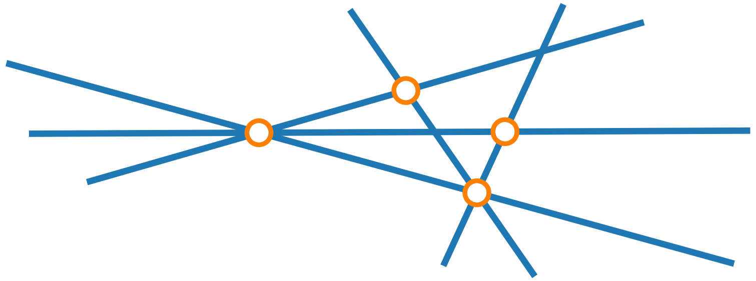
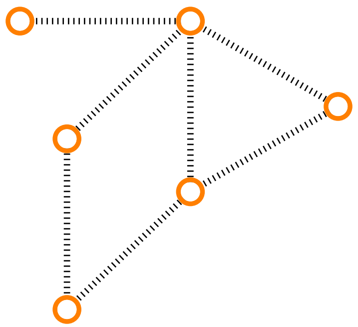
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 11:
The Crossing Lemma
and Its Applications





Part I:
Definition and Hanani–Tutte

Alexander Wolff

Crossing Number and Topological Graphs

For a graph G, the **crossing number** $\mathrm{cr}(G)$ is the smallest number of edge crossings in a drawing of G (in the plane).

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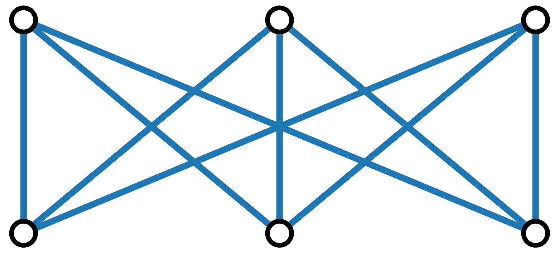
Example.
cr$(K_{3,3}) =$ 9?



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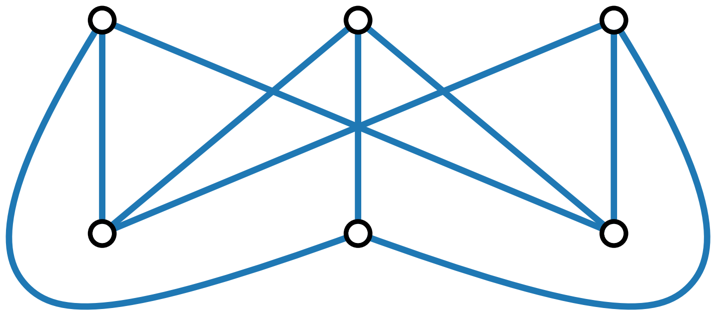
$\mathrm{cr}(K_{3,3}) = 5$?



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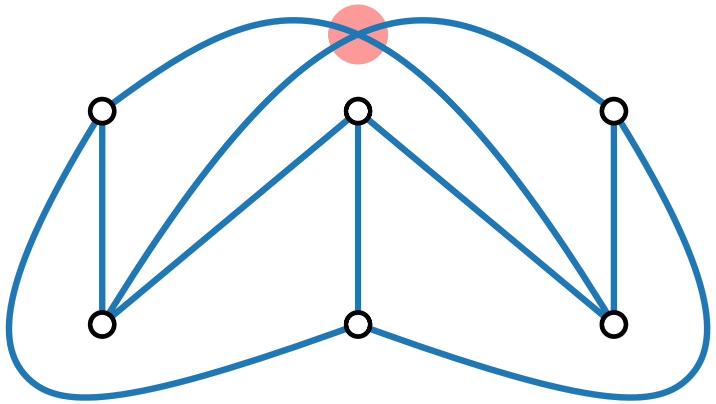

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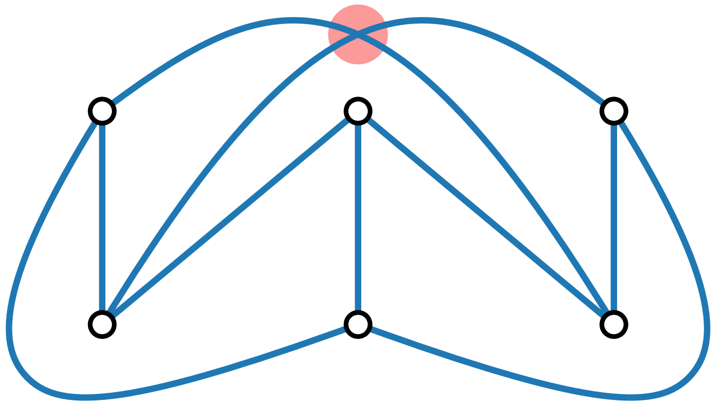

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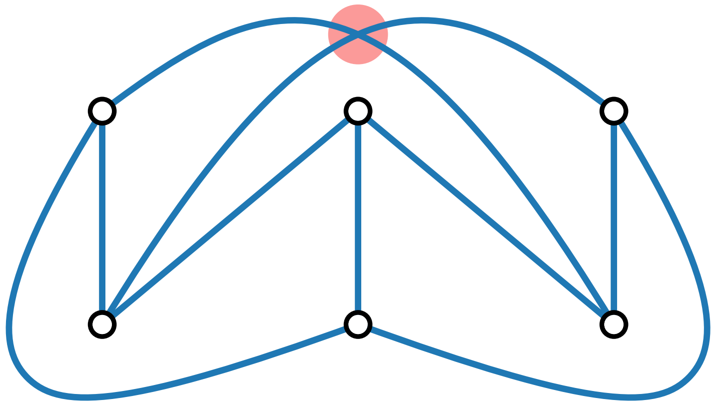

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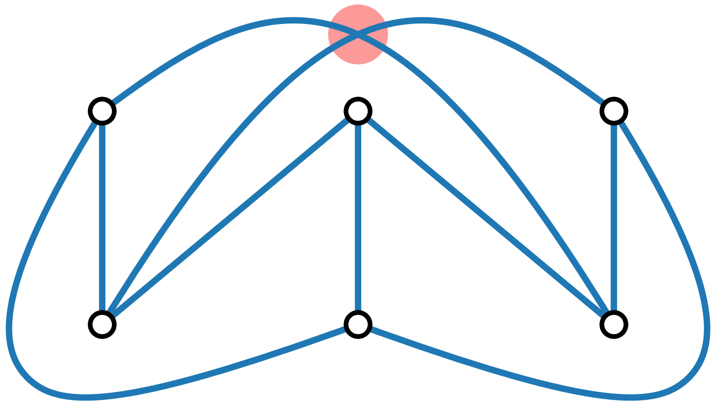

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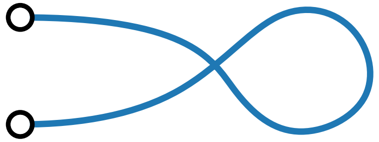

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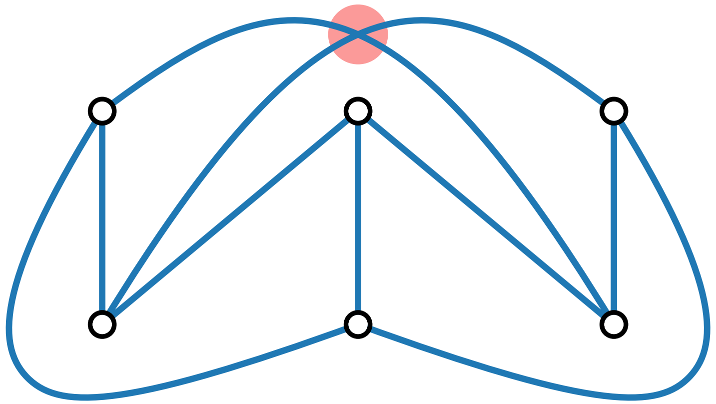

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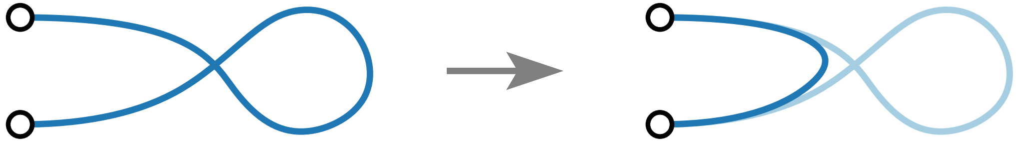

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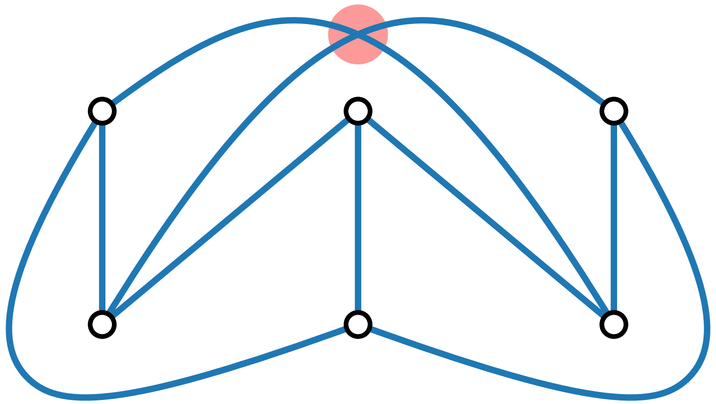

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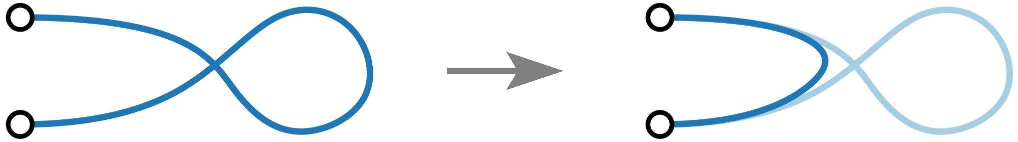

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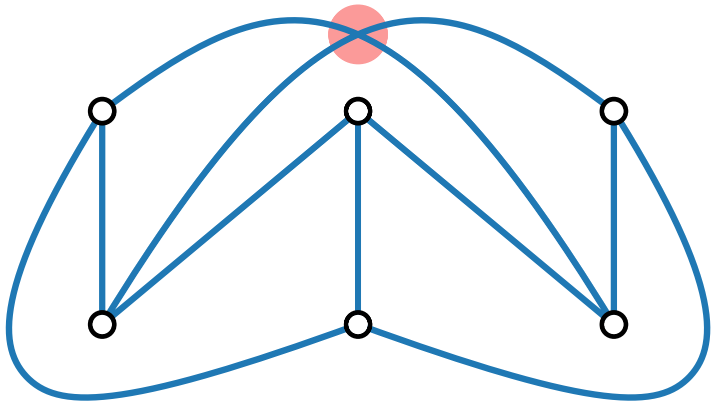

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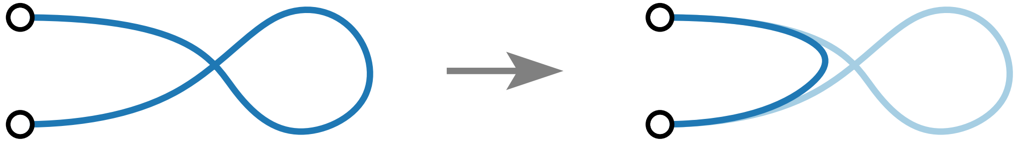

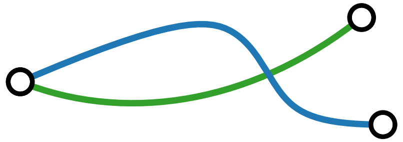

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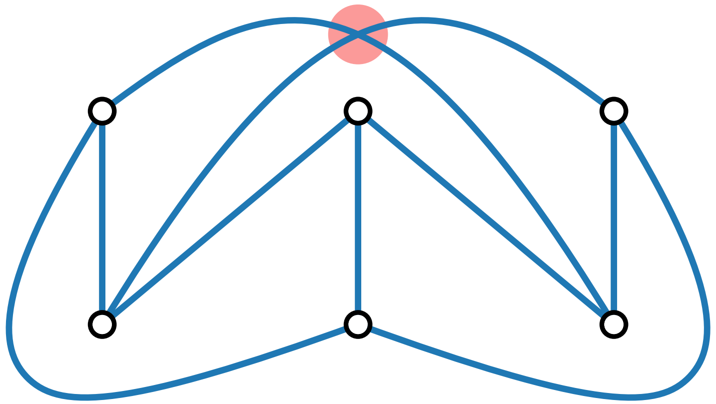

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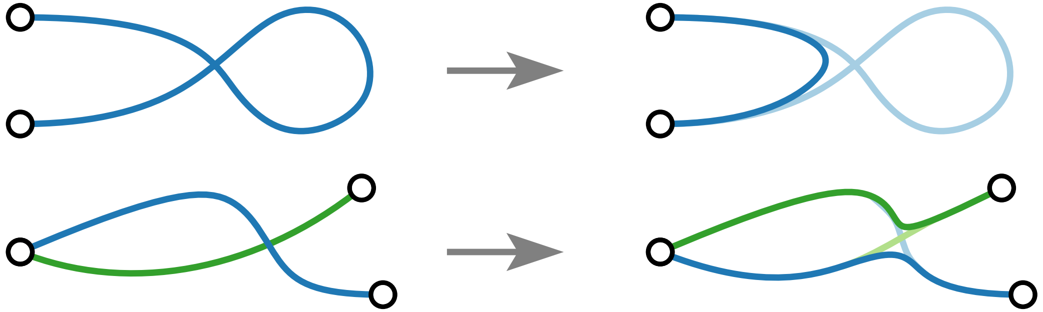

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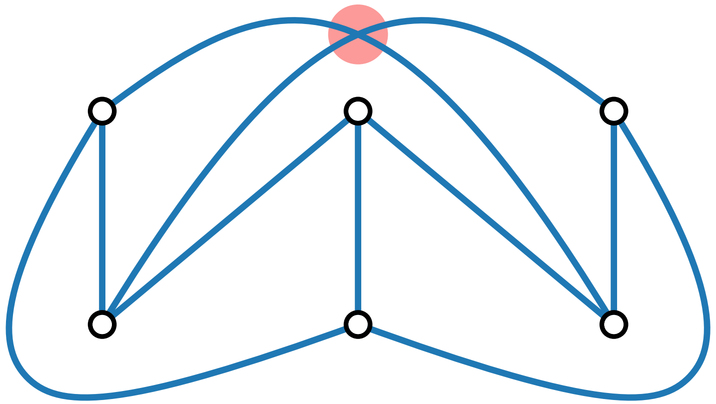

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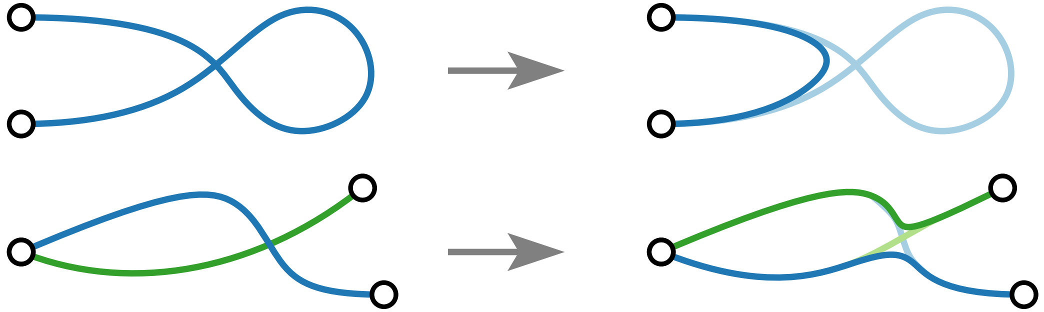

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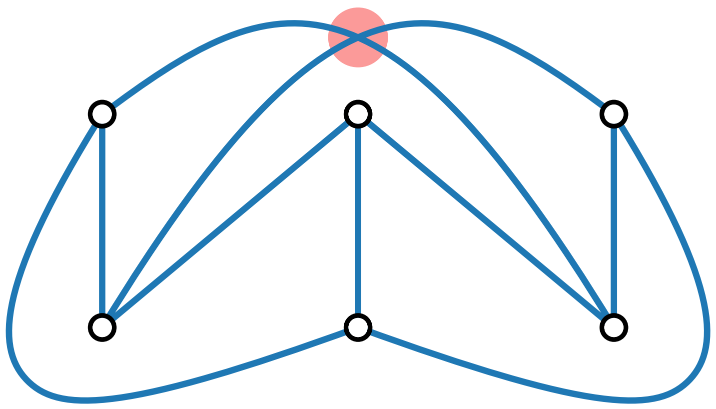

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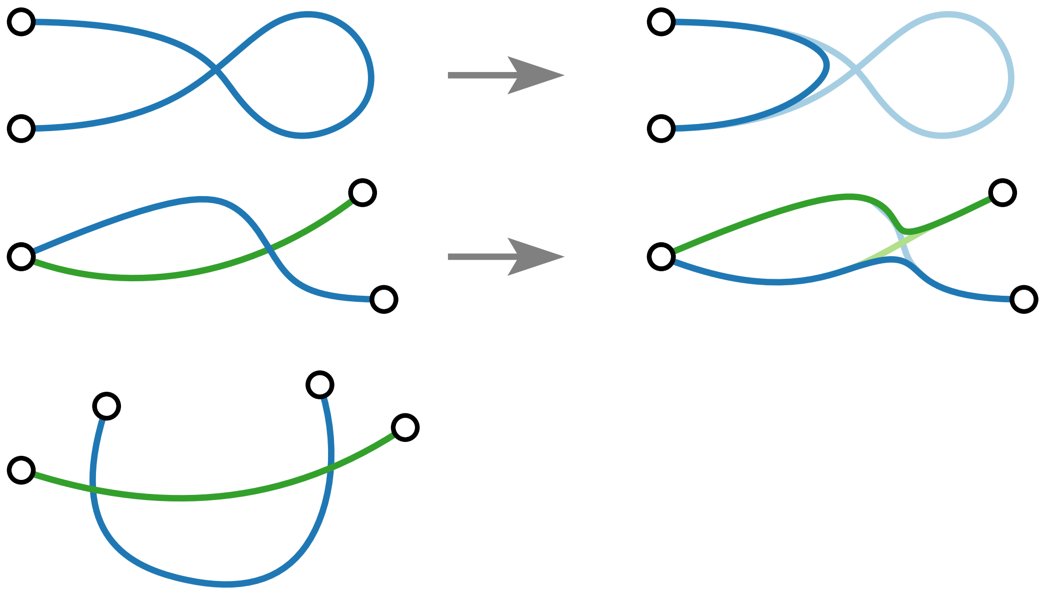

