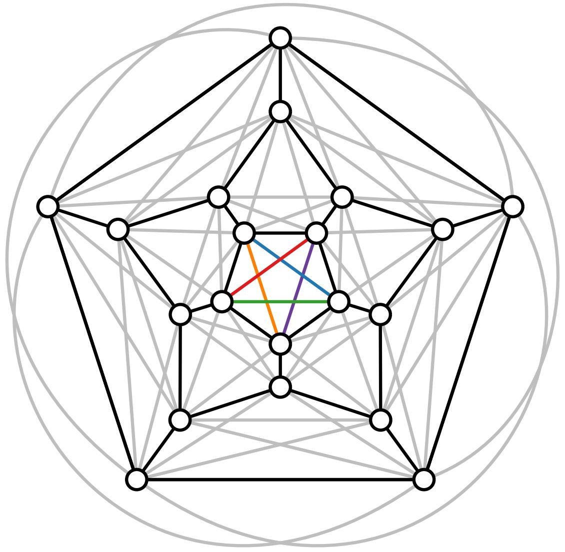
Edges per face:

Total:

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]



optimal 2-planar

$n - m + f = 2$
$m = c \cdot f$?

Planar structure:
$\frac{5}{3}(n-2)$ edges
$\frac{2}{3}(n-2)$ faces

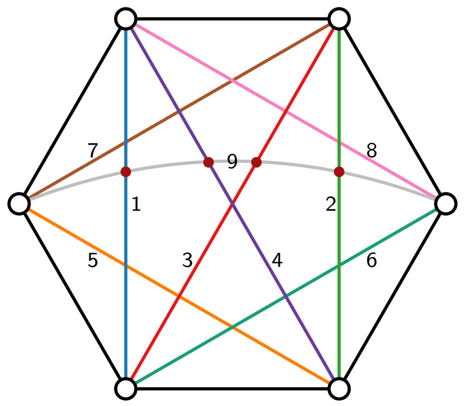
Edges per face: 5 edges

Total: $5(n-2)$ edges

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]



optimal 3-planar

Density of k-Planar Graphs

Theorem.
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0	$3(n-2)$	Euler's formula
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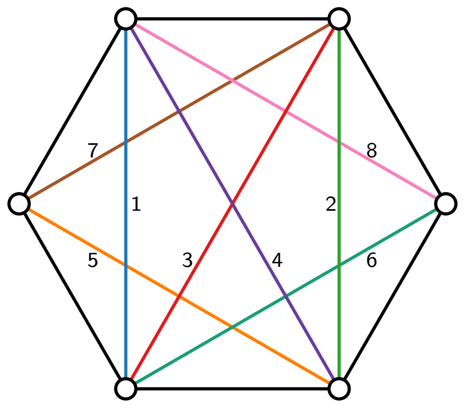

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Density of k-Planar Graphs

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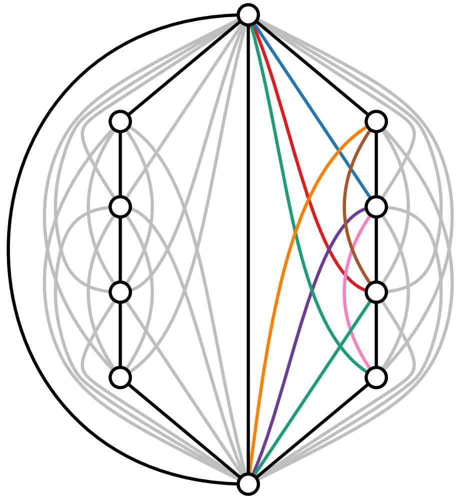


optimal 3-planar

Planar structure:
$\frac{3}{2}(n-2)$ edges
$\frac{1}{2}(n-2)$ faces

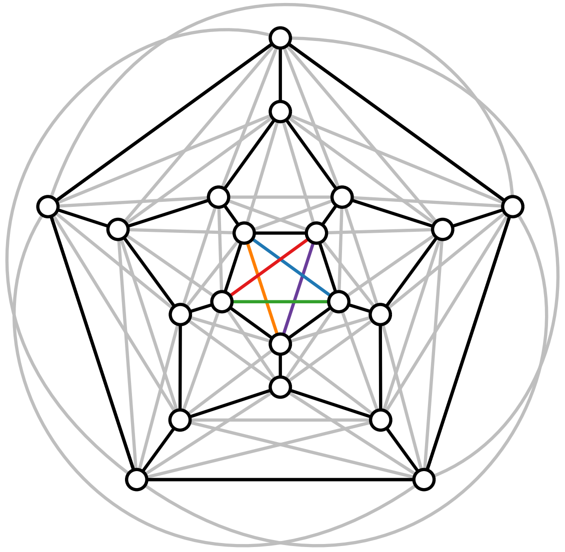
Edges per face: 8 edges

Total: $5.5(n-2)$ edges

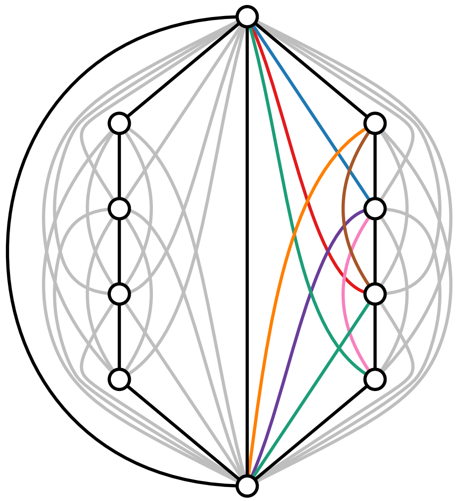
Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]
4	$6(n-2)$	[Ackerman 2015]
> 4	$4.108\sqrt{kn}$	[Pach and Tóth 1997]

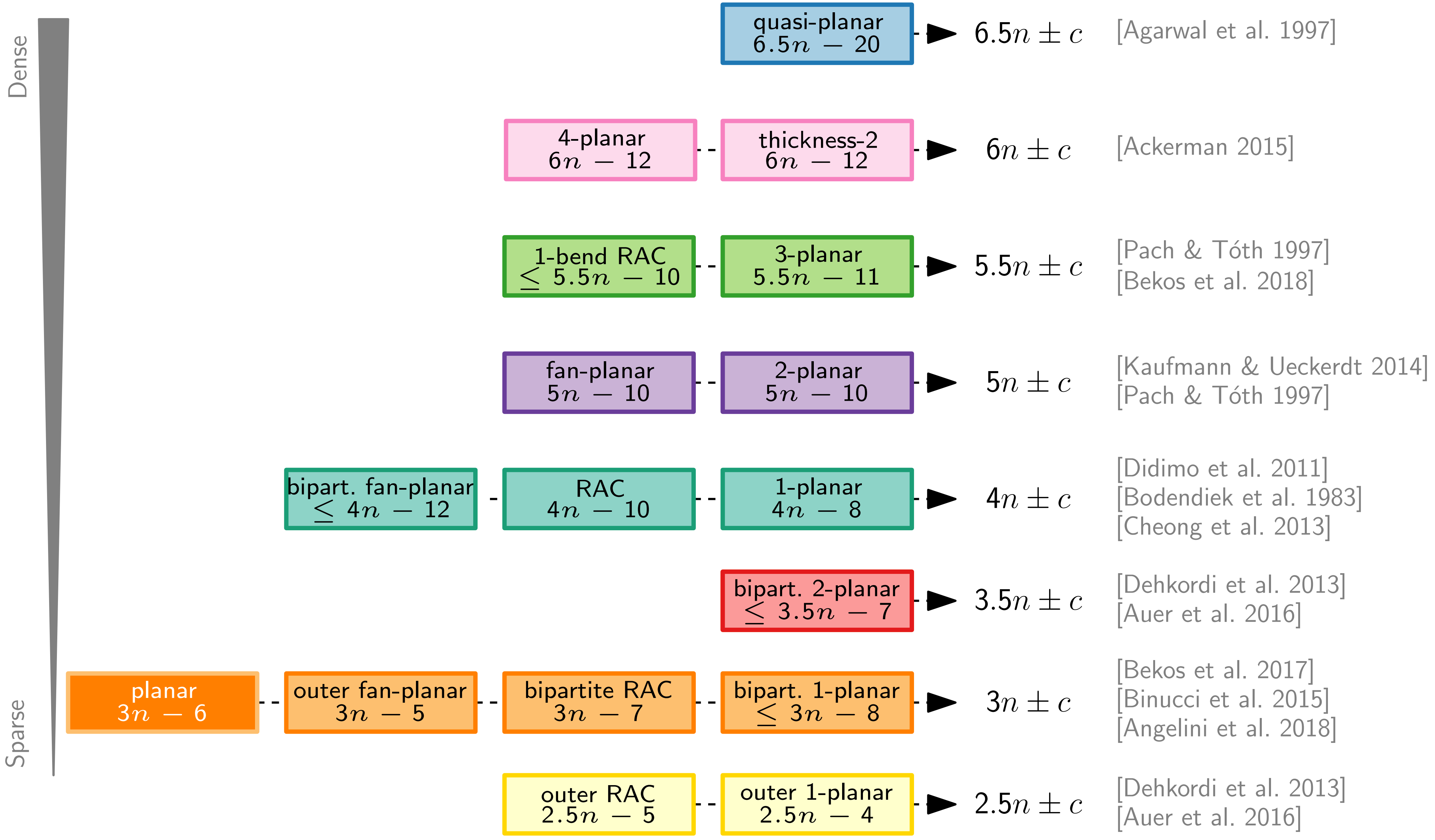


optimal 2-planar

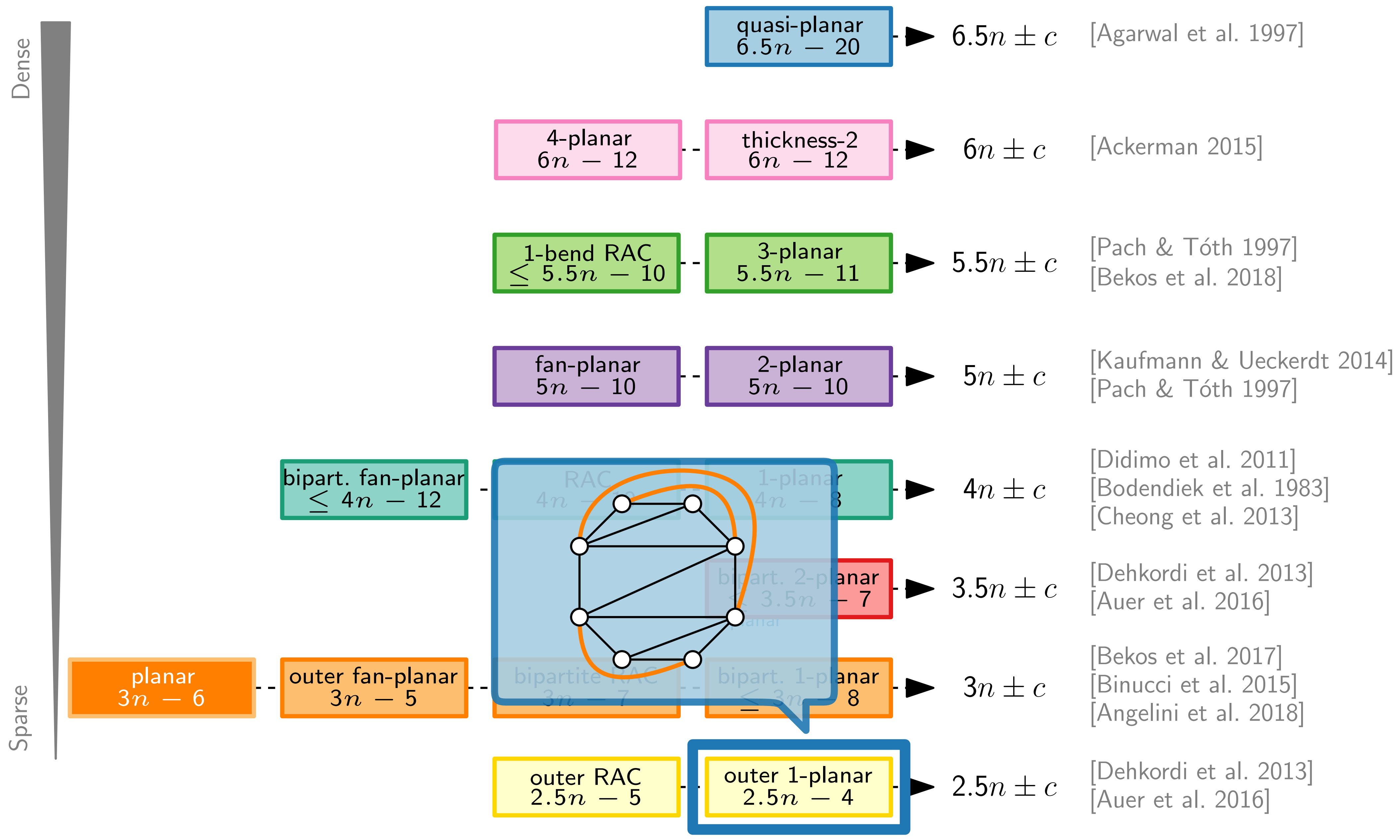


optimal 3-planar

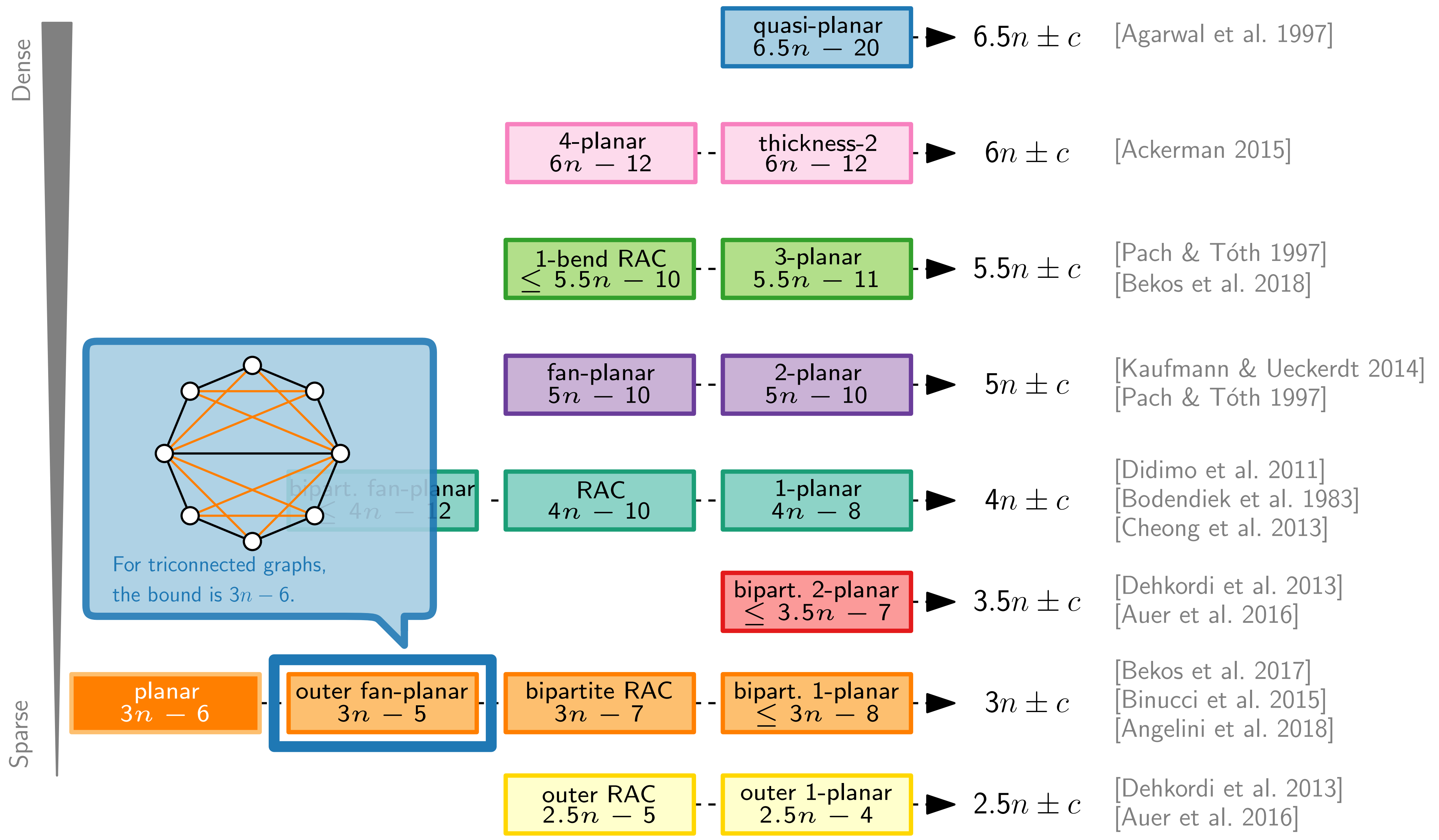
GD Beyond Planarity: a Hierarchy



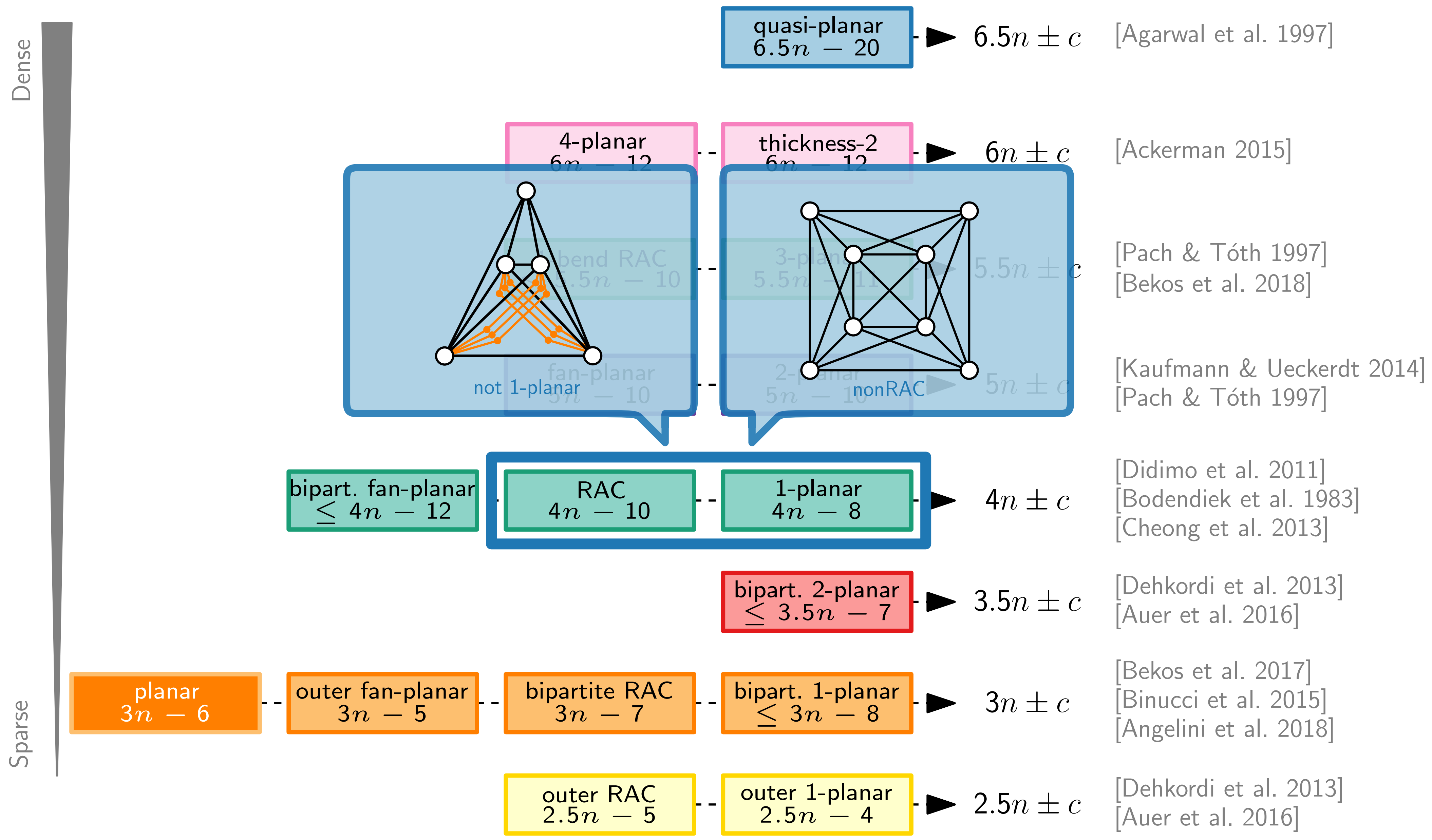
GD Beyond Planarity: a Hierarchy



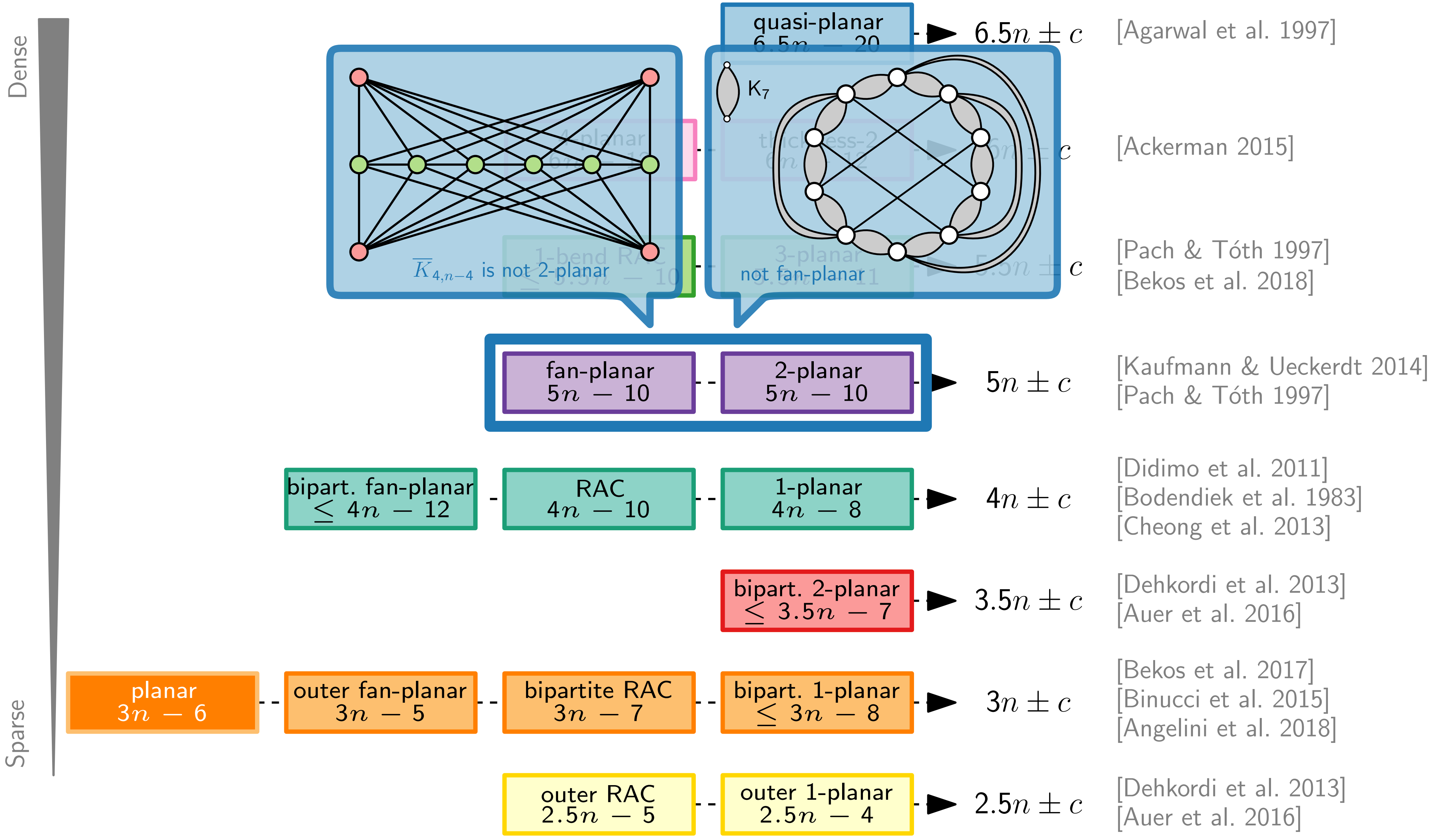
GD Beyond Planarity: a Hierarchy



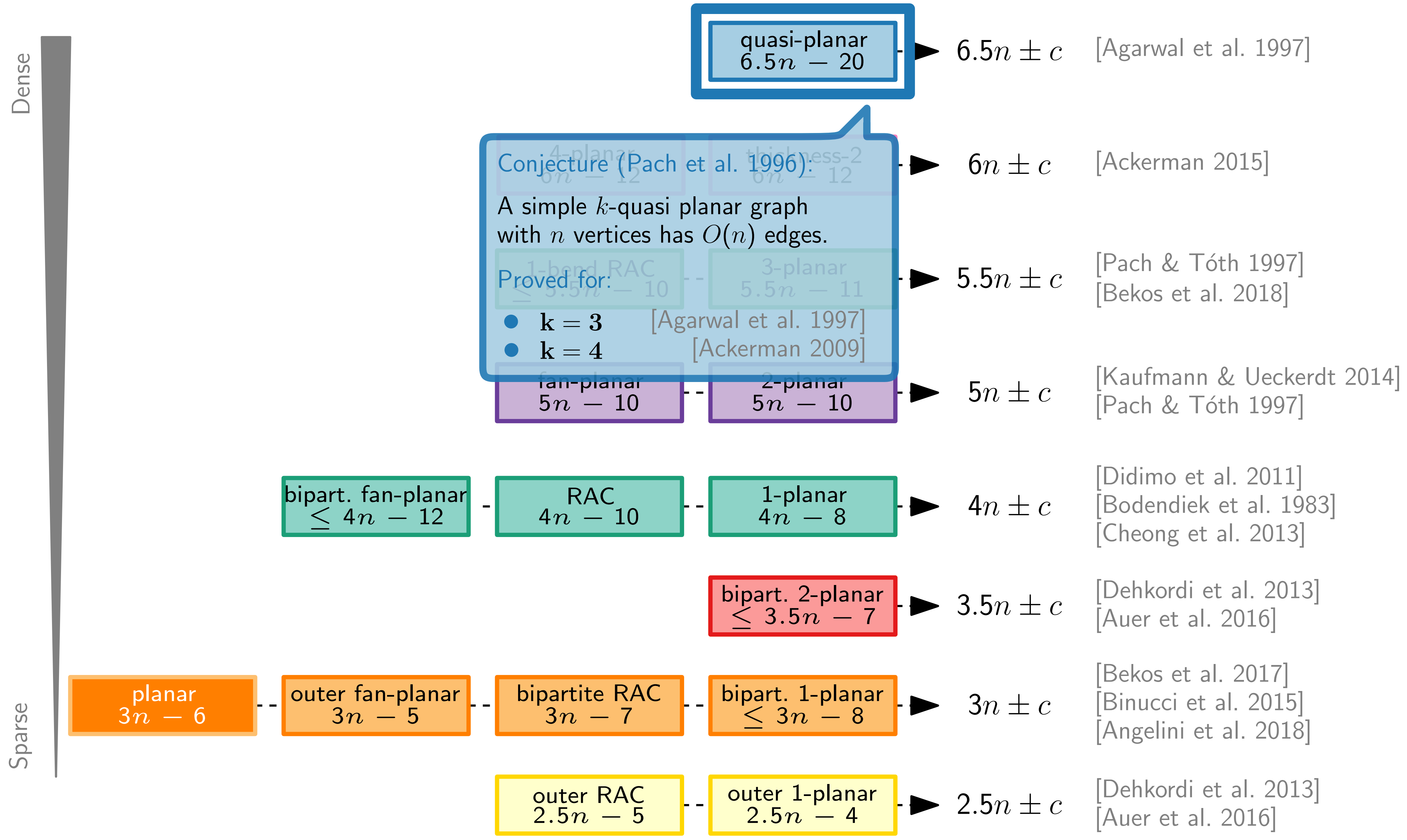
GD Beyond Planarity: a Hierarchy



GD Beyond Planarity: a Hierarchy



GD Beyond Planarity: a Hierarchy



Conjecture (Pach et al. 1996):

A simple k-quasi planar graph with n vertices has $O(n)$ edges.

