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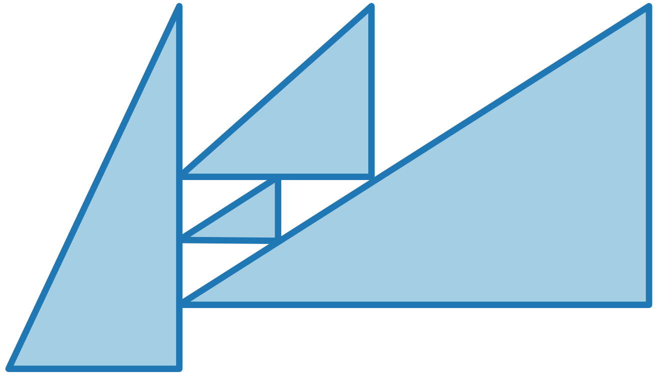
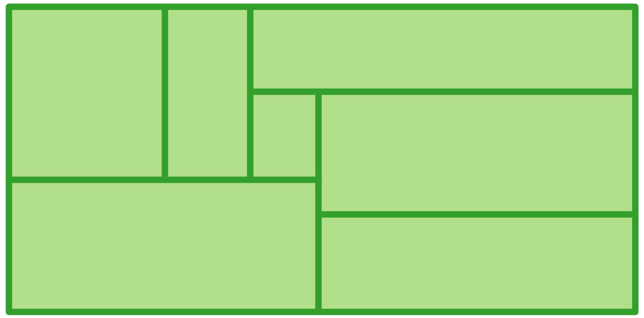
Visualization of Graphs

Lecture 7:
Contact Representations of Planar Graphs:
Triangle Contacts and Rectangular Duals



Part I:
Geometric Representations

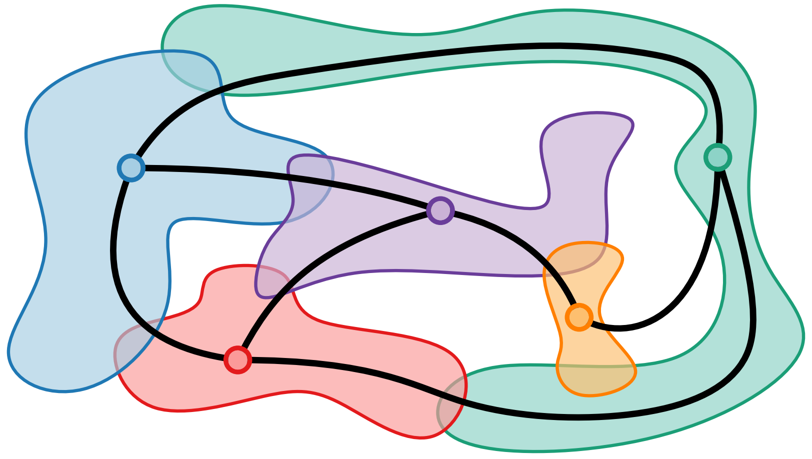


Alexander Wolff

Intersection Representation of Graphs

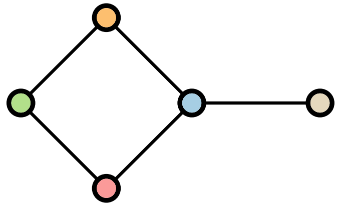
In an **intersection representation** of a graph,
- each vertex is represented by a set
- such that two sets intersect $\Leftrightarrow$ the corresponding vertices are adjacent.

For a collection $\mathcal{S}$ of sets,
the **intersection graph** $G(\mathcal{S})$ of $\mathcal{S}$
has vertex set $\mathcal{S}$ and edge set
$\{\{S, S'\} \colon S, S' \in \mathcal{S}, S \neq S',$ and $S \cap S' \neq \emptyset\}$.



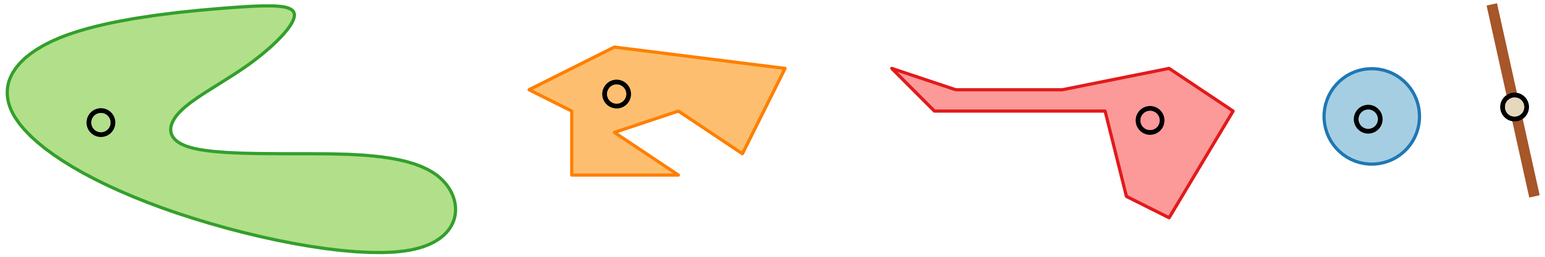
Contact Representation of Graphs

Let G be a graph.

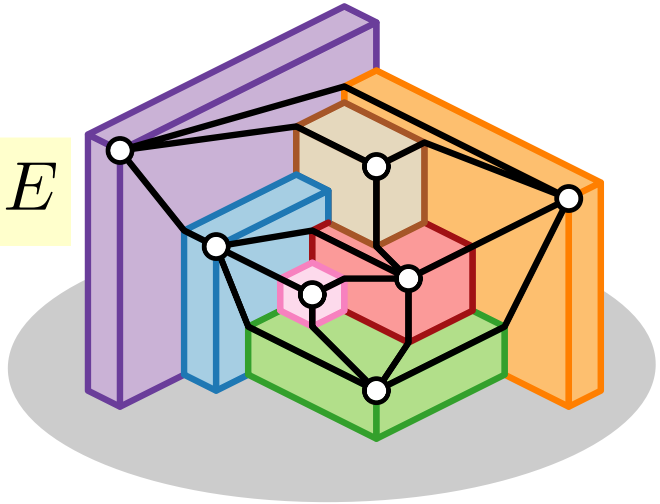


Let $\mathcal{S}$ be a family of geometric objects (e.g., disks).

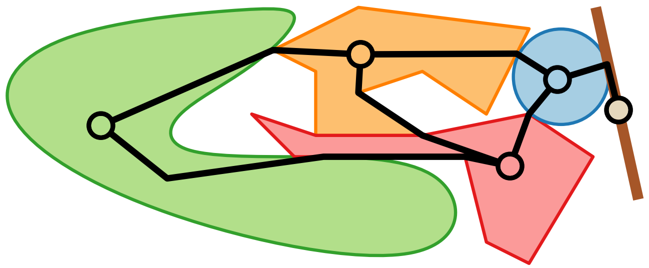
Represent each vertex v by a geometric object $S(v) \in \mathcal{S}$

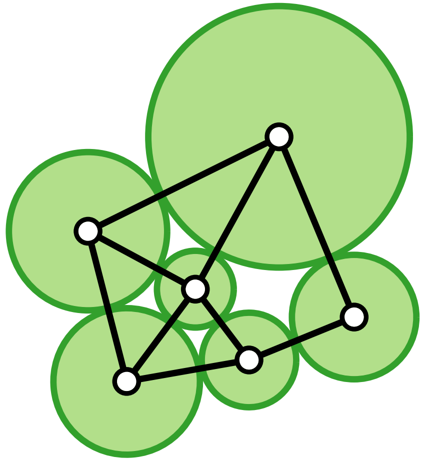


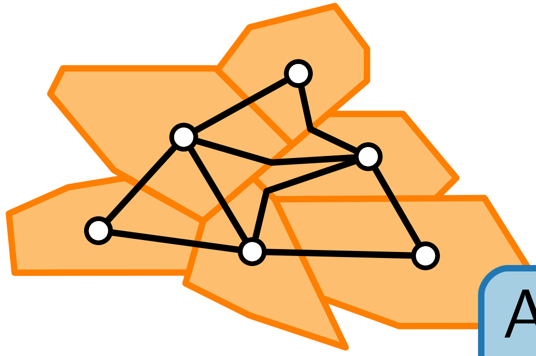
rectangular cuboids



In an $\mathcal{S}$-**contact representation** of G, $S(u)$ and $S(v)$ *touch* iff $uv \in E$







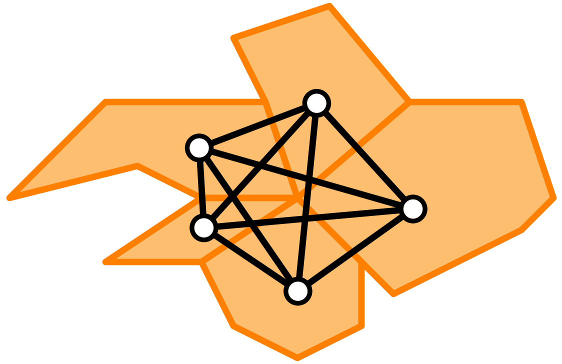
G is planar $\xrightarrow[\text{[Koebe 1936]}]{}$ disks $\longrightarrow$ polygons

A contact representation is an intersection representation with interior-disjoint sets.

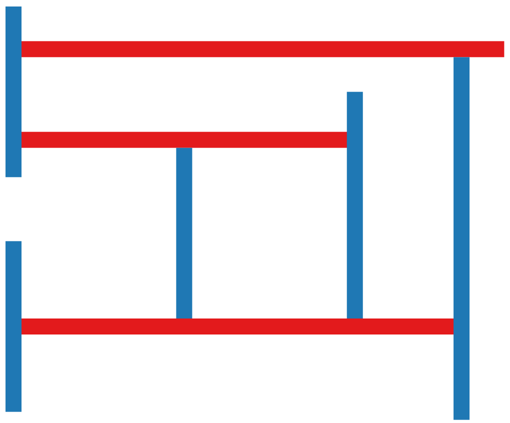
Contact Representation of Planar Graphs

Is the intersection graph of a contact representation always planar?

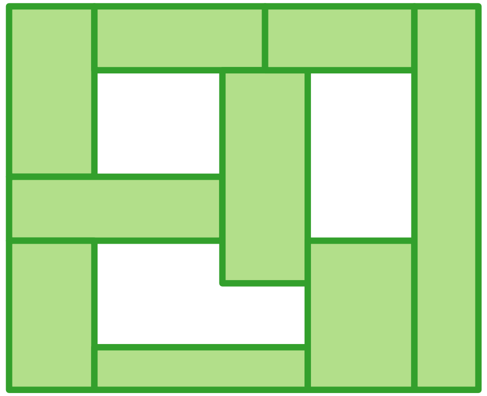
- No, not even for connected object types.



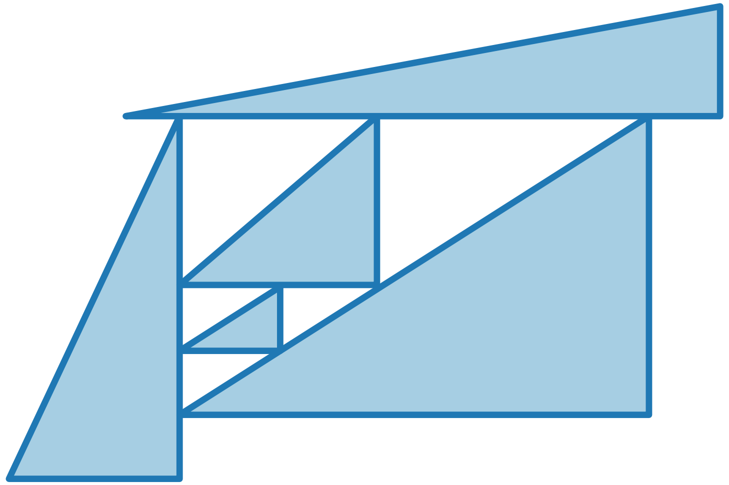
Some object types are used to represent special classes of planar graphs:



bipartite graphs



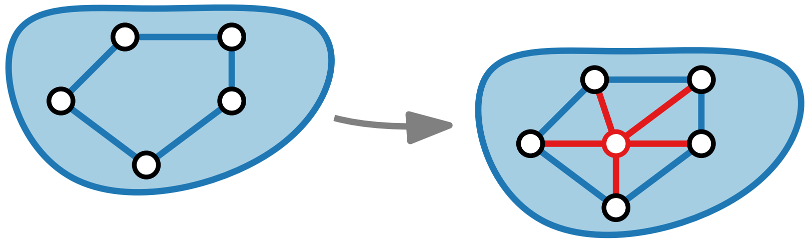
max. triangle-free graphs



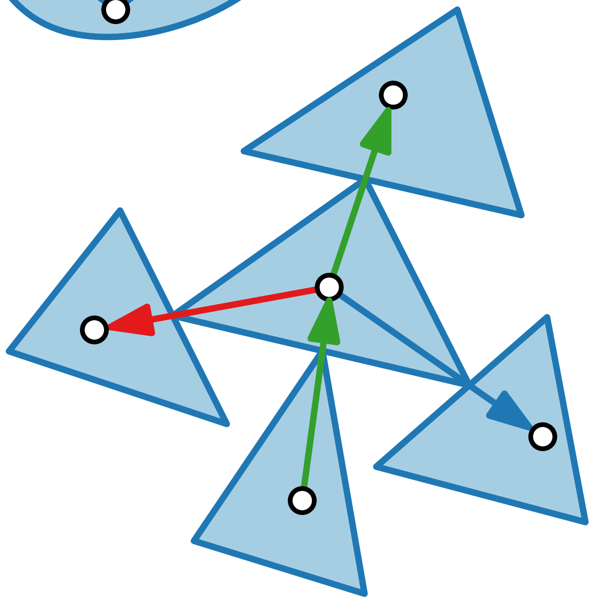
planar triangulations

General Approach

How to compute a contact representation of a given graph G?