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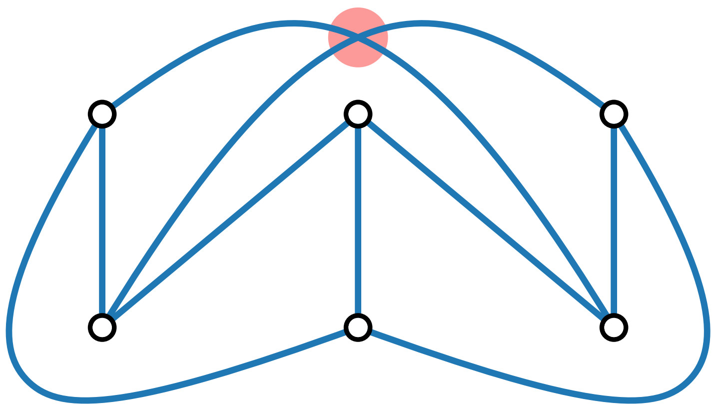

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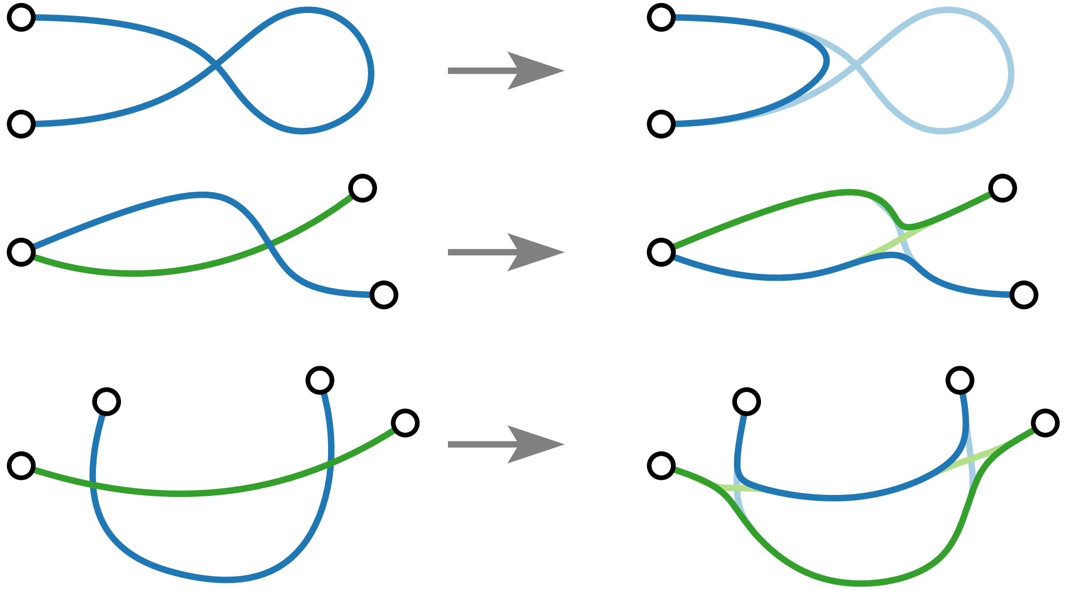

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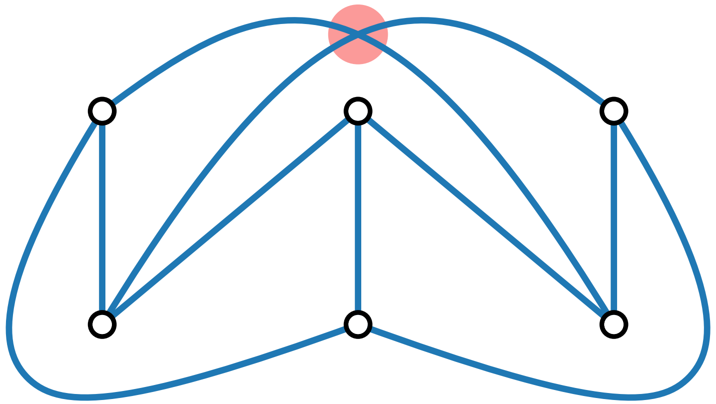

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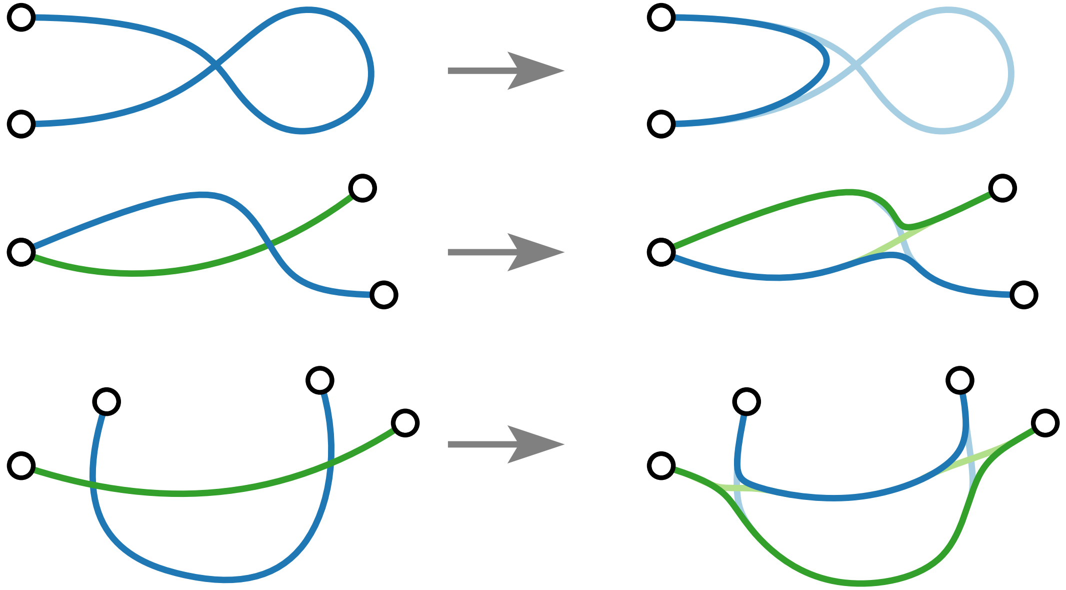

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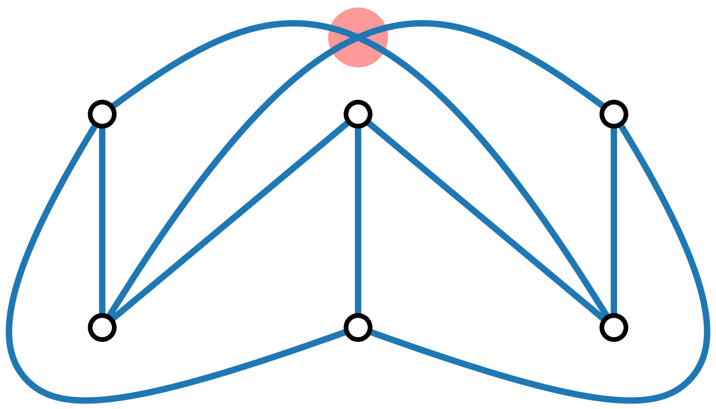

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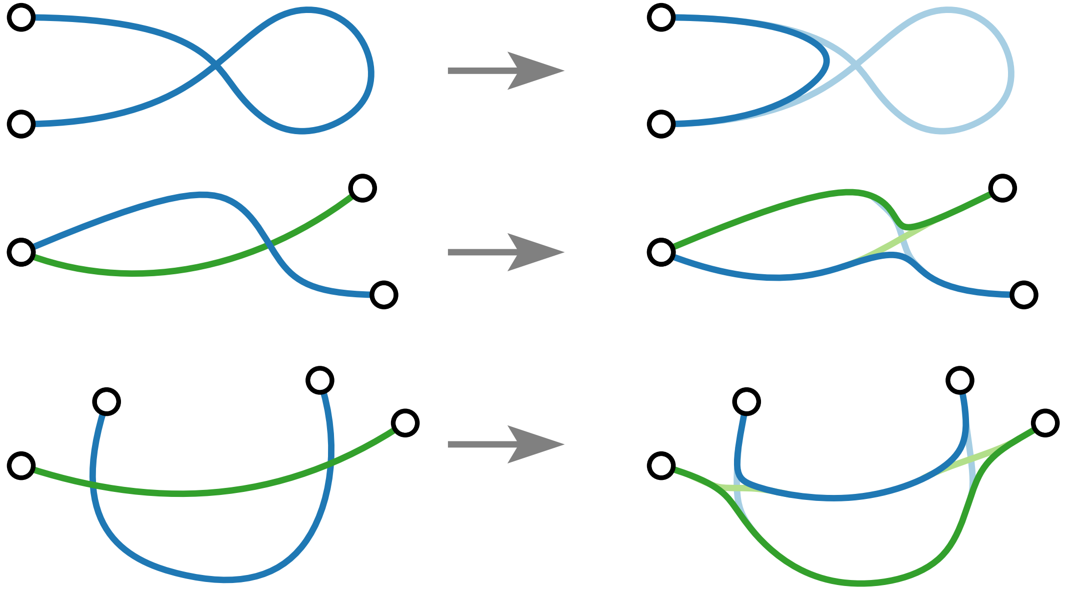

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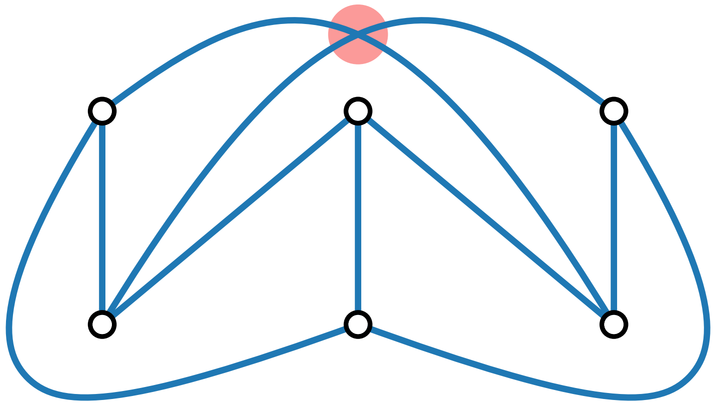

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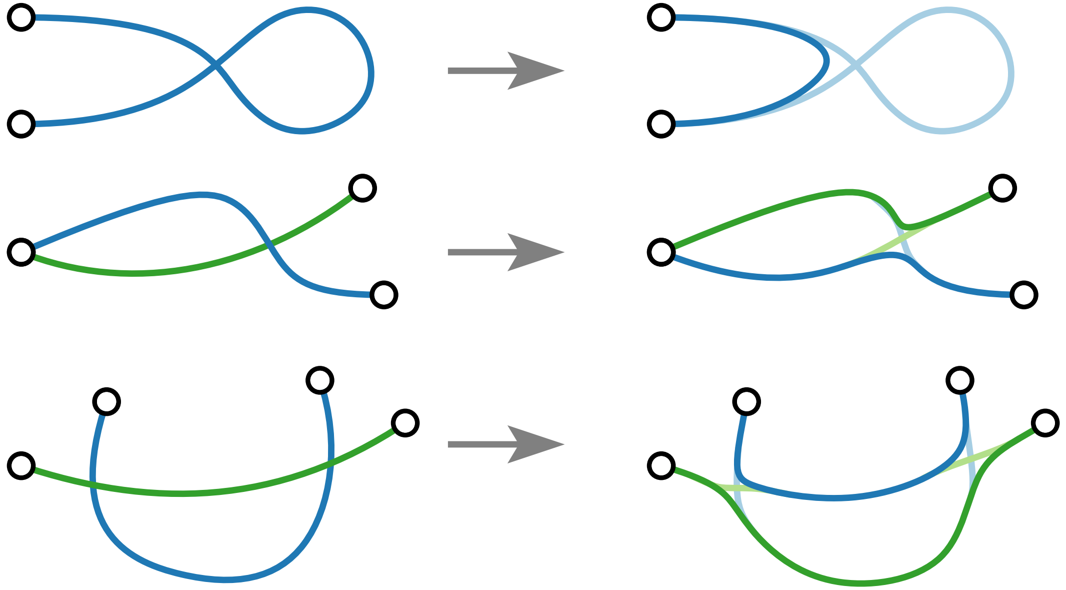

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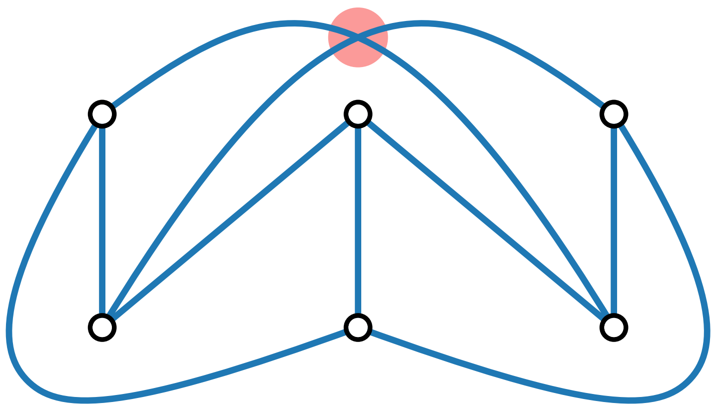

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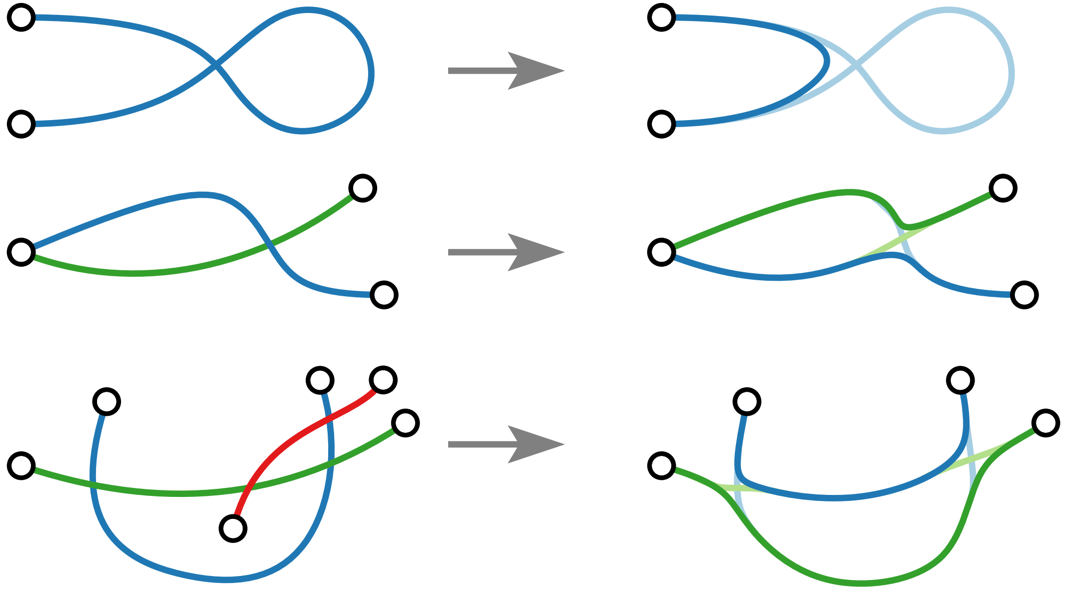

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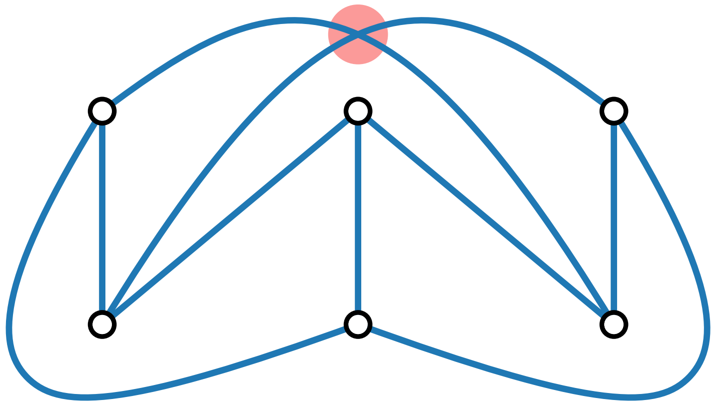

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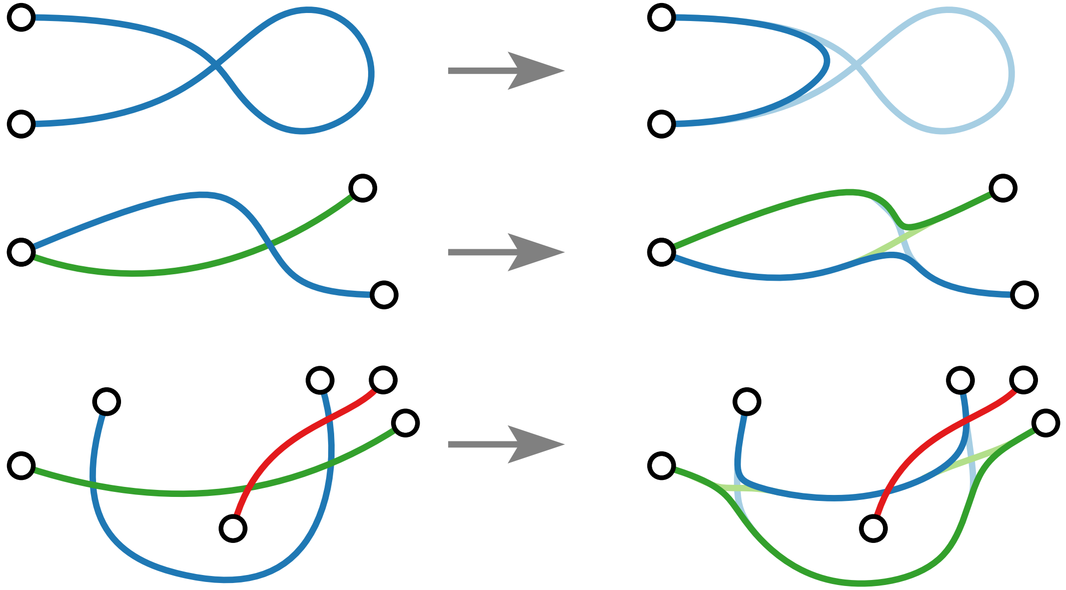