Proved for:

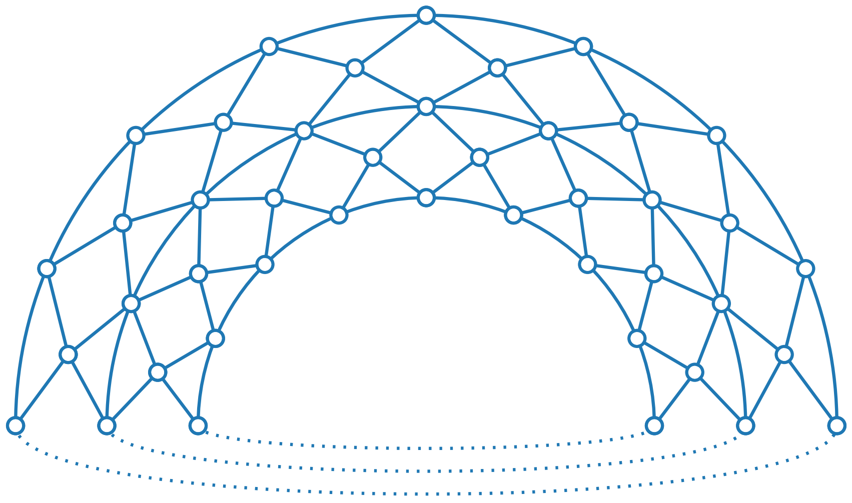
- $\mathbf{k = 3}$ [Agarwal et al. 1997]
- $\mathbf{k = 4}$ [Ackerman 2009]

Crossing Numbers

The **k-planar crossing number** $\mathrm{cr}_{k\text{-pl}}(G)$ of a graph G is the number of crossings required in any k-planar drawing of G.

- $\mathrm{cr}_{1\text{-pl}}(G) \leq n - 2$
- $\mathrm{cr}(G) = 1 \Rightarrow \mathrm{cr}_{1\text{-pl}}(G) = 1$

Theorem. [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph G with $n = 11\ell + 2$ vertices such that $\mathrm{cr}(G) = 2$ and $\mathrm{cr}_{1\text{-pl}}(G) = n - 2$.



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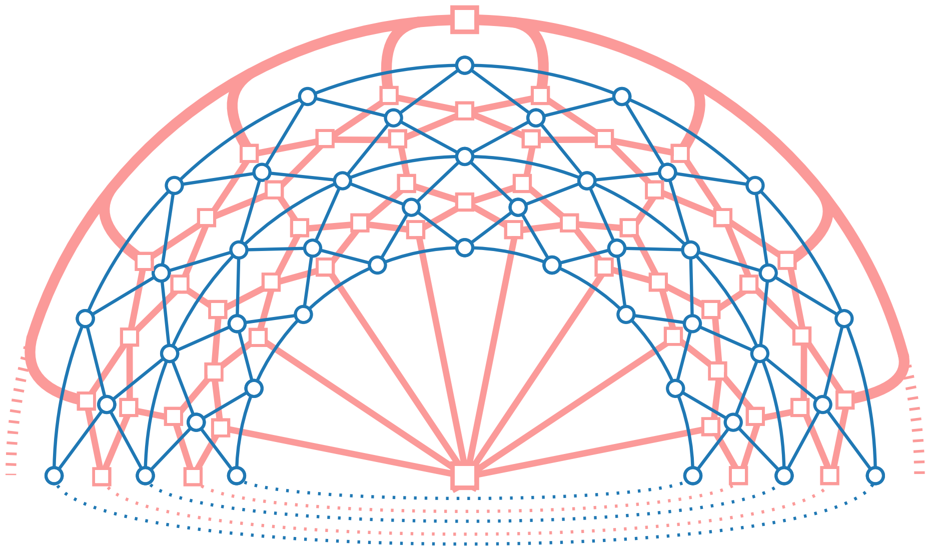

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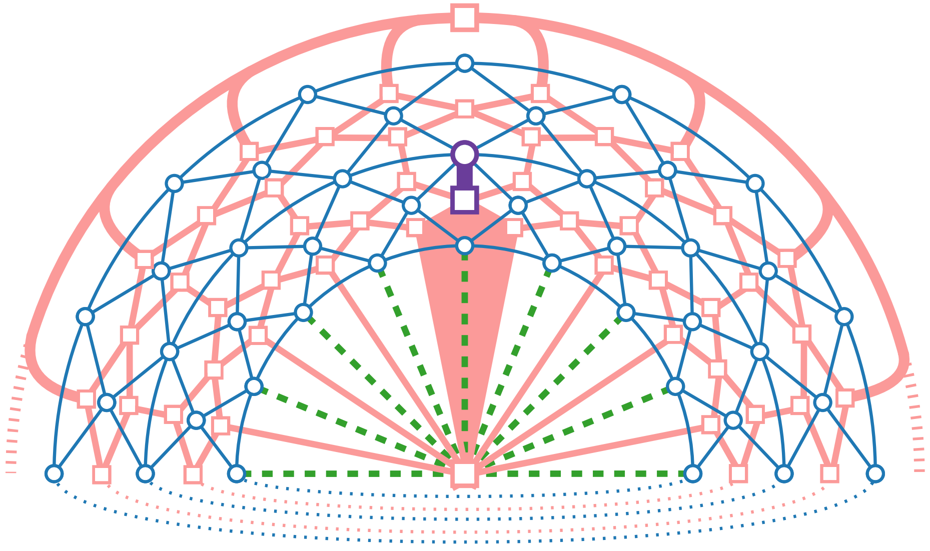
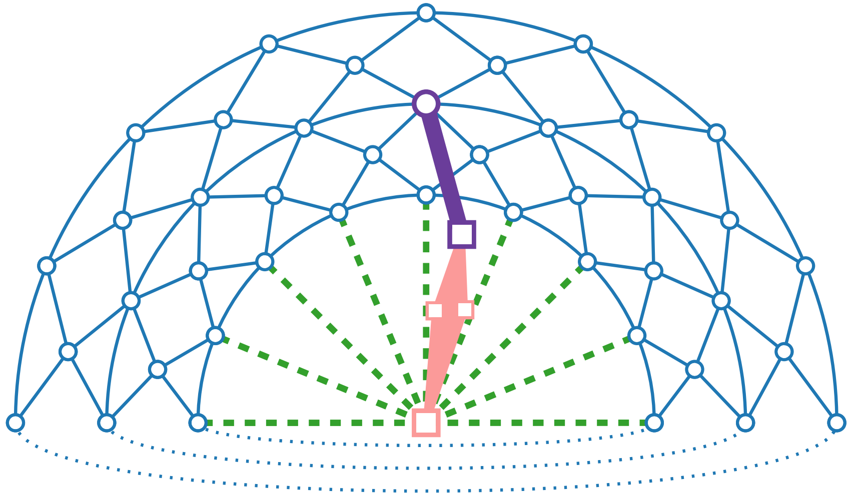
Crossing ratio
$\rho_{1\text{-pl}}(n) = (n-2)/2$



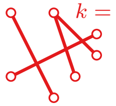
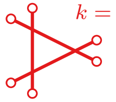
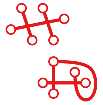
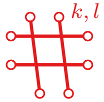
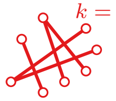
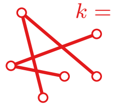
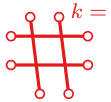
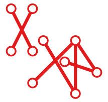
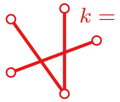
$\text{cr}_{1\text{-pl}}(G) = n - 2$



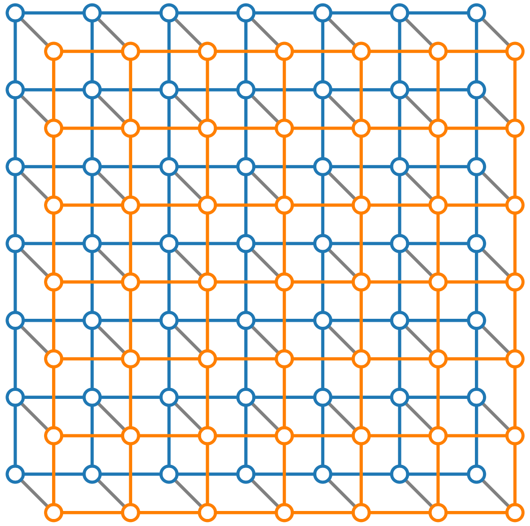
$\text{cr}(G) = 2$

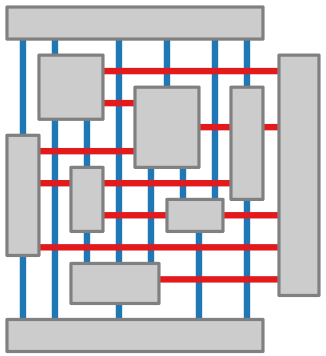
Crossing Ratios

Table from "Crossing Numbers of Beyond-Planar Graphs Revisited" [van Beusekom, Parada & Speckmann 2021]

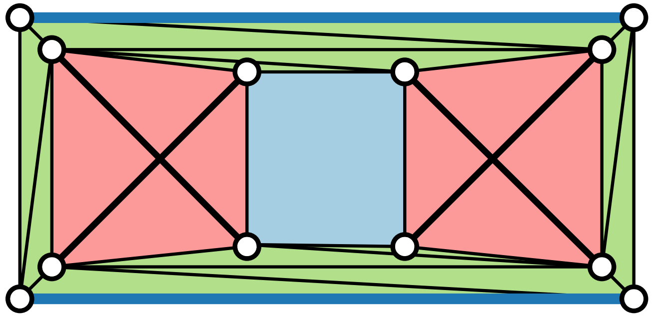
Family	**Forbidden Configurations**		**Lower**	**Upper**
k-planar	An edge crossed more than k times	 $k = 2$	$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(k\sqrt{k}n)}$
k-quasi-planar	k pairwise crossing edges	 $k = 3$	$\Omega(n/k^3)$	$f(k)n^2 \log^2 n$
Fan-planar	Two independent edges crossing a third or two adjacent edges crossing another edge from different "side"		$\Omega(n)$	$O(n^2)$
(k, l)-grid-free	Set of k edges such that each edge crosses each edge from a set of l edges.	 $k, l = 2$	$\Omega\left(\frac{n}{kl(k+l)}\right)$	$g(k, l)n^2$
k-gap-planar	More than k crossings mapped to an edge in an optimal mapping	 $k = 1$	$\boldsymbol{\Omega(n/k^3)}$	$\boldsymbol{O(k\sqrt{k}n)}$
Skewness-k	Set of crossings not covered by at most k edges	 $k = 1$	$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(kn + k^2)}$
k-apex	Set of crossings not covered by at most k vertices	 $k = 1$	$\Omega(n/k)$	$O(k^2n^2 + k^4)$
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge		$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$
k-fan-crossing-free	An edge that crosses k adjacent edges	 $k = 2$	$\boldsymbol{\Omega(n^2/k^3)}$	$\boldsymbol{O(k^2n^2)}$
Straight-line RAC	Two edges crossing at an angle $< \frac{\pi}{2}$		$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$

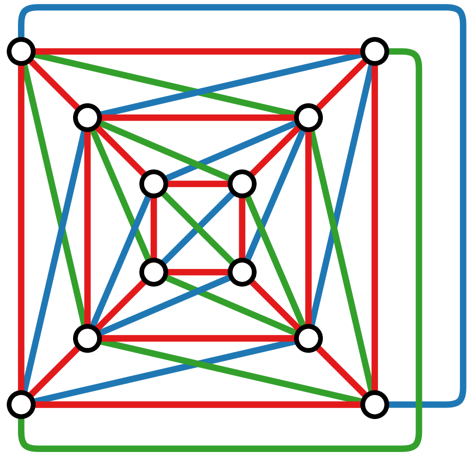
Visualization of Graphs





Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings

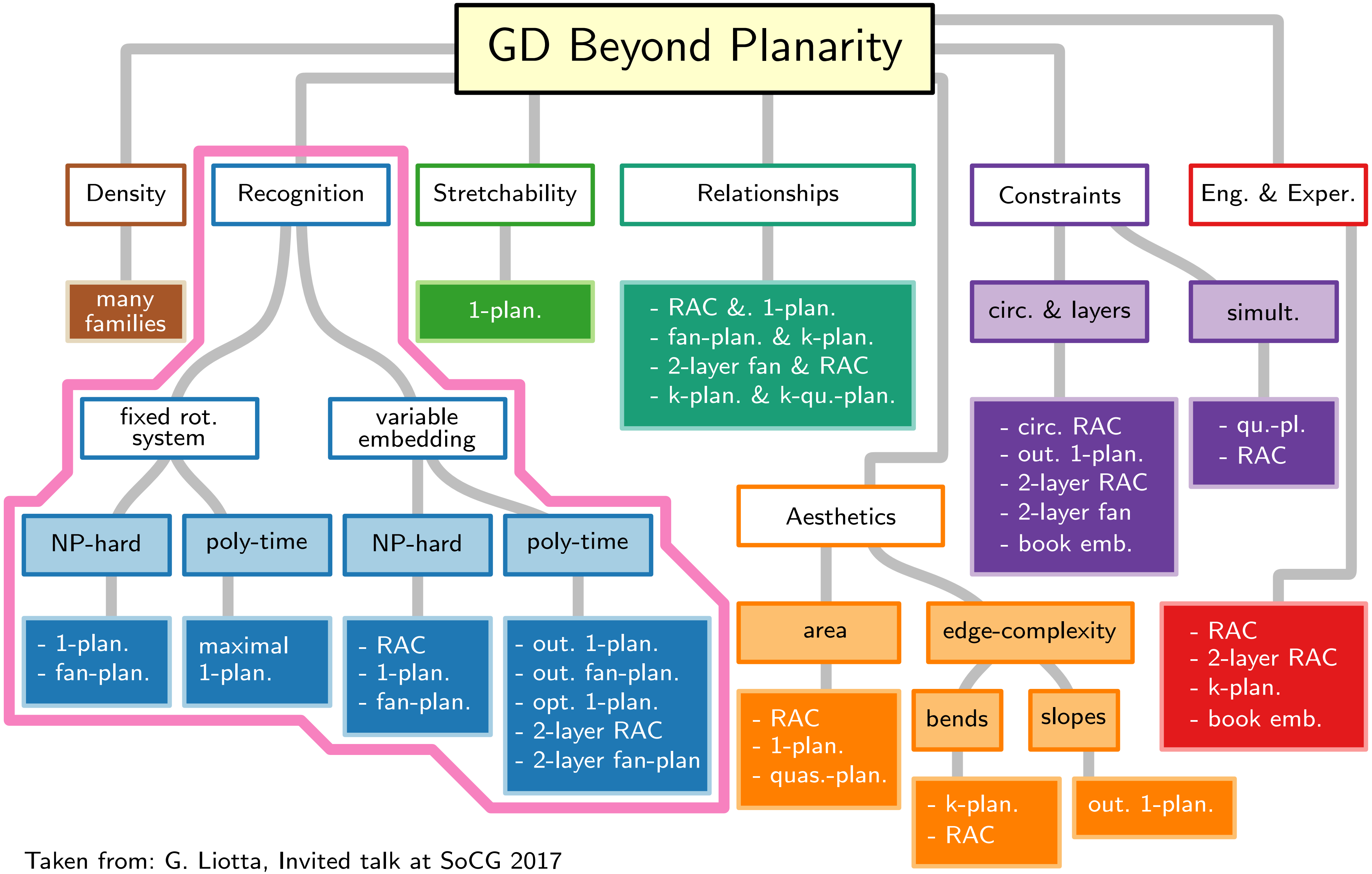




Part III:
Recognition

Alexander Wolff

GD Beyond Planarity: a Taxonomy



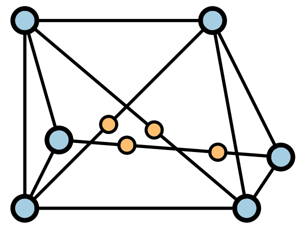
Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

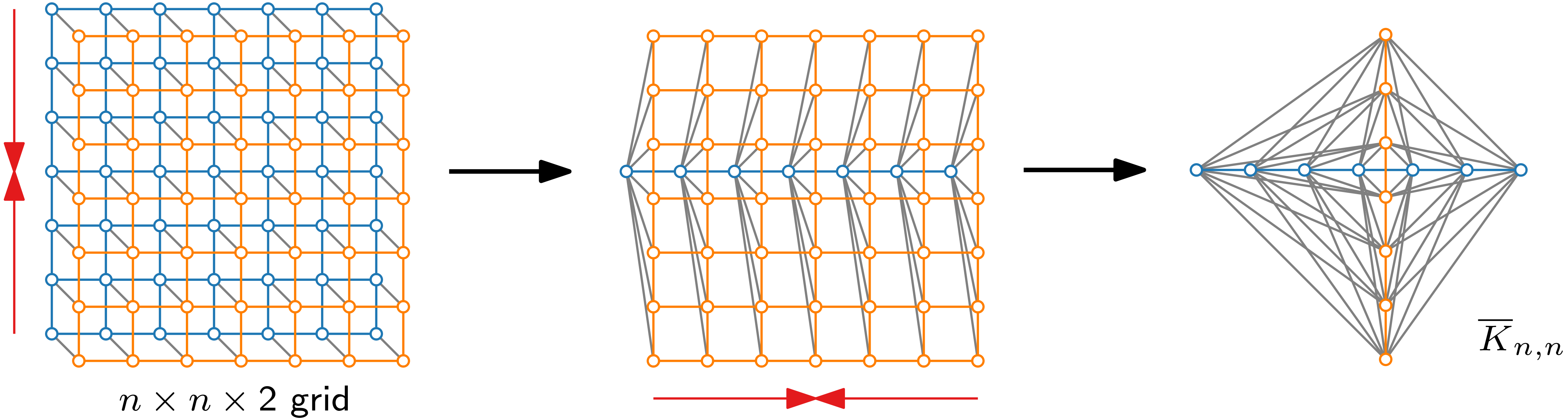
Minors of 1-Planar Graphs

Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$ neither K_5 nor $K_{3,3}$ minor of G

Theorem. [Chen & Kouno 2005]
The class of 1-planar graphs is not closed under edge contraction.

For every graph there is a 1-planar subdivision.



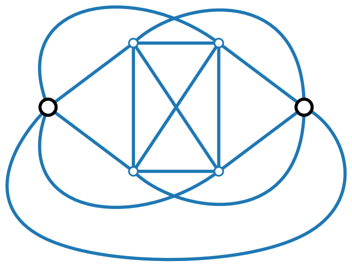


Theorem. [Korzhik & Mohar 2013]
For any n, there exist $\Omega(2^n)$ distinct graphs that are not 1-planar but all their proper subgraphs are 1-planar.

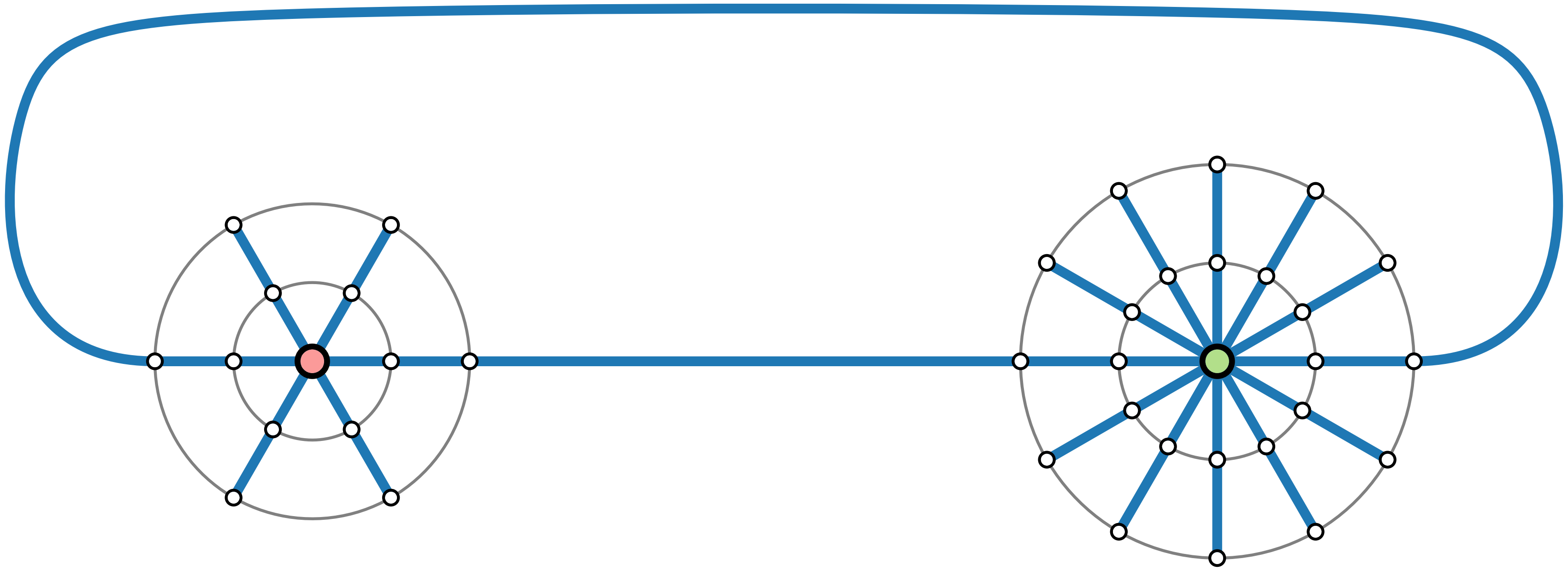
Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

Proof.
Reduction from 3-Partition.