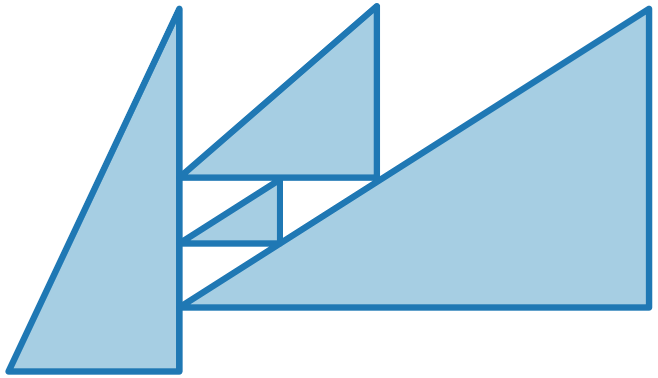
- Consider only inner triangulations (or maximal bipartite graphs, etc.)
  - Triangulate by adding vertices, not by adding edges
- Describe contact representation combinatorically.
  - Which objects touch each other in which way?
- Compute combinatorical description.
- Show that combinatorical description can be used to construct drawing.



This Lecture

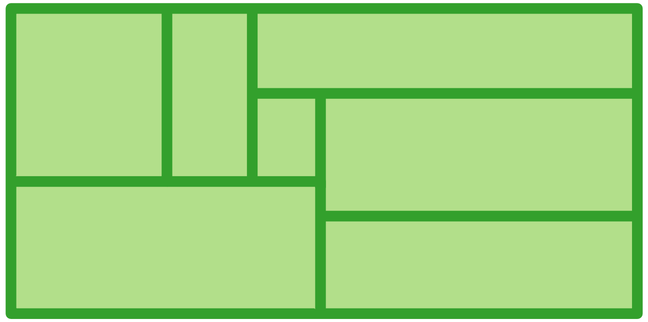
Representation with right-triangles and corner contact:

- Use Schnyder realizer to describe contacts between triangles.
- Use canonical order to compute drawing.



Representation with dissection of a rectangle, called **rectangular dual**:

- Find a description similar to a Schnyder realizer for rectangles.
- Construct drawing via st-digraphs, duals, and topological sorting.



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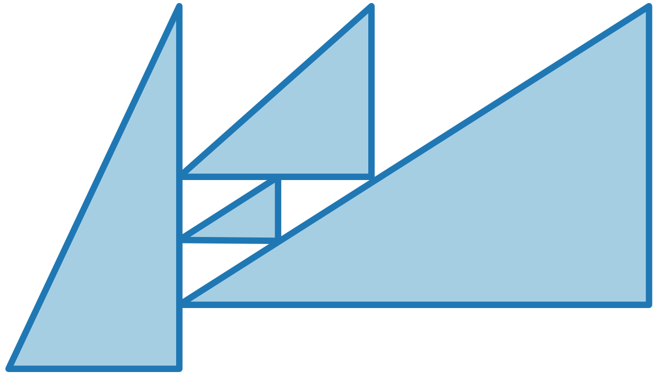

Part II:
Triangle Contact Representations

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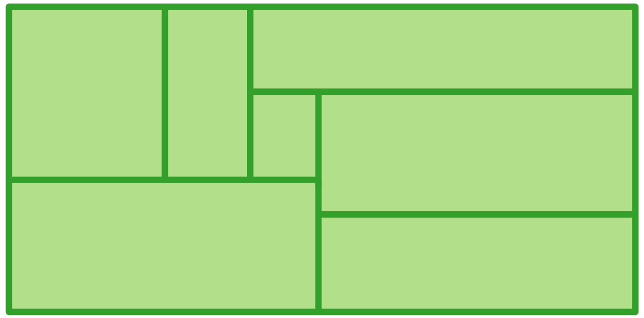
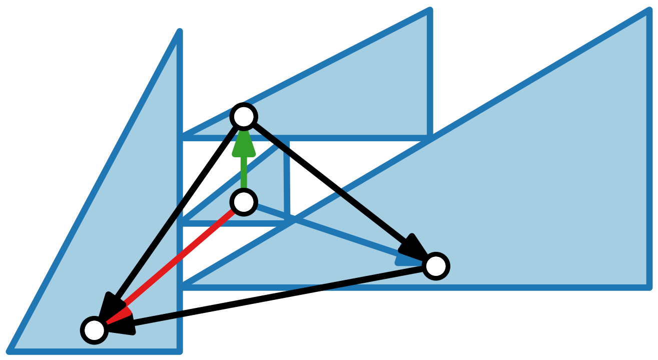


Triangle Corner Contact Representation

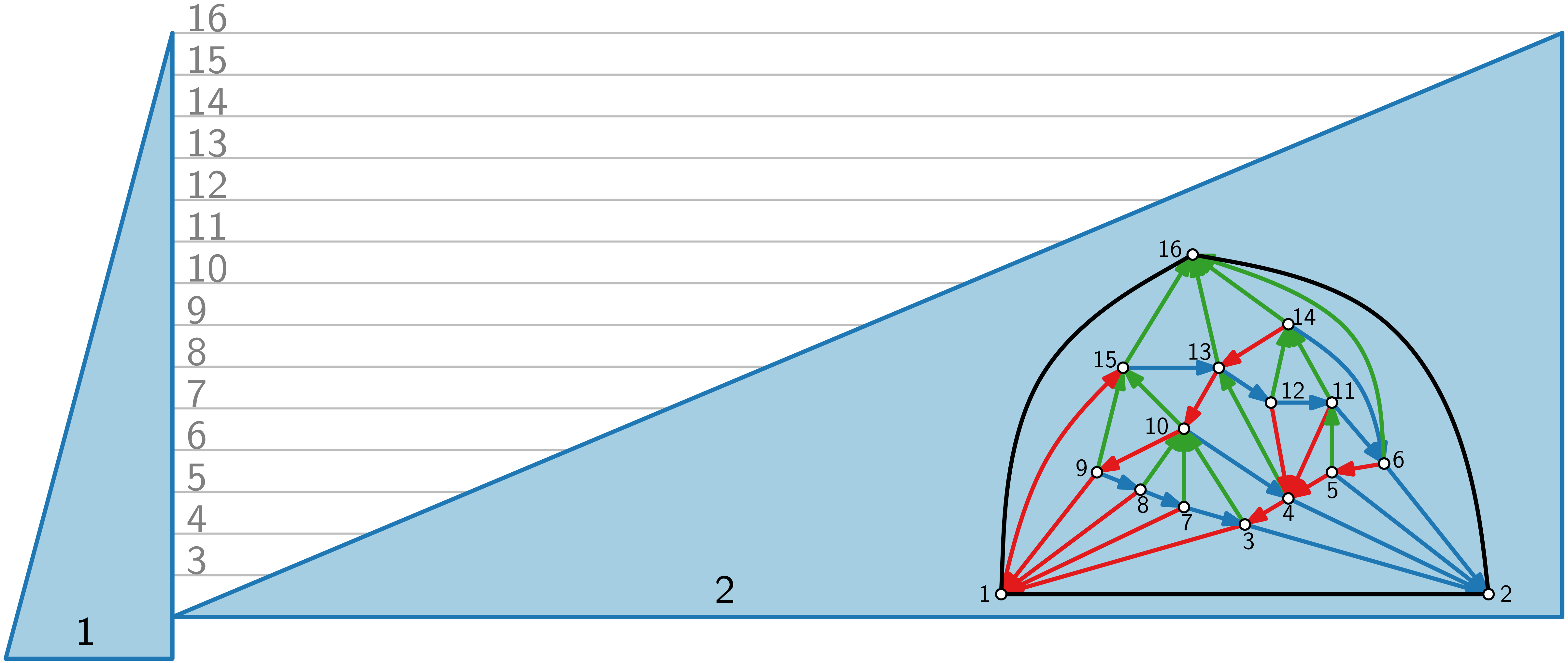


Idea.
Use canonical order and Schnyder realizer to find coordinates for triangles.

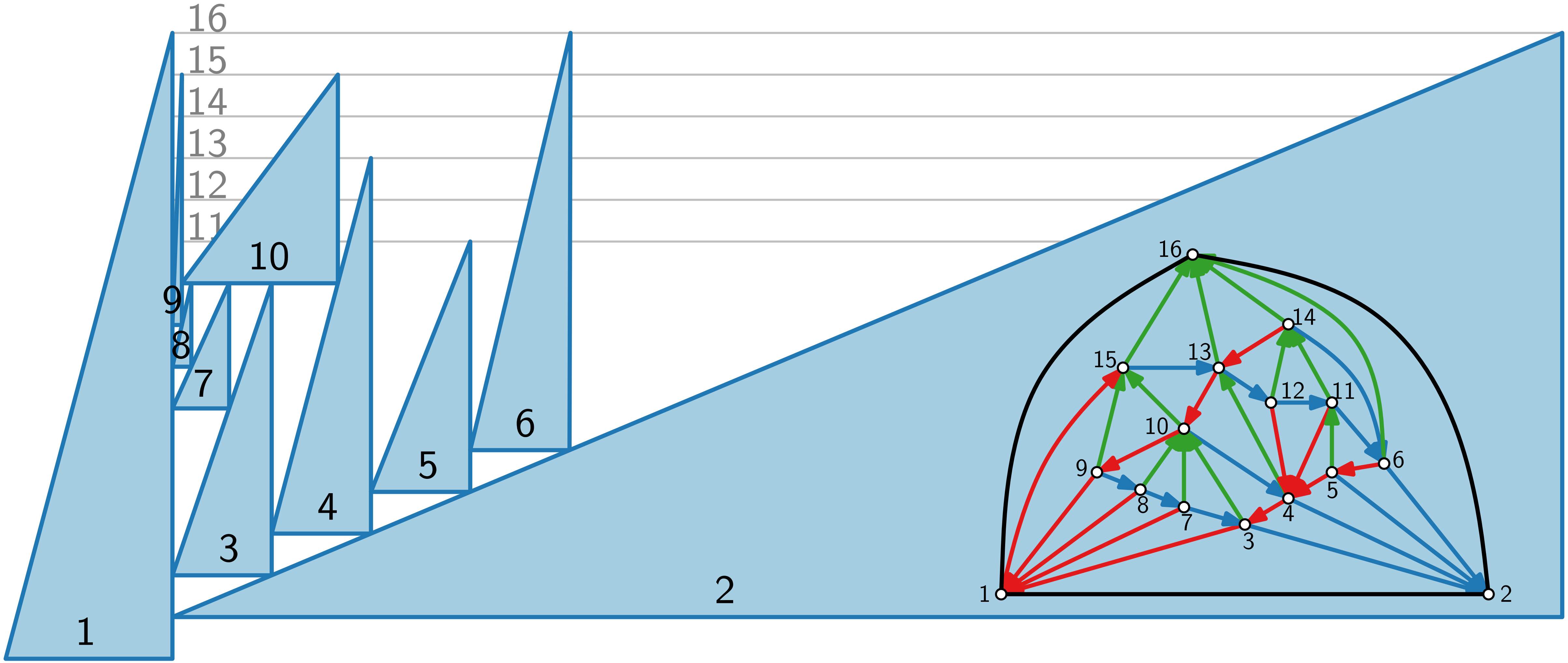
Observation.

- Can set base of triangle at height equal to position in canonical order.
- Triangle tip is precisely at base of triangle corresponding to cover neighbor.
- Outgoing edges in Schnyder forest indicate corner contacts.