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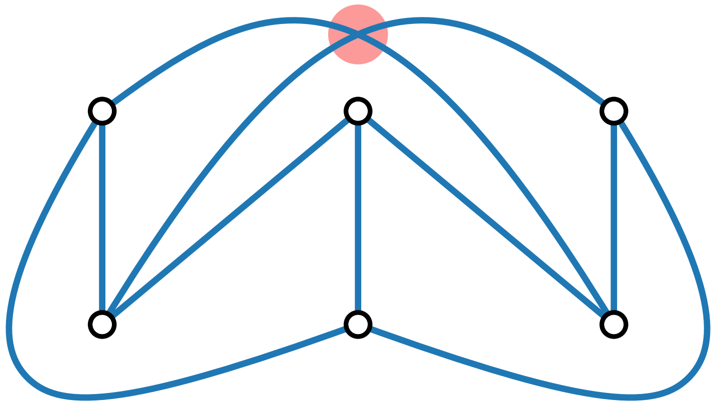

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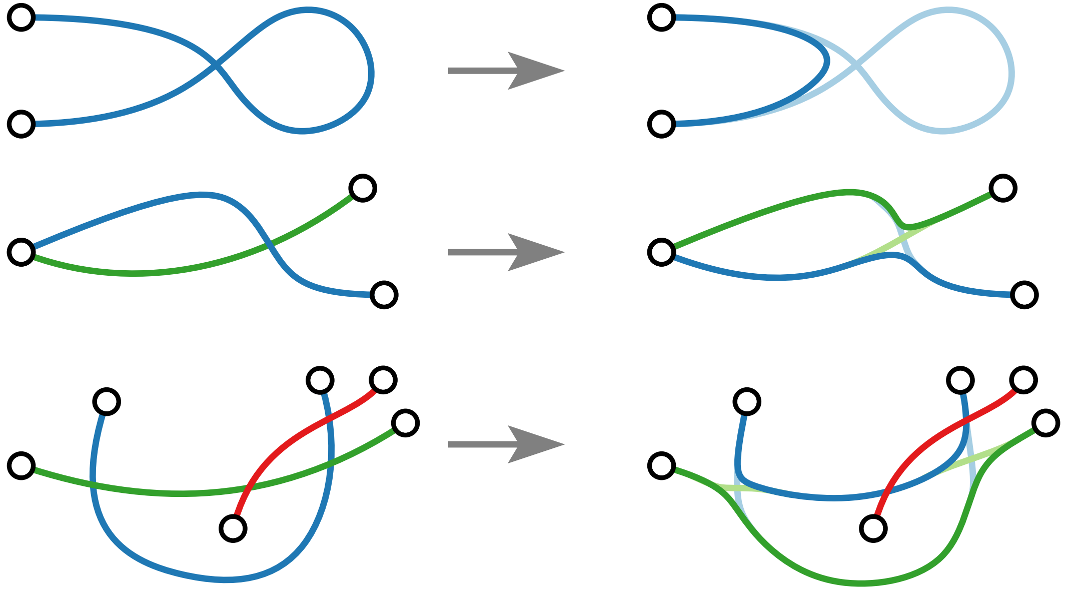

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Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

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Proof sketch.
Hanani showed that every drawing of K_5 and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

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Every non-planar graph has K_5 or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.

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Hence, there must be two edges on these paths that cross an odd number of times. $\square$

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The **odd crossing number** $\mathrm{ocr}(G)$ of G is the smallest number of pairs of edges that cross oddly in a drawing of G.

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Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with ocr$(G) <$ cr$(G) \leq 10$

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If Γ is a drawing of G and E_0 is the set of edges with only even numbers of crossings in Γ, then G can be drawn such that no edge in E_0 is involved in any crossings.

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Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
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The **pairwise crossing number** pcr(G) of G is the smallest number of pairs of edges that cross in a drawing of G.

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By definition ocr$(G) \leq$ pcr$(G) \leq$ cr(G)

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By definition ocr$(G) \leq$ pcr$(G) \leq$ cr(G)
Is pcr$(G) =$ cr(G)?

Hanani–Tutte Theorem

Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The **odd crossing number** $\mathrm{ocr}(G)$ of G is the smallest number of pairs of edges that cross oddly in a drawing of G.

Corollary. $\mathrm{ocr}(G) = 0 \Rightarrow \mathrm{cr}(G) = 0$

Is $\mathrm{ocr}(G) = \mathrm{cr}(G)$? No!

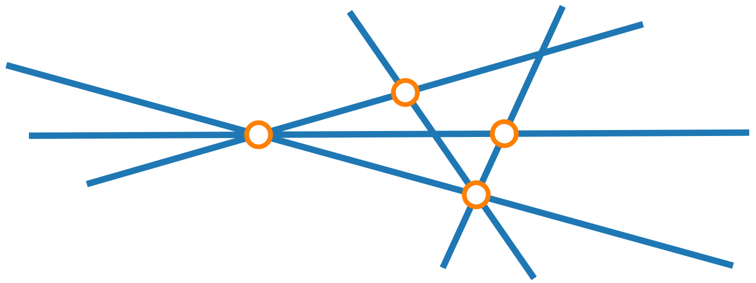
Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with $\mathrm{ocr}(G) < \mathrm{cr}(G) \leq 10$

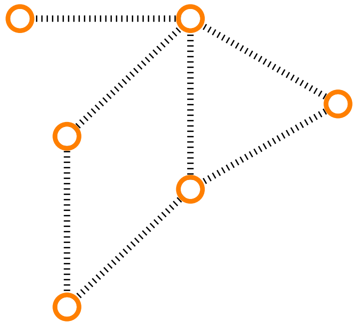
The **pairwise crossing number** $\mathrm{pcr}(G)$ of G is the smallest number of pairs of edges that cross in a drawing of G.

By definition $\mathrm{ocr}(G) \leq \mathrm{pcr}(G) \leq \mathrm{cr}(G)$
Is $\mathrm{pcr}(G) = \mathrm{cr}(G)$? Open!

Visualization of Graphs

Lecture 11: The Crossing Lemma and its Applications





Part II: Computation & Variations

Alexander Wolff

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- cr(G) is a measure of how far G is from being planar.
- Planarization, where we replace crossings with dummy vertices, also uses only heuristics.

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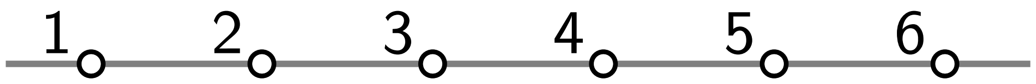

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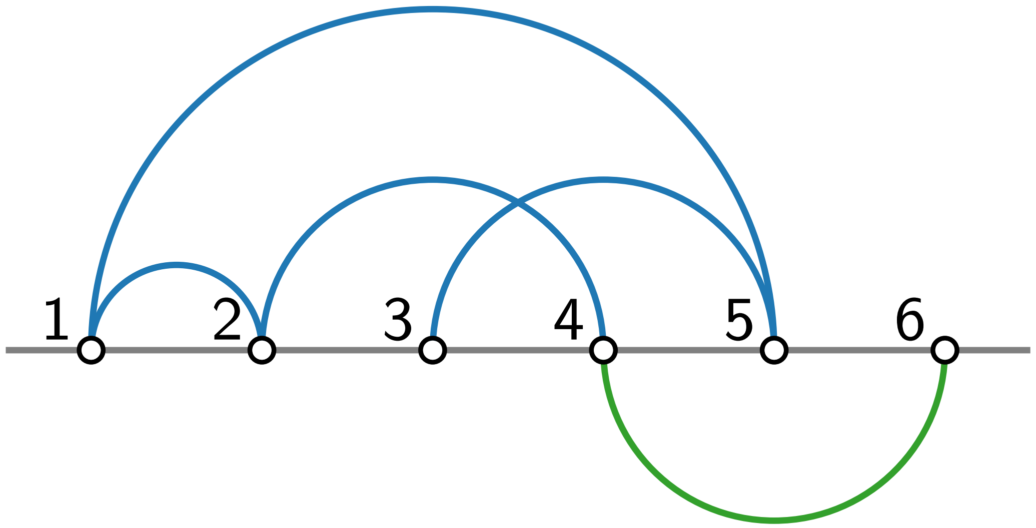

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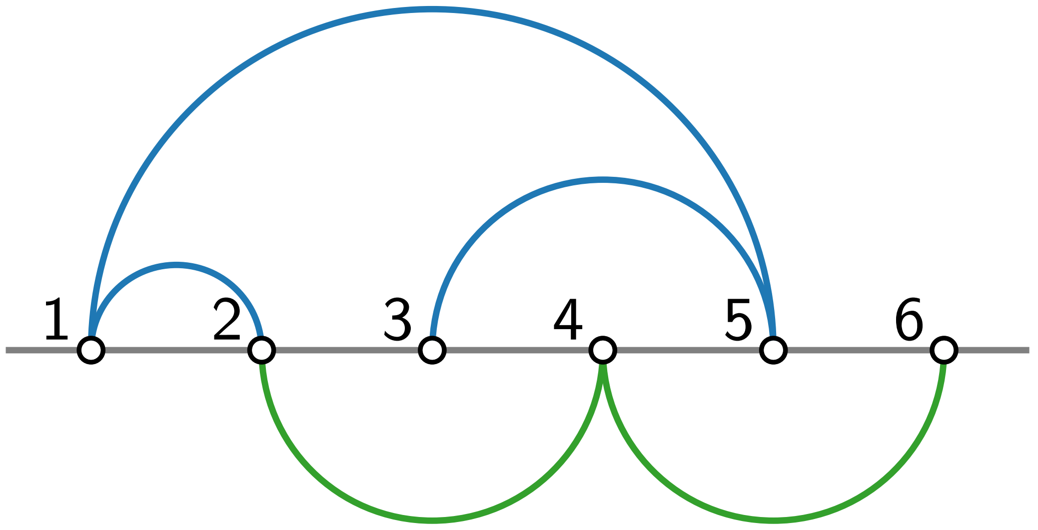

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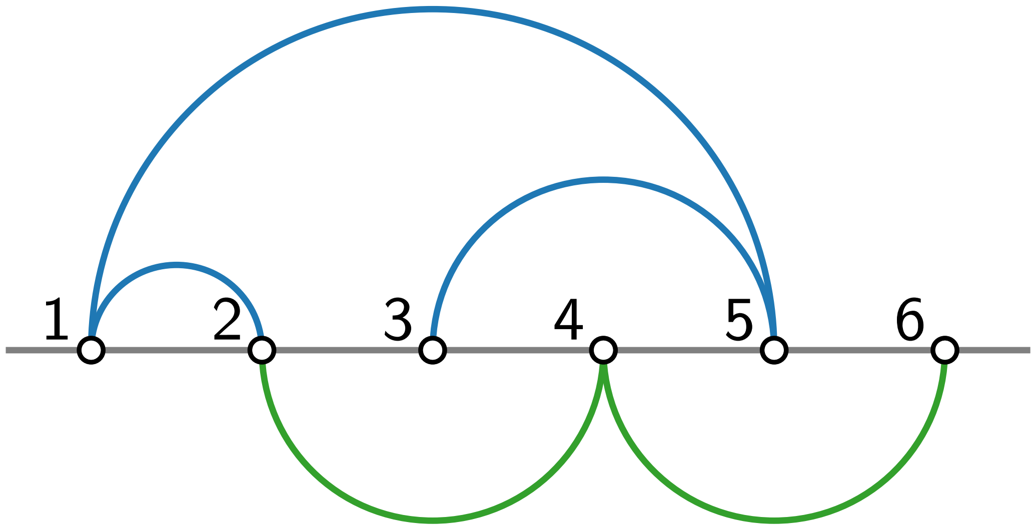

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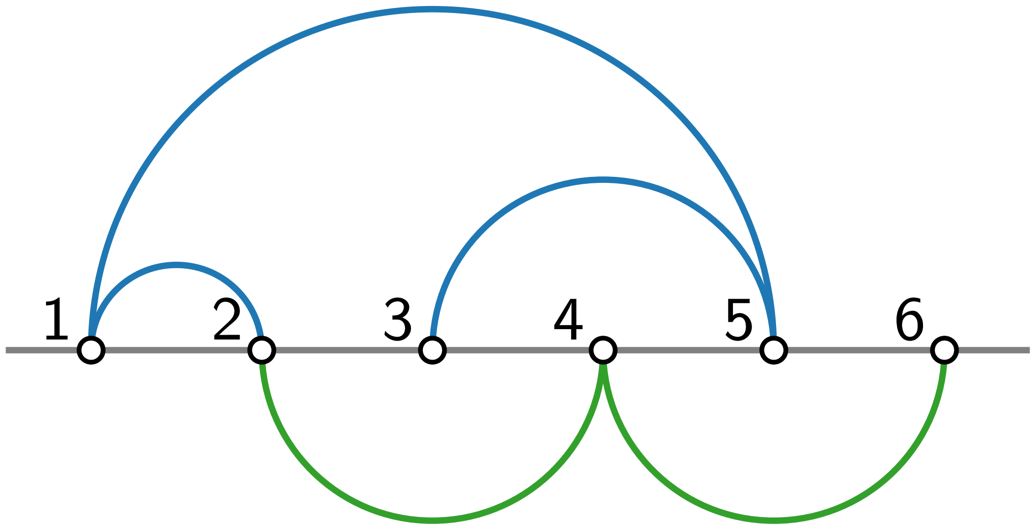

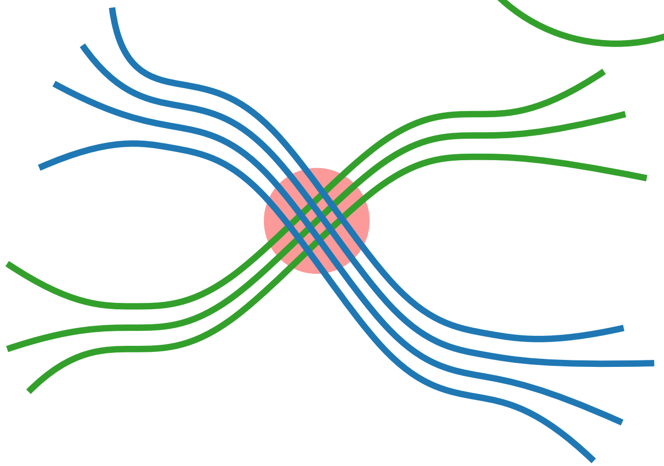

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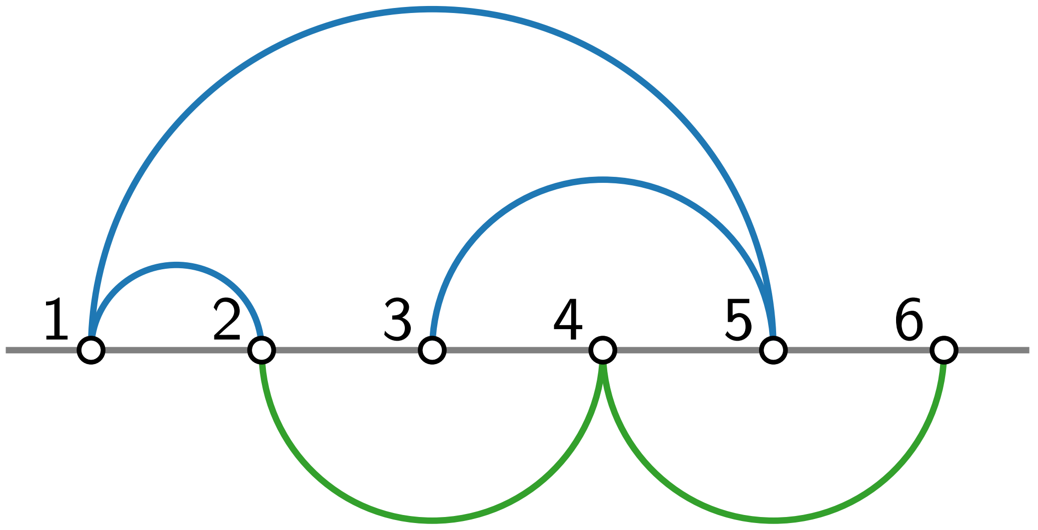

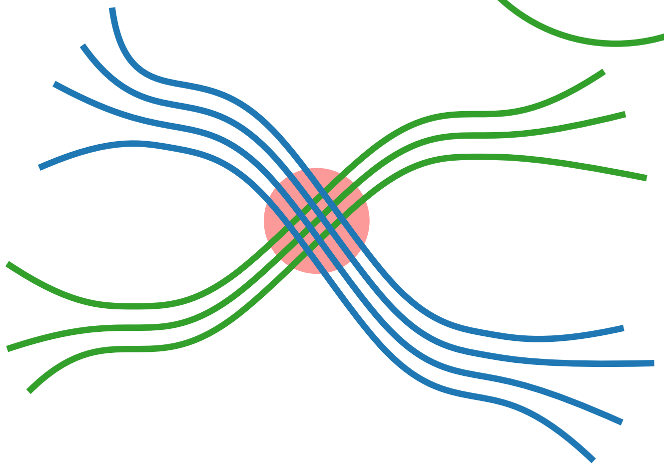

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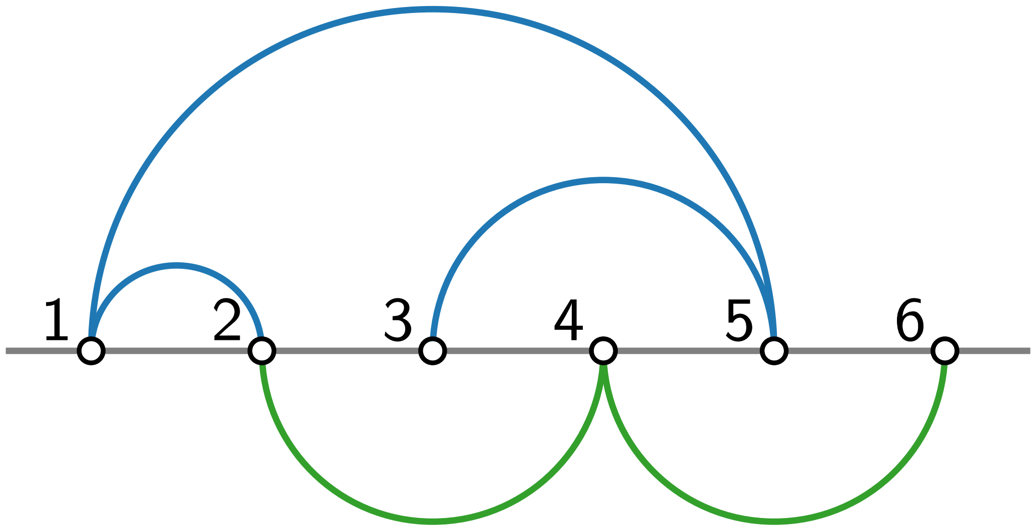