Only 1-planar embedding of K_6



Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
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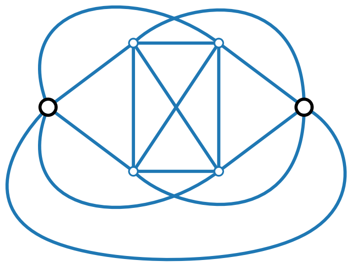
Proof.
Reduction from 3-Partition.

$$A = \{\overbrace{1, 3, 2}^{6}, \overbrace{4, 1, 1}^{6}\}$$



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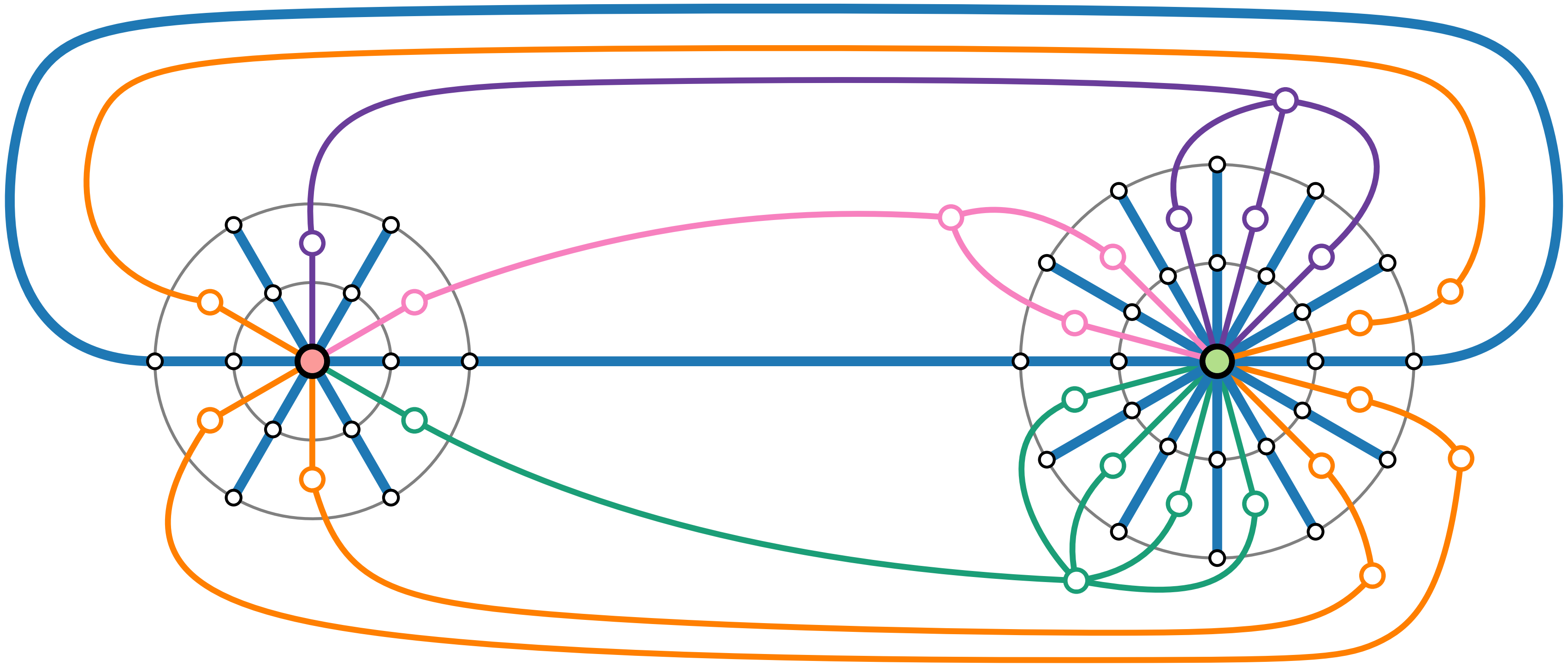


Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

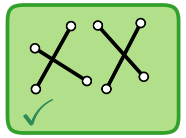
Theorem. [Cabello & Mohar 2013]
Testing 1-planarity is NP-complete –
even for almost planar graphs, i.e., planar graphs plus one edge.

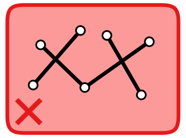
Theorem. [Bannister, Cabello & Eppstein 2018]
Testing 1-planarity is NP-complete –
even for graphs of bounded bandwidth (pathwidth, treewidth).

Theorem. [Auer, Brandenburg, Gleißner & Reislhuber 2015]
Testing 1-planarity is NP-complete –
even for 3-connected graphs with a fixed rotation system.

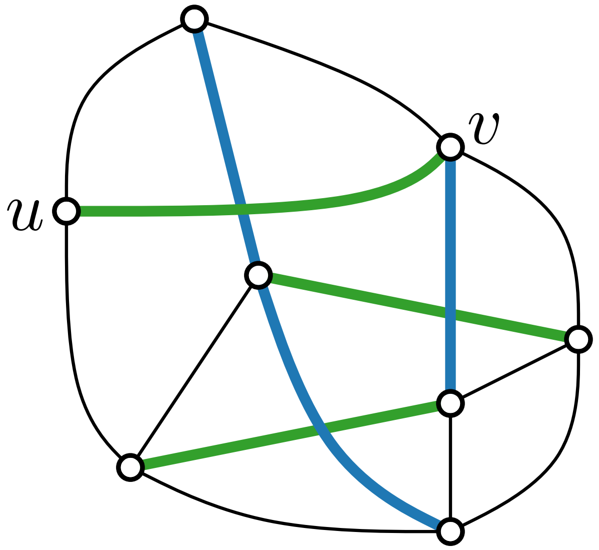
Recognition of IC-Planar Graphs

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.

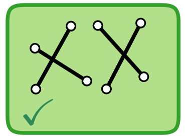


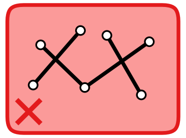


Proof.
Reduction from 1-planarity testing.



Recognition of IC-Planar Graphs





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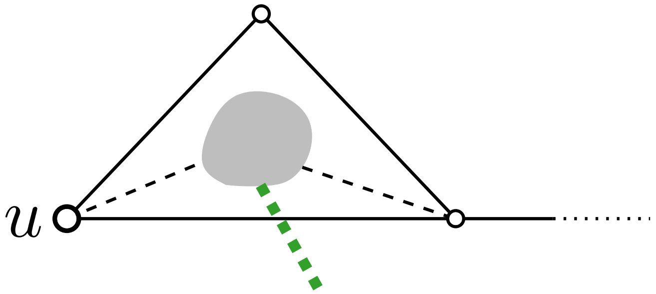
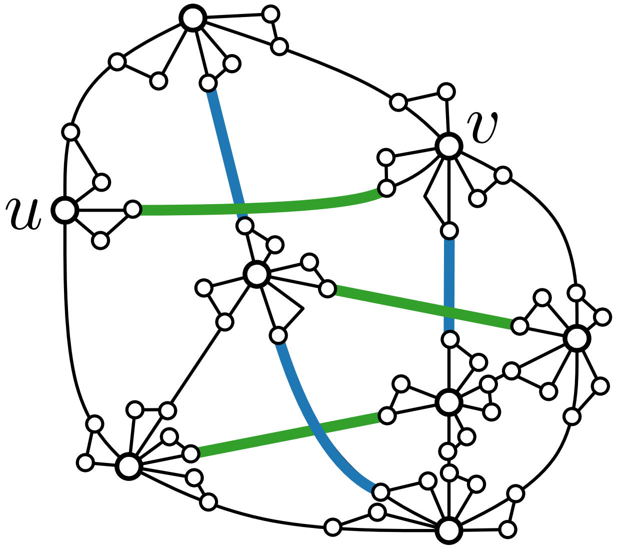
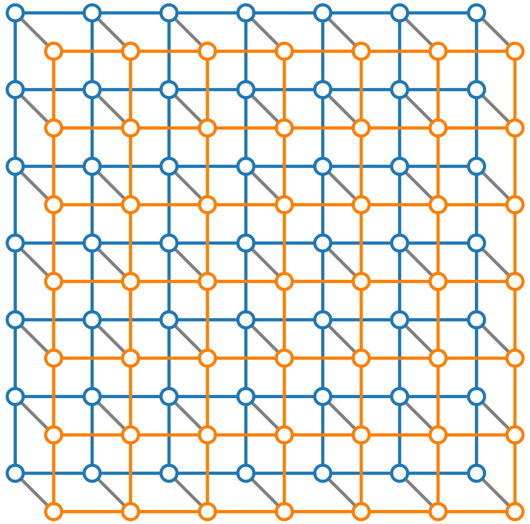
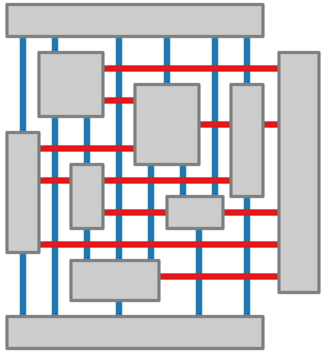


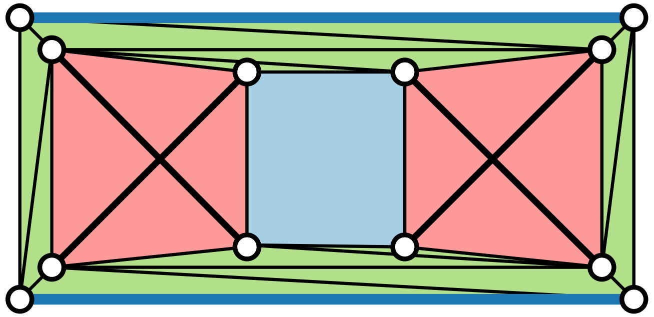


Visualization of Graphs

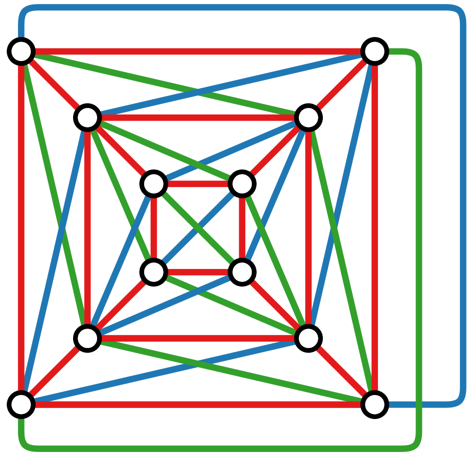




Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings

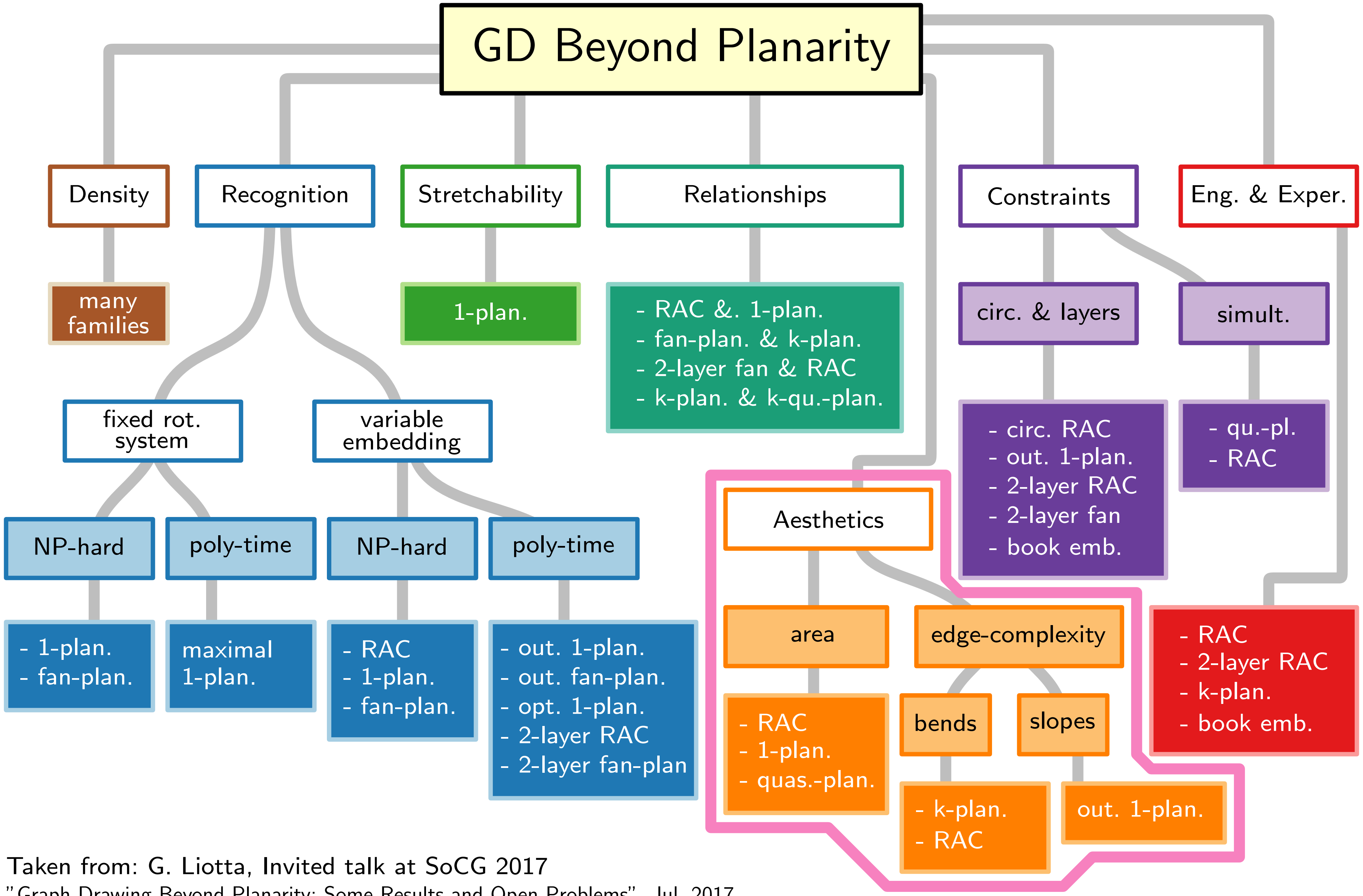


Part IV:
RAC Drawings



Alexander Wolff

GD Beyond Planarity: a Taxonomy



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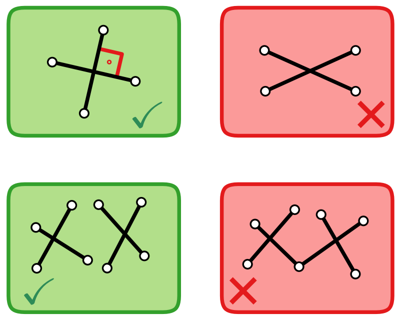
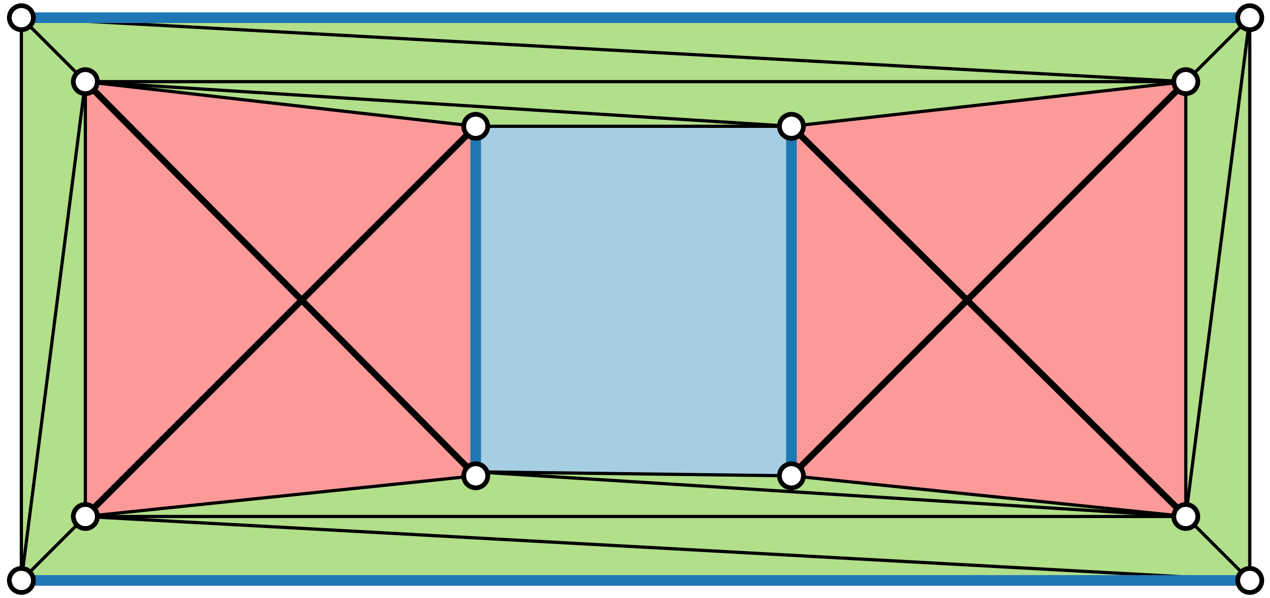
Area of Straight-Line RAC Drawings



Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Some IC-planar straight-line RAC drawings require exponential area.



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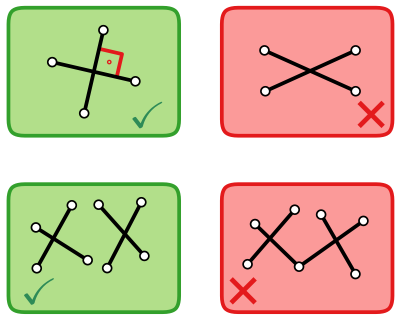
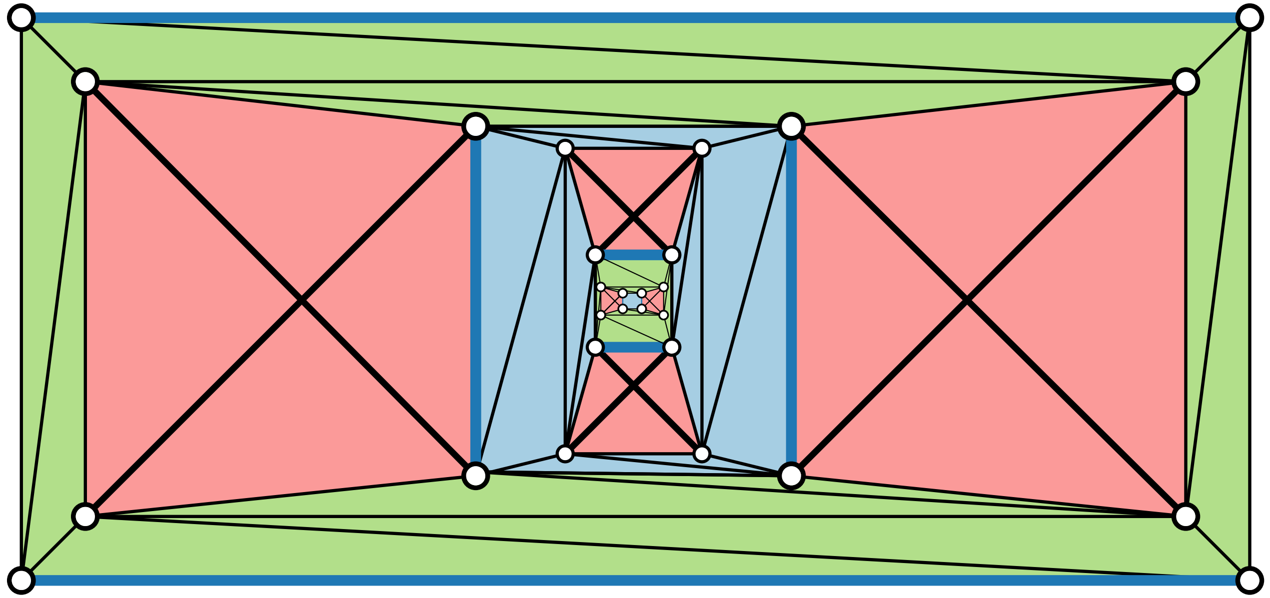
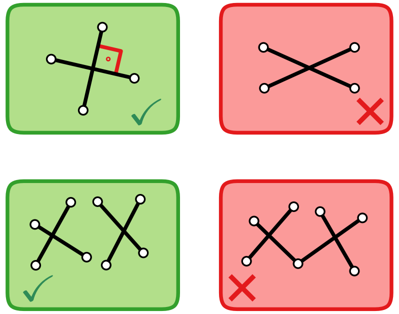
Some IC-planar straight-line RAC drawings require exponential area.



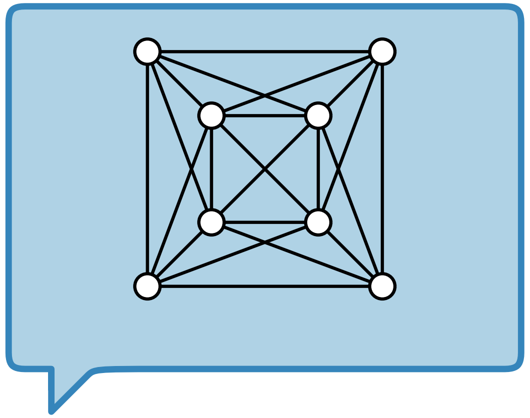


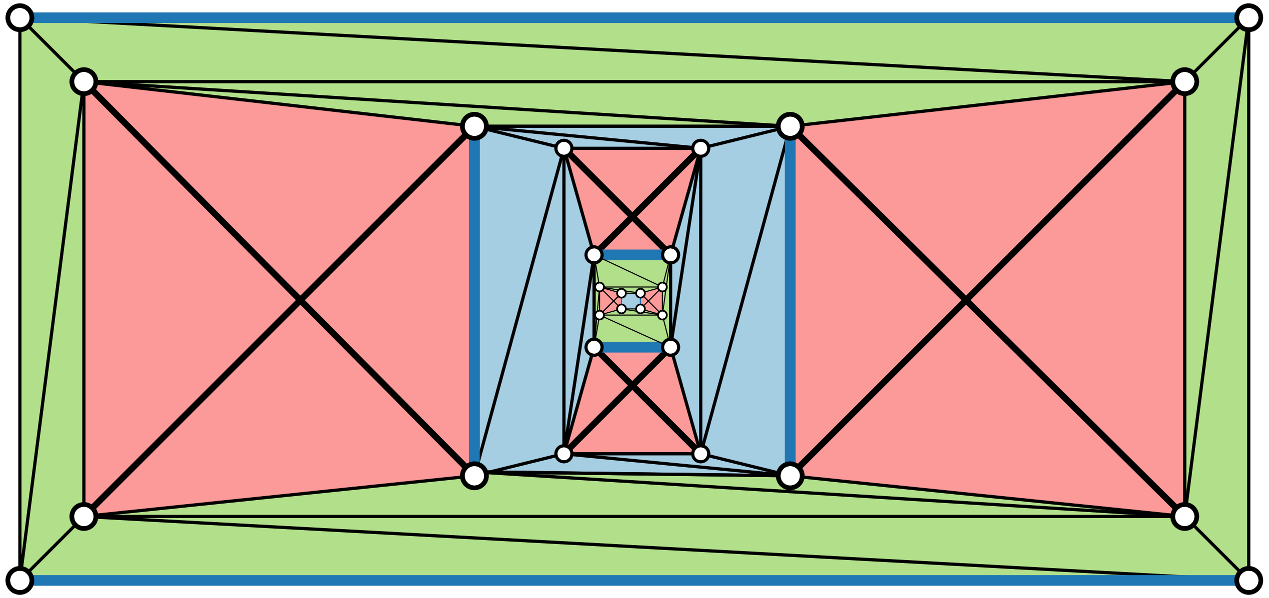
Area of Straight-Line RAC Drawings

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Some IC-planar straight-line RAC drawings require exponential area.