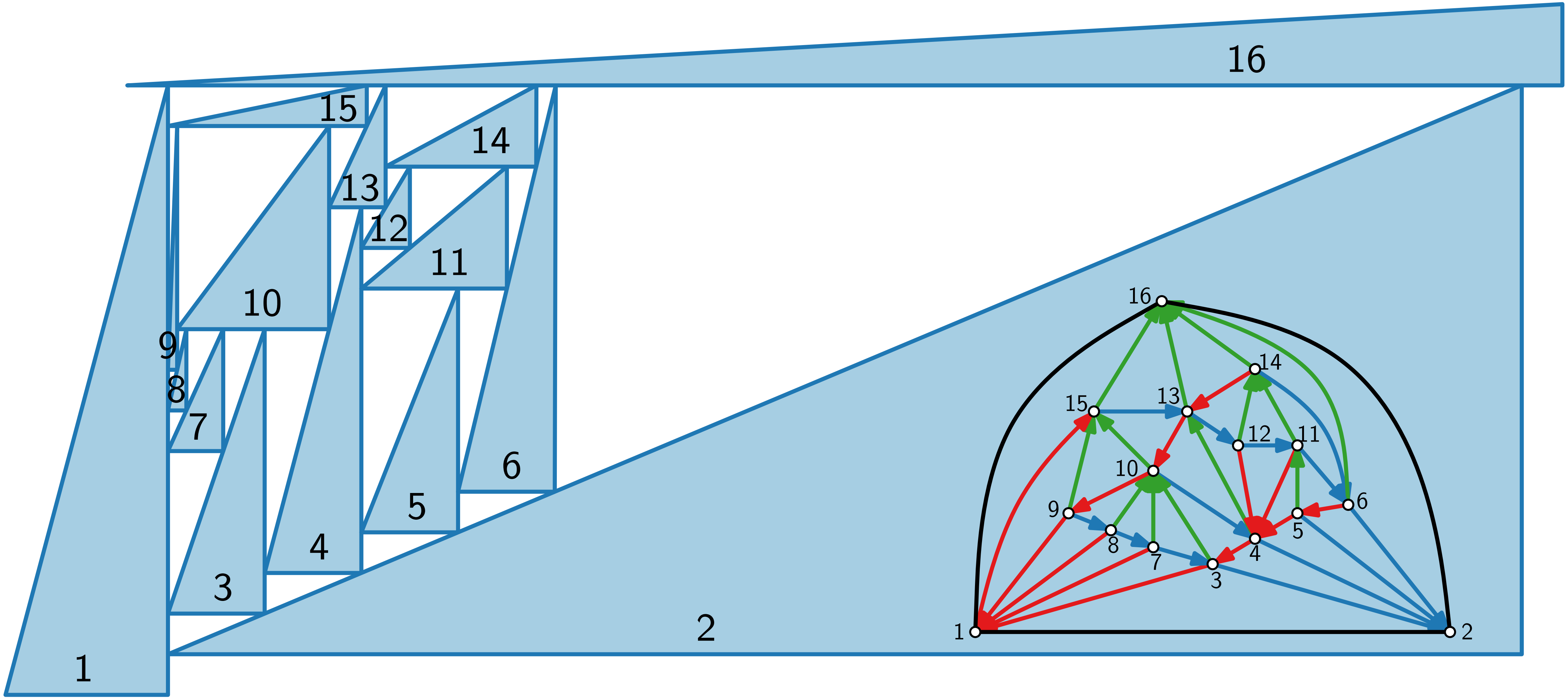
Triangle Contact Representation Example



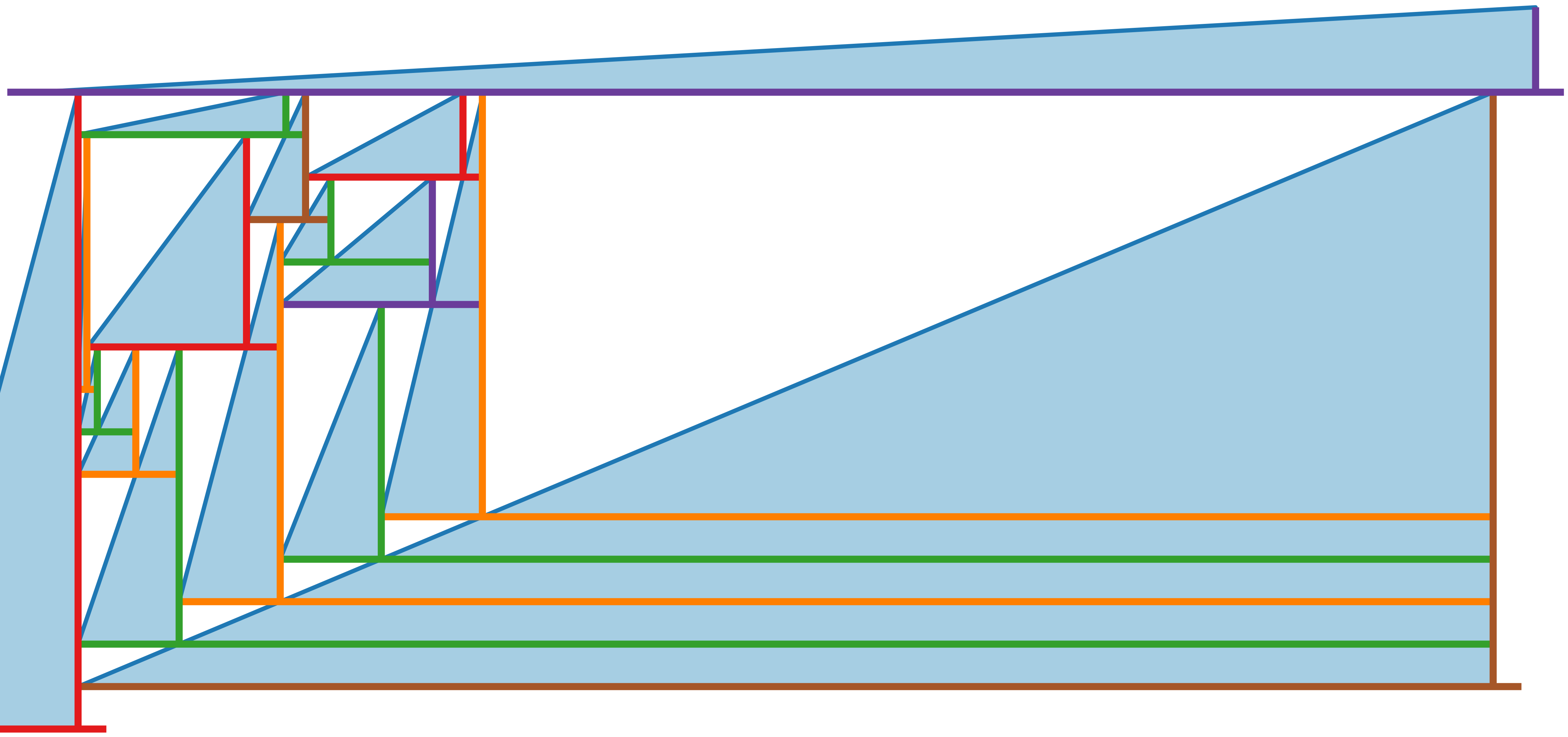
Triangle Contact Representation Example



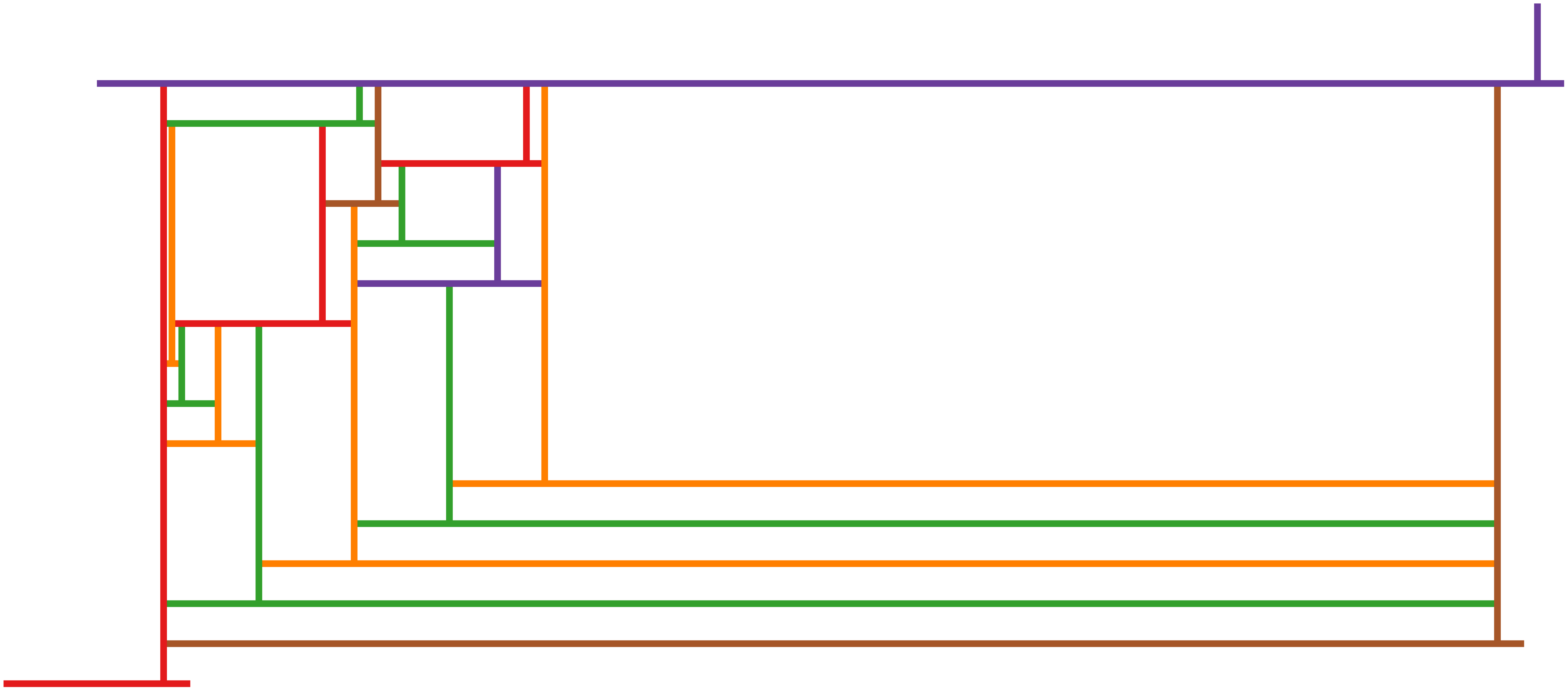
Triangle Contact Representation Example



T-shape Contact Representation



T-shape Contact Representation



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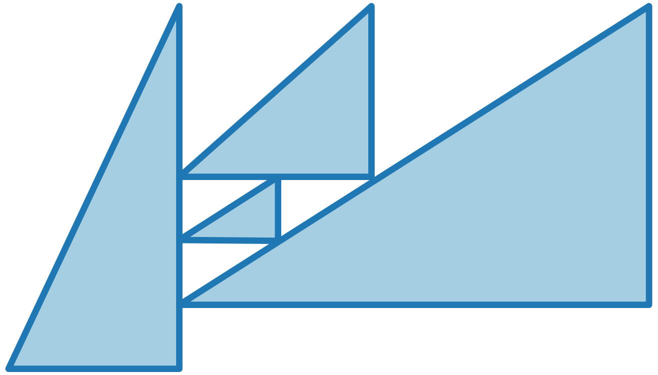
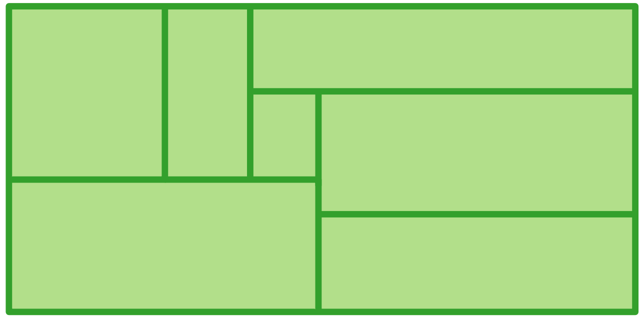
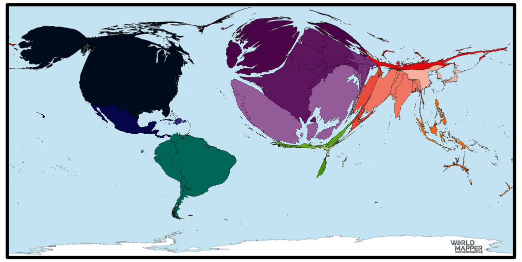