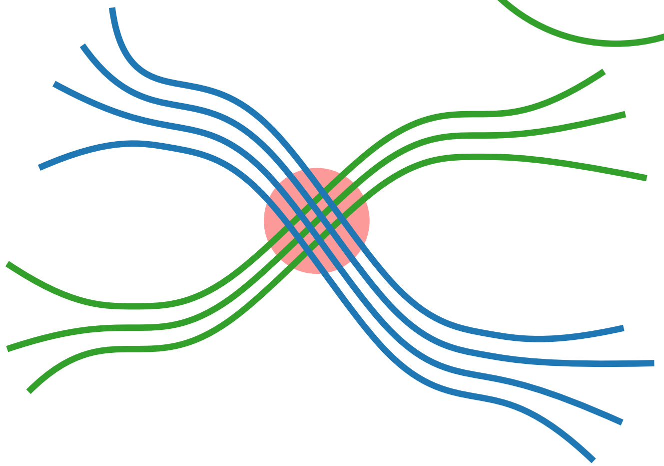

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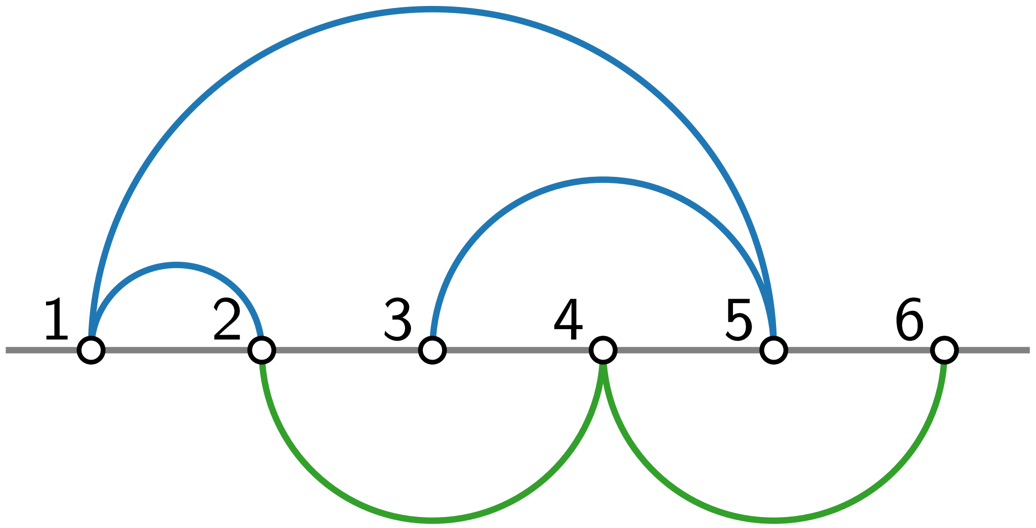

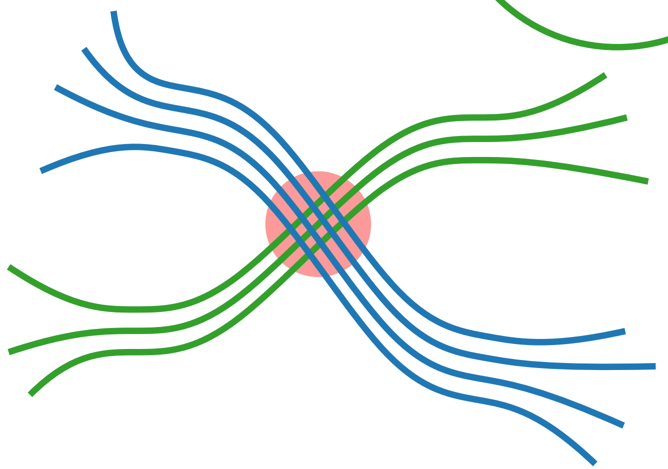

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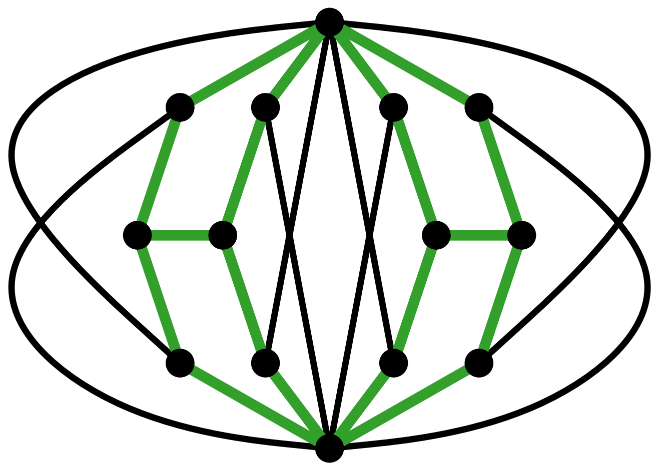

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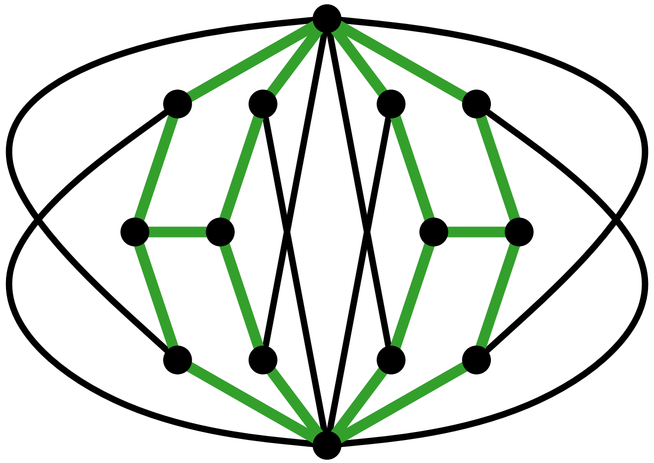

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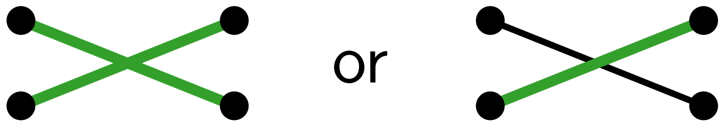

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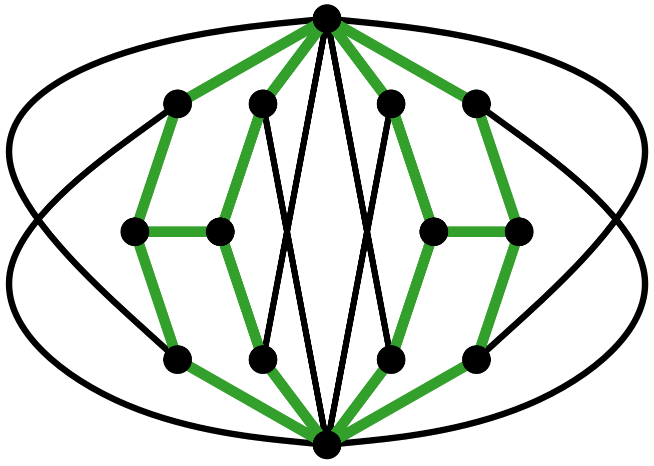

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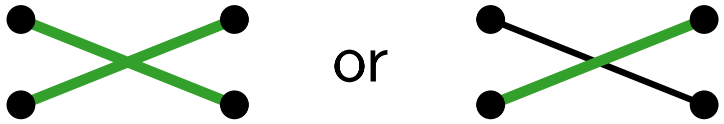
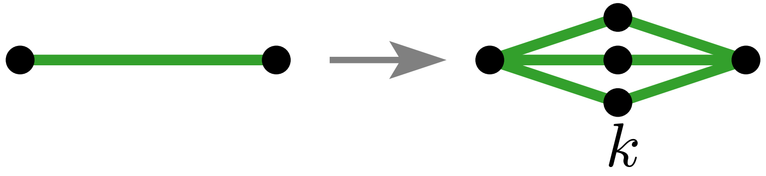


- From G_1 to G_k do



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Visualization of Graphs

Lecture 11:
The Crossing Lemma
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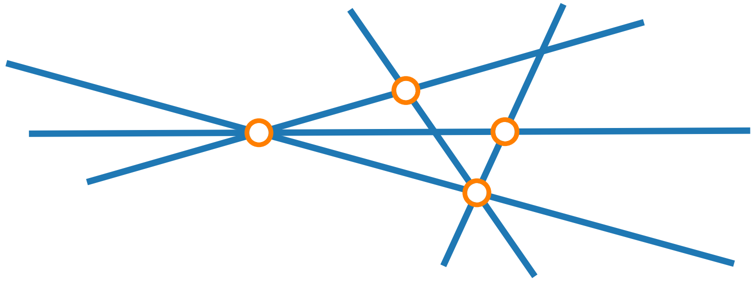

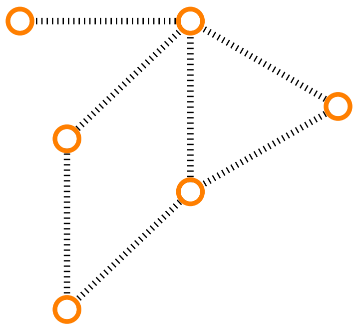

Part III:
First Bounds

Alexander Wolff

Bounds for Complete Graphs

Theorem. [Guy '60]

$$\mathrm{cr}(K_n) \leq \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil = \frac{3}{8}\binom{n}{4} + O(n^3)$$

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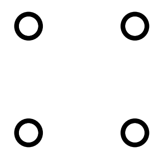

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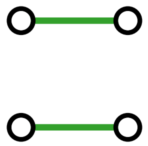

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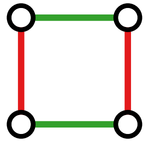

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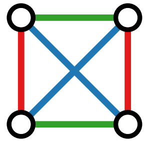

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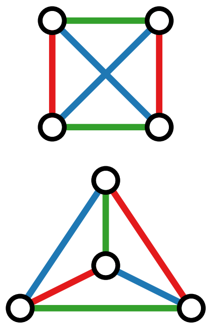

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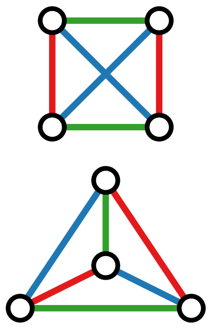

Sylvester's four-point problem

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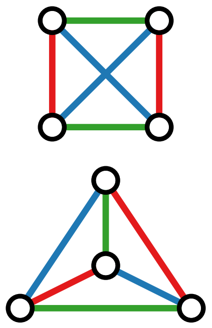

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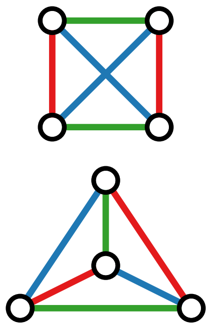

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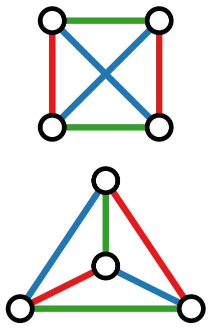

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Theorem. [Zarankiewicz '54, Urbaník '55]

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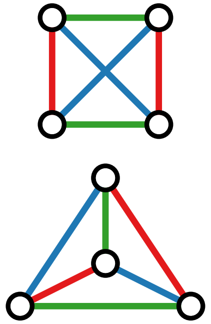

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Pál Turán
*1910 – 1976
Budapest, Hungary

Turán's brick factory problem (1944)

© TruckinTim

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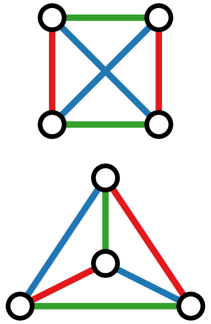
$$\mathrm{cr}(K_n) \underset{=}{\not\leq} \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is tight for $n \leq 12$.



Sylvester's four-point problem

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Turán's brick factory problem (1944)

Pál Turán
*1910 – 1976
Budapest, Hungary

© TruckinTim

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Theorem. [Lovász et al. '04, Aichholzer et al. '06]

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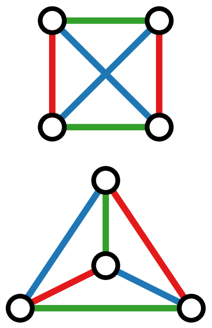

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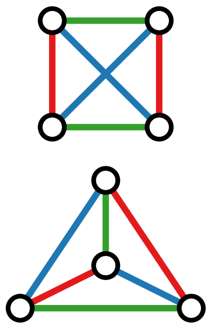

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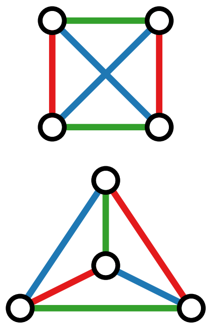

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Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`!

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Lemma 2.
For a graph G with n vertices and m edges,

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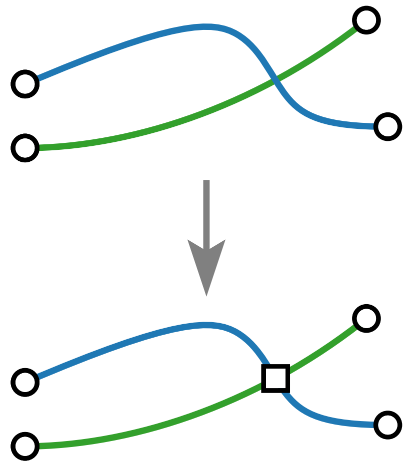

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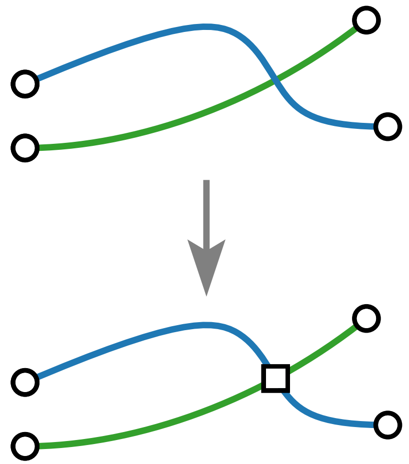

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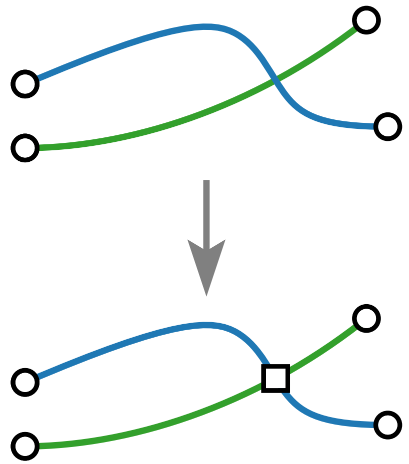

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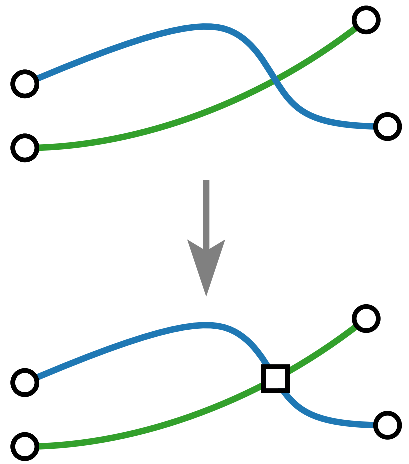

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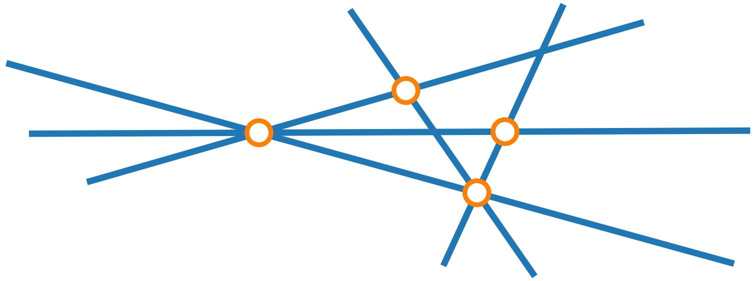
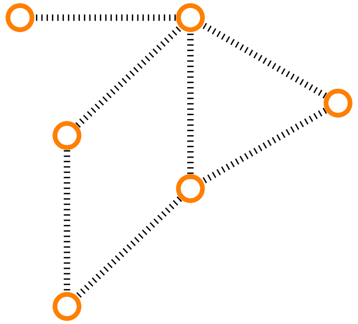
Visualization of Graphs

Lecture 11: The Crossing Lemma

and its Applications





Part IV: The Crossing Lemma

Alexander Wolff

The Crossing Lemma

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- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.

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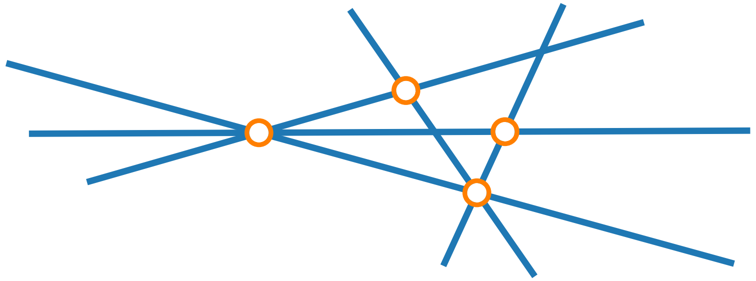
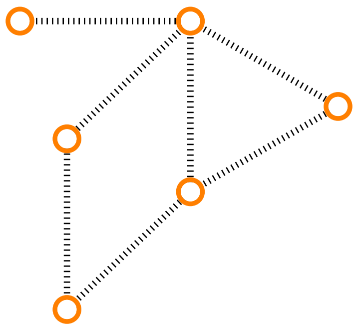
Visualization of Graphs

Lecture 11: The Crossing Lemma and its Applications





Part V: Applications

Alexander Wolff

Application 1: Point–Line Incidences

- For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L})$ = number of point–line incidences in $(P, \mathcal{L})$.