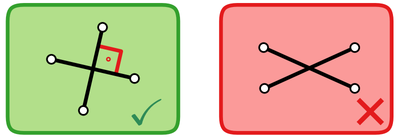


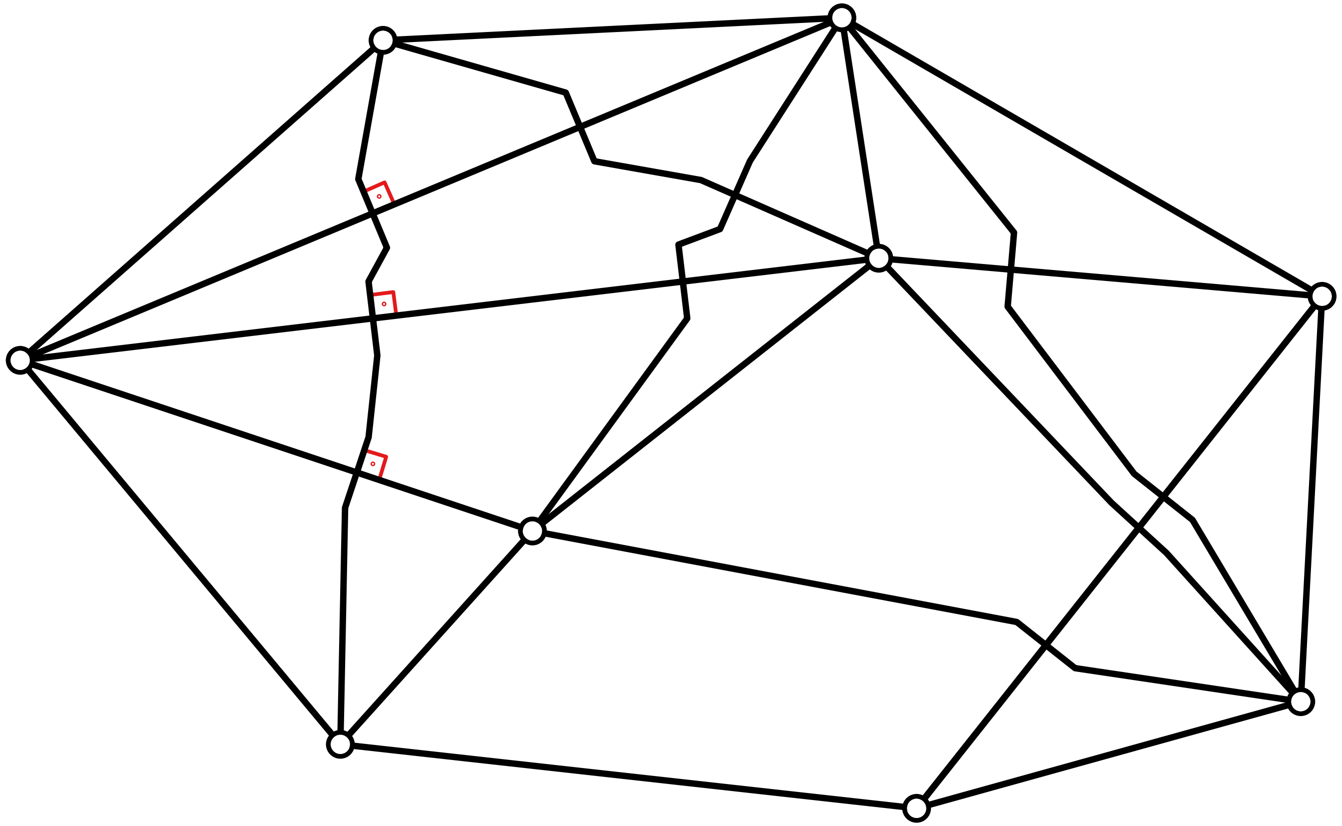
Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Every IC-planar graph has an IC-planar straight-line RAC drawing, and such a drawing can be found in polynomial time.





RAC Drawings With Enough Bends



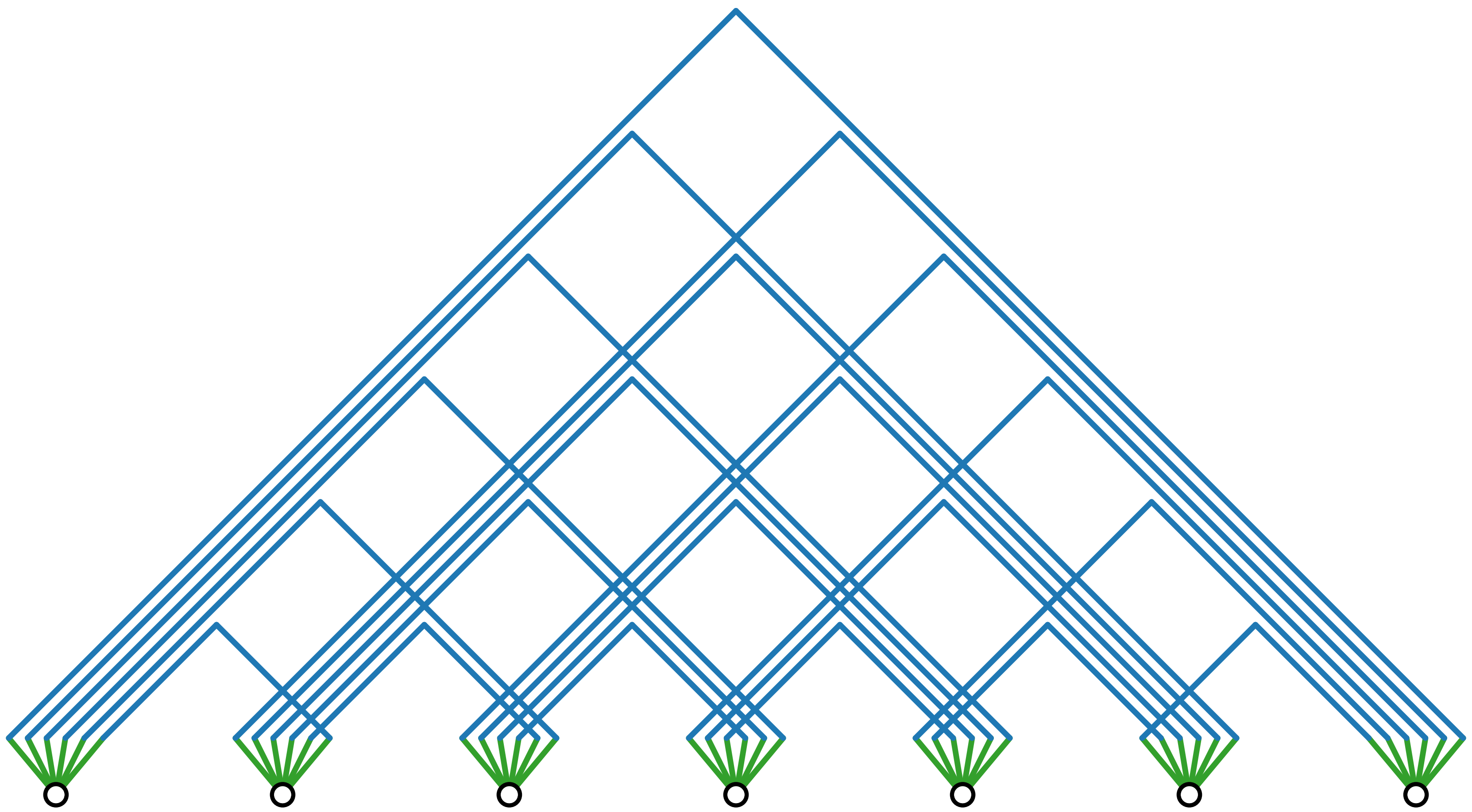


Every graph admits a RAC drawing . . .
. . . if we use enough bends.

How many do we need at most in total or per edge?

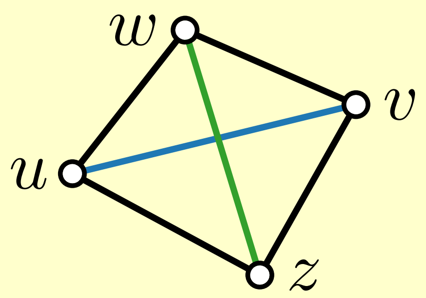
3-Bend RAC Drawings

Theorem. [Didimo, Eades & Liotta 2017]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most 3 bends.

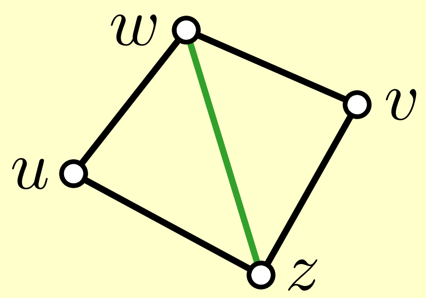


Kite Triangulations

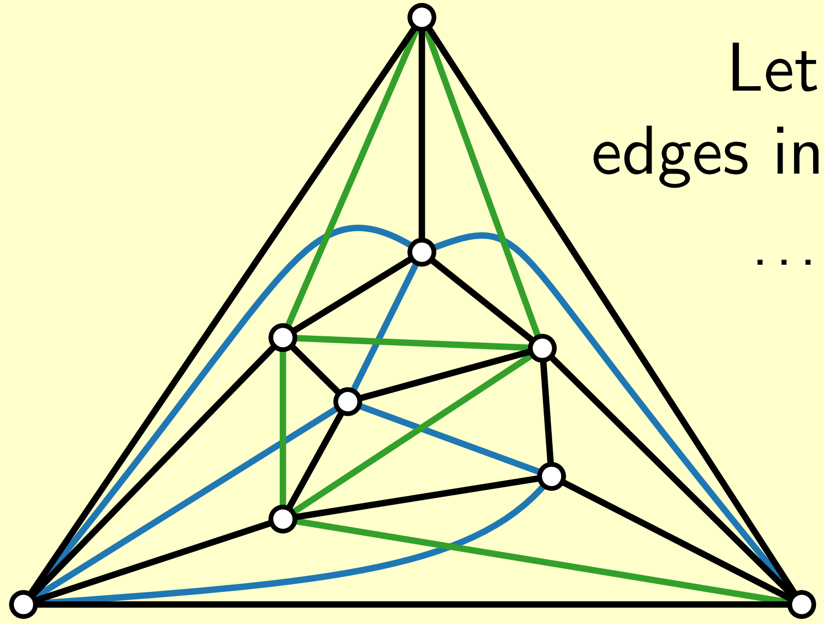
This is a **kite**:



u and v are **opposite** w.r.t. $\{z, w\}$



Let G' be a plane triangulation.



Let $S \subset E(G')$ s.t. no two edges in S lie on the same face

. . . and their opposite vertices do not have an edge in $E(G')$.

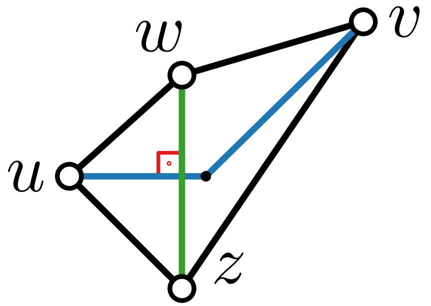
Add set T of edges connecting opposite vertices.

The resulting graph G is a **kite-triangulation**.

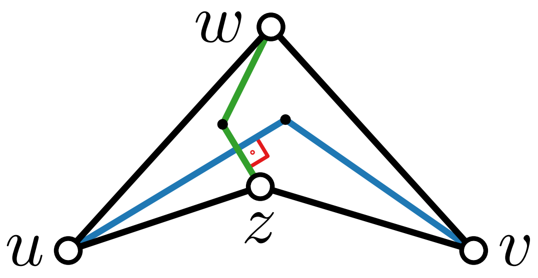
Note: optimal 1-planar graphs $\subset$ kite-triangulations.

Theorem. [Angelini et al. '11]
Every kite-triangulation G on n vertices admits a 1-planar 1-bend RAC drawing, which can be constructed in $\mathcal{O}(n)$ time.

Proof.
Let G' be the underlying plane triangulation of G. Let $G'' = G' - S$.
Construct straight-line drawing of G''.
Fill faces as follows:

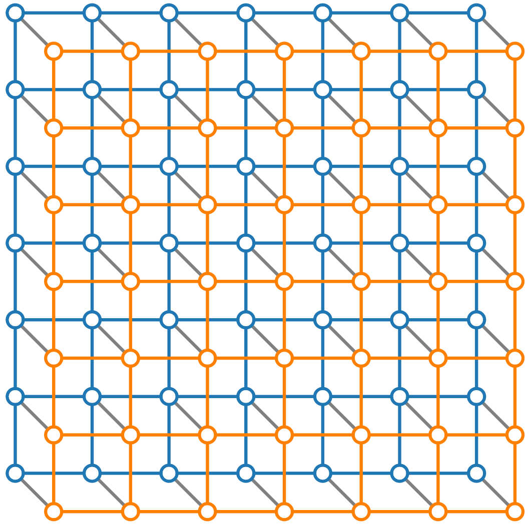


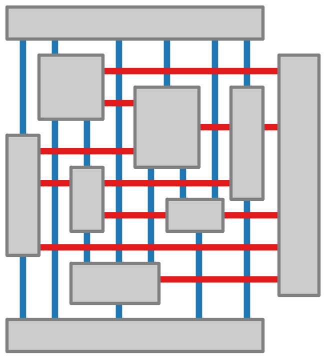
strictly convex face



otherwise

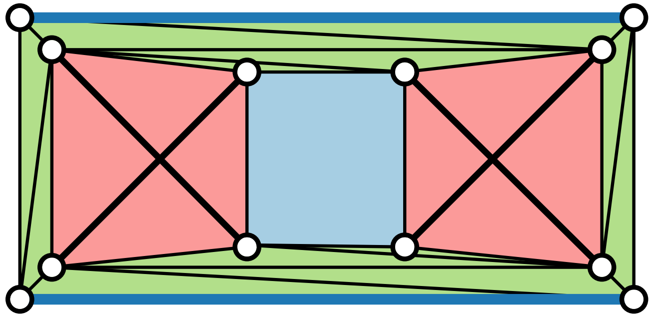
Visualization of Graphs

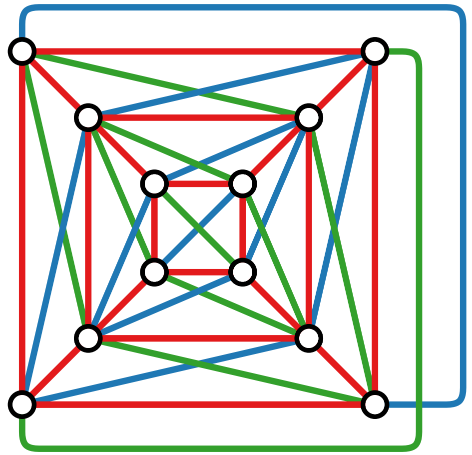




Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings

Part V:
1-Planar 1-Bend RAC Drawings



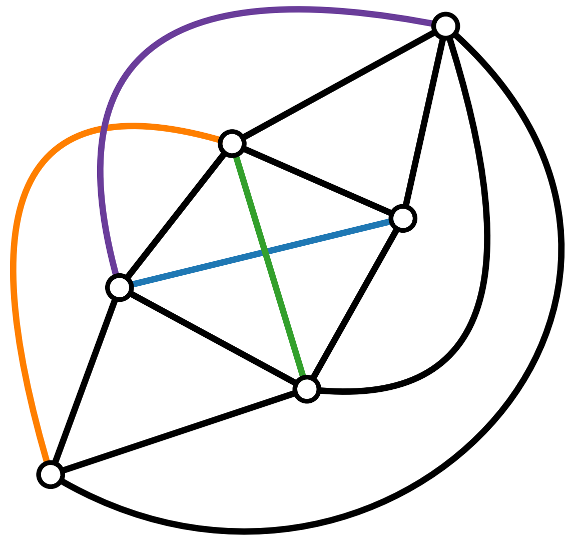


Alexander Wolff

1-Planar 1-Bend RAC Drawings

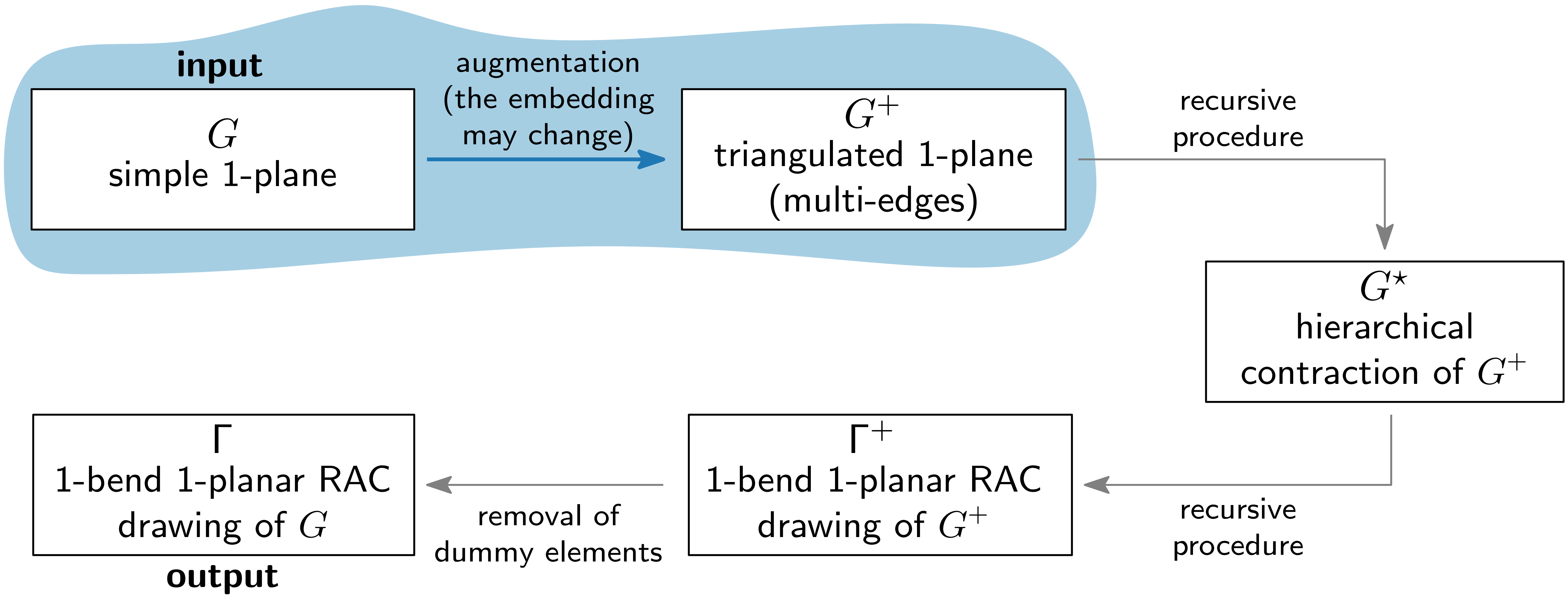
Theorem. [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph G admits a 1-planar 1-bend RAC drawing.
If a 1-planar embedding of G is given as part of the input, such a drawing can be computed in linear time.

Observation.
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of G forms a(n empty) kite, except for at most one pair if their crossing point is on the outer face of G.



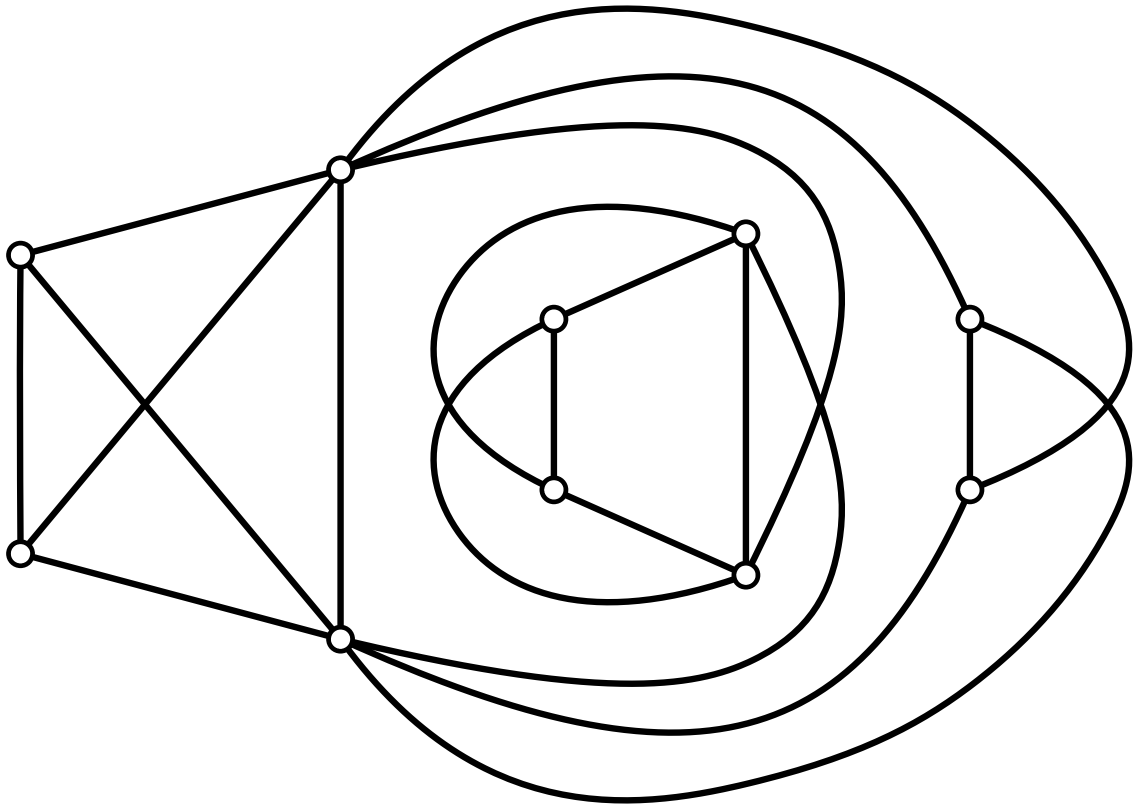
Theorem. [Chiba, Yamanouchi & Nishizeki 1984]
For every plane graph G with outer face C_k and every convex k-gon P, there exists a strictly convex planar straight-line drawing of G whose outer face coincides with P. Such a drawing can be computed in linear time.