Part III:
Rectangular Duals



Alexander Wolff

Cartograms



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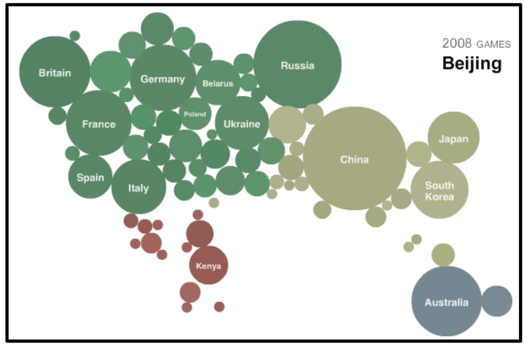

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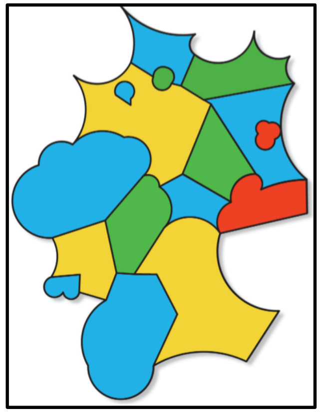

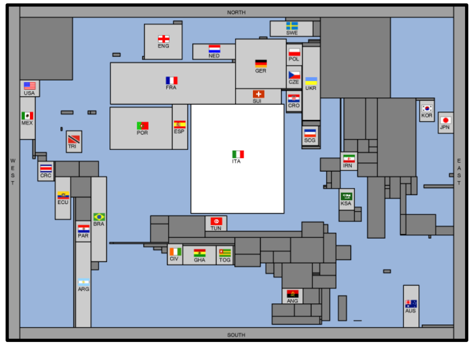

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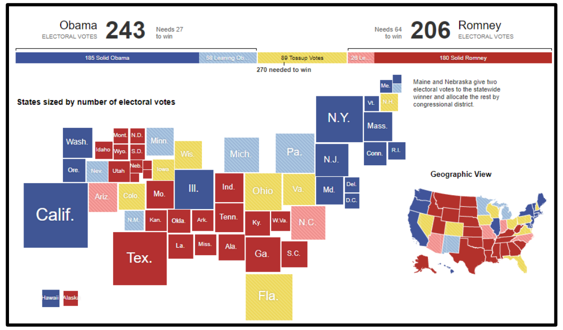

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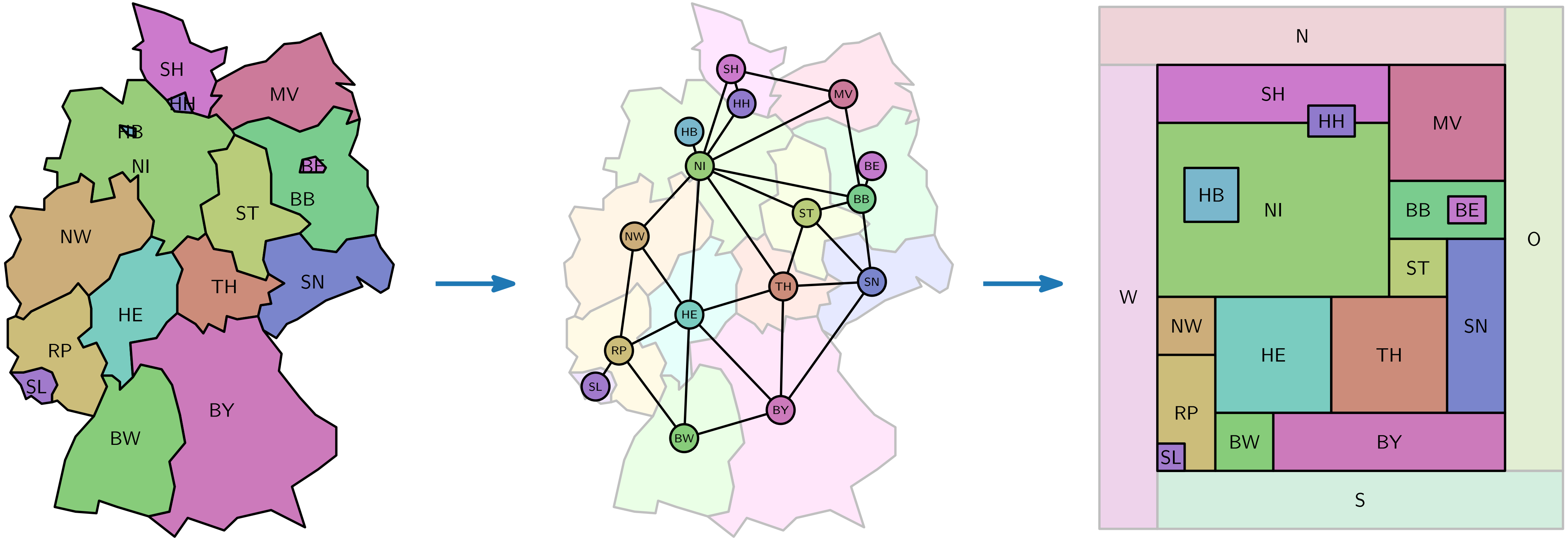
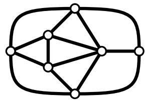
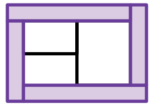


Rectangular Dual

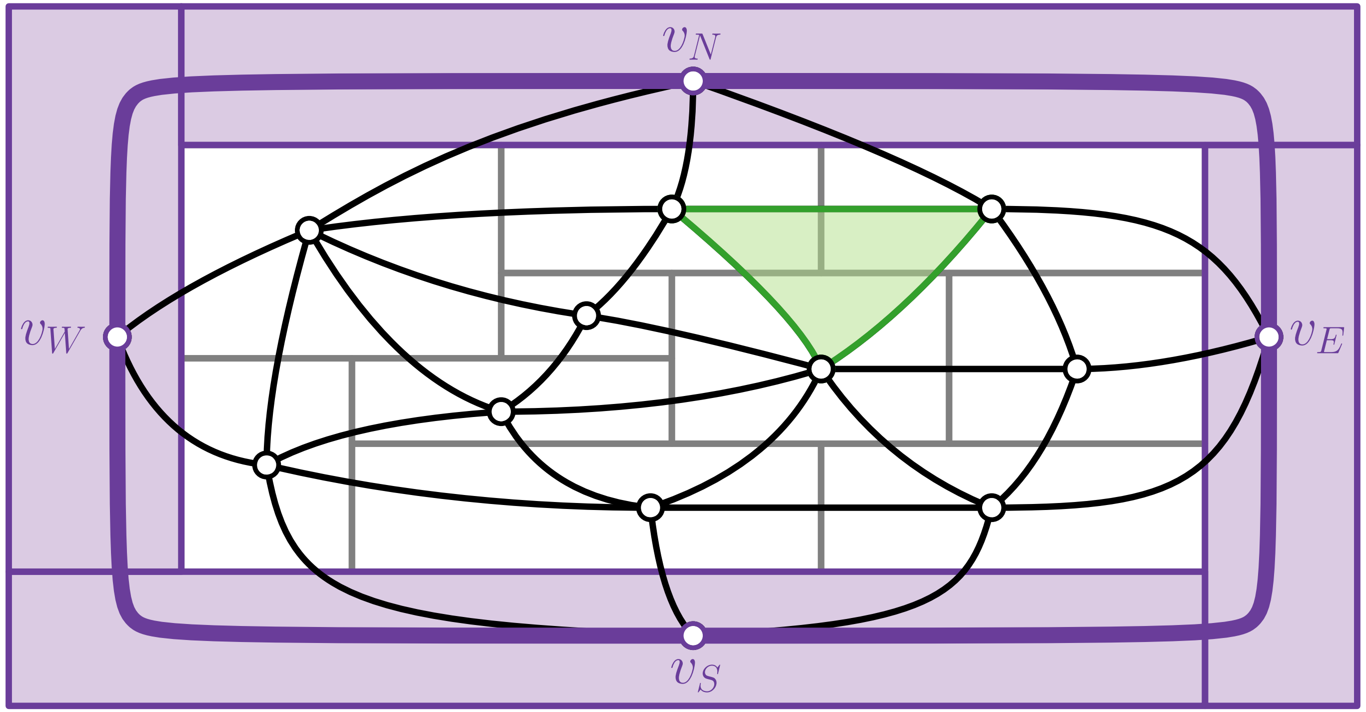


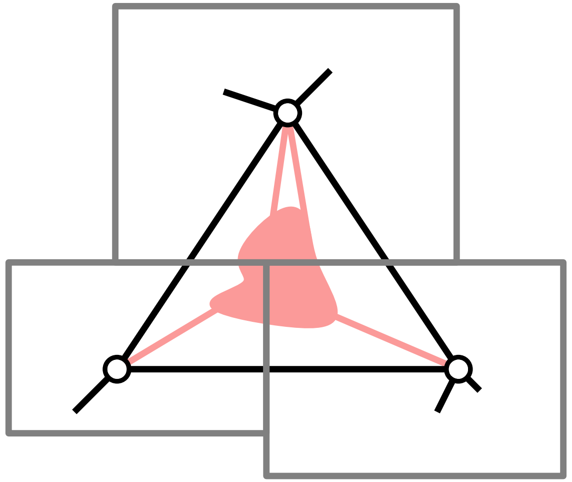
Properly Triangulated Planar Graph G



Rectangular Dual $\mathcal{R}$

Exactly 4 vertices on outer face

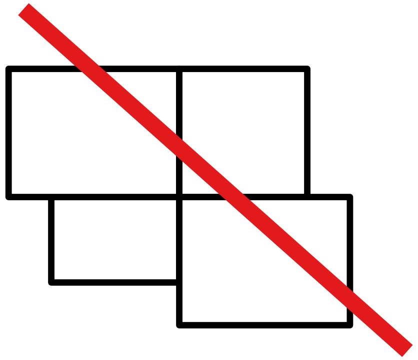




no separating triangle

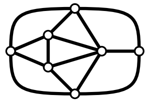
A **rectangular dual** of a graph G is a contact representation with axis-aligned rectangles s.t.

- no four rectangles share a point, and
- the union of all rectangles is a rectangle

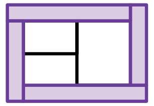


Theorem. [Koźmiński, Kinnen '85]
A graph G has a rectangular dual $\mathcal{R}$ if and only if G is a PTP graph.