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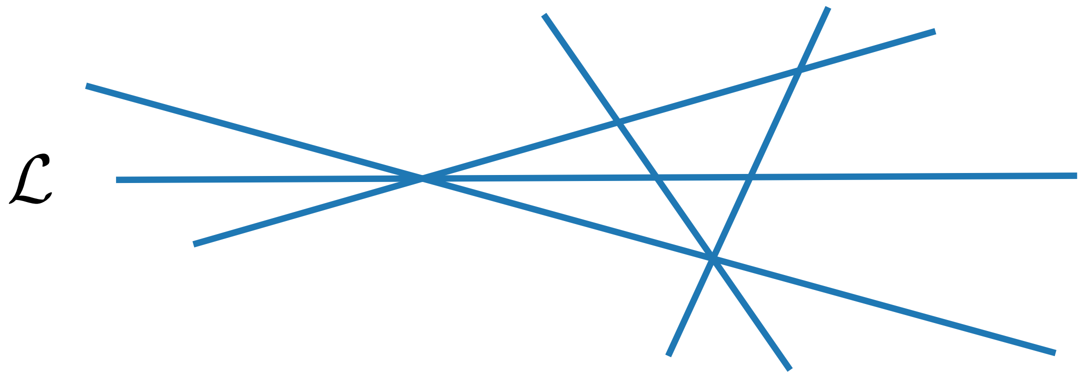

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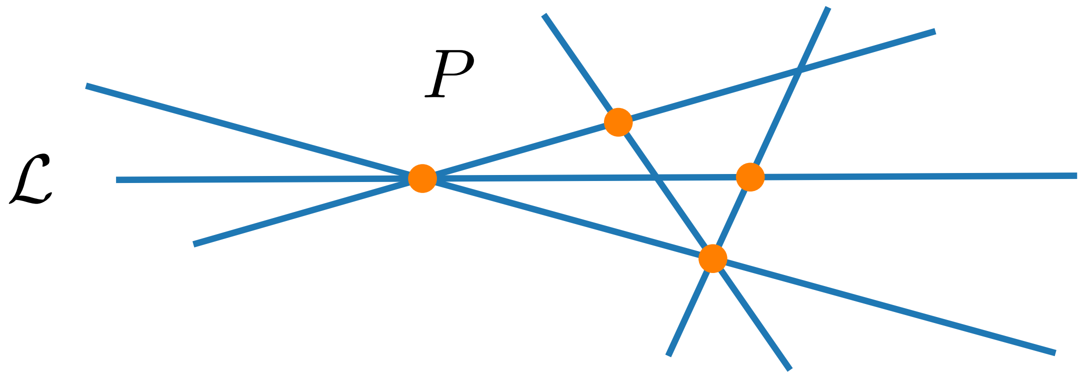

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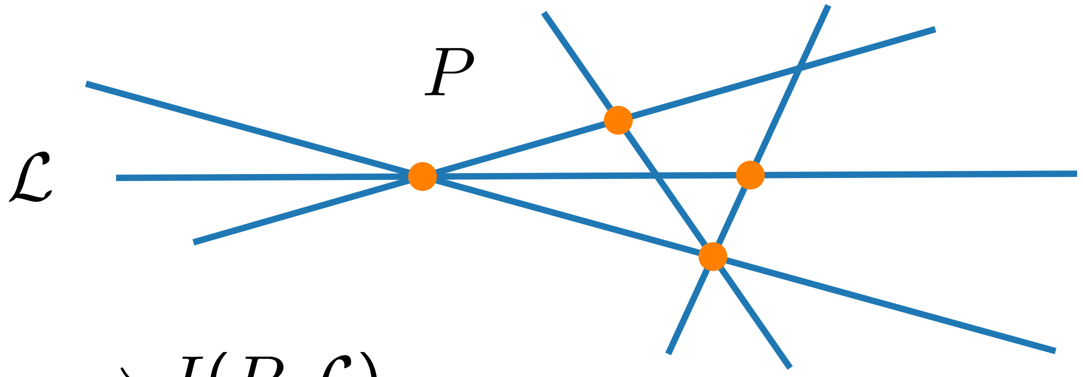

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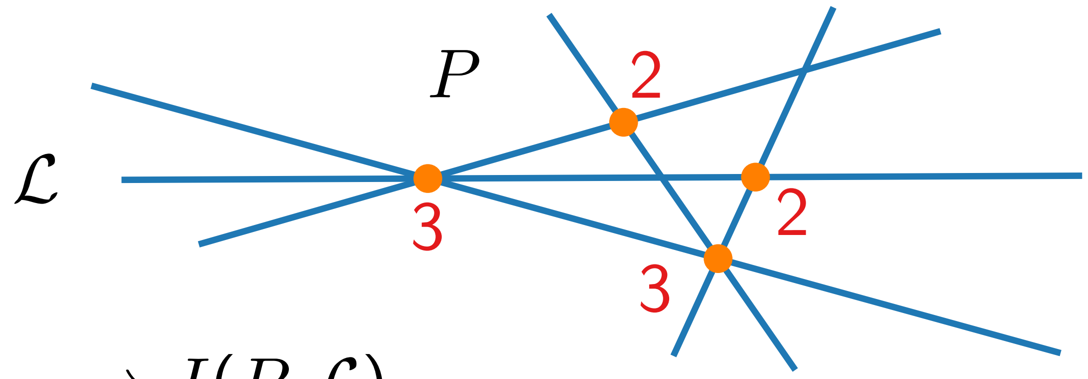

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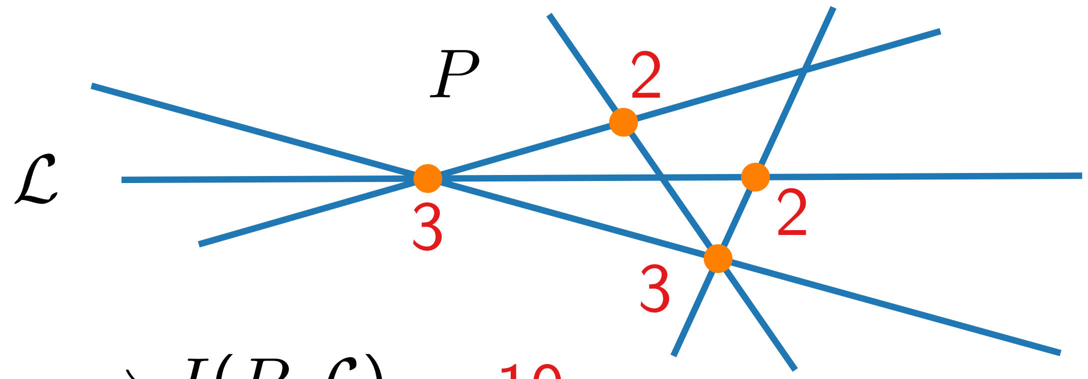

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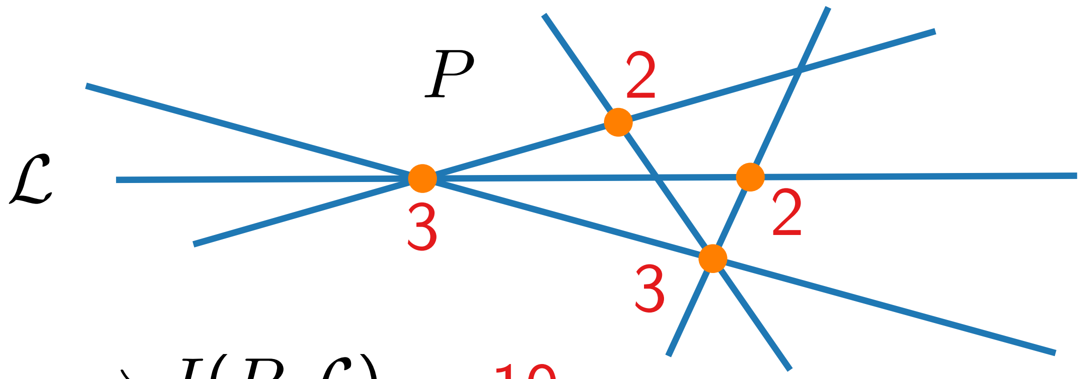

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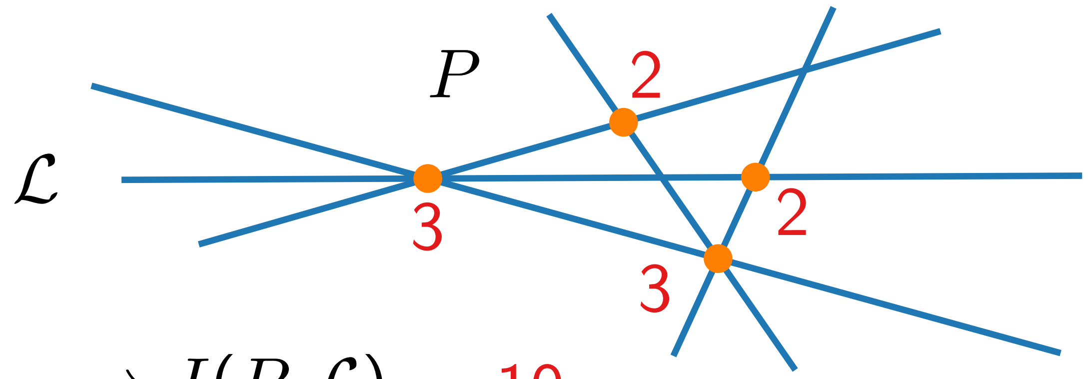

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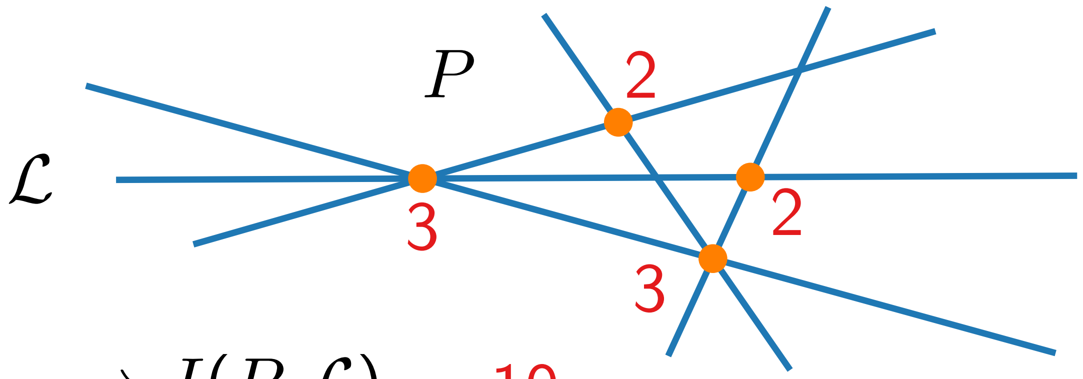

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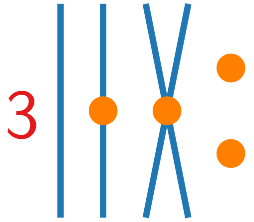

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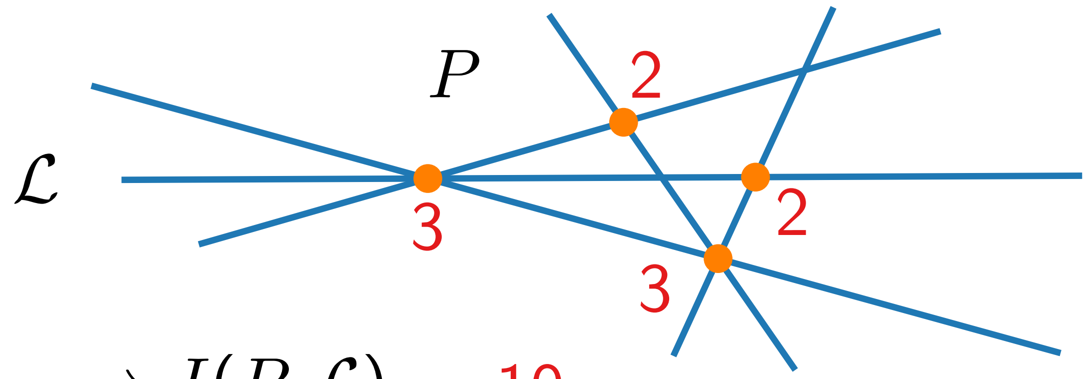

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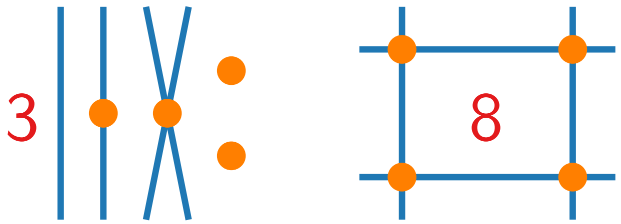

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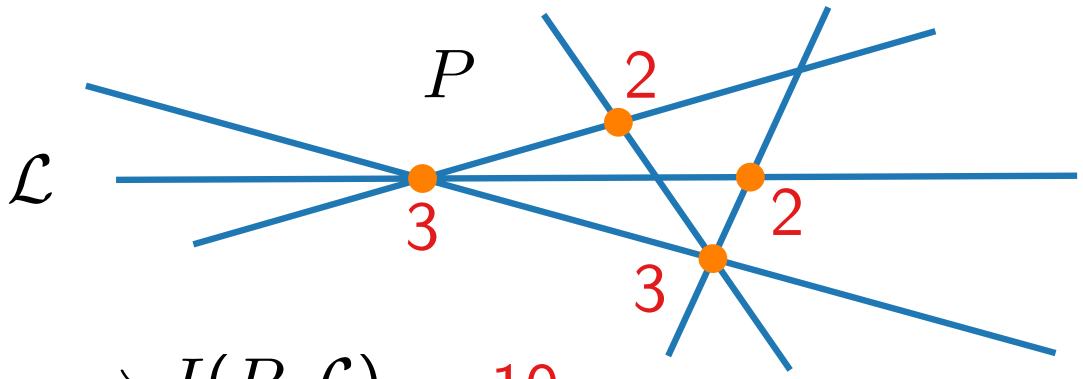

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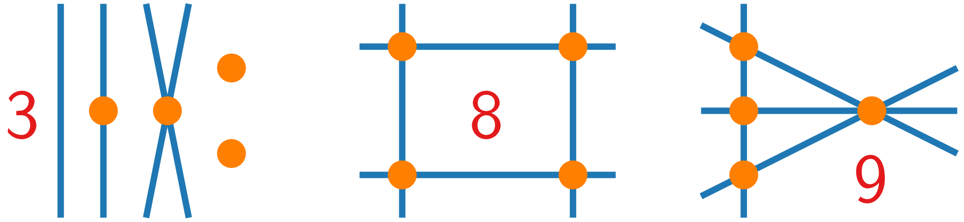

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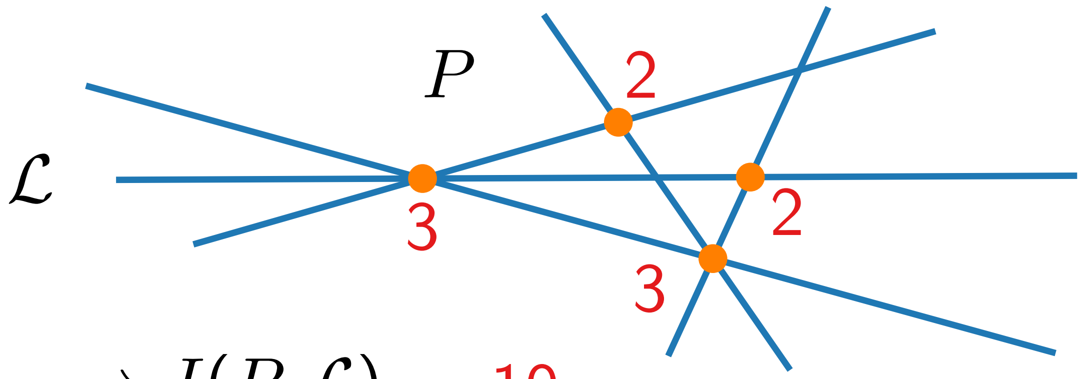

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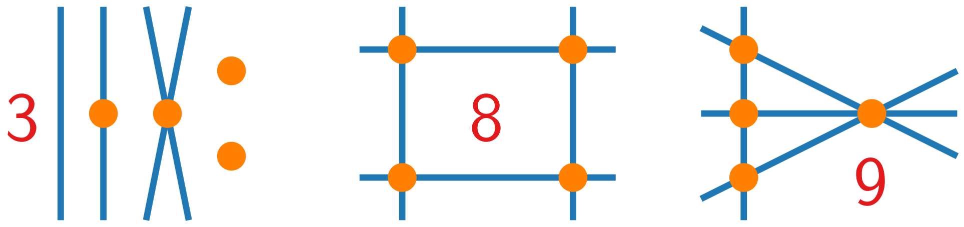

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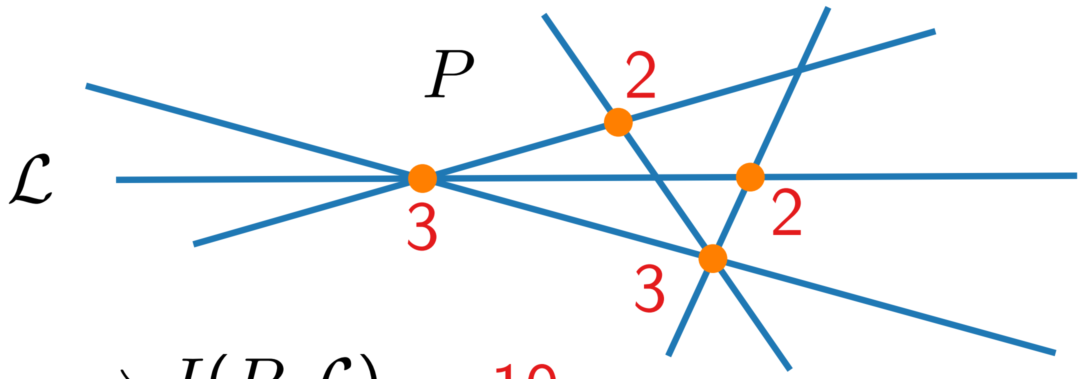