Algorithm Outline



Algorithm Step 1: Augmentation

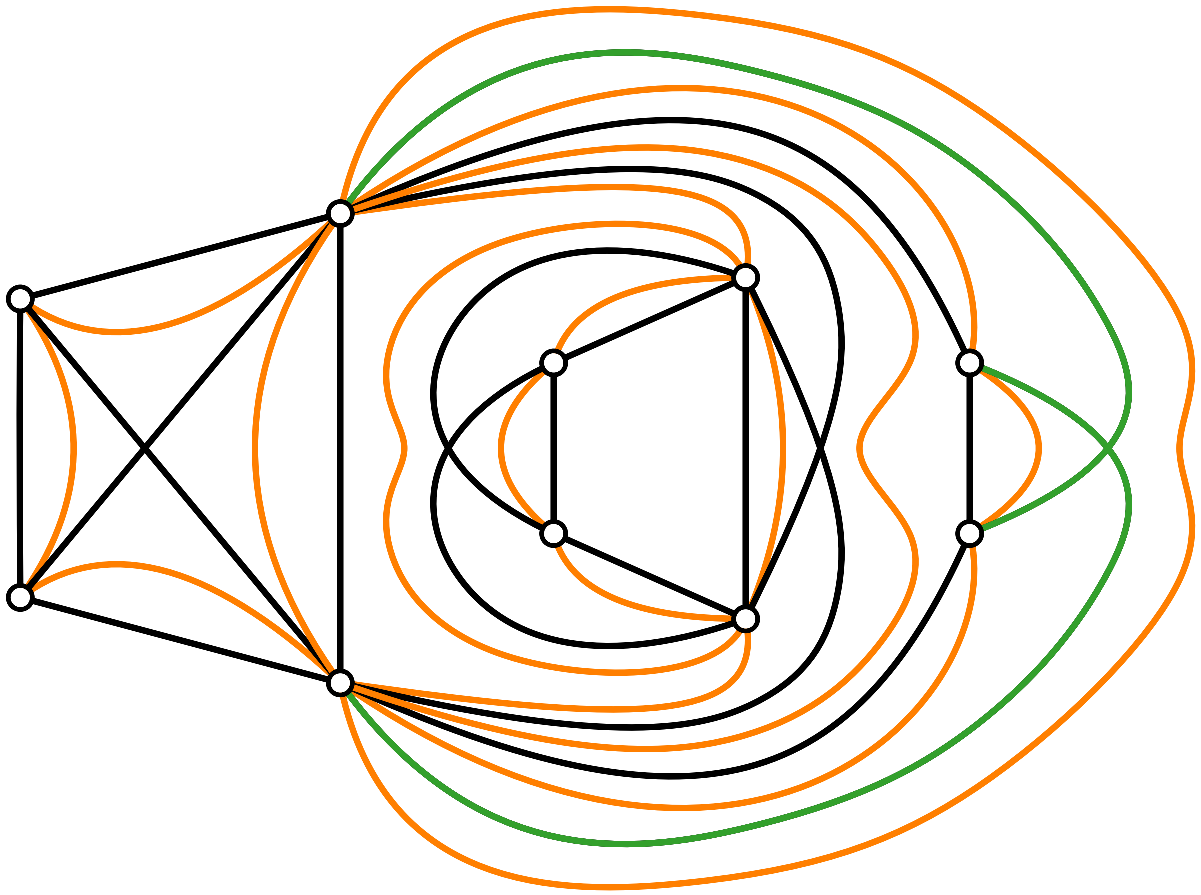
G: simple 1-plane graph



Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

G: simple 1-plane graph

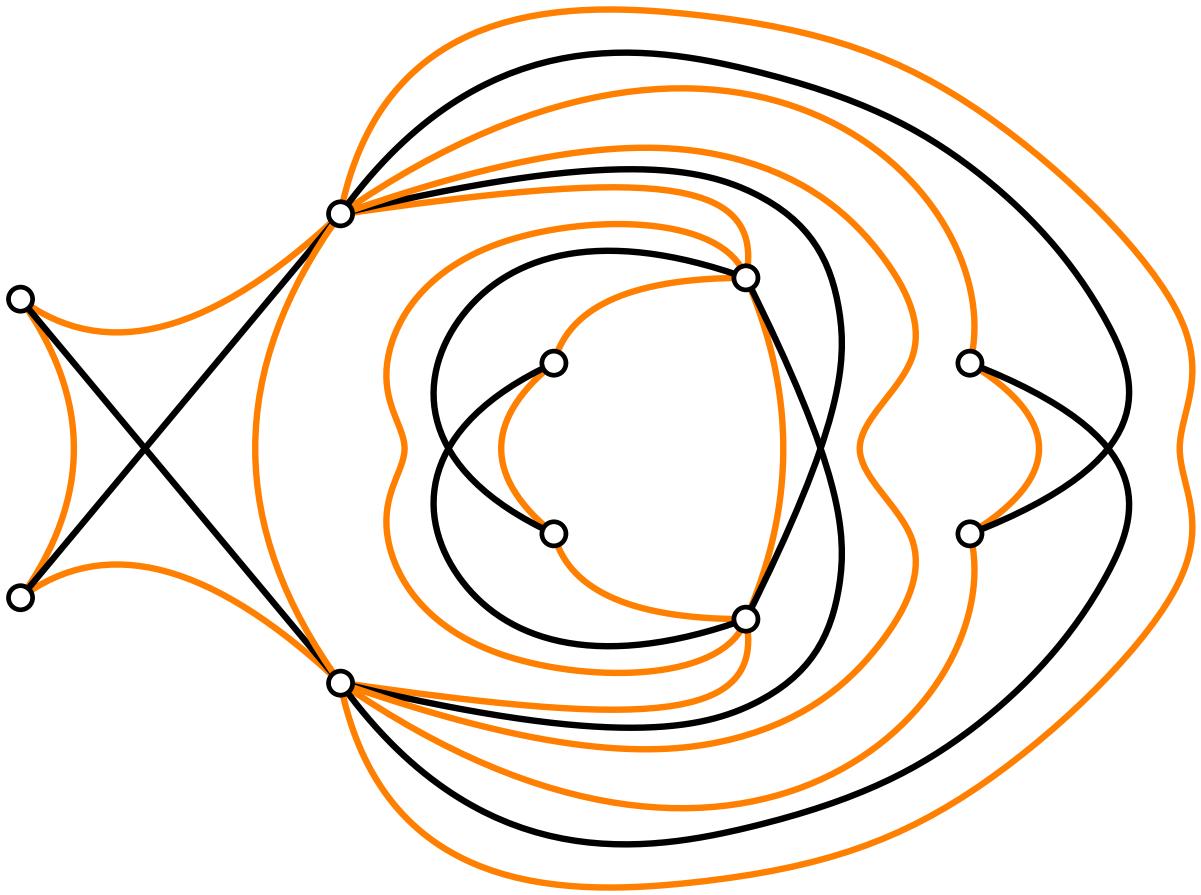


Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to G.

G: simple 1-plane graph

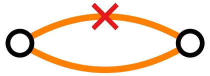


Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

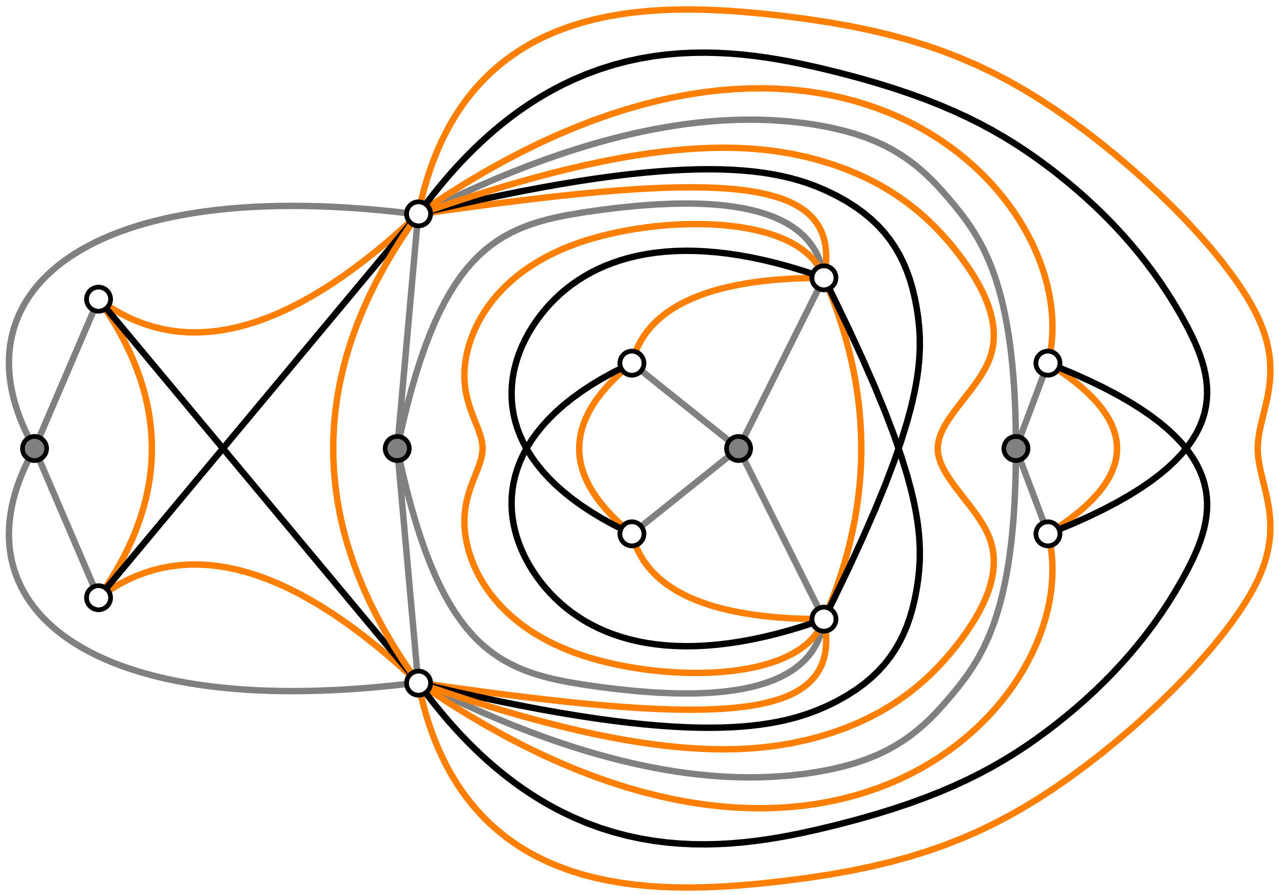
2. Remove those multiple edges that belong to G.

3. Remove one (multiple) edge from each face of degree two (if any).

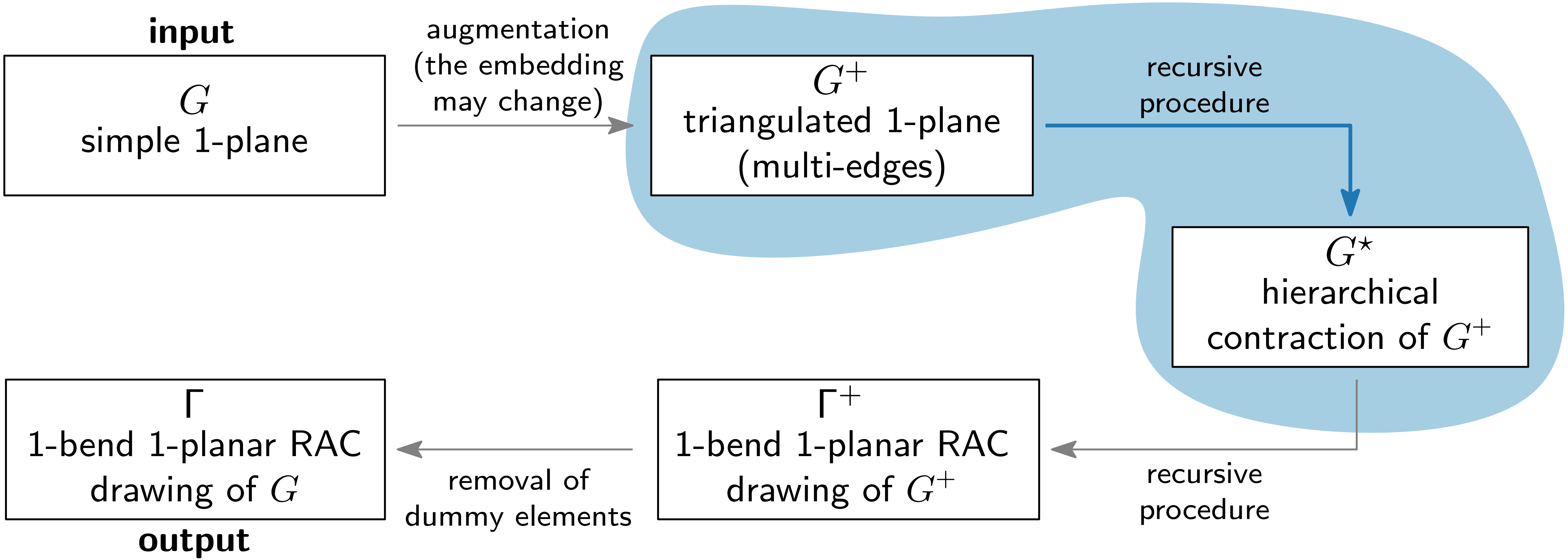


4. Triangulate faces of degree > 3 by inserting a star inside them.

G: simple 1-plane graph → G^+: triangulated 1-plane (multi-edges)



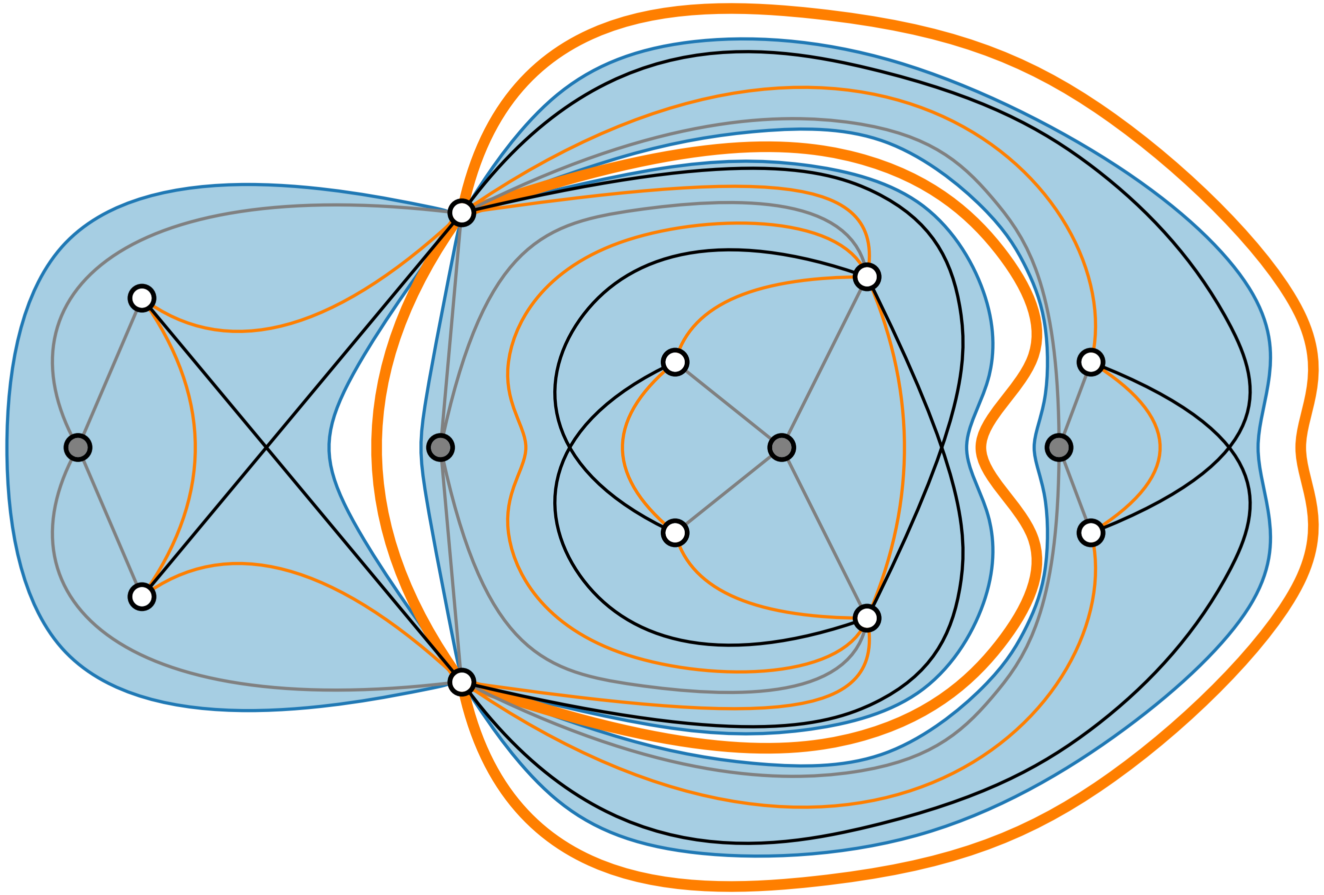
Algorithm Outline



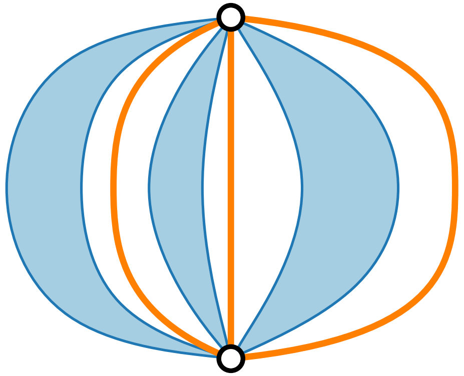
Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



$\Rightarrow$

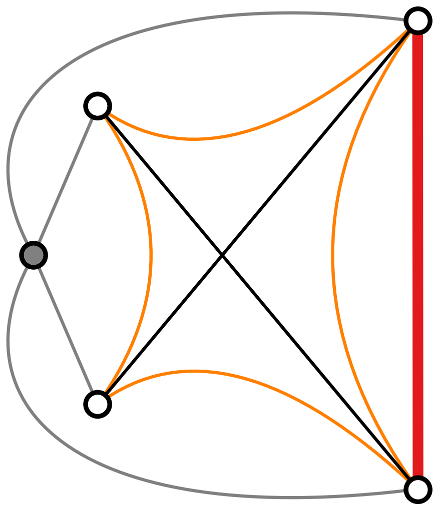


structure of each separation pair

Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



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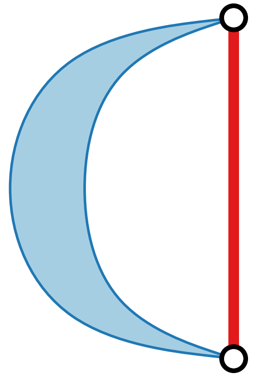

structure of each separation pair

Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
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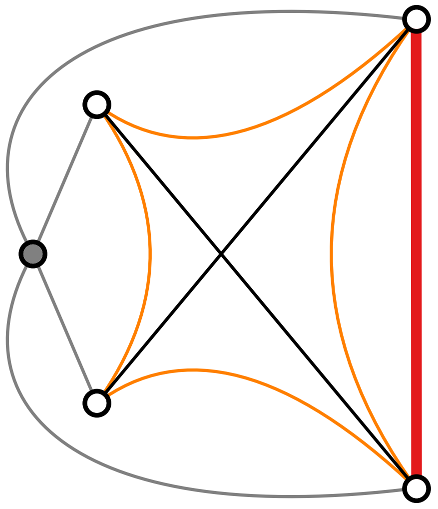
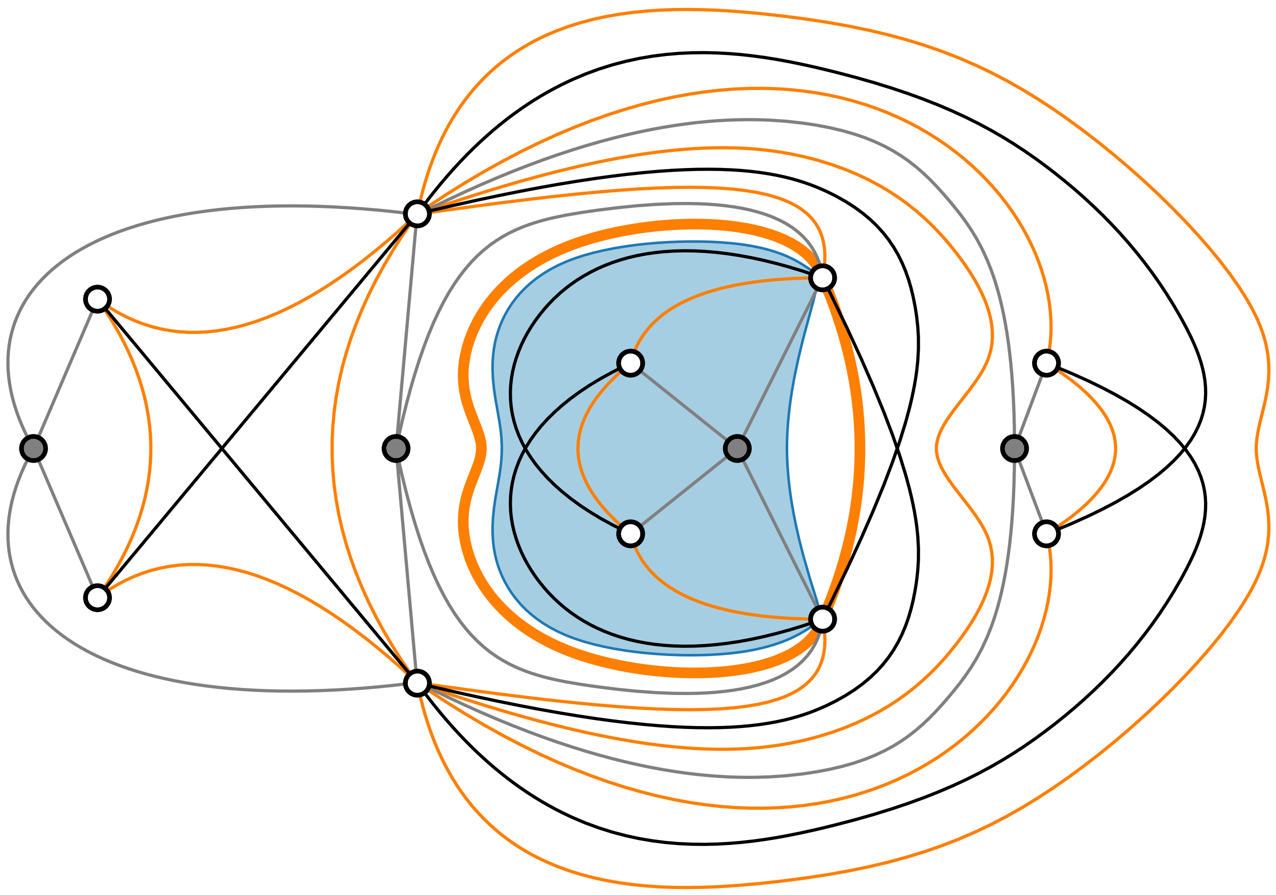
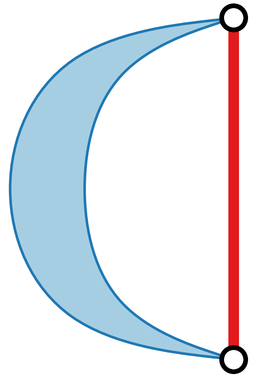




$\Rightarrow$



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Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

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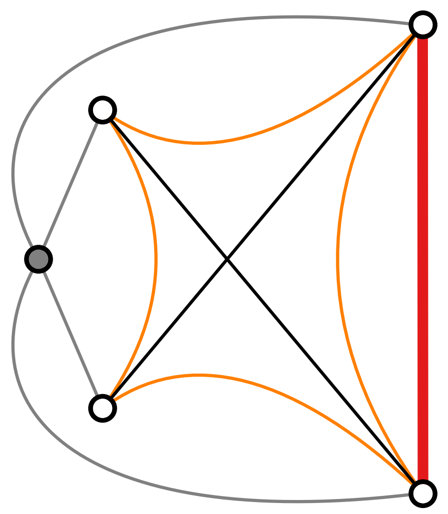
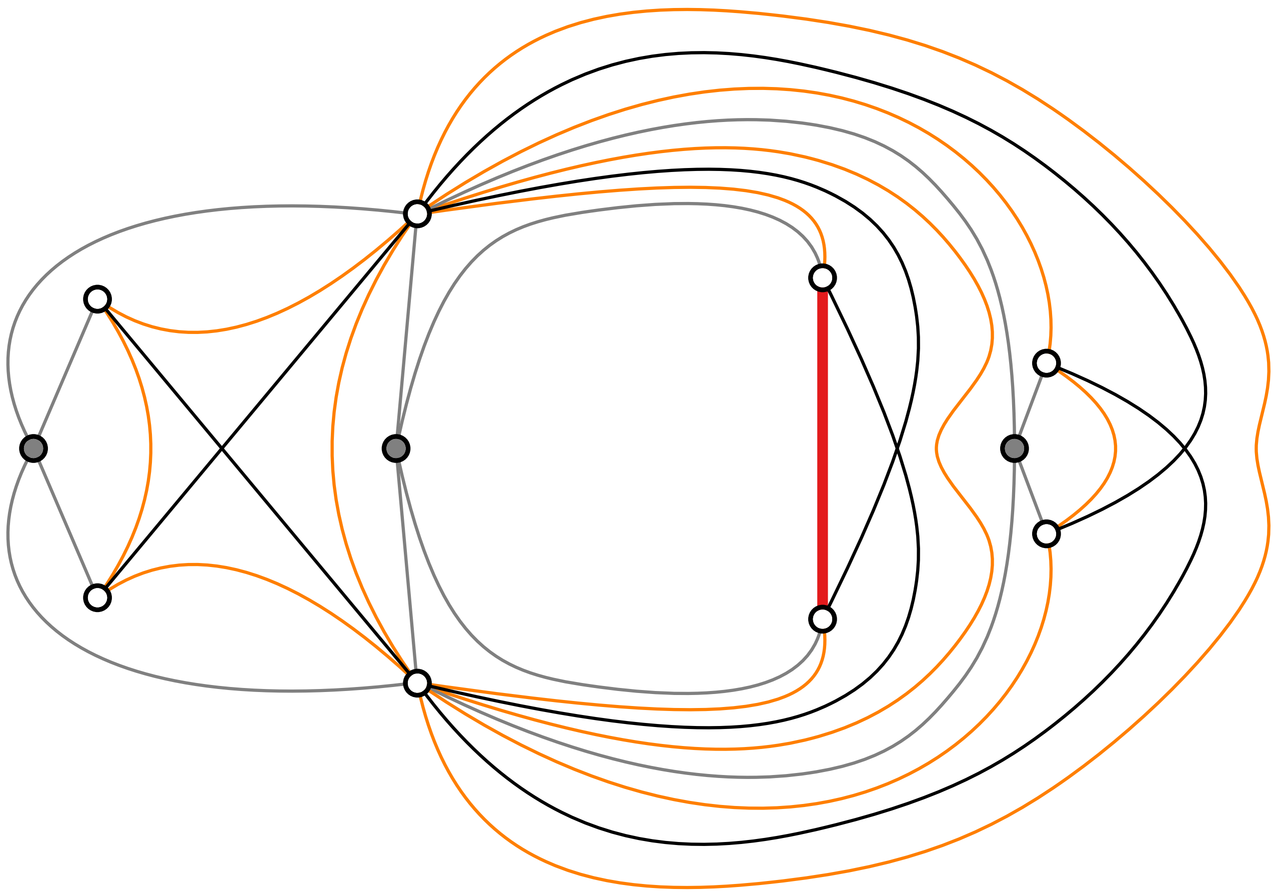
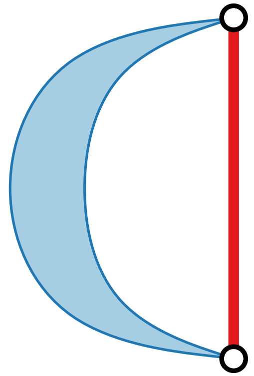