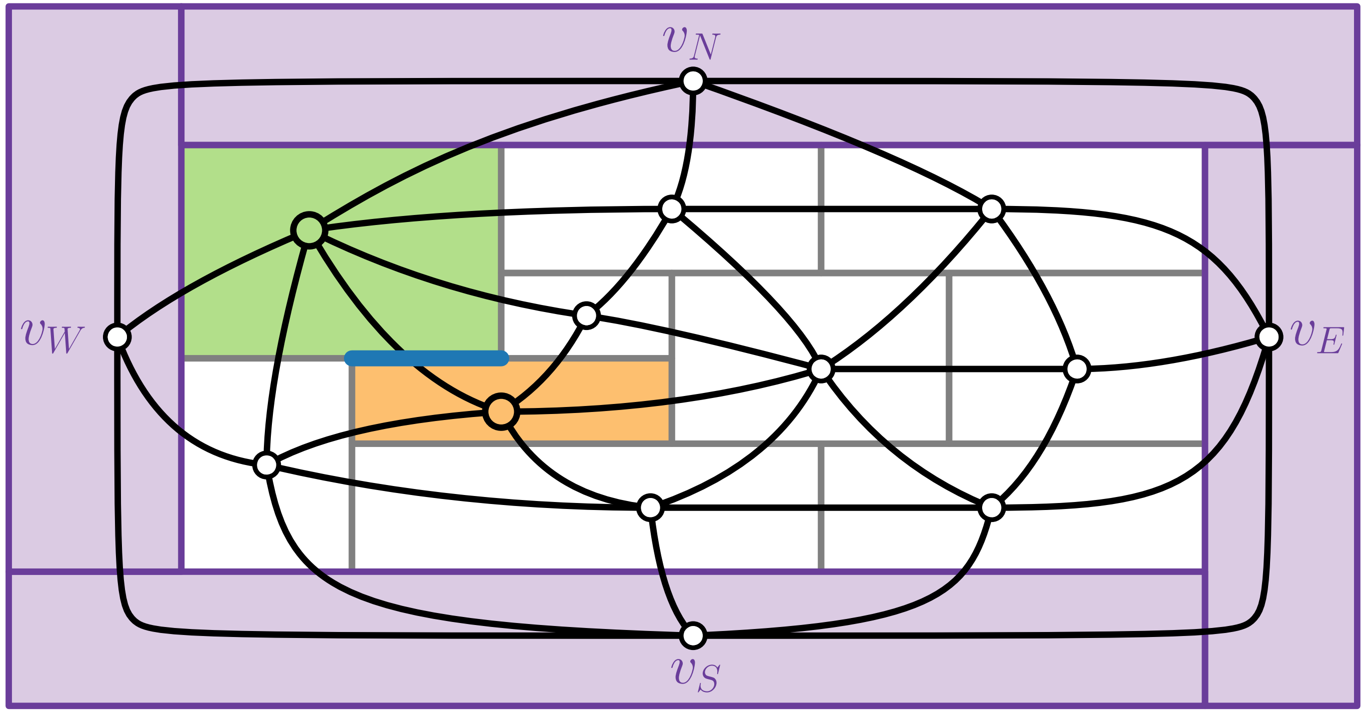
Regular Edge Labeling


PTP

Properly Triangulated Planar Graph G


RD

Rectangular Dual $\mathcal{R}$



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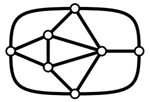
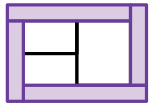
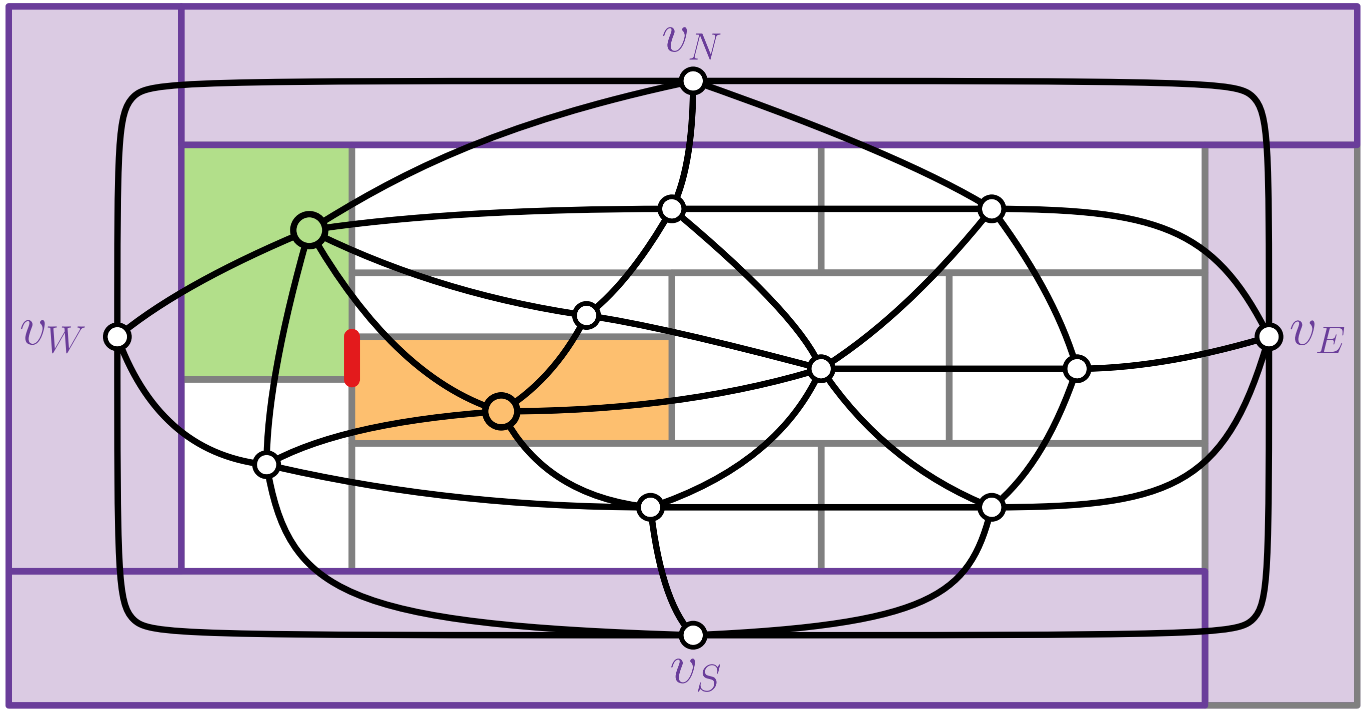


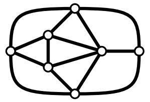
Properly Triangulated Planar Graph G



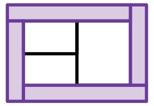
Rectangular Dual $\mathcal{R}$



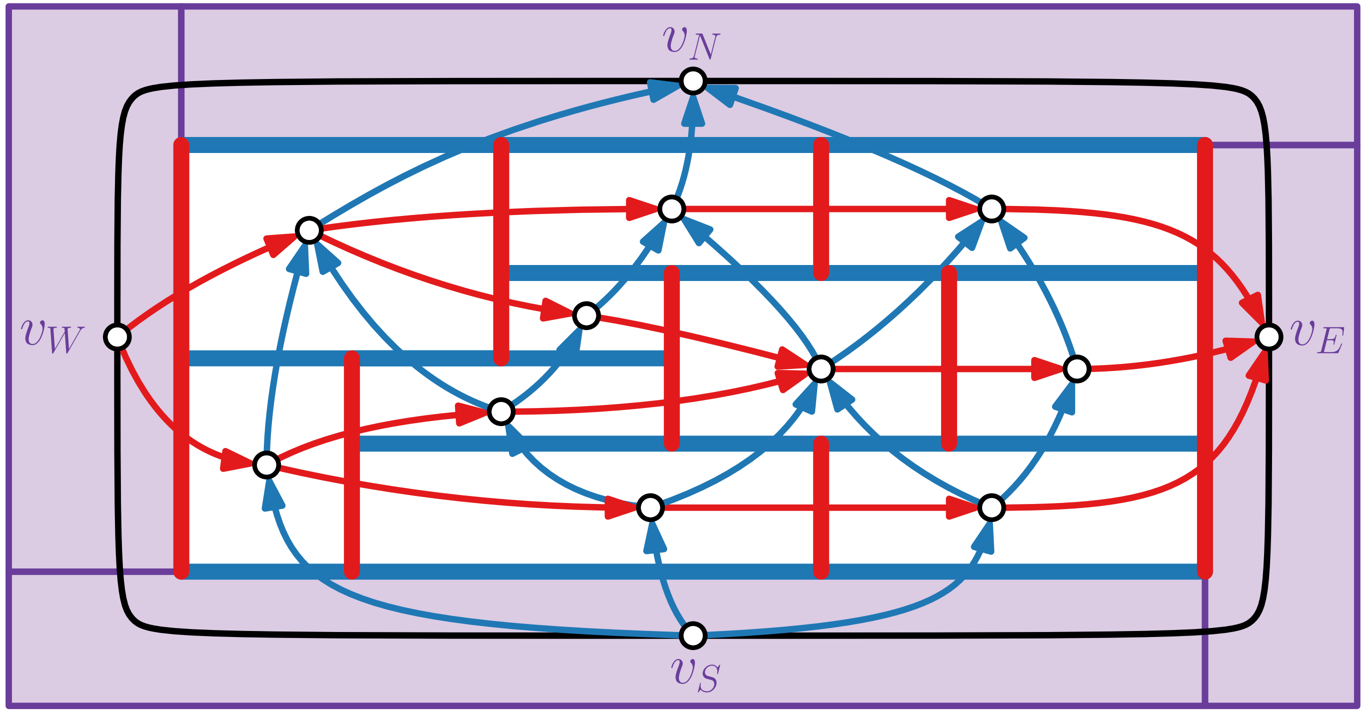
Regular Edge Labeling



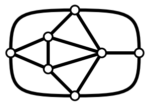
Properly Triangulated Planar Graph G



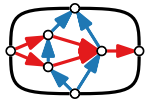
Rectangular Dual $\mathcal{R}$



Regular Edge Labeling


PTP

Properly Triangulated Planar Graph G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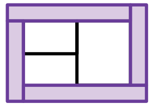
Regular Edge Labeling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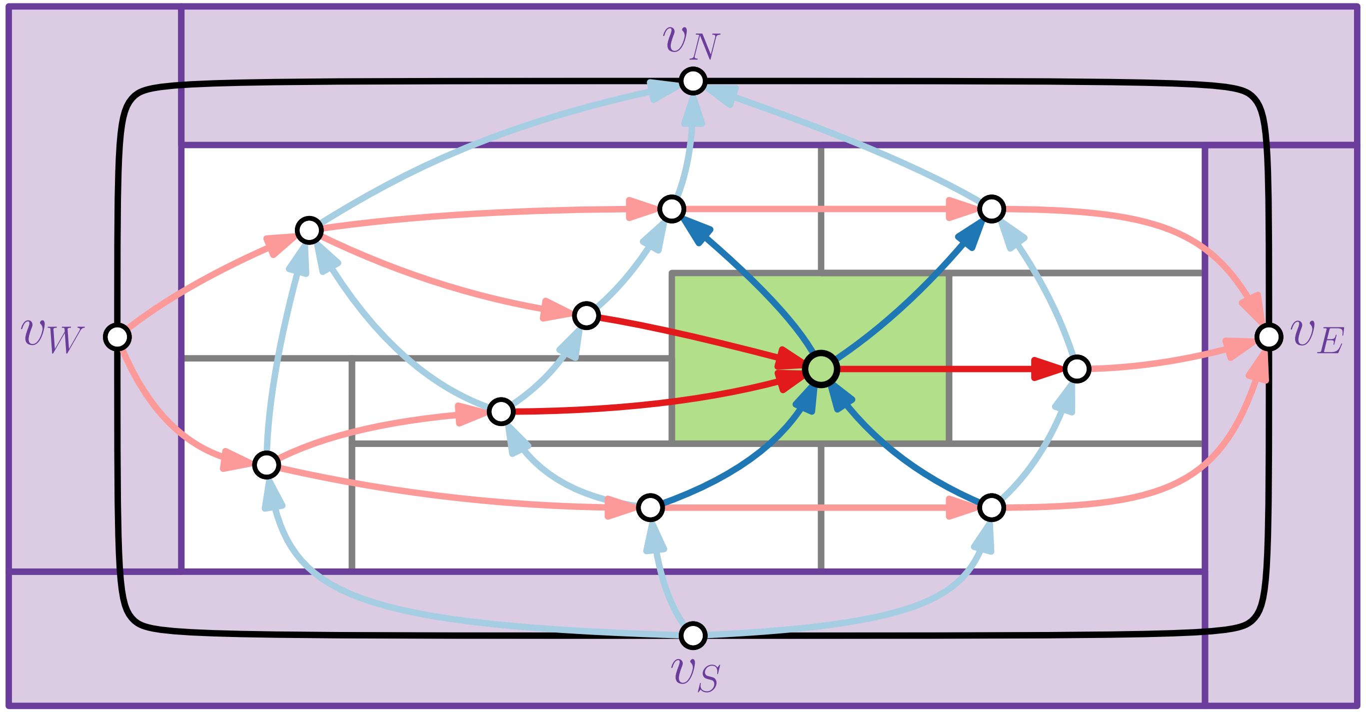
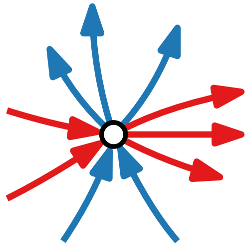
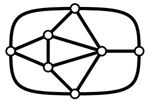




for every inner vertex

Regular Edge Labeling



PTP

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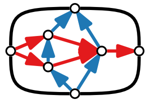
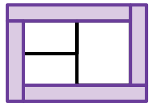