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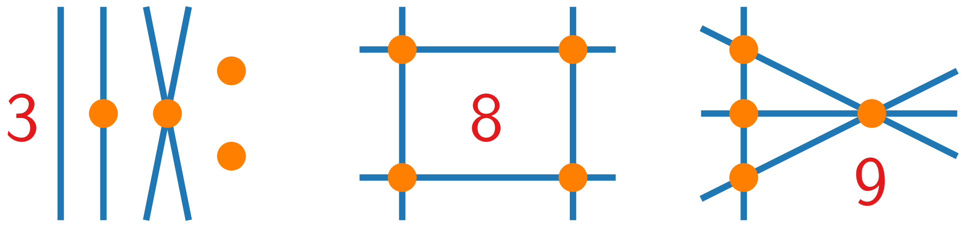

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[Szemerédi, Trotter '83, Székely '97]
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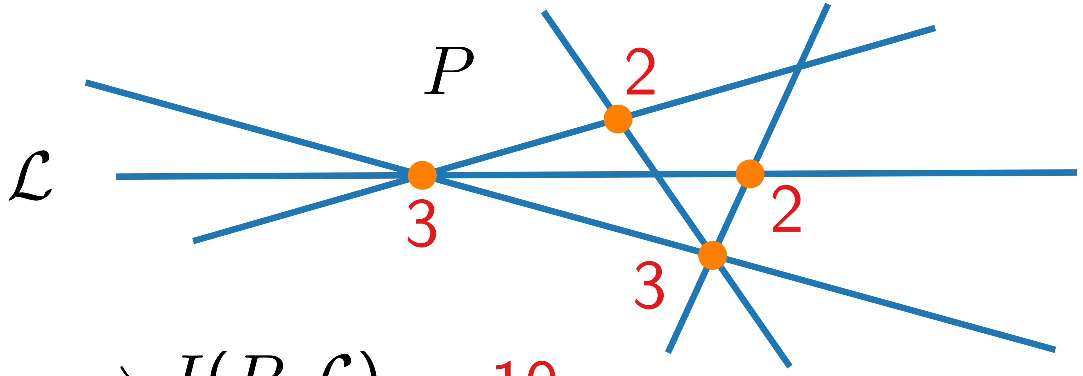

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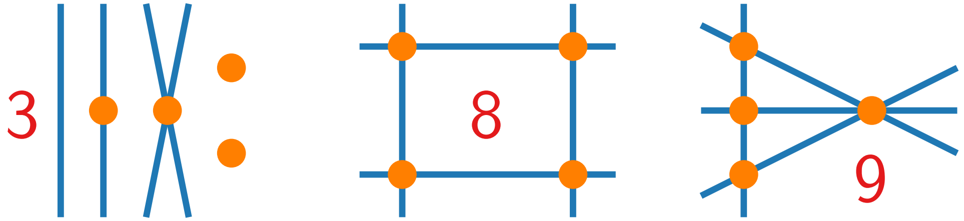

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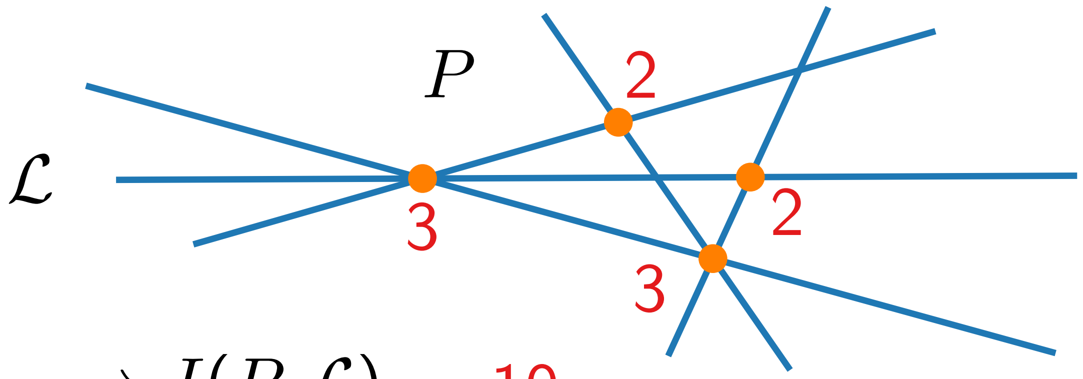

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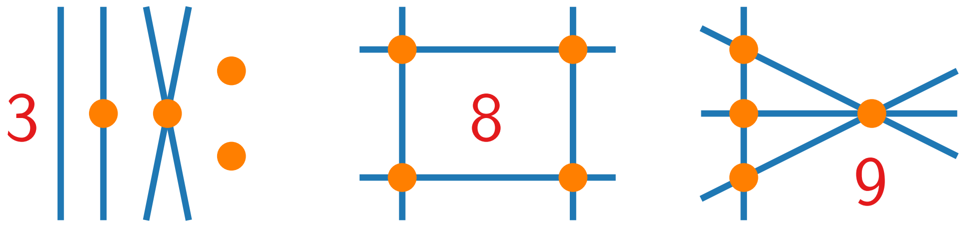

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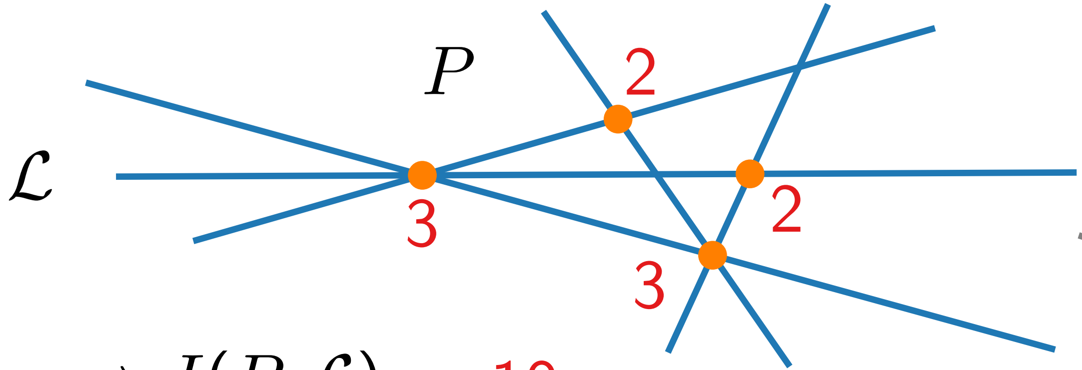

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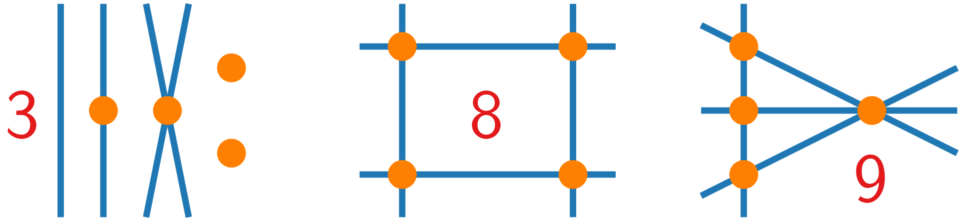

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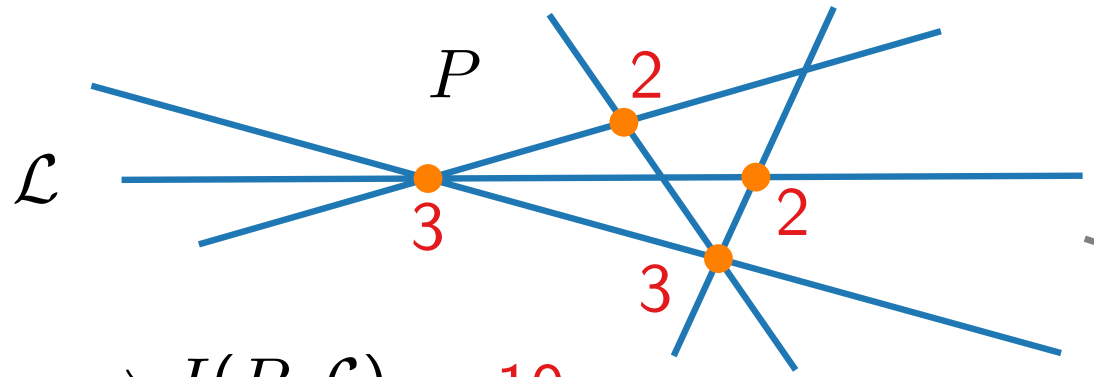

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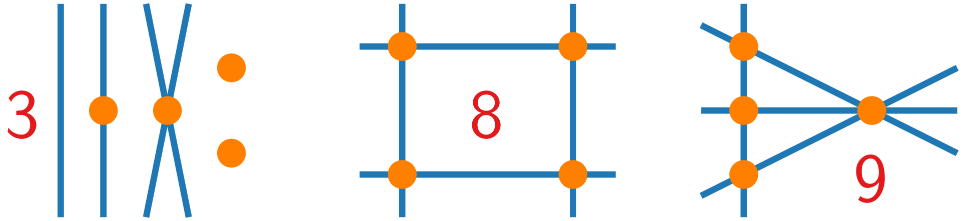

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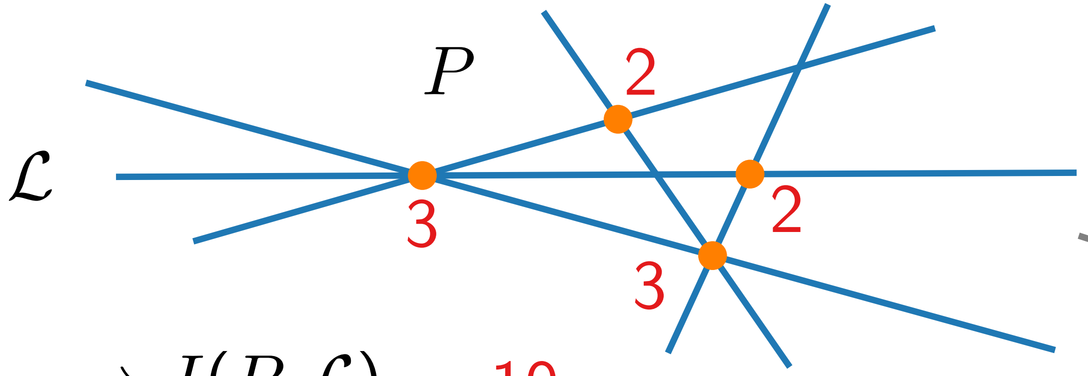

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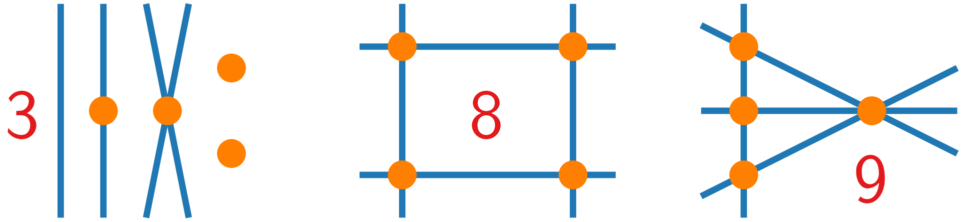

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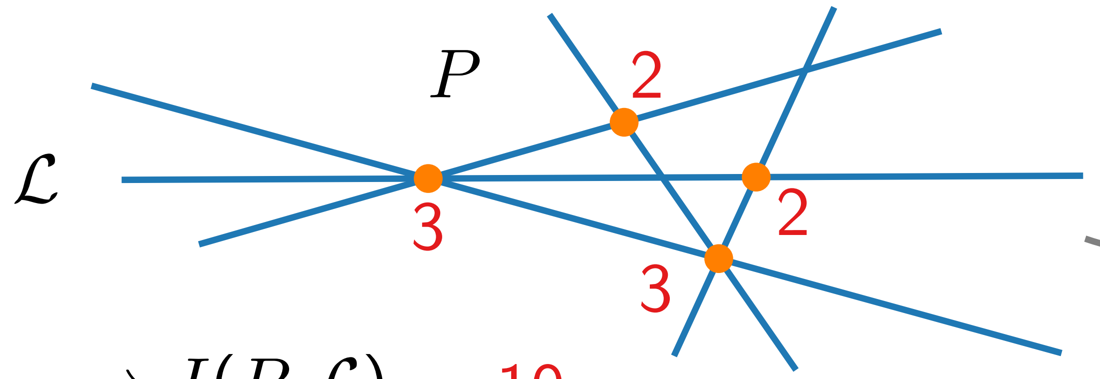

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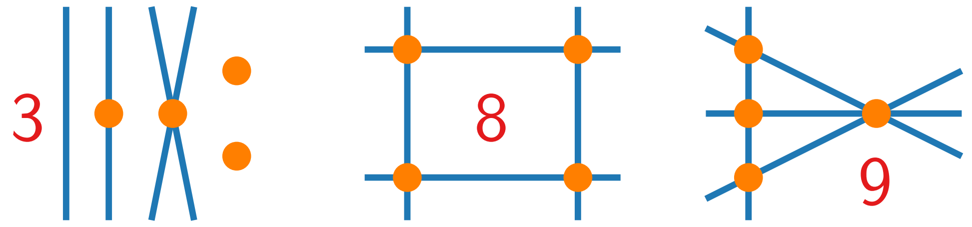

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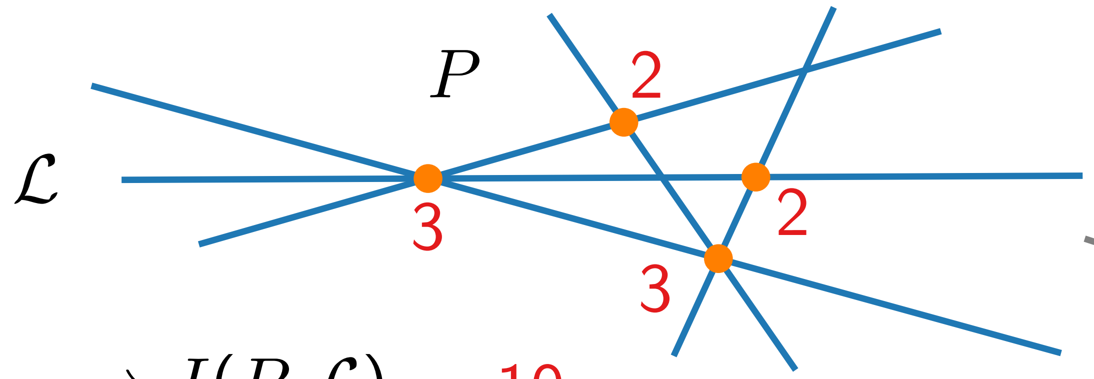

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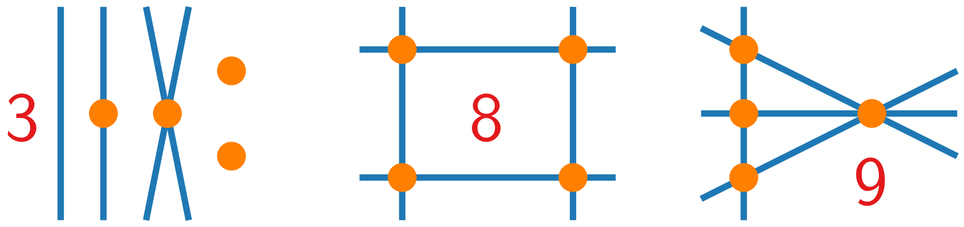

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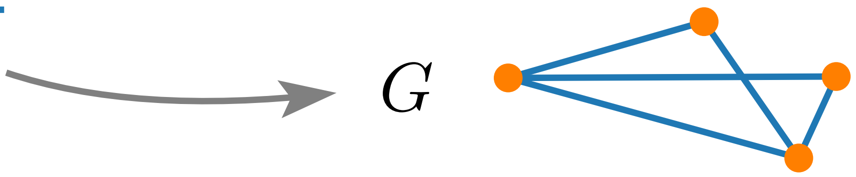

- $\mathrm{cr}(G) \le k^2$
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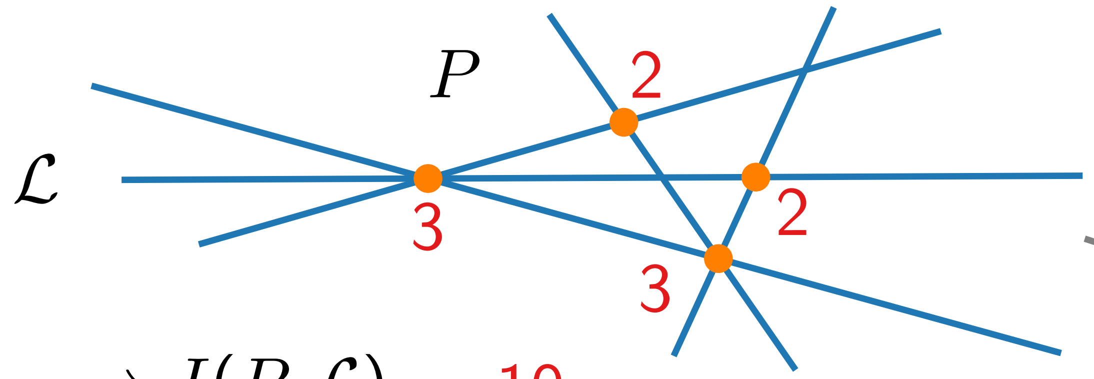

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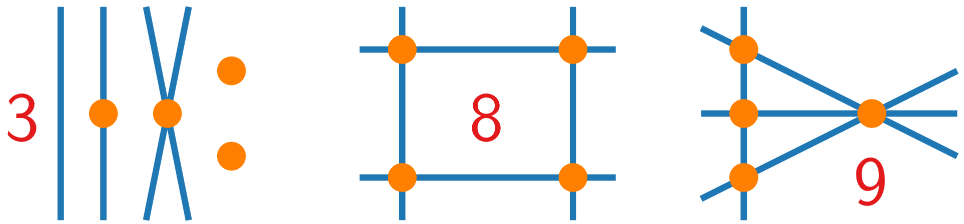