$\Rightarrow$



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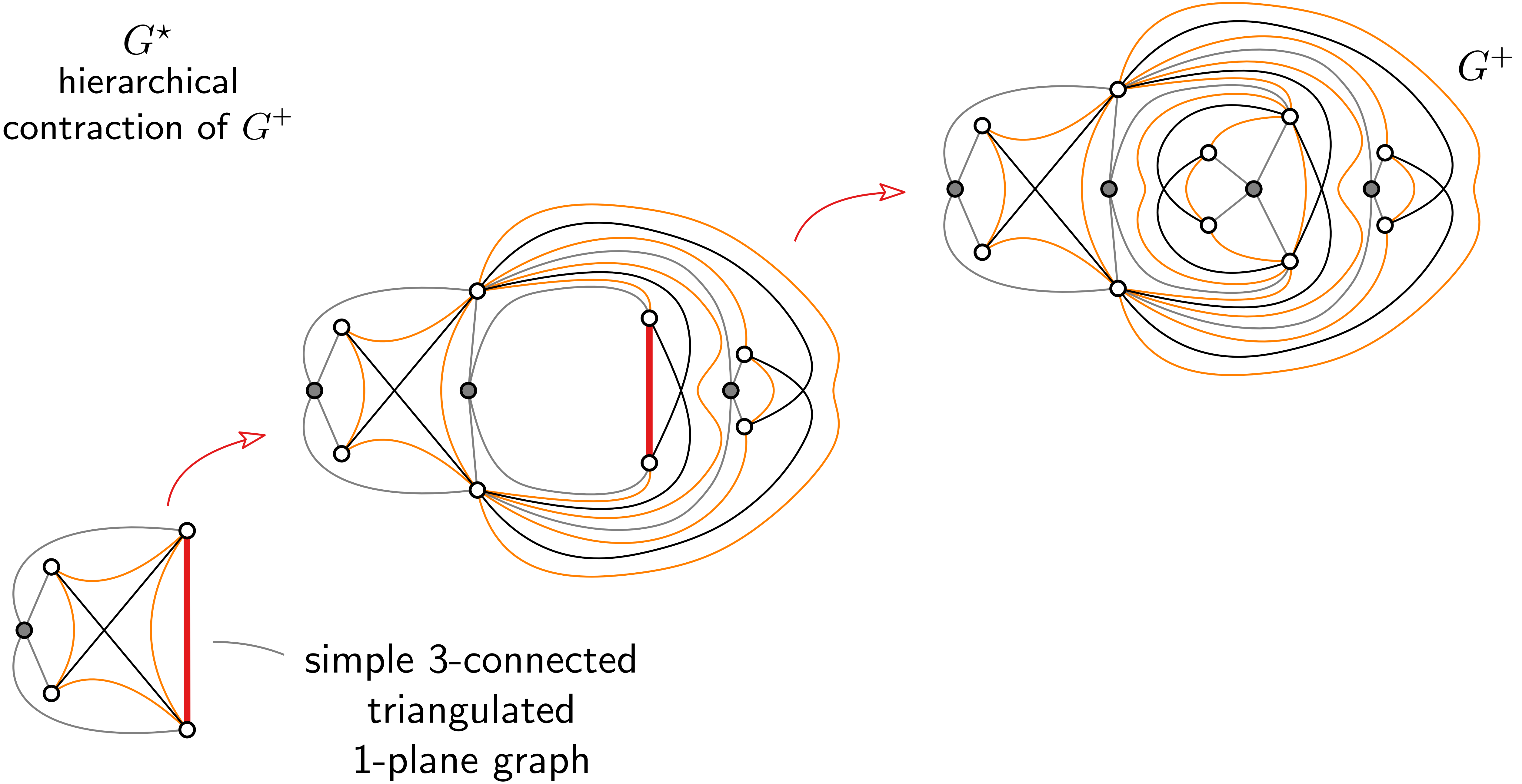
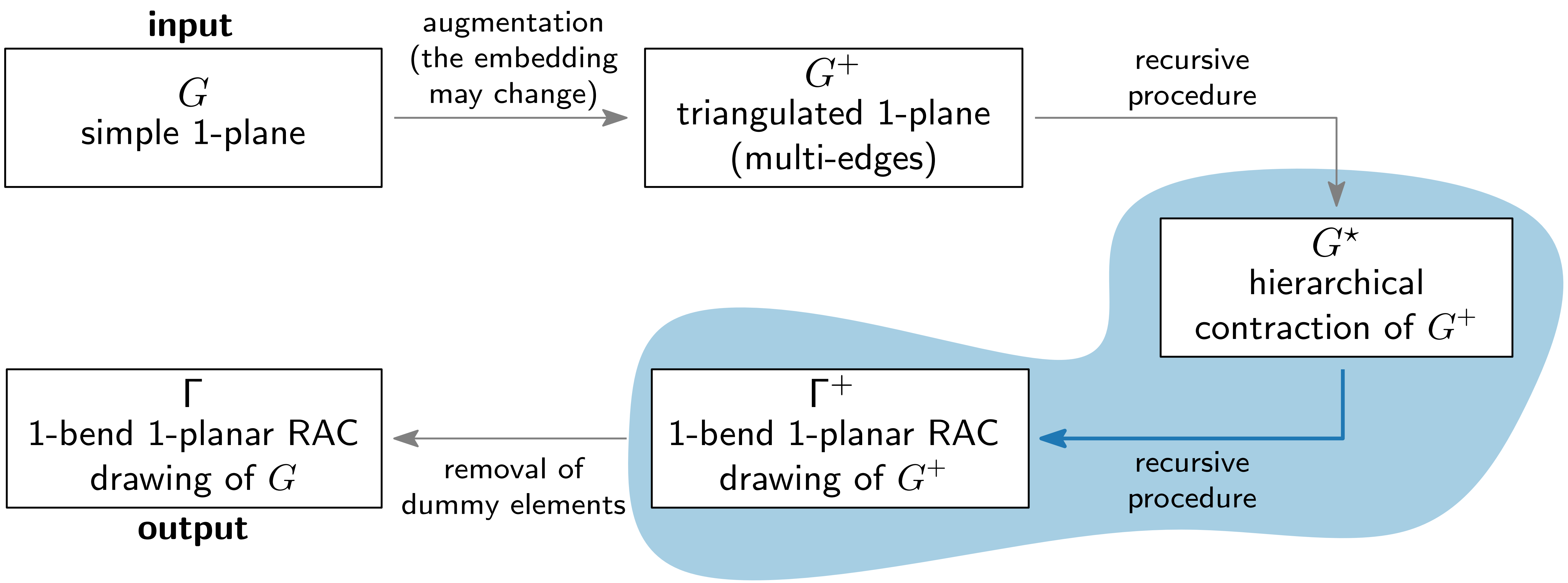
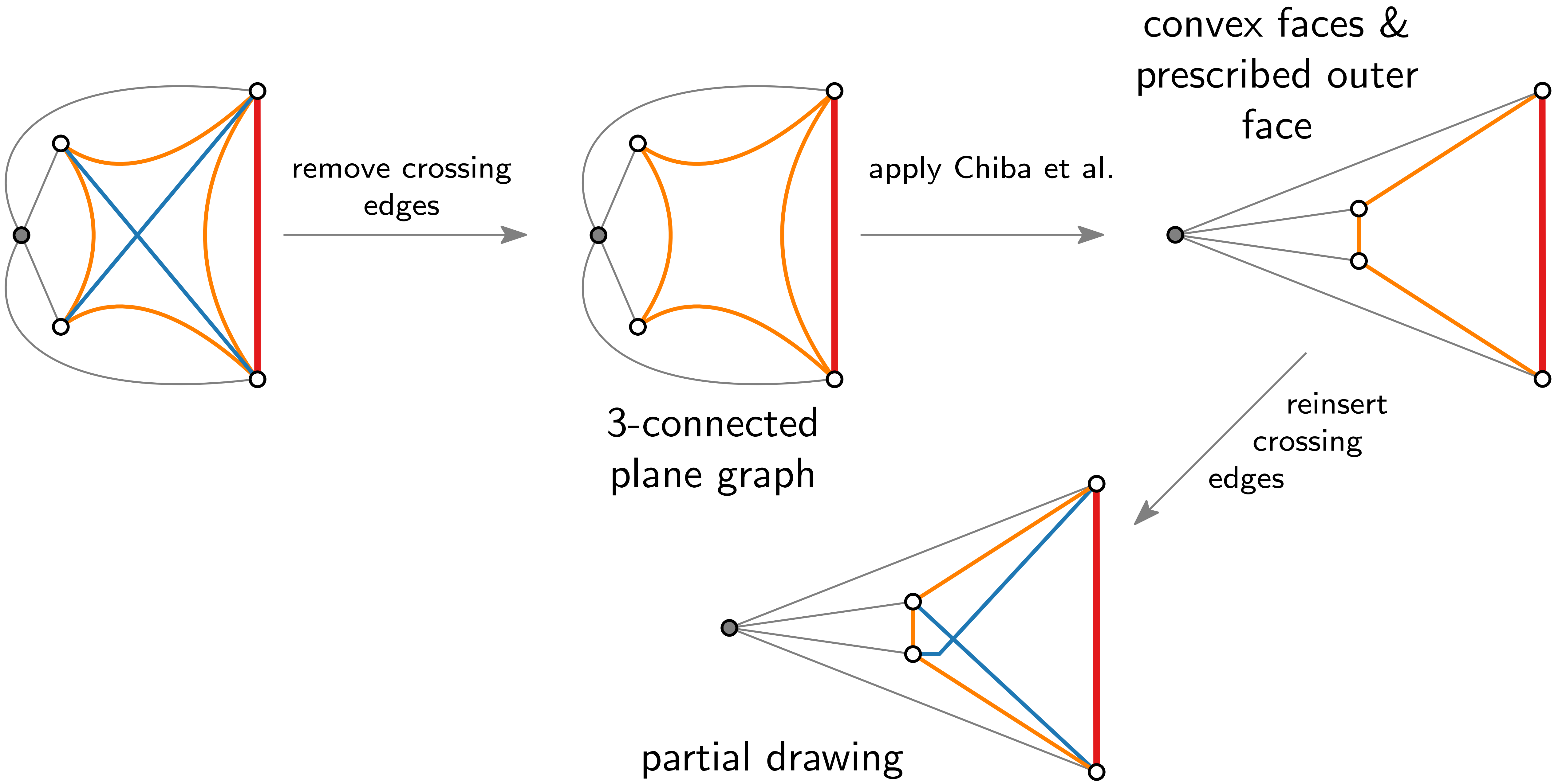
Algoritm Step 2: Hierarchical Contractions