REL

Regular Edge Labeling



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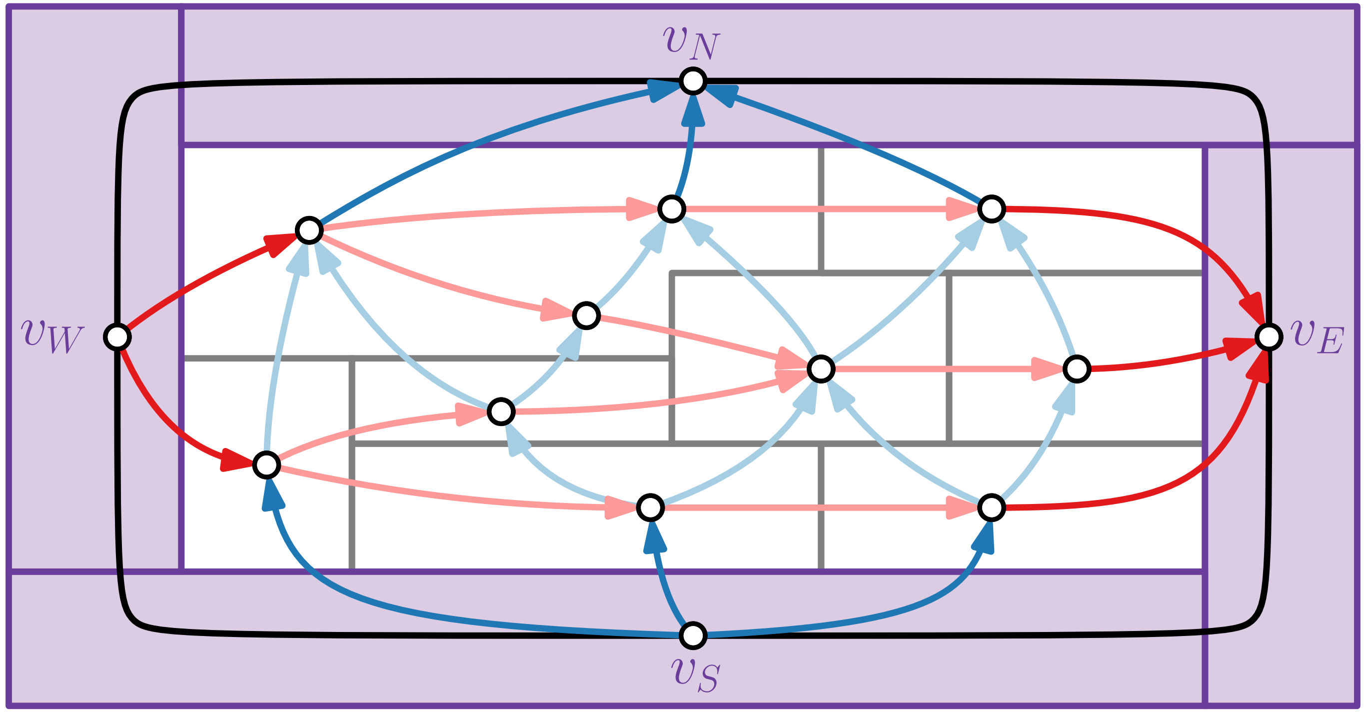
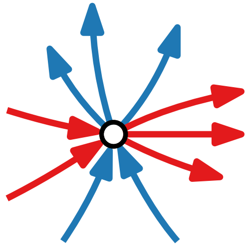
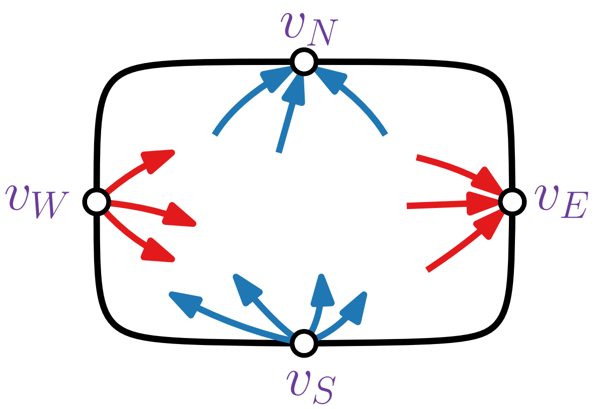


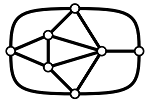


for every inner vertex

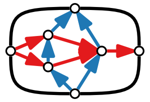


for four outer vertices

Regular Edge Labeling


PTP

Properly Triangulated Planar Graph G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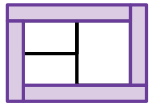
Regular Edge Labeling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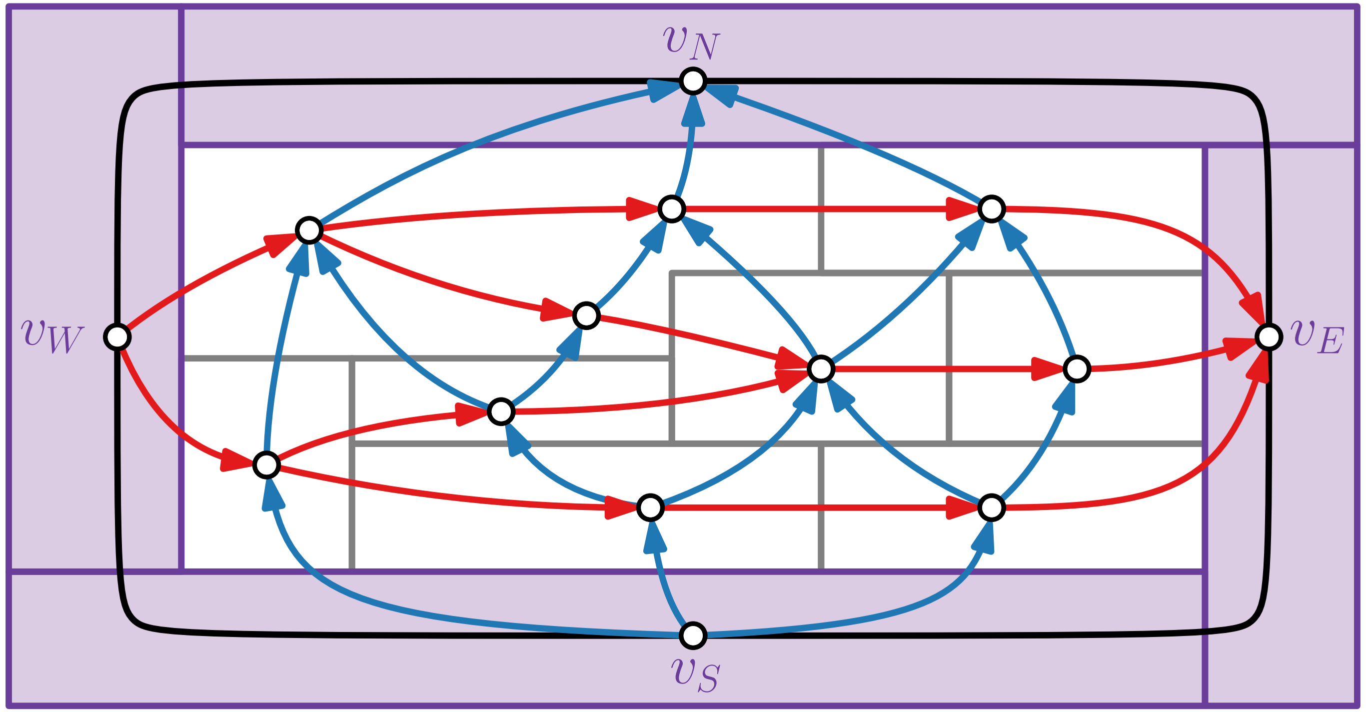
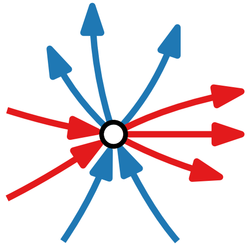
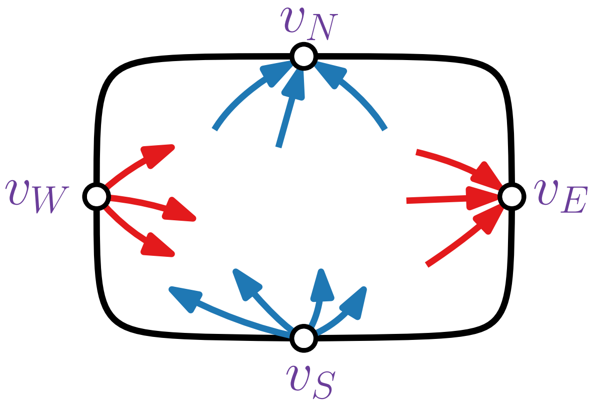
Rectangular Dual $\mathcal{R}$



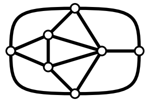
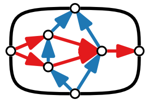
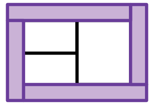


for every inner vertex



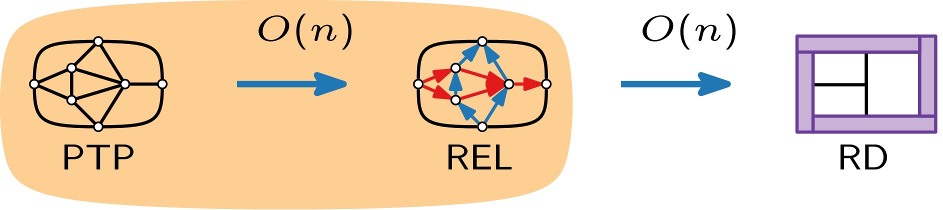
for four outer vertices

[Kant, He '94]: In linear time

 $\xrightarrow{O(n)}$  $\xrightarrow{O(n)}$ 

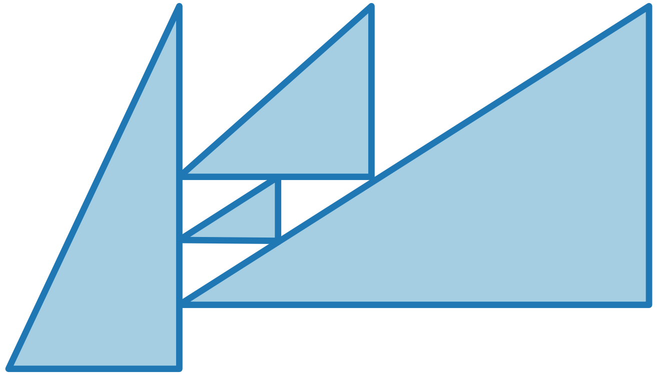
PTP REL RD

[Kant, He '94]:



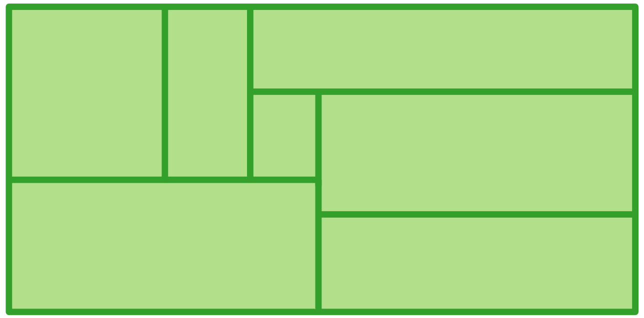
Visualization of Graphs

Lecture 7:
Contact Representations of Planar Graphs:
Triangle Contacts and Rectangular Duals