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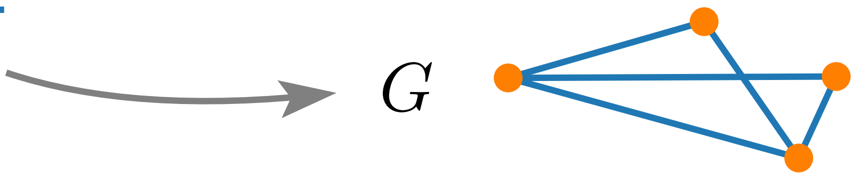

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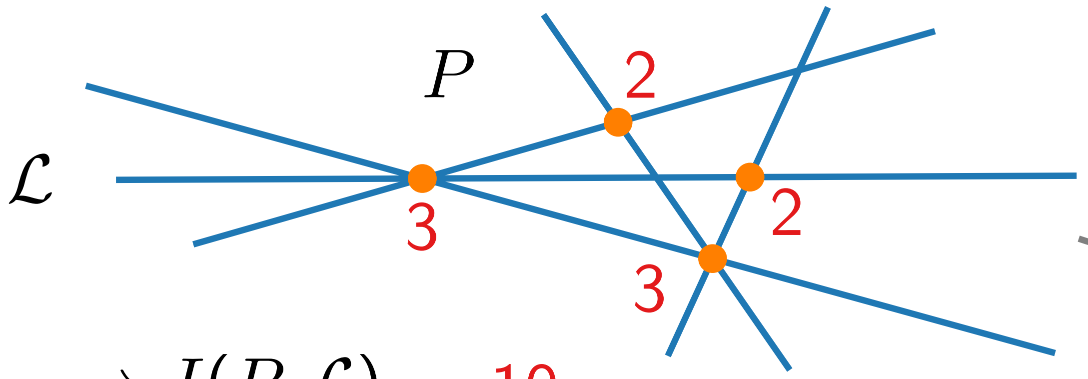

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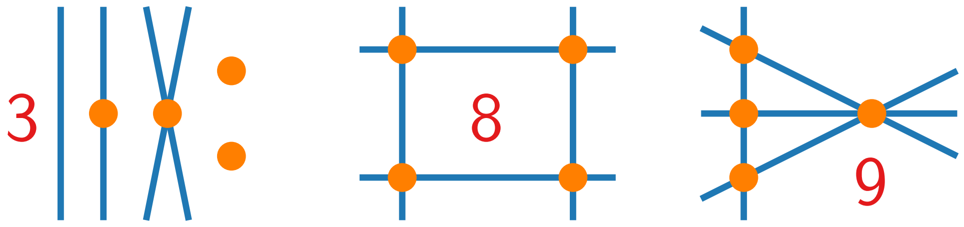

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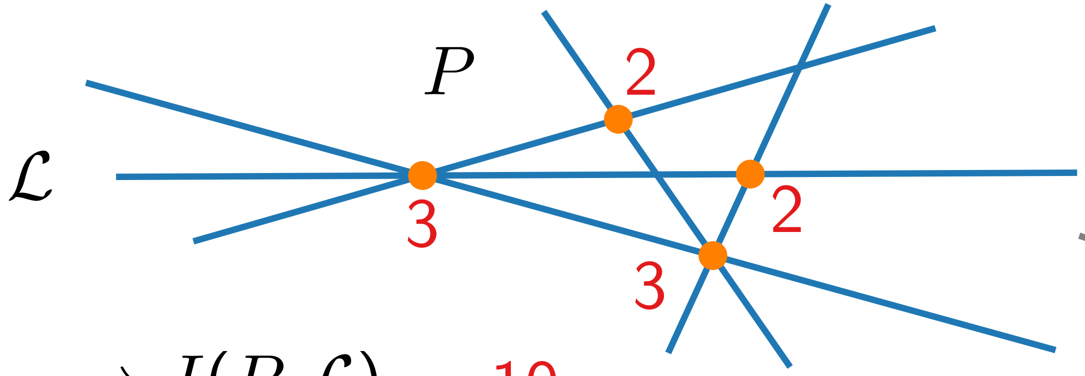

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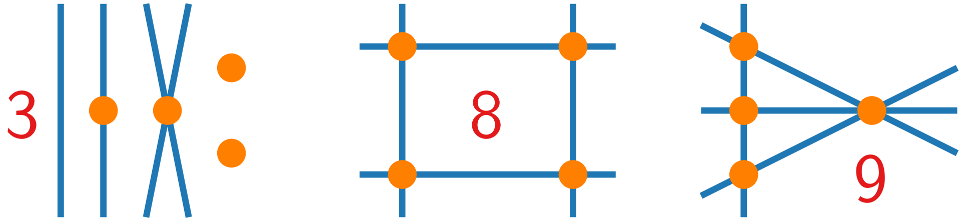

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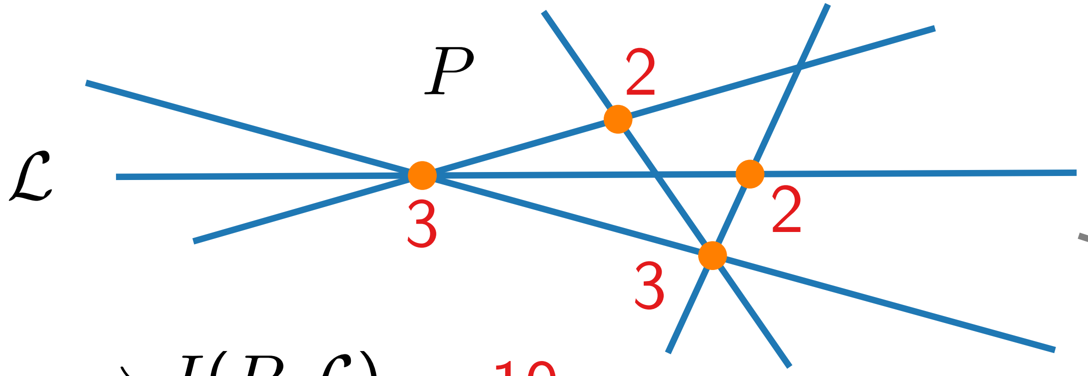

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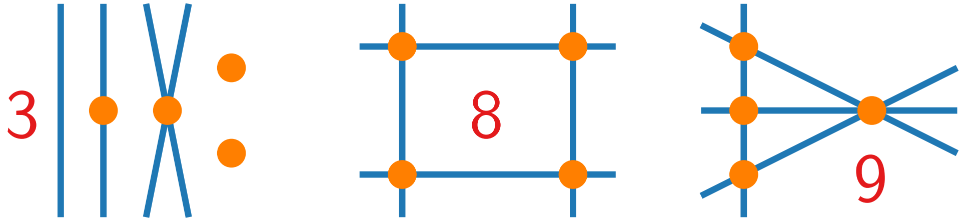

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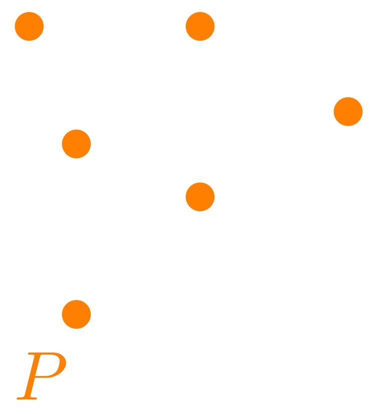

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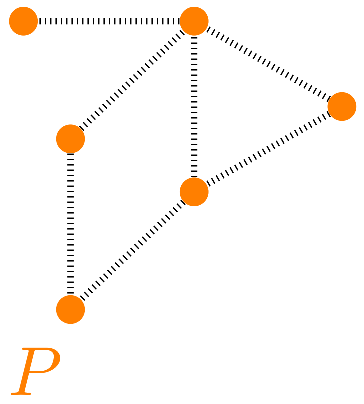

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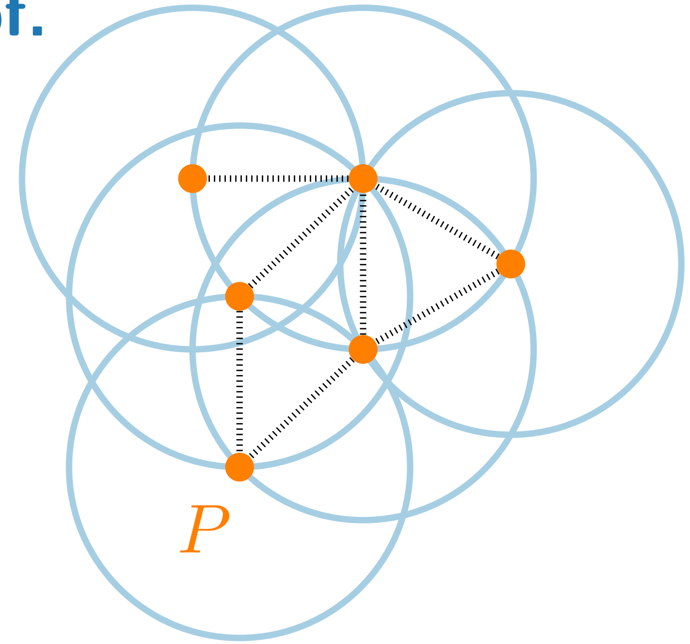

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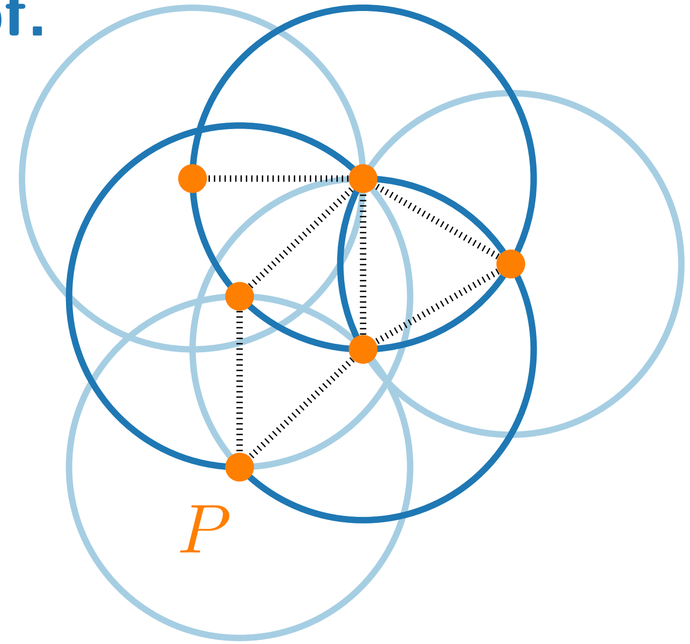

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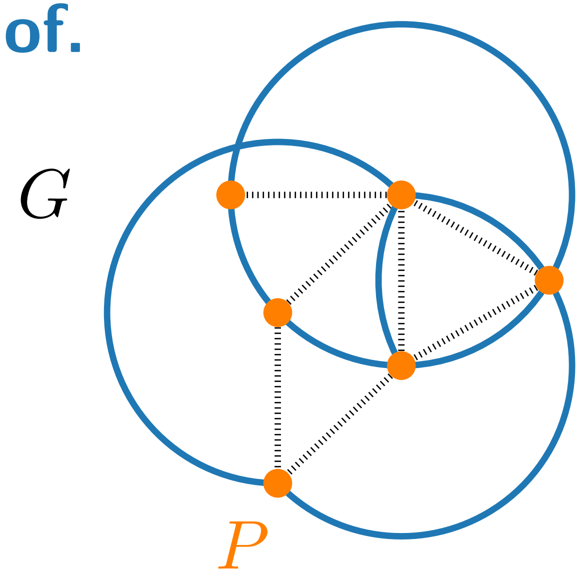

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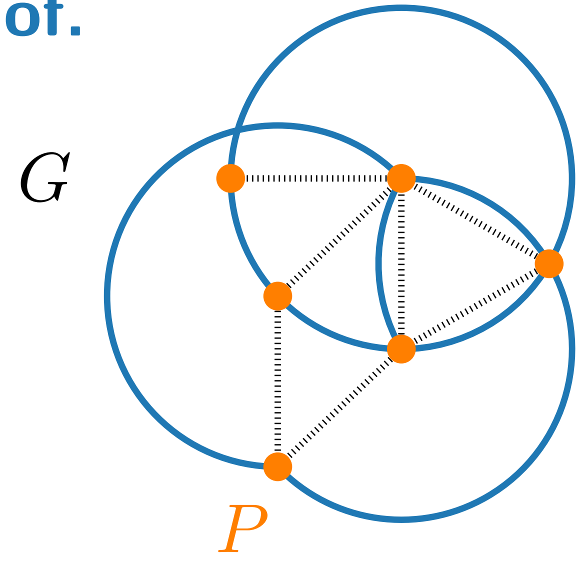

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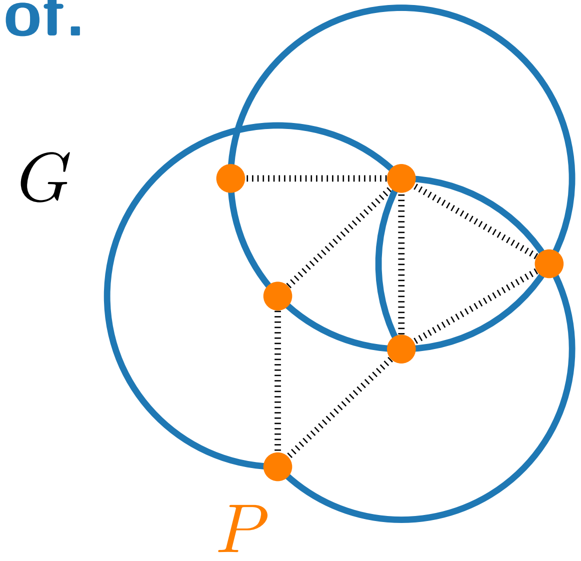

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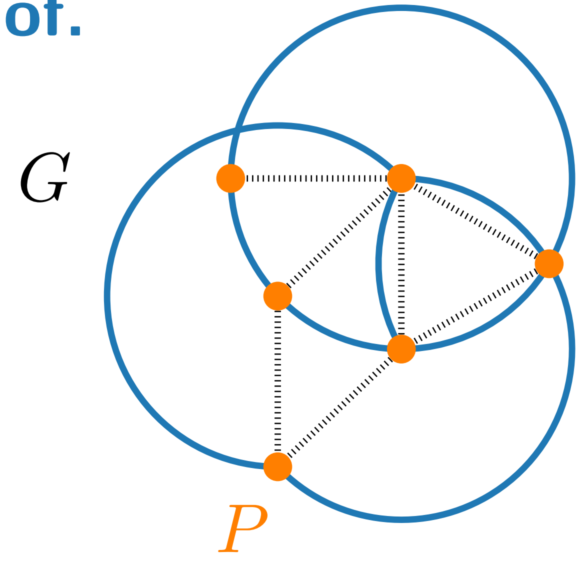

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$\square$

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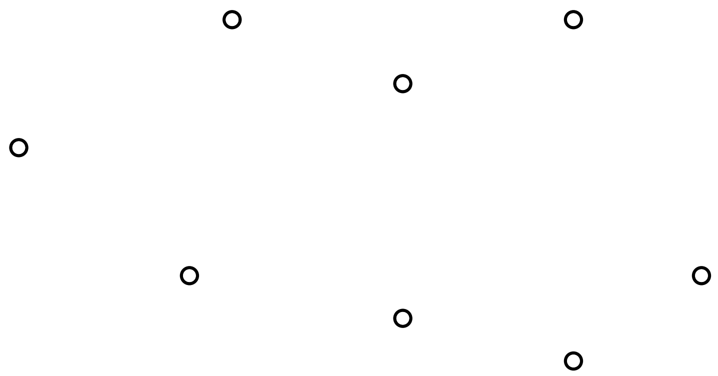
Given set of n points (in general position, n even) –



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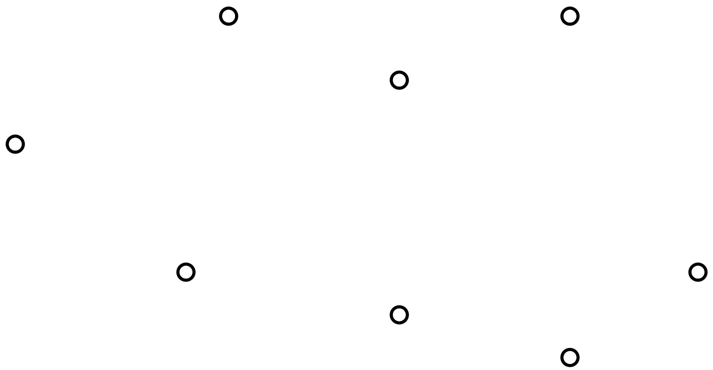

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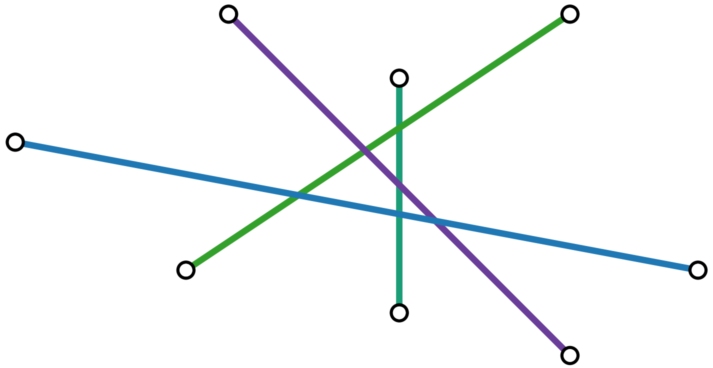

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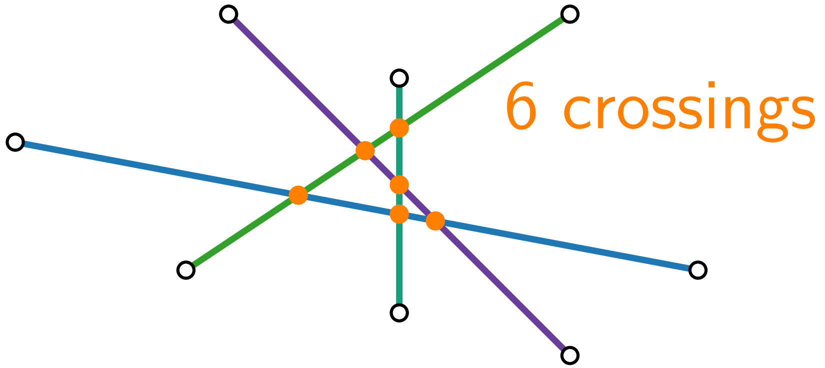