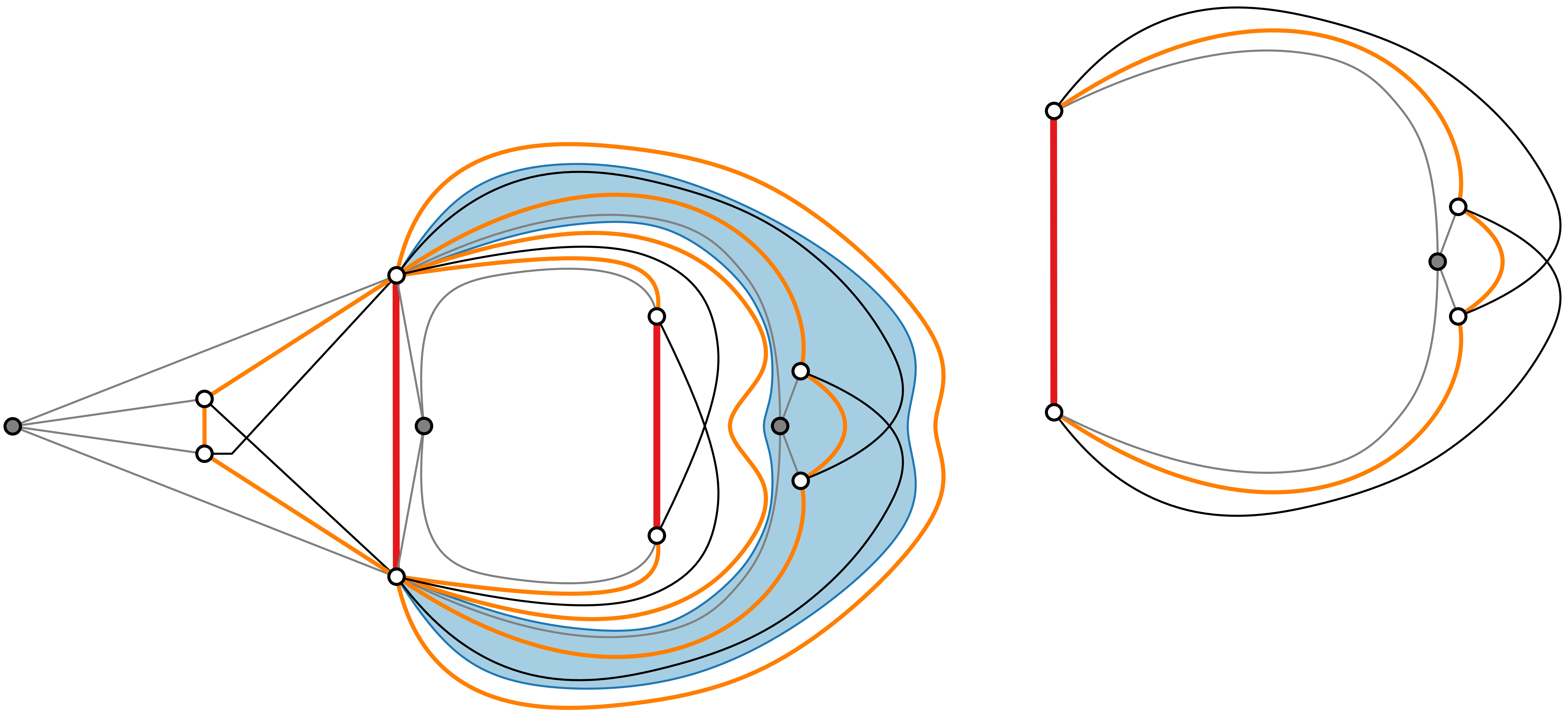
Algorithm Outline



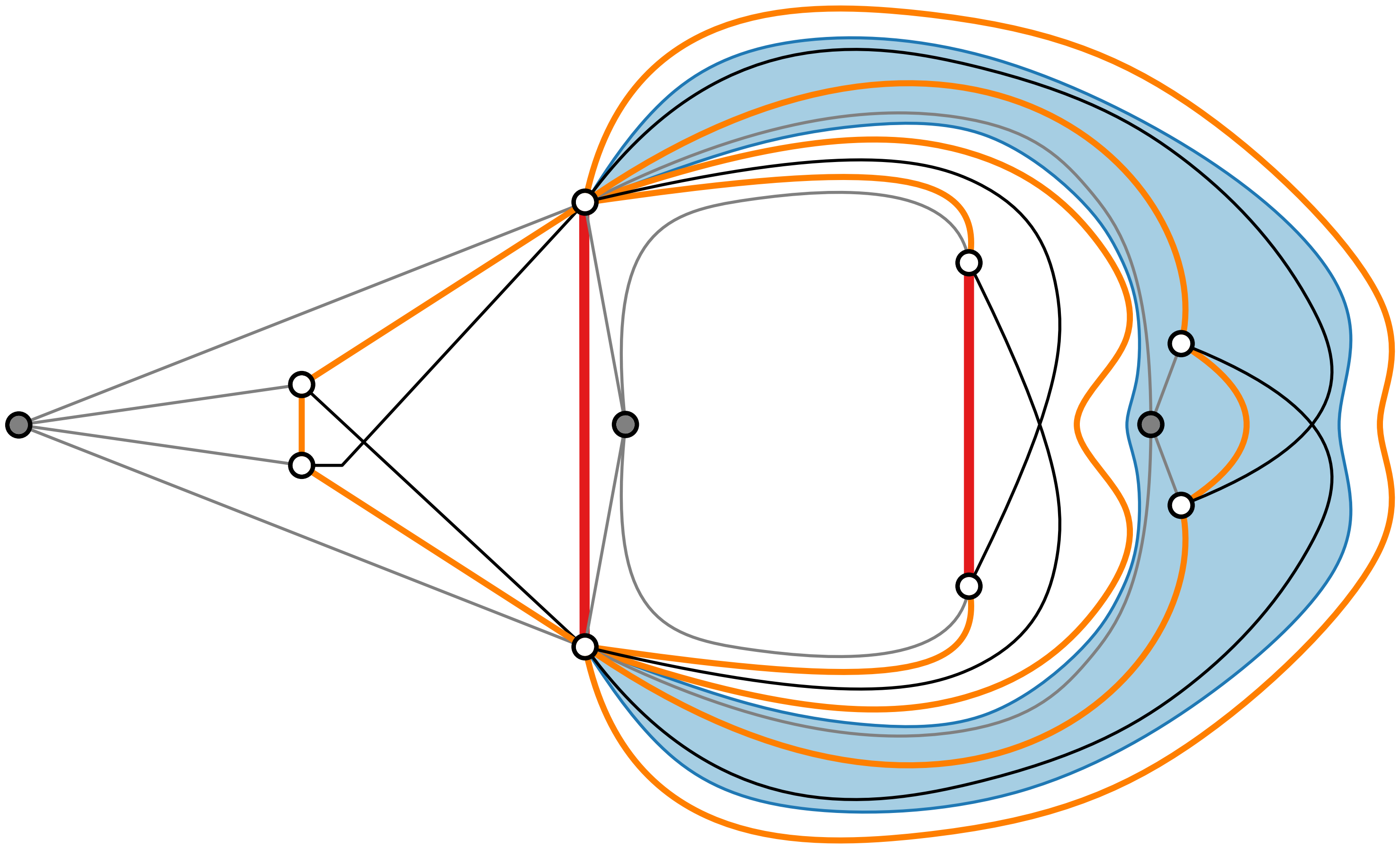
Algorithm Step 3: Drawing Procedure

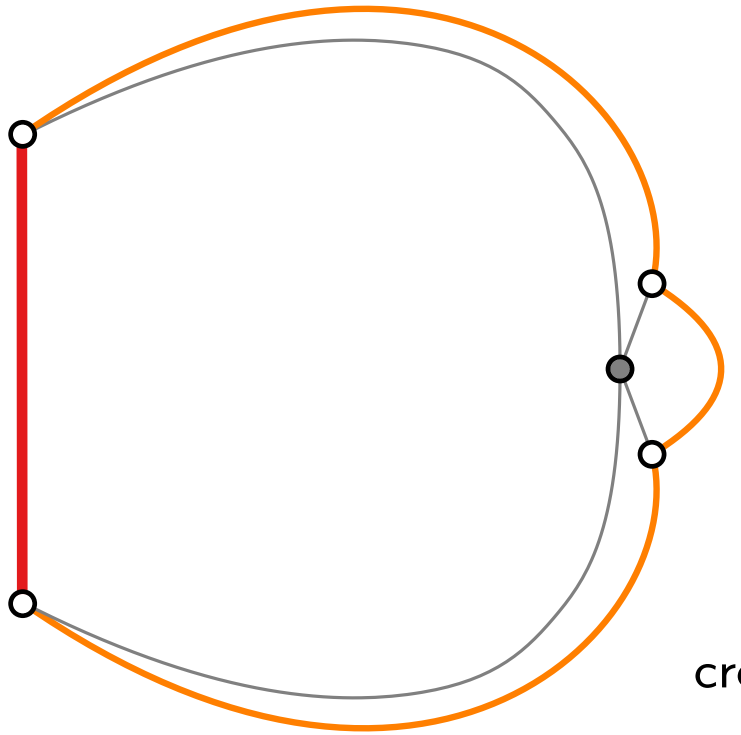


Algorithm Step 3: Drawing Procedure



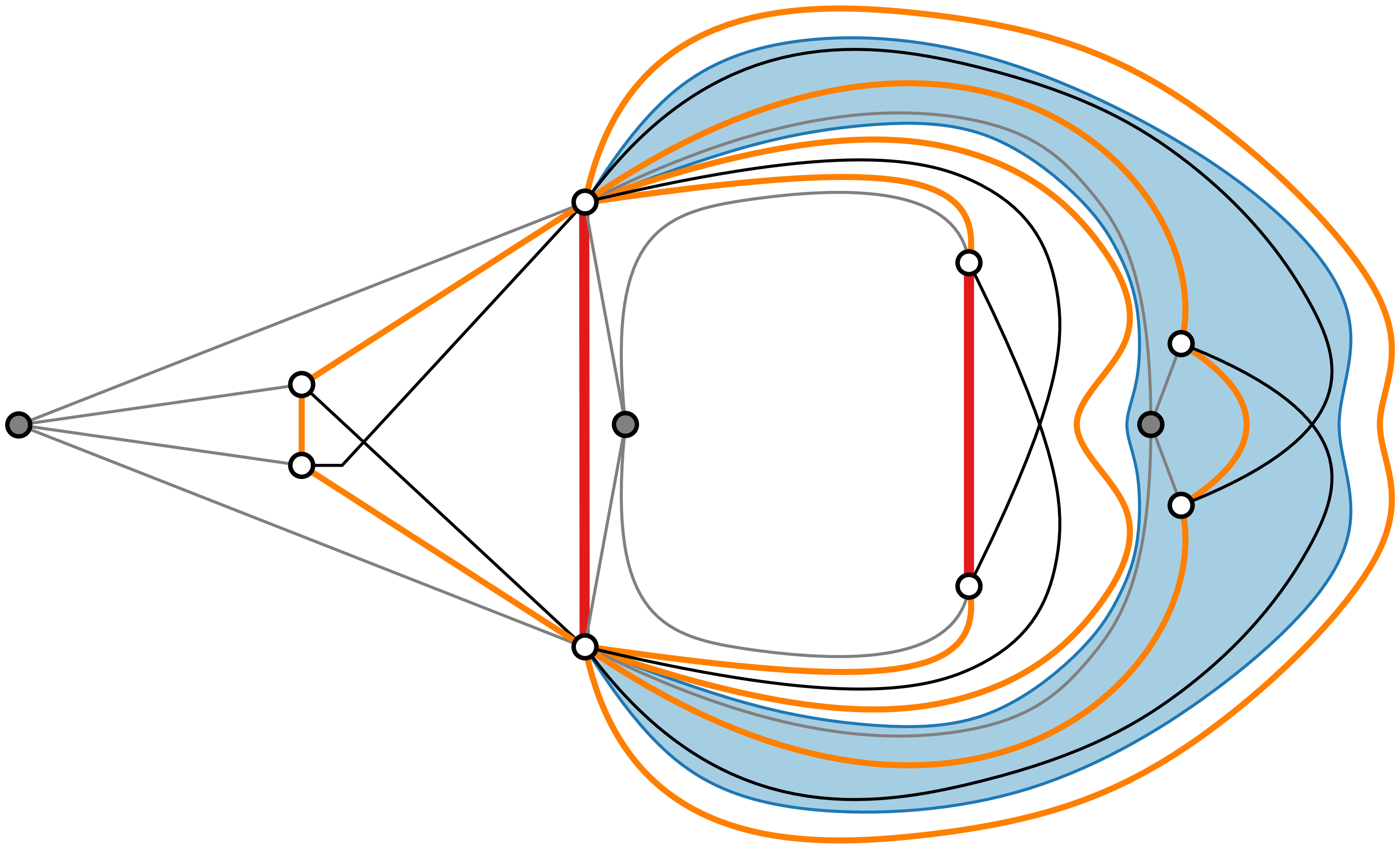
Algorithm Step 3: Drawing Procedure

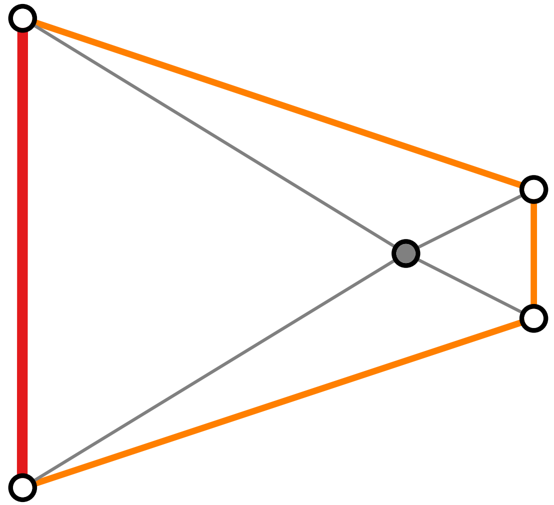




remove
crossing edges

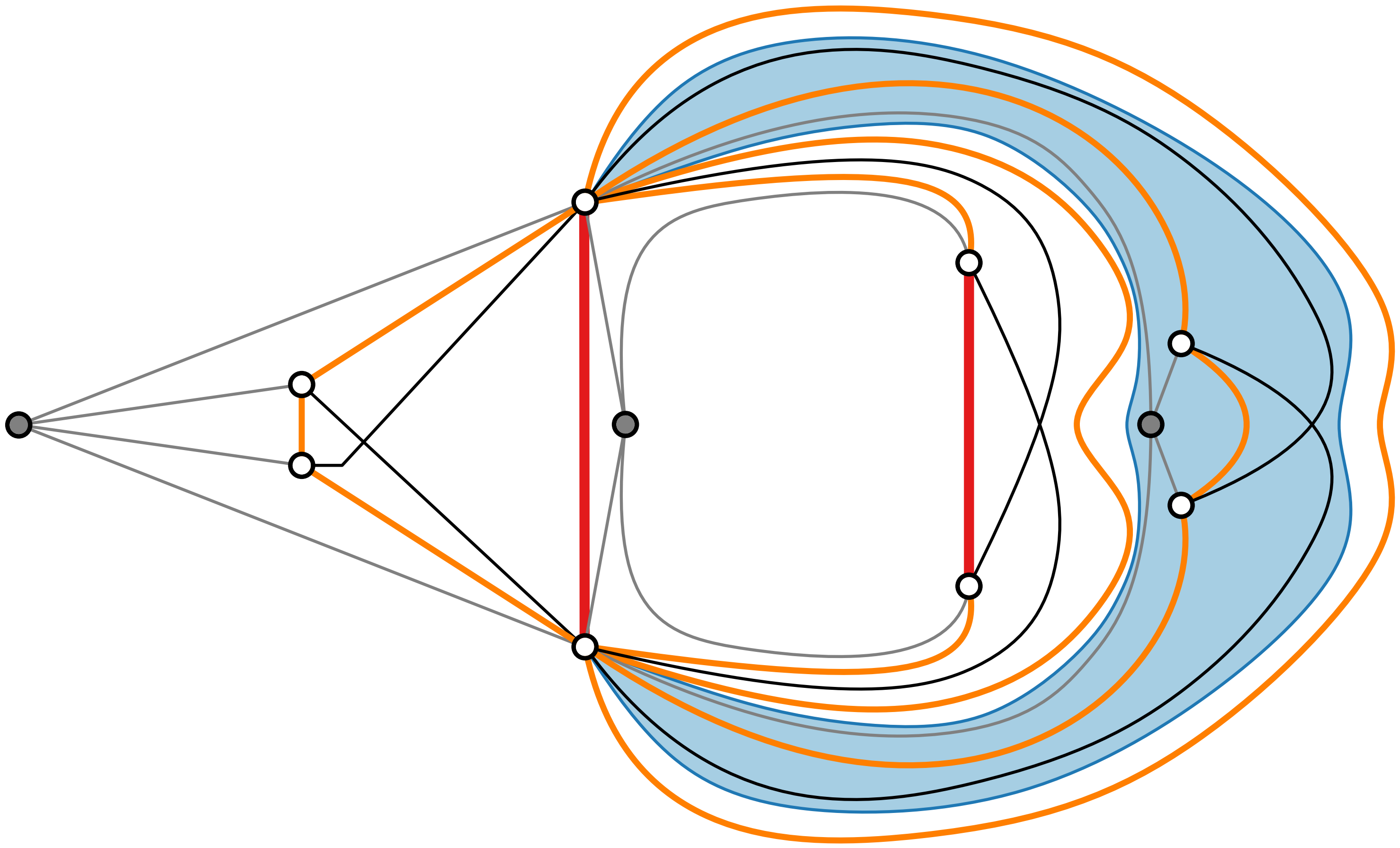
Algorithm Step 3: Drawing Procedure

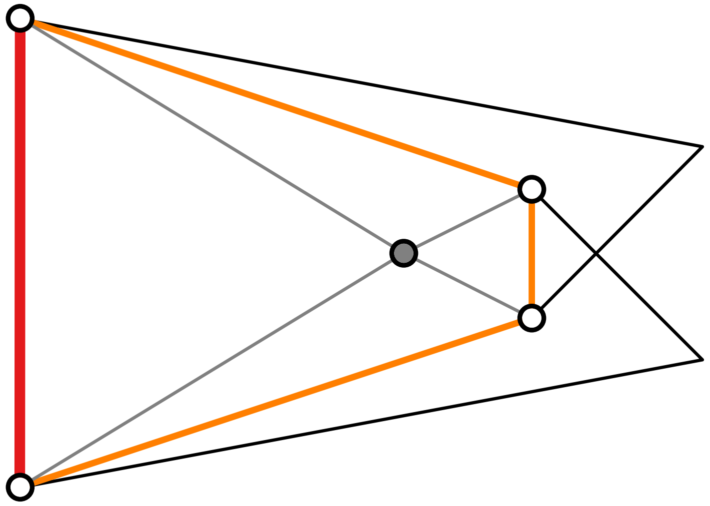




apply Chiba et al.

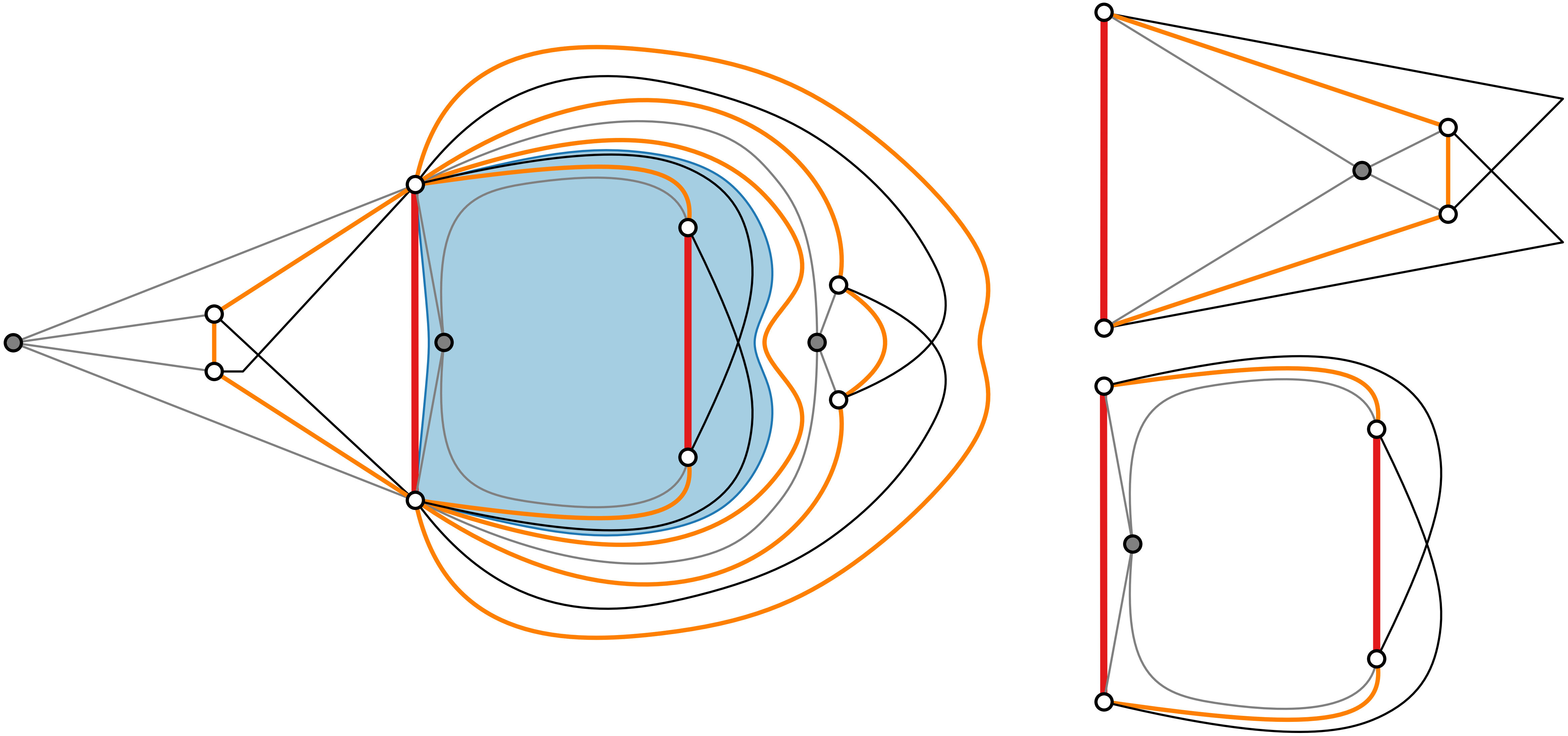
Algorithm Step 3: Drawing Procedure



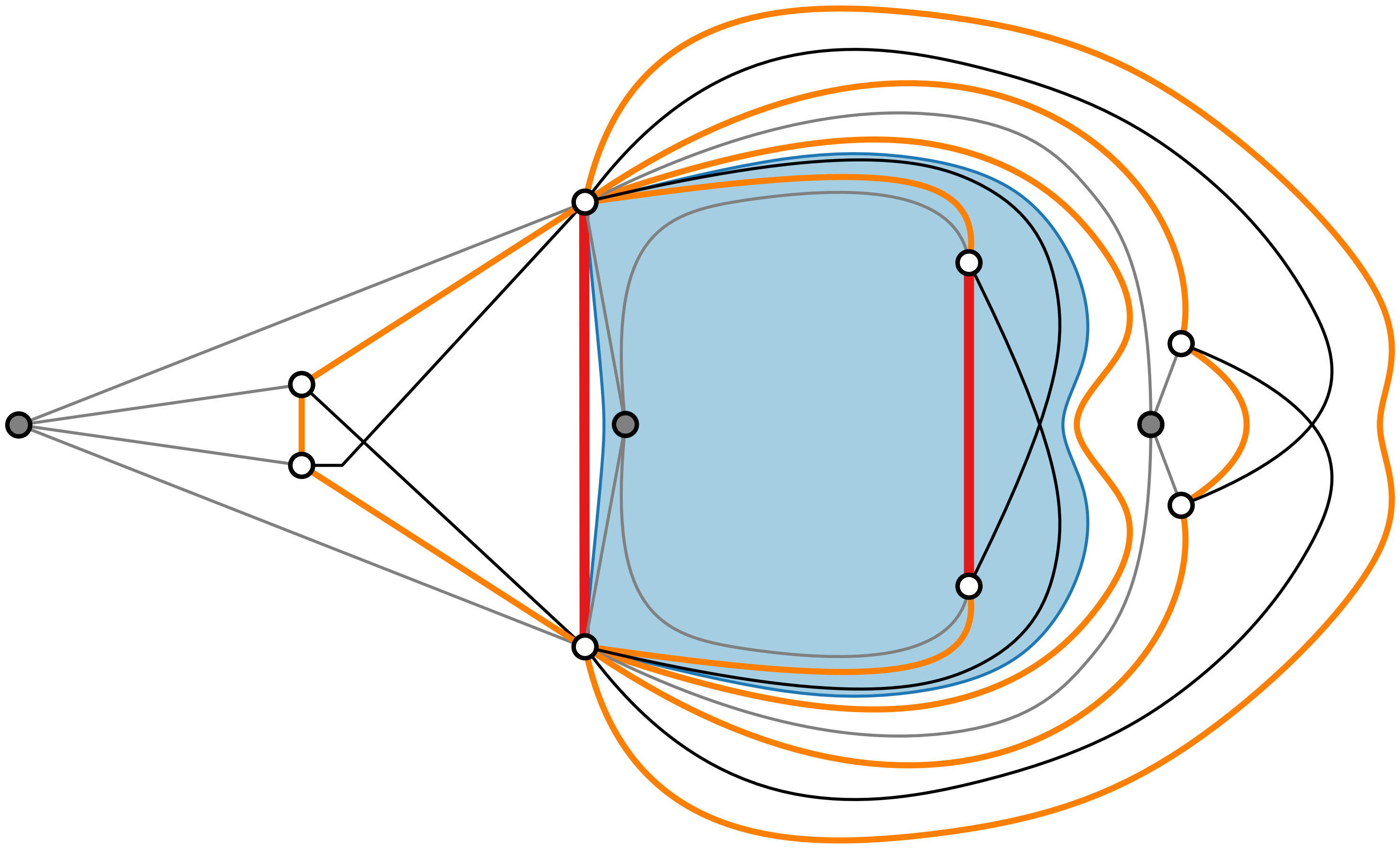


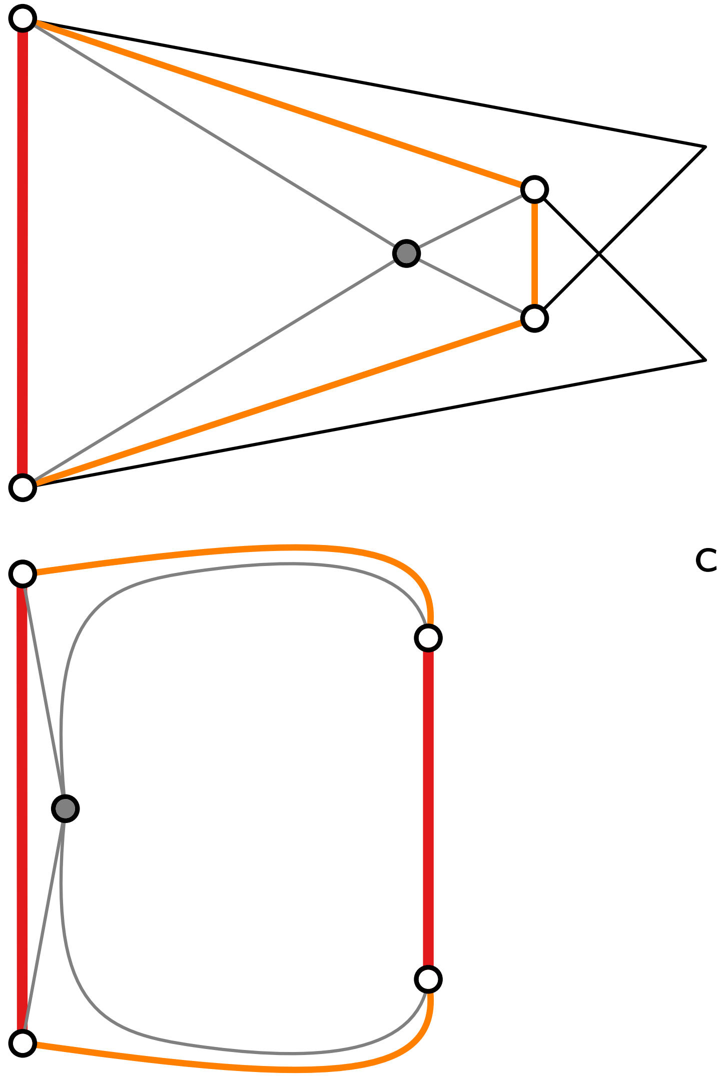
reinsert
crossing edges

Algorithm Step 3: Drawing Procedure



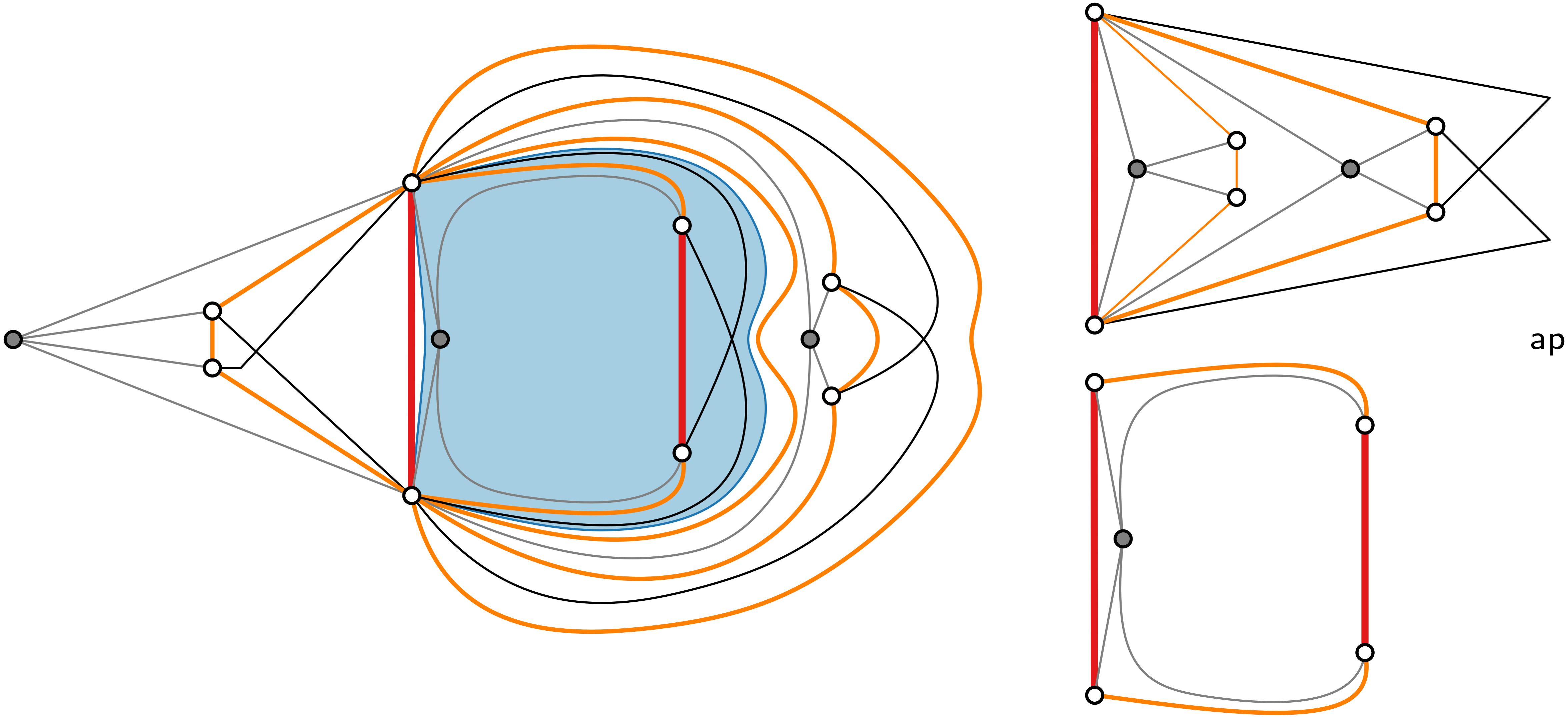
Algorithm Step 3: Drawing Procedure





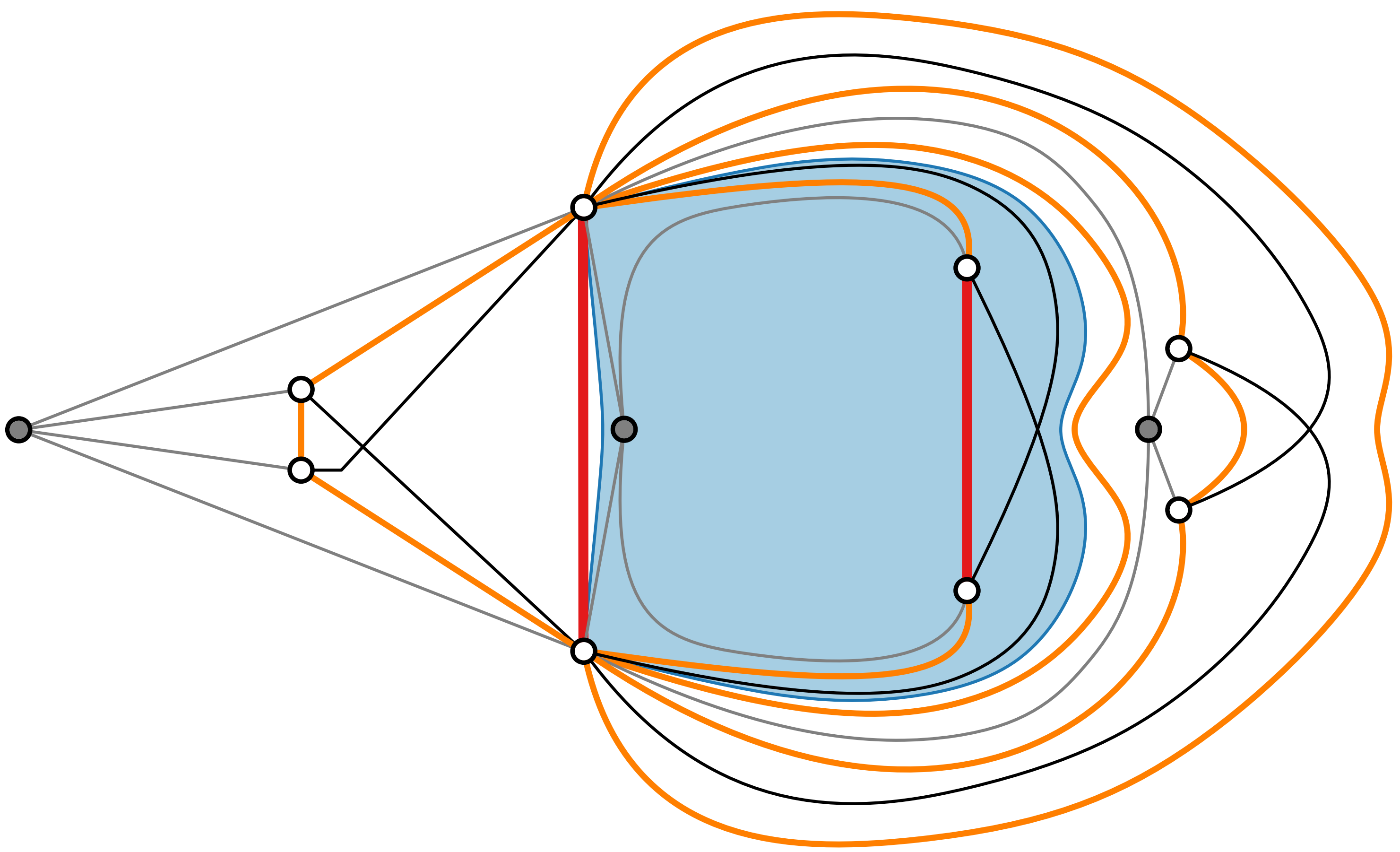
remove
crossing edges

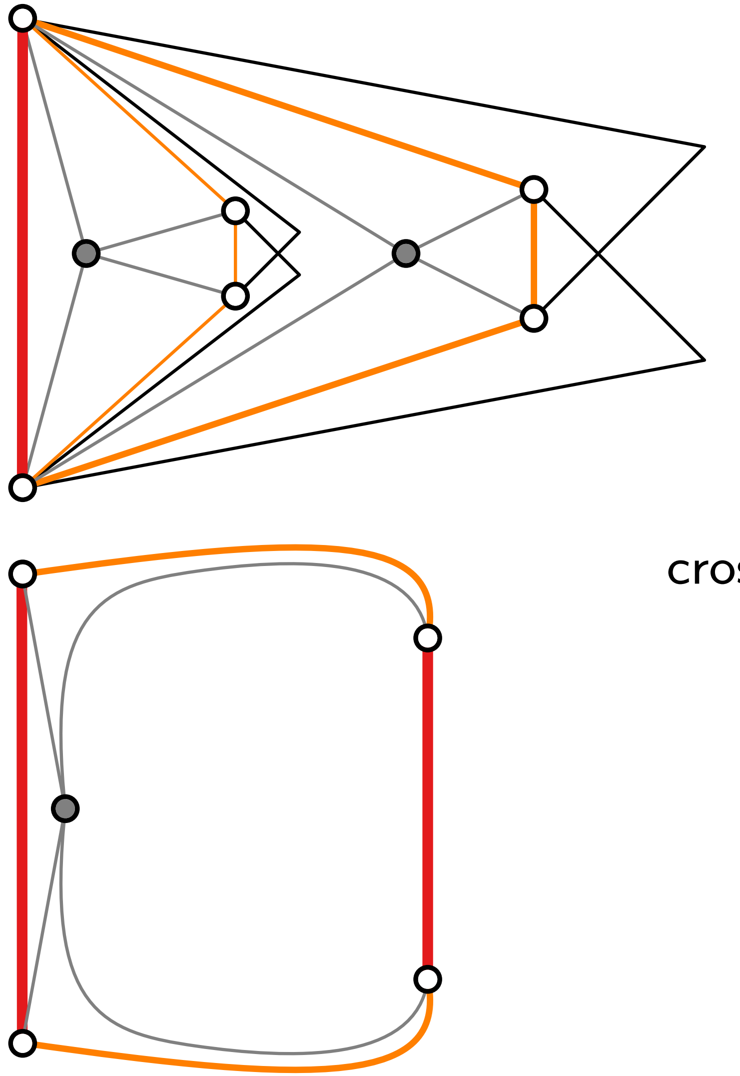
Algorithm Step 3: Drawing Procedure



apply Chiba et al.