Part IV:
Computing a REL

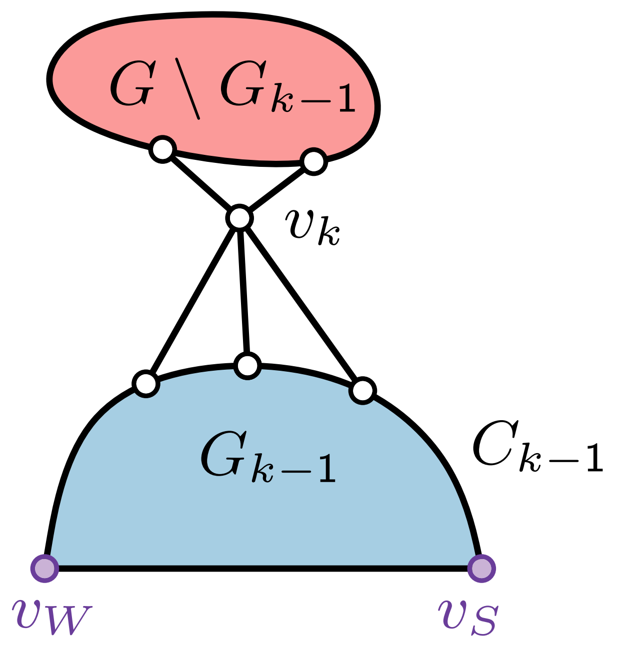
Alexander Wolff



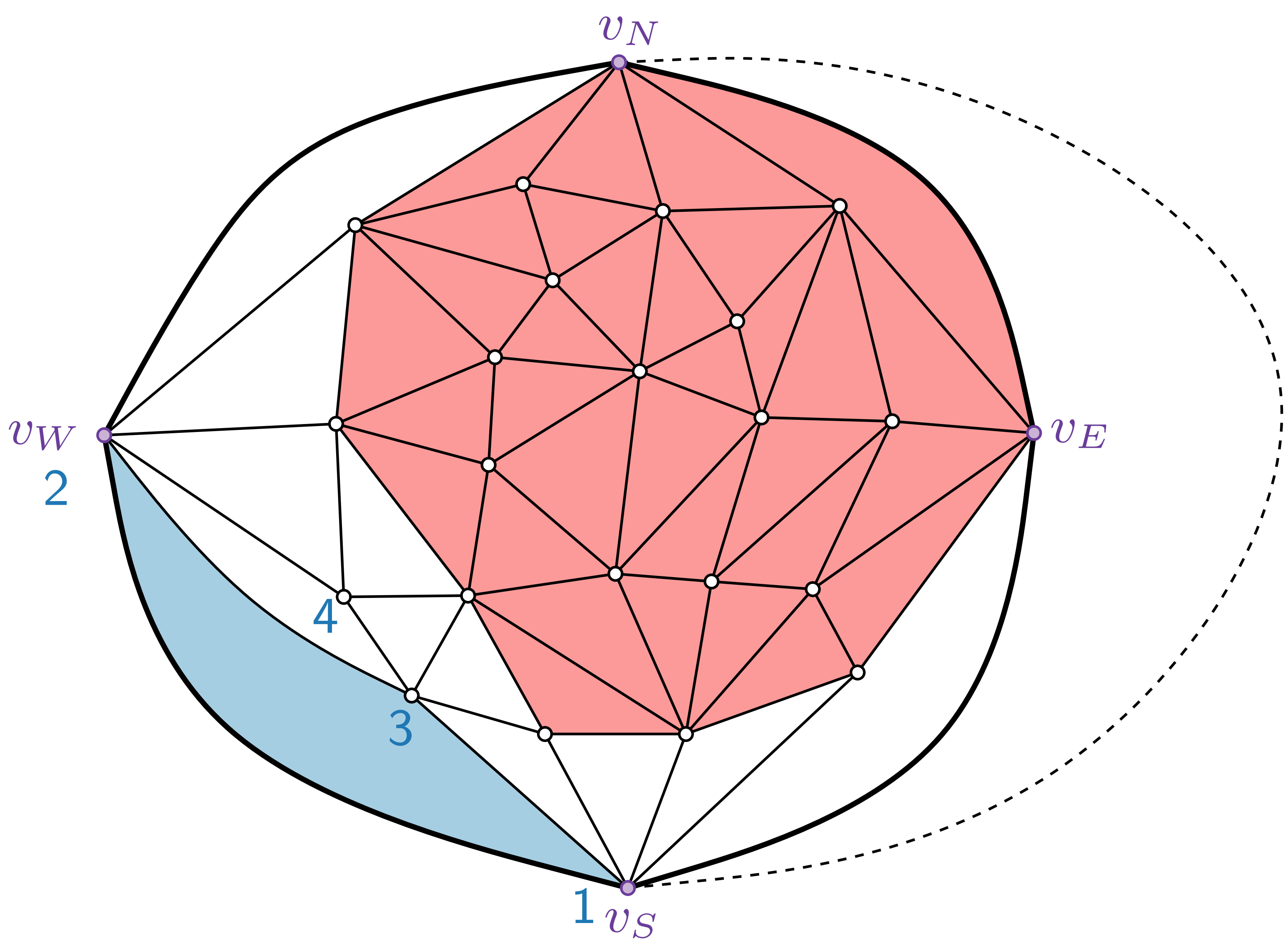
Refined Canonical Order

Theorem.
Let G be a PTP graph. There exists a labeling $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ of the vertices of G such that for every $4 \leq k \leq n$:

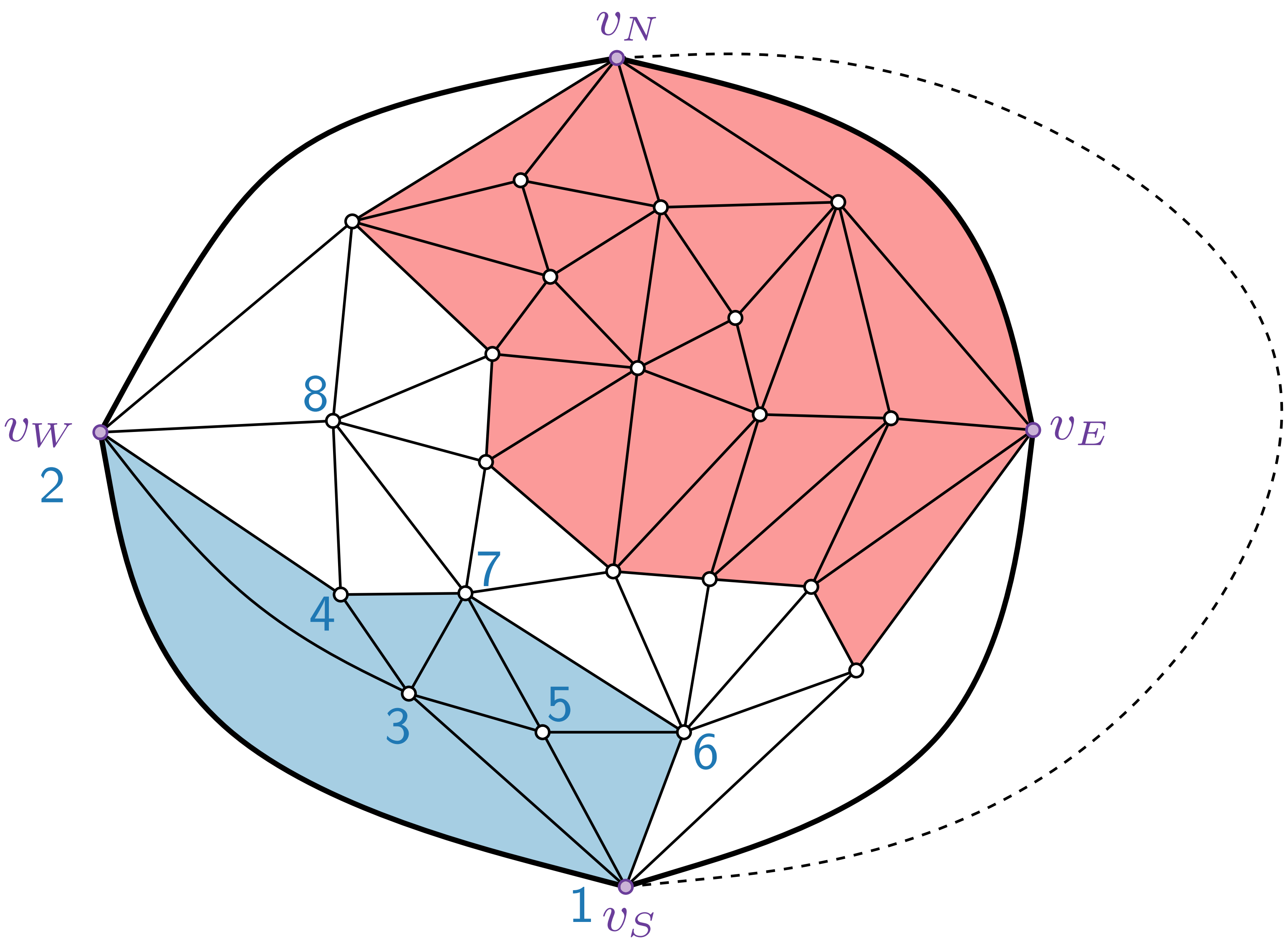
- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and the boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form a (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$.
- If $k \leq n-2$, then v_k has at least 2 neighbors in $G \setminus G_{k-1}$.