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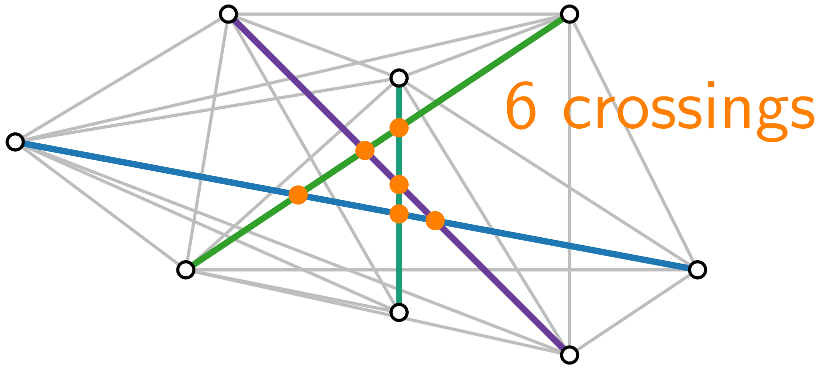

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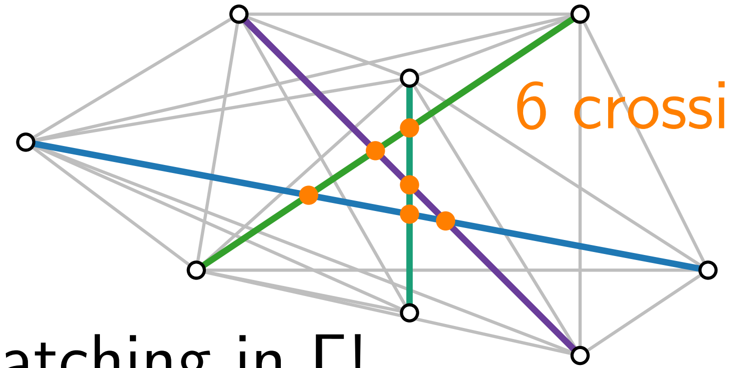

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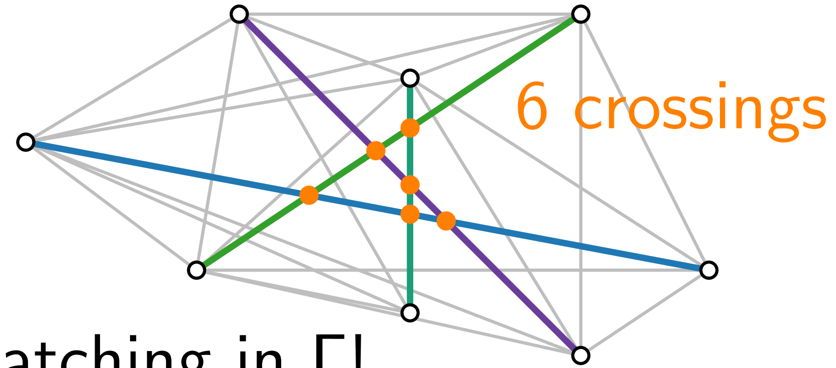

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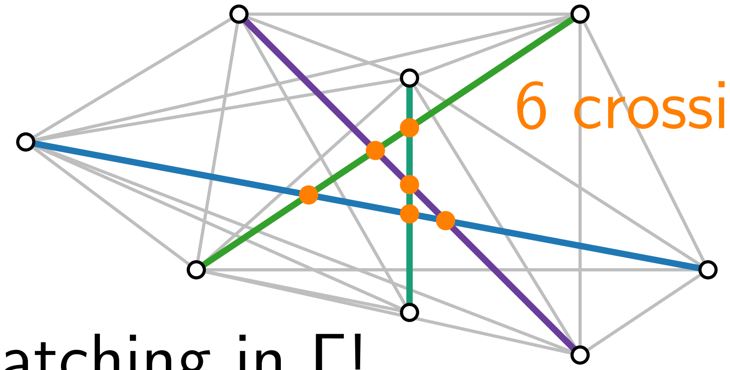

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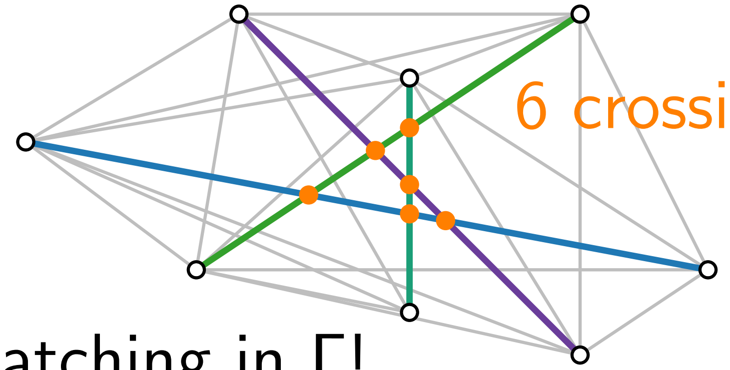

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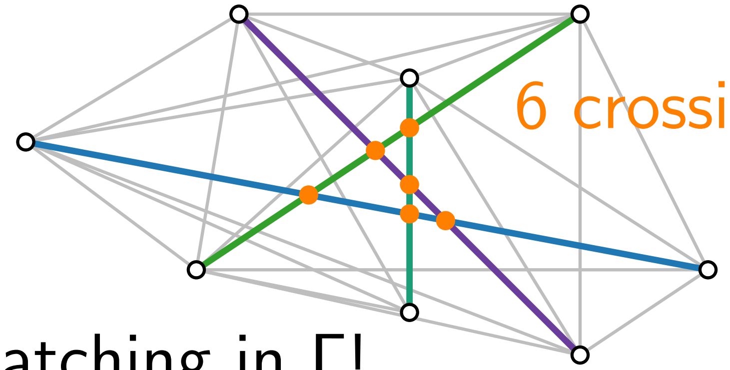

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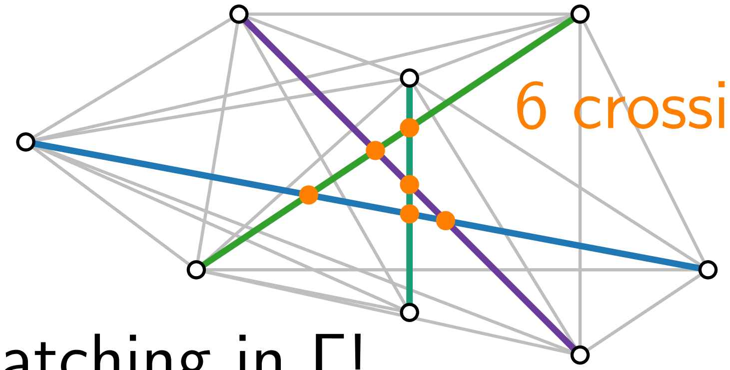

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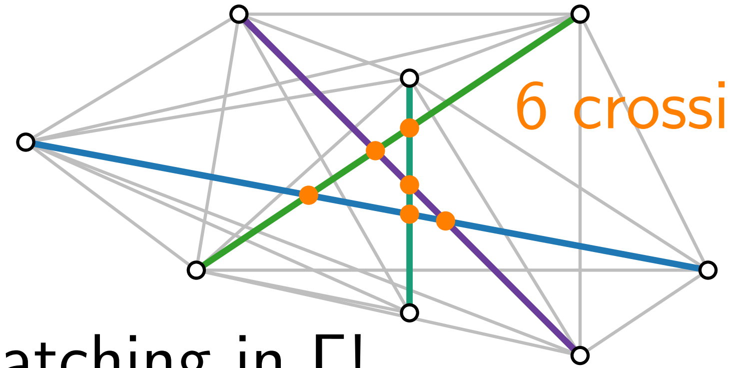

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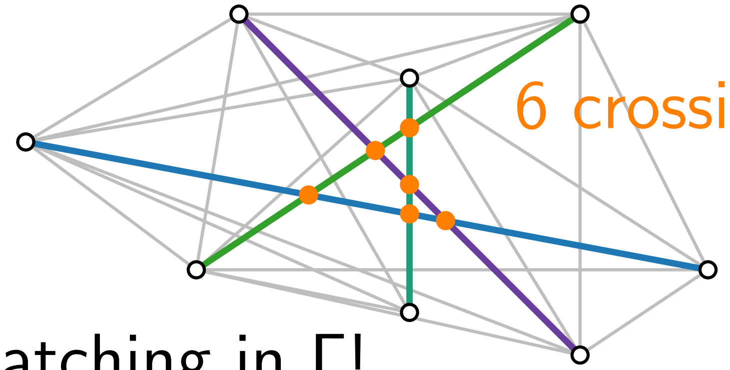

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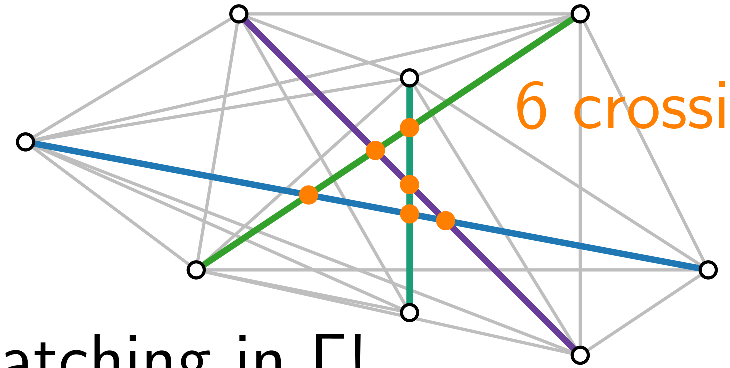

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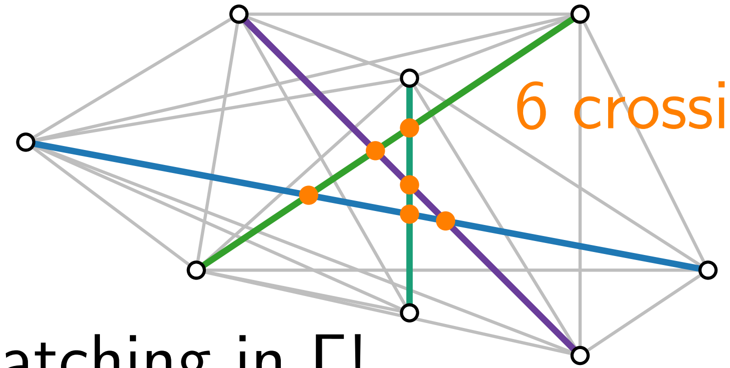

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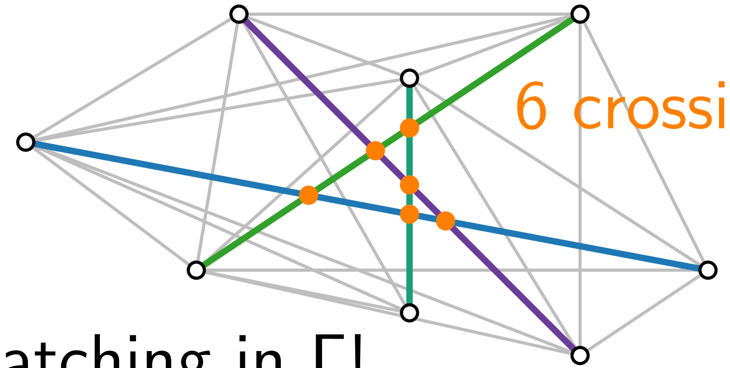
Application 3: Expected Number of Crossings in a Matching



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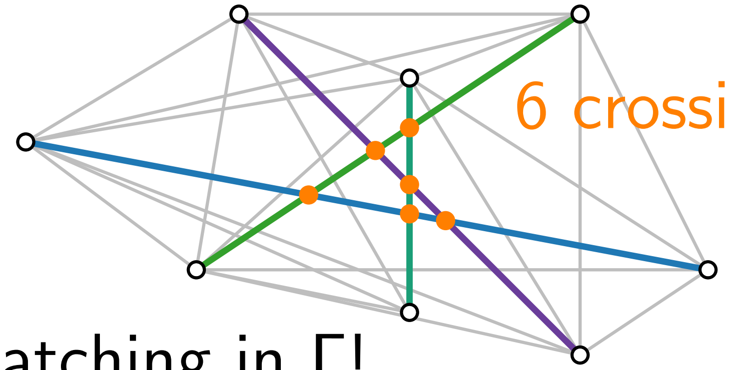

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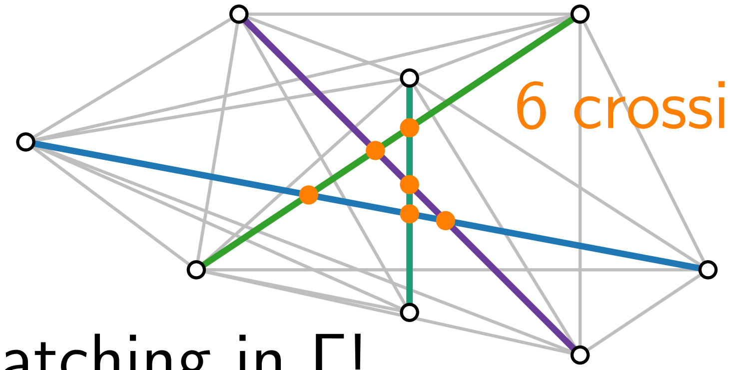

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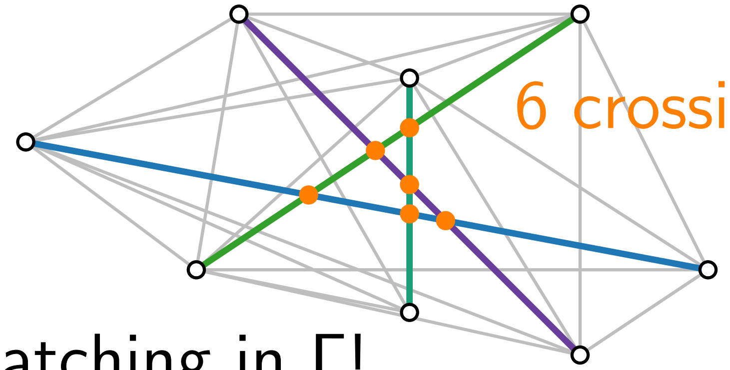

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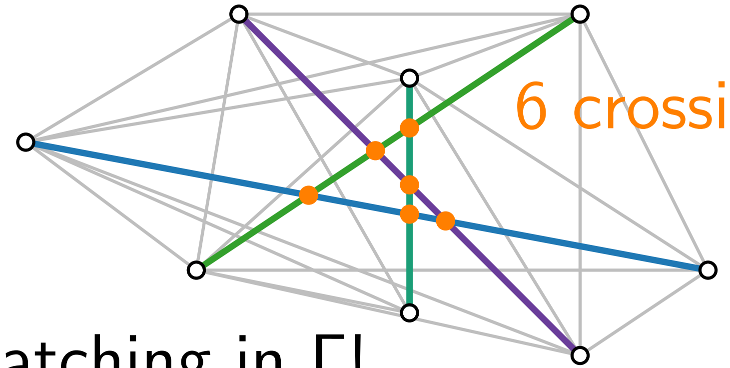

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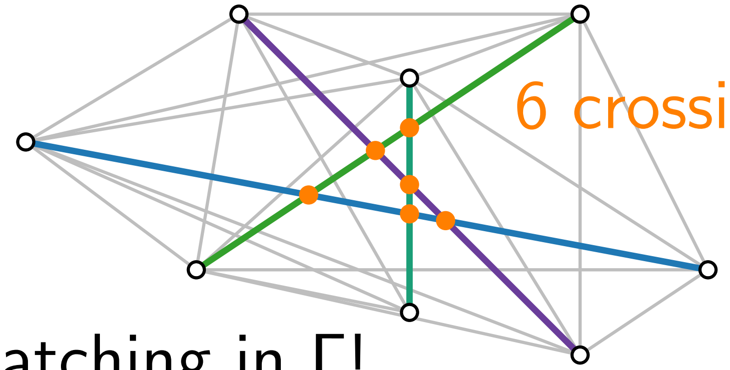

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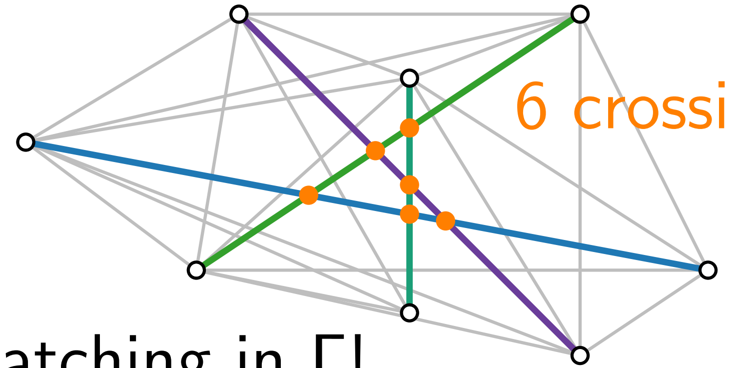

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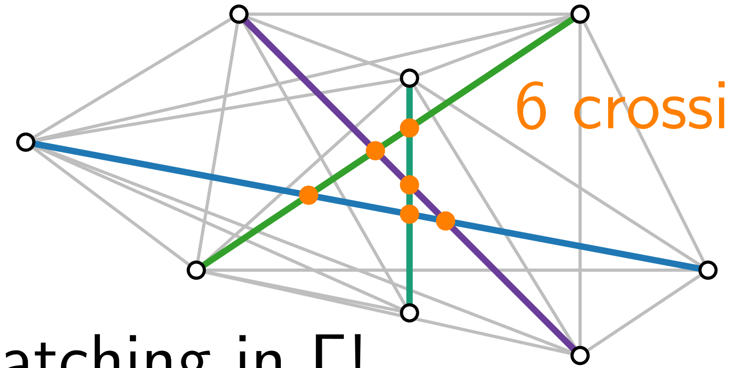

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Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao's blog post "The crossing number inequality" from 2007
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Székely '97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography "N Is a Number: A Portrait of Paul Erdős"
- Exact computations of crossing numbers: `http://crossings.uos.de`