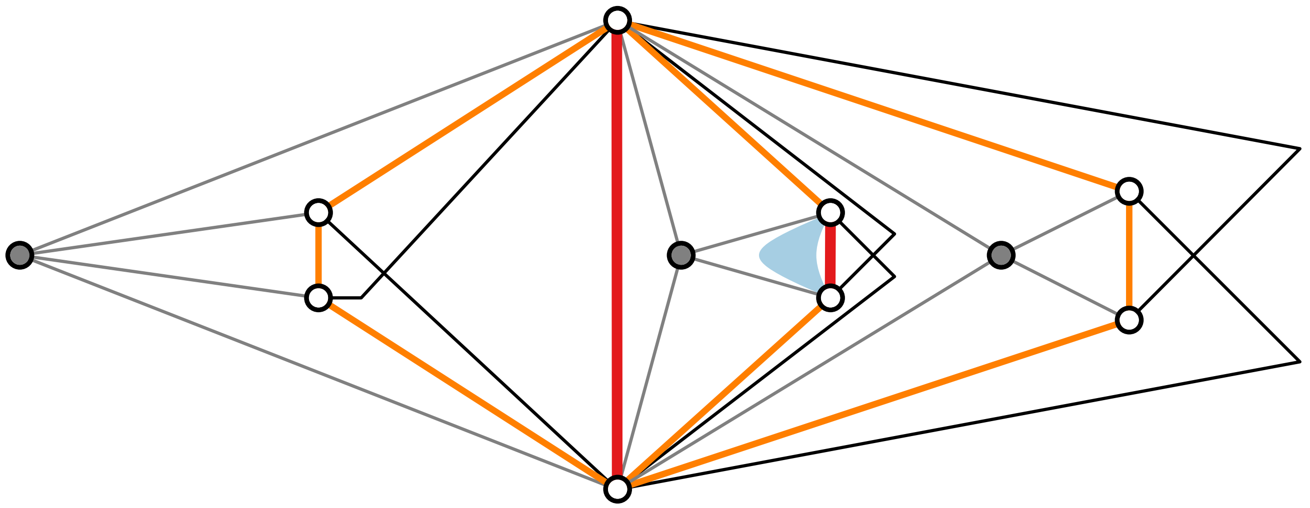
Algorithm Step 3: Drawing Procedure

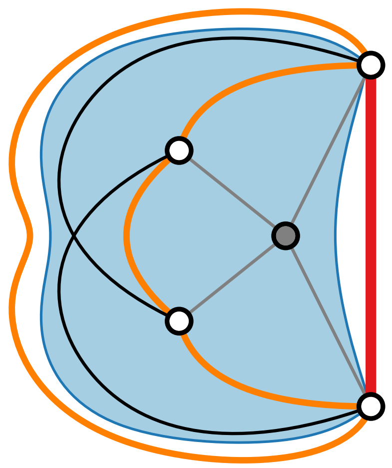




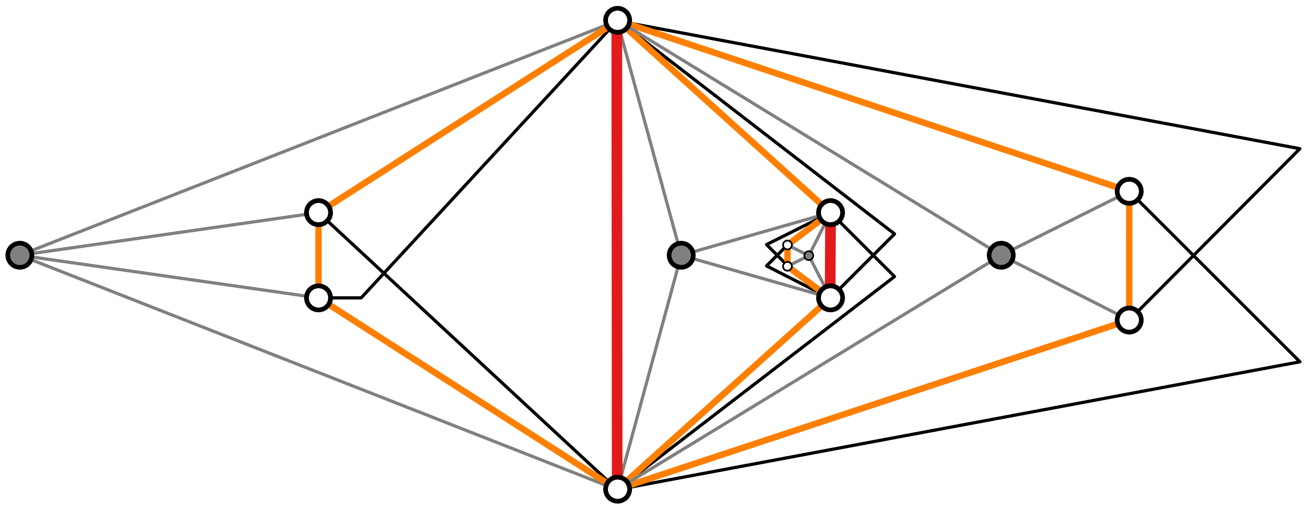
reinsert crossing edges

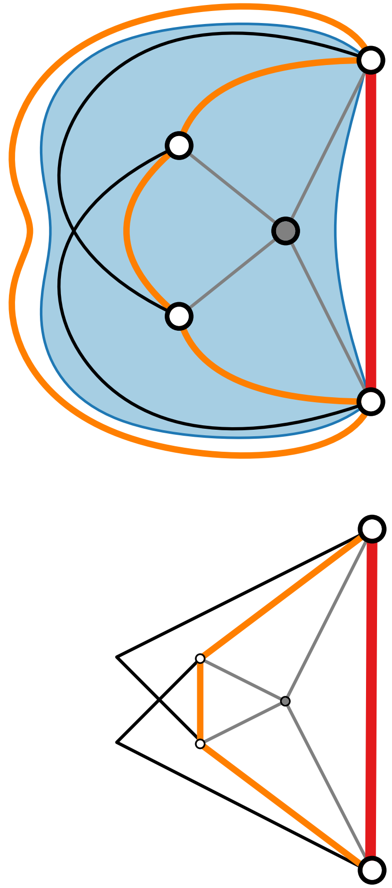
Algorithm Step 3: Drawing Procedure





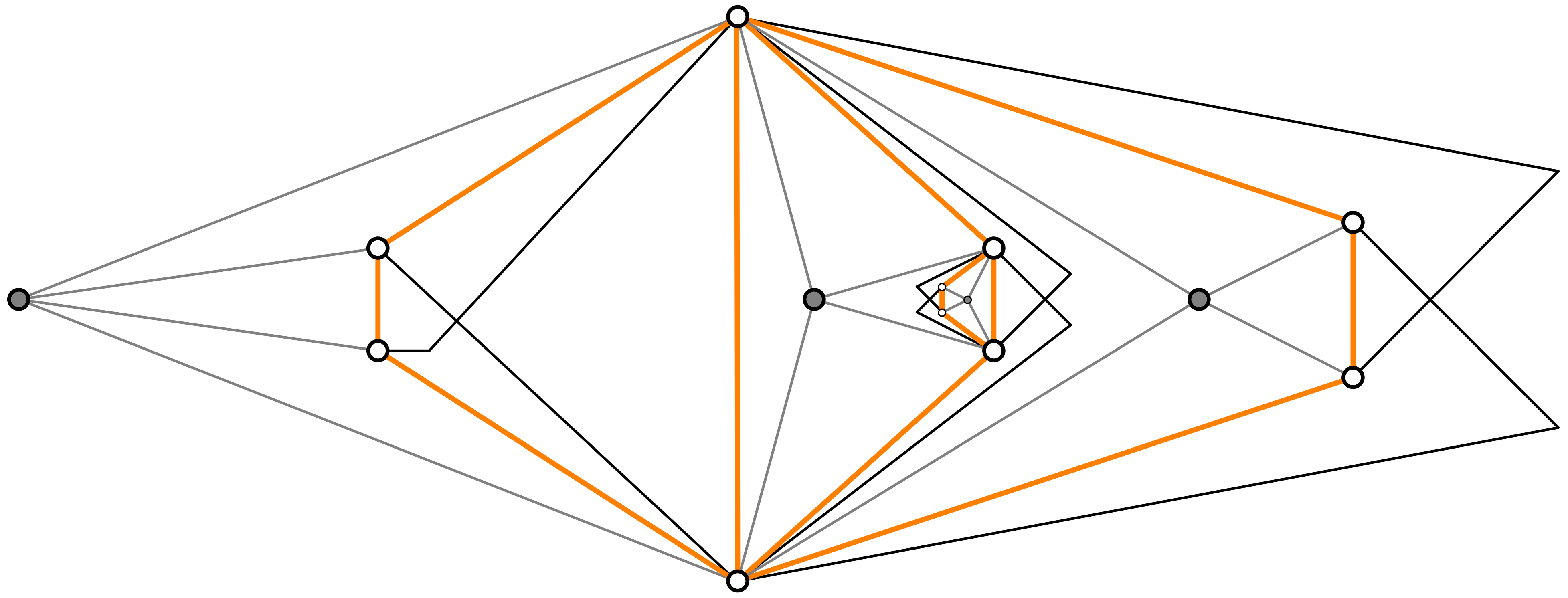
Algorithm Step 3: Drawing Procedure



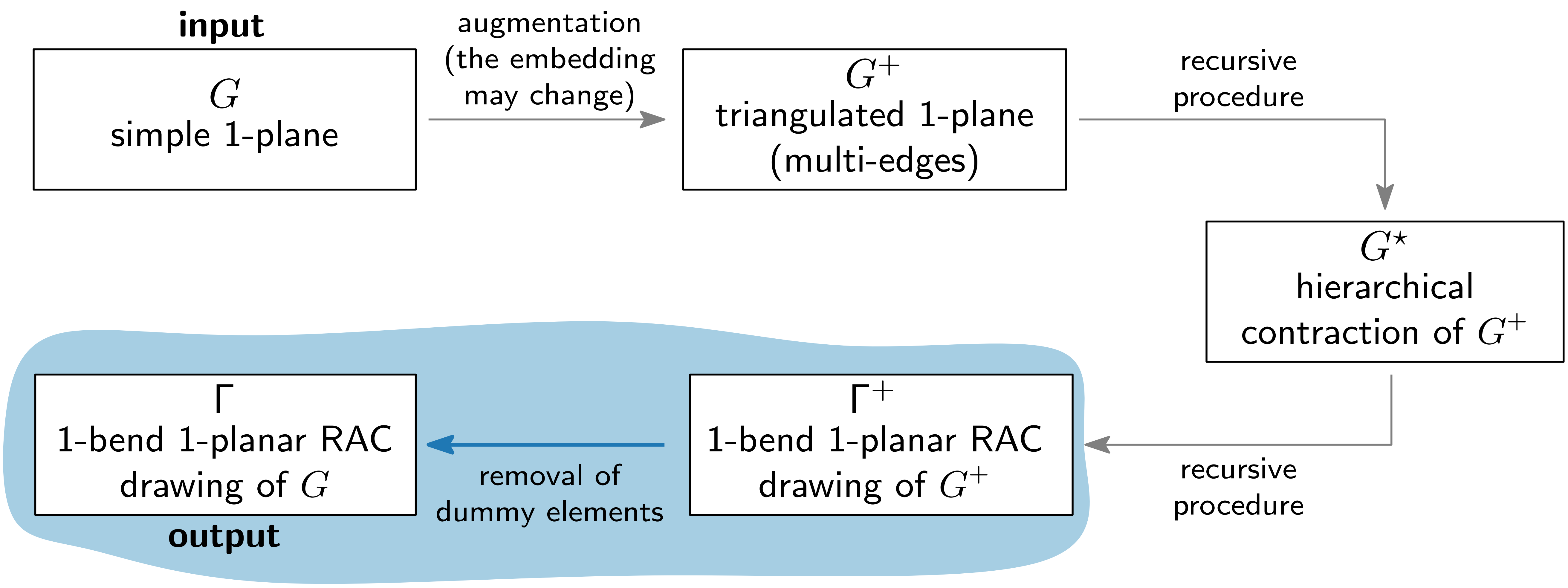


Algorithm Step 3: Drawing Procedure

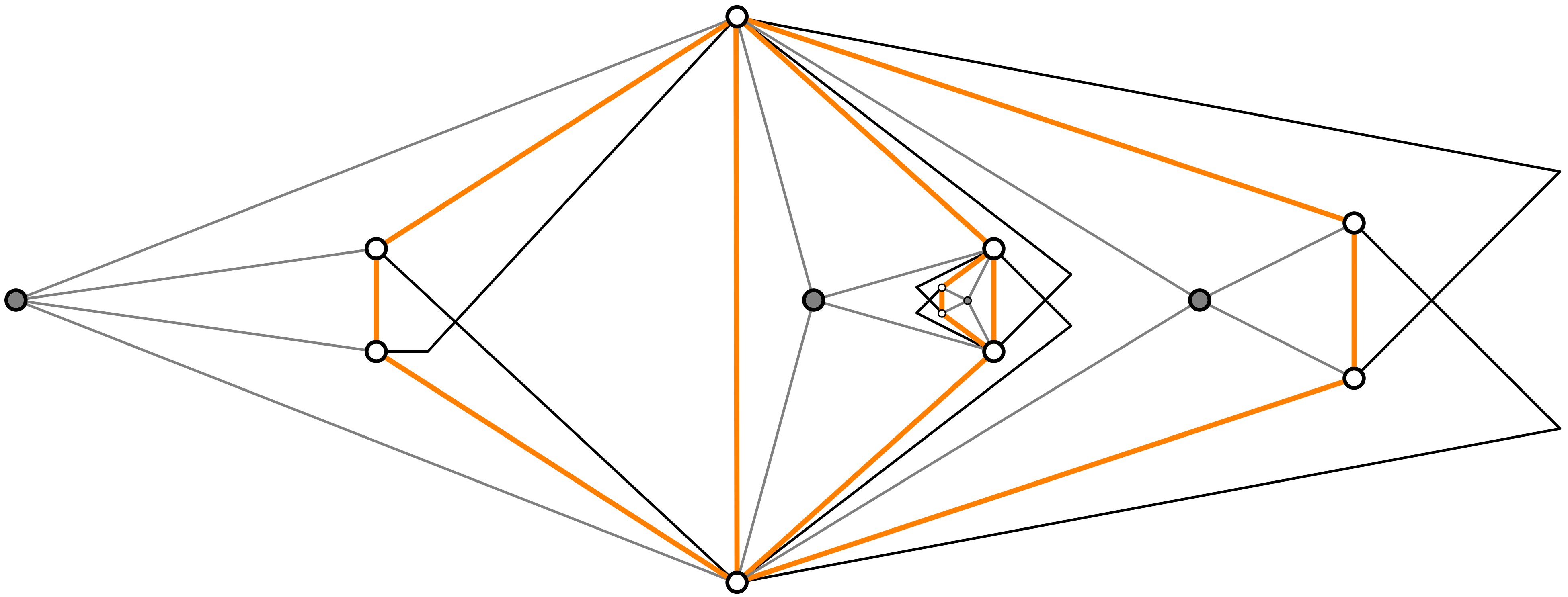
Γ^+: 1-bend 1-planar RAC drawing of G^+



Algorithm Outline

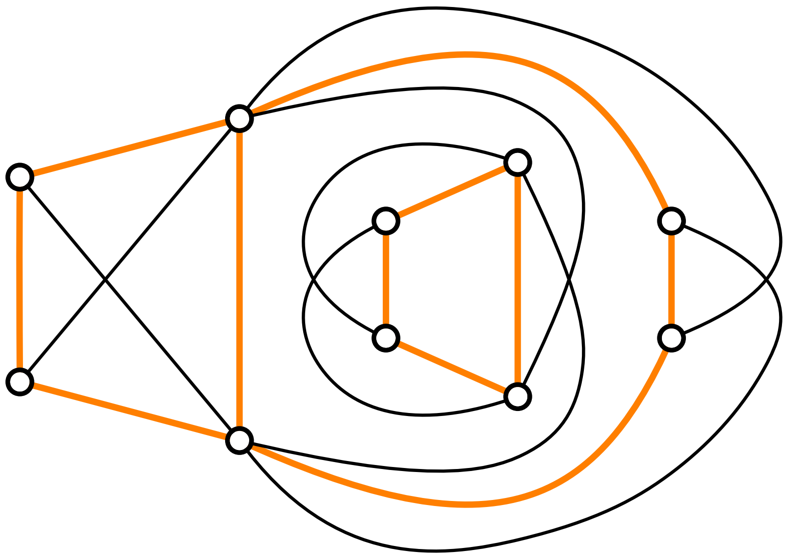


Algorithm Step 4: Removal of Dummy Vertices

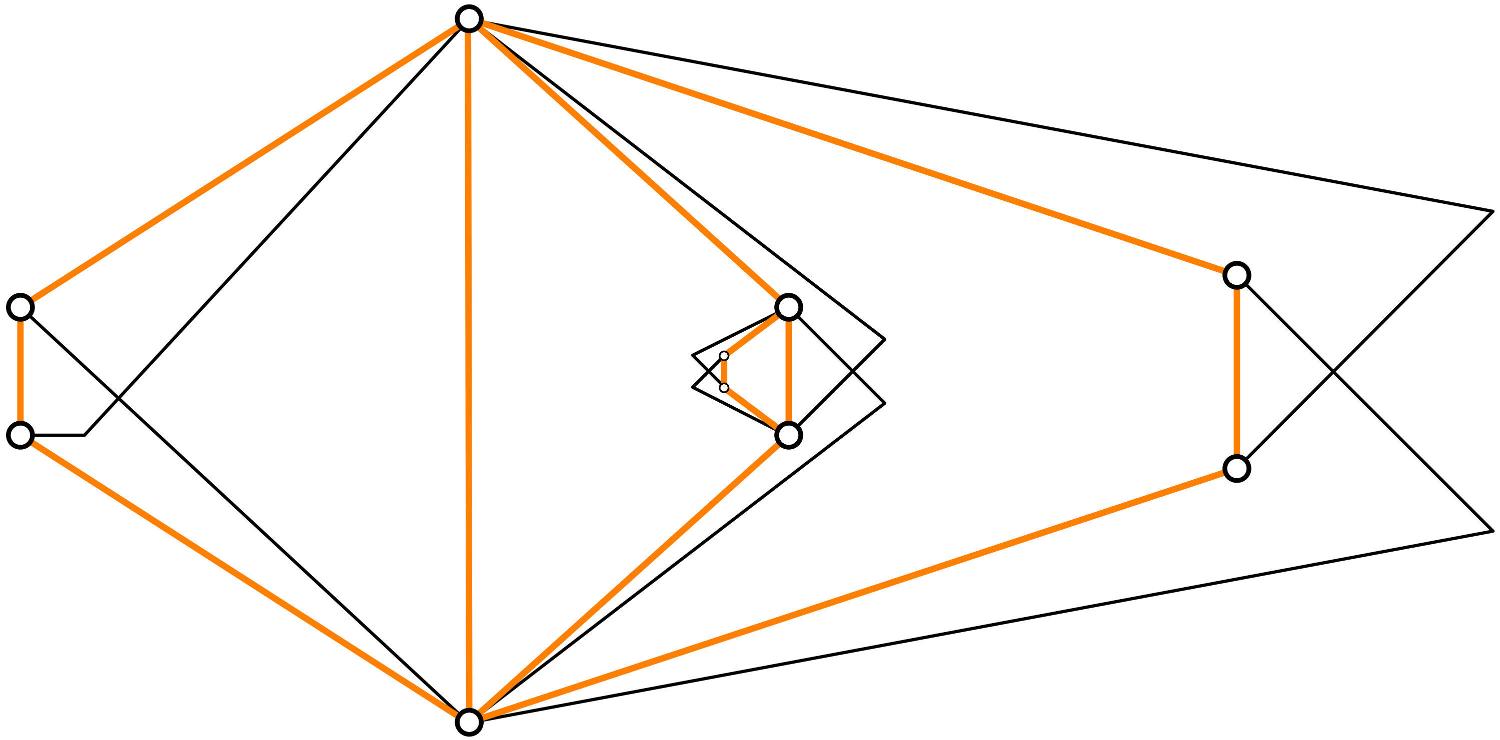


Algorithm Step 4: Removal of Dummy Vertices

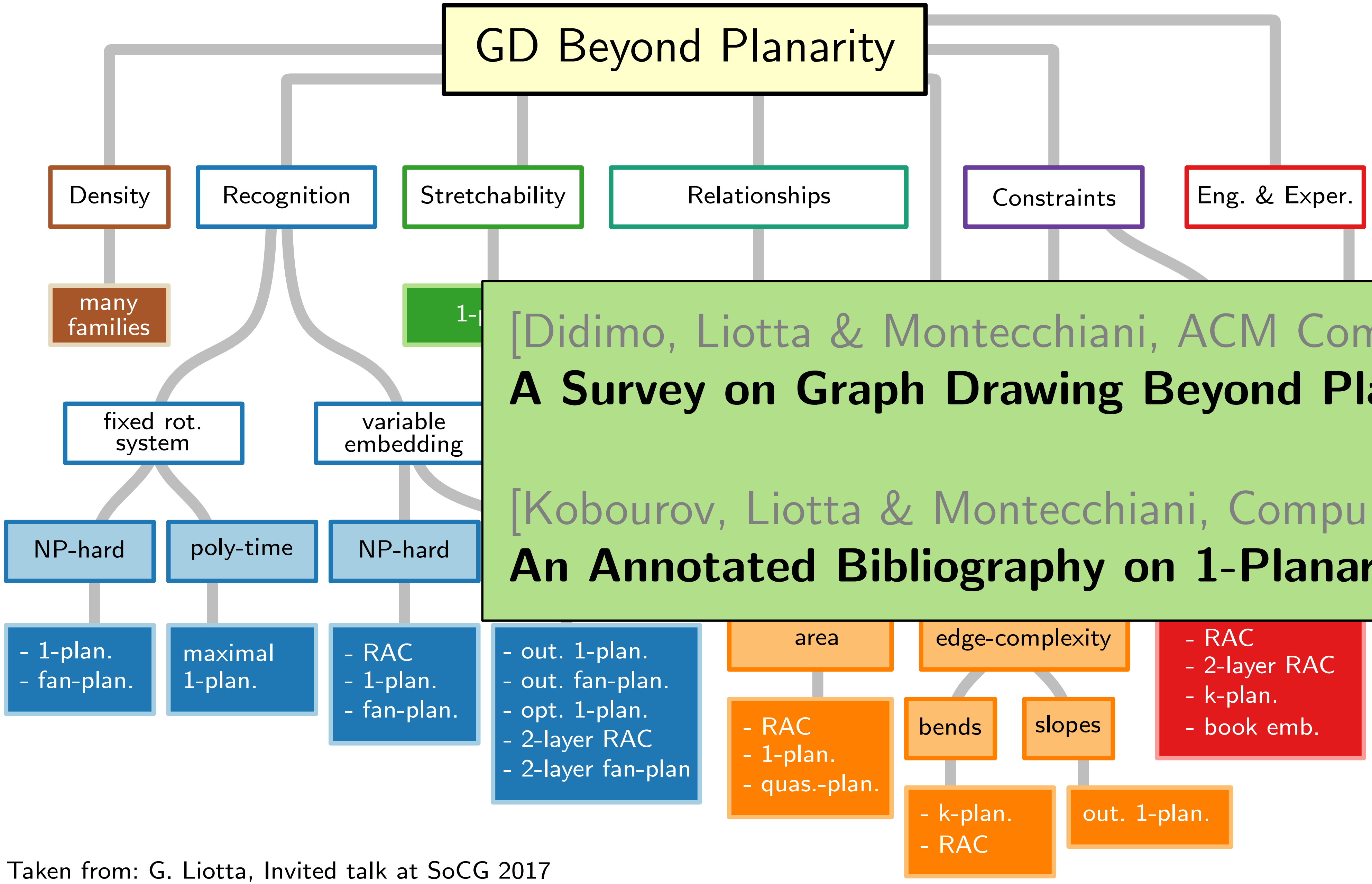
G: simple 1-plane graph



Γ: 1-bend 1-planar RAC drawing of G



GD Beyond Planarity: a Taxonomy



[Didimo, Liotta & Montecchiani, ACM Comput. Surv. 2019]
A Survey on Graph Drawing Beyond Planarity.

[Kobourov, Liotta & Montecchiani, Compu. Sci. Review 2017]
An Annotated Bibliography on 1-Planarity.

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Literature

Books and surveys:

- [Didimo, Liotta & Montecchiani 2019] A Survey on Graph Drawing Beyond Planarity
- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
- [Hong and Tokuyama, editors '20] Beyond Planar Graphs

Some references for proofs:

- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
- [Chimani, Kindermann, Montecchani, Valtr '19] Crossing Numbers of Beyond-Planar Graphs
- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angelini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
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