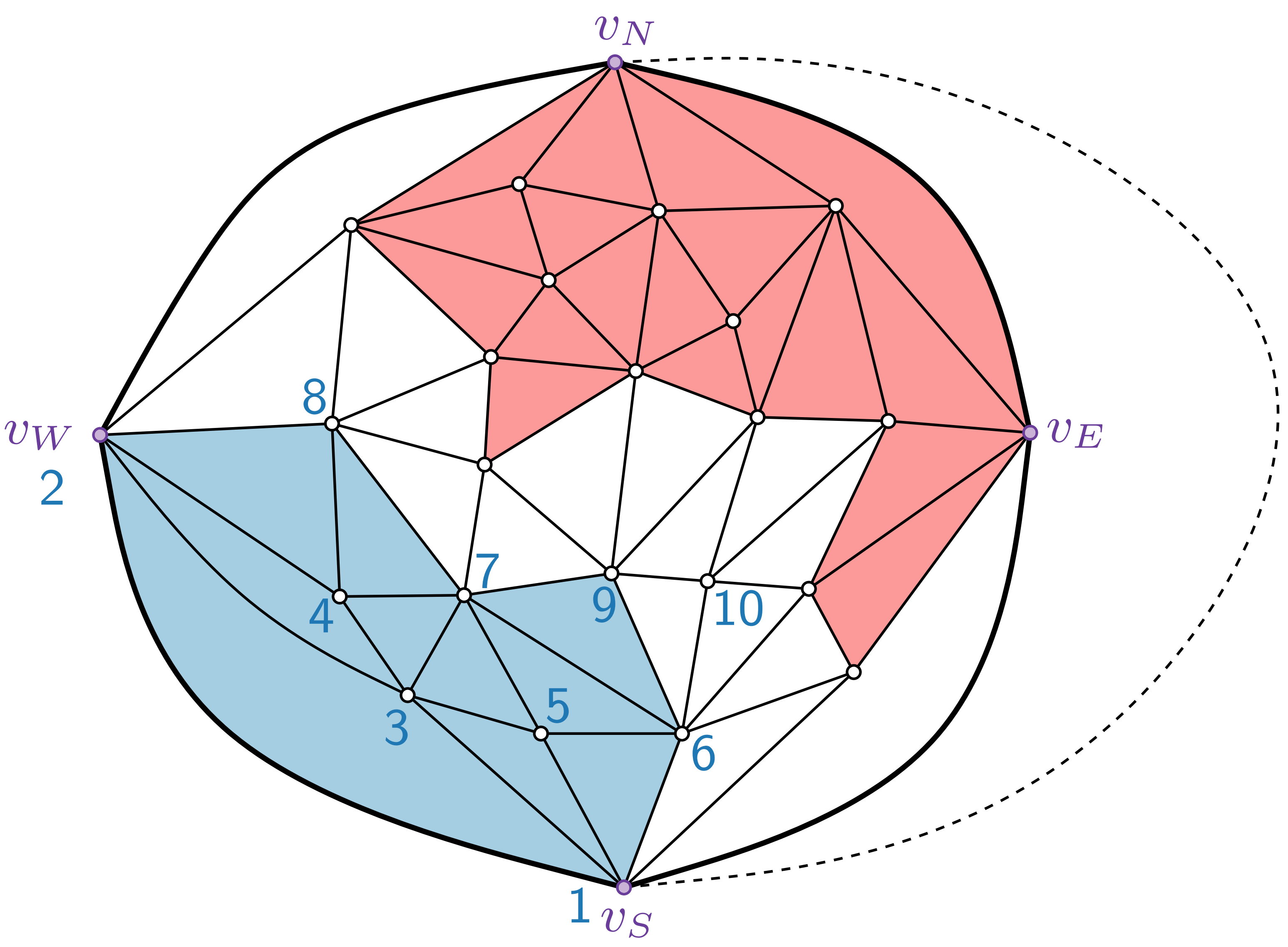
Refined Canonical Order Example



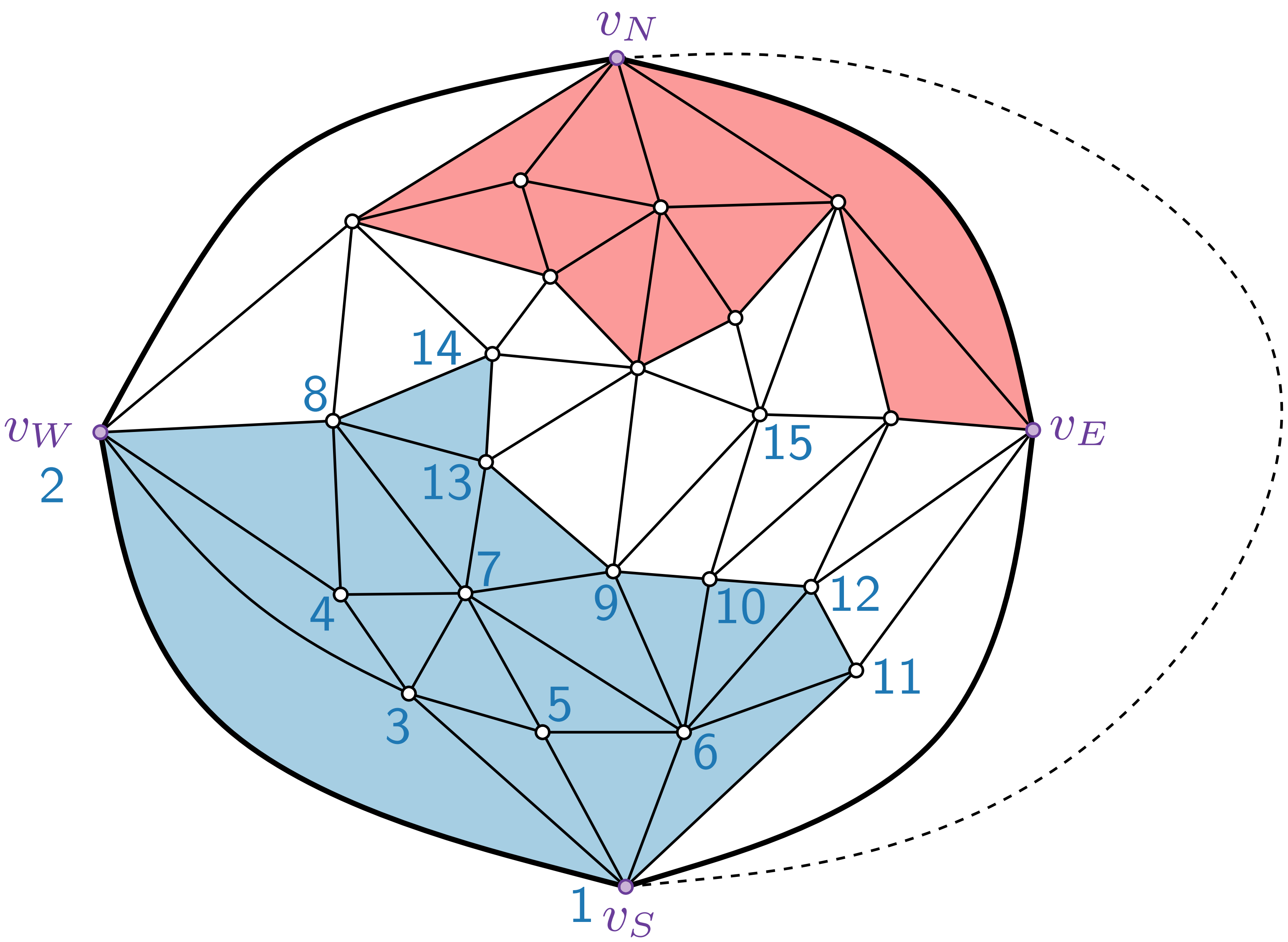
Refined Canonical Order Example



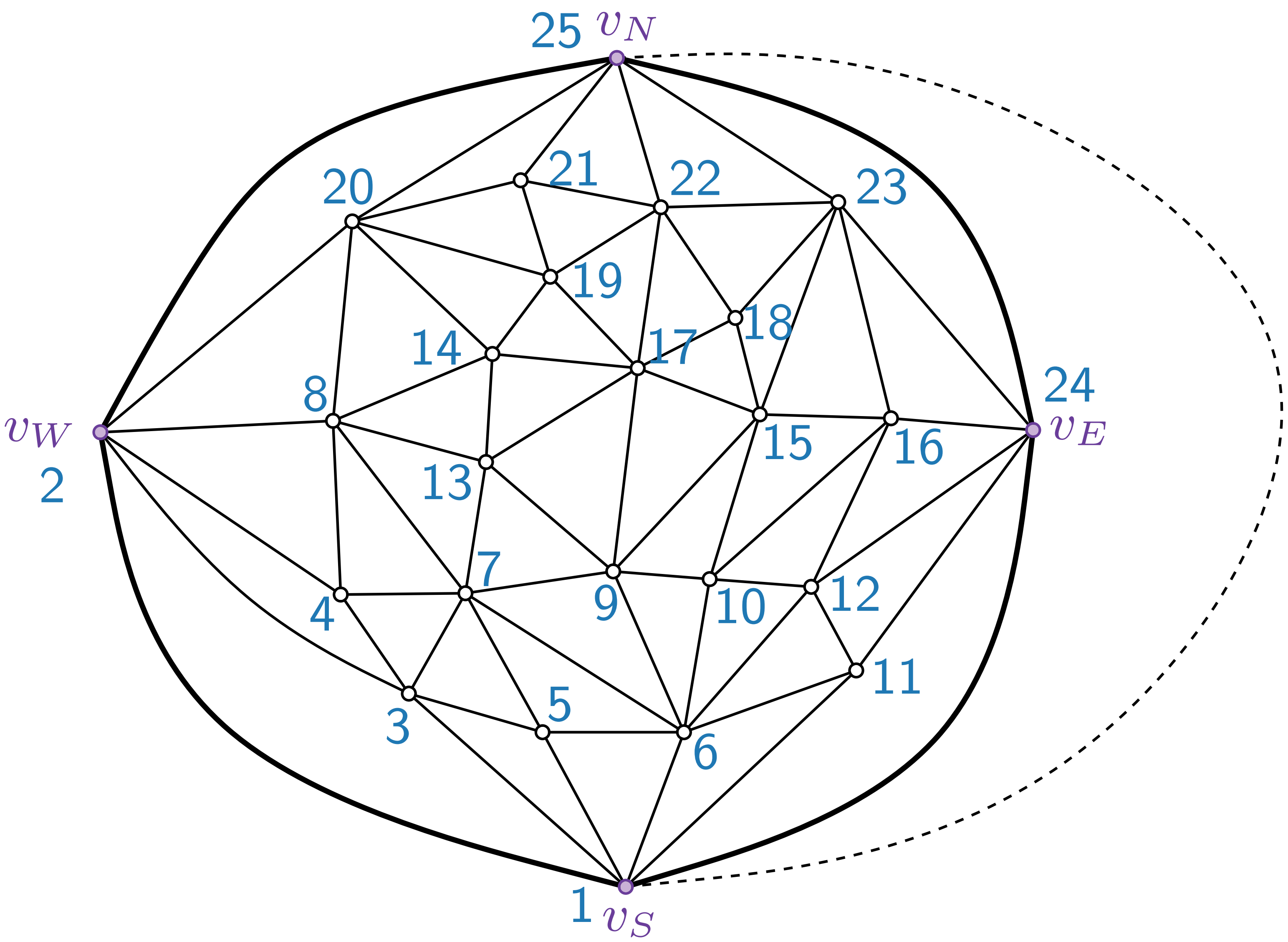
Refined Canonical Order Example



Refined Canonical Order Example



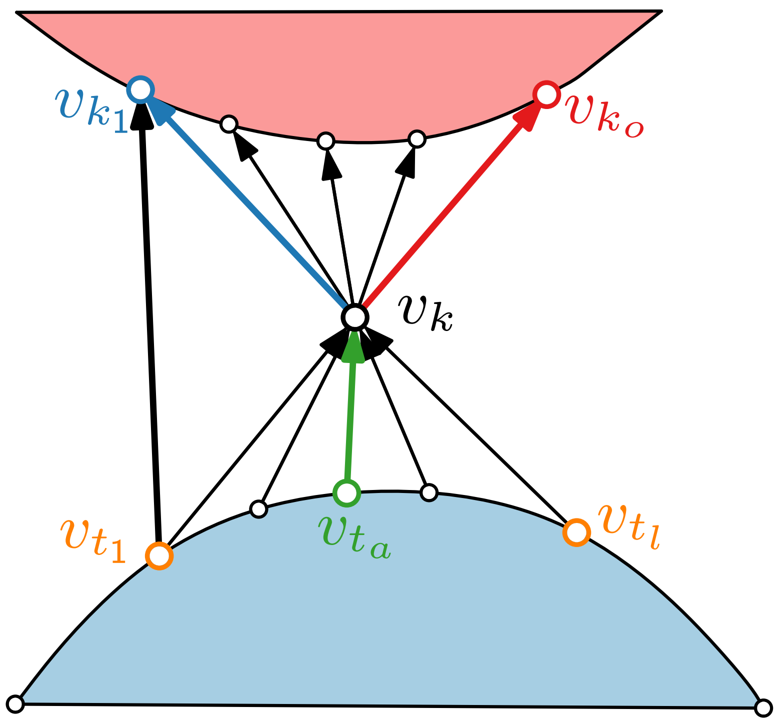
Refined Canonical Order Example



Refined Canonical Order $\rightarrow$ REL

We construct a REL as follows:

- For $i < j$, orient (v_i, v_j) from v_i to v_j;
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is **left point** of v_k and v_{t_l} is **right point** of v_k.
- **Base edge** of v_k is (v_{t_a}, v_k), where $t_a < k$ is minimal.
- If $v_{k_1}, \ldots, v_{k_o}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) **left edge** and (v_k, v_{k_o}) **right edge**.

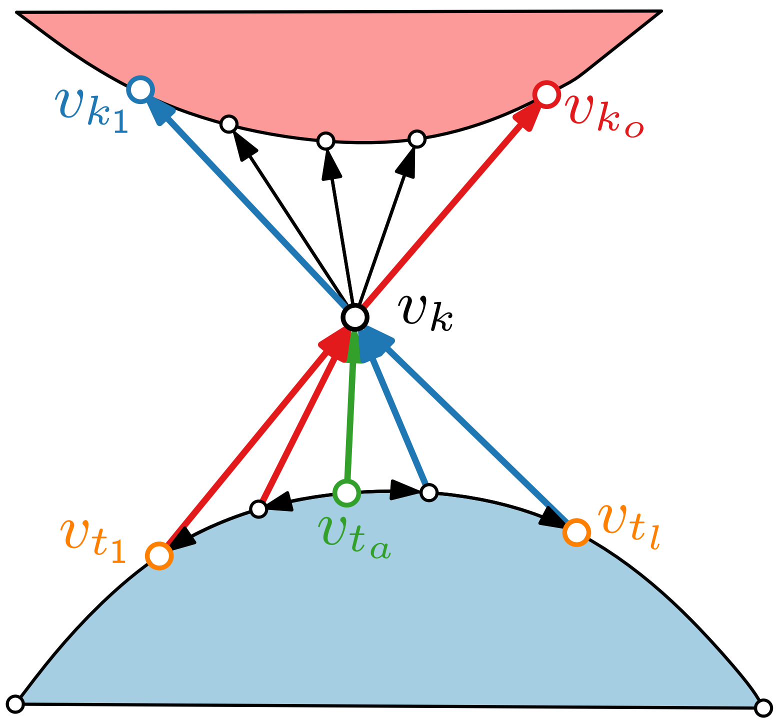


Lemma 1.
A left edge or right edge cannot be a base edge.

Proof. Suppose left edge (v_k, v_{k_1}) is base edge of v_{k_1}.
Since G triangulated, $(v_{t_1}, v_{k_1}) \in E(G)$.
Contradiction since $k > t_1$.

Refined Canonical Order $\rightarrow$ REL

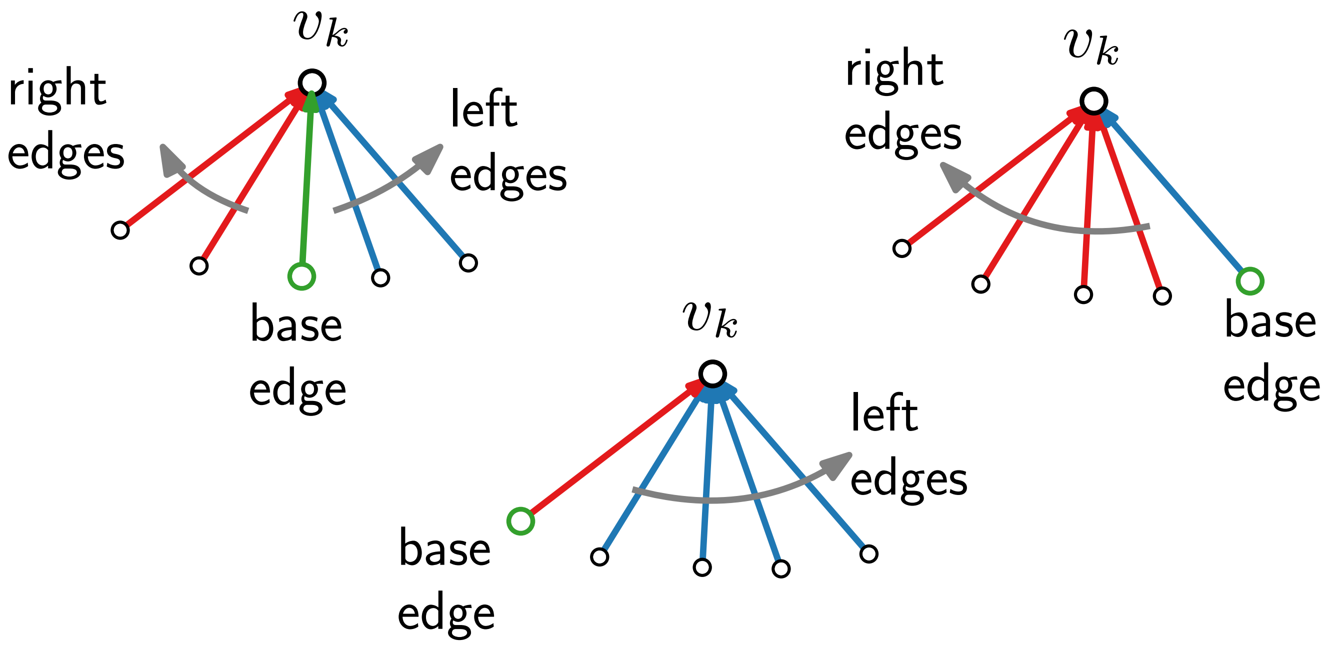
Lemma 2.
An edge is either a left edge, a right edge or a base edge.



Proof.

- Exclusive "or" follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$.
  - v_{t_i} has at least two higher-numbered neighbors.
  - One of them is v_k; the other one is either $v_{t_{i-1}}$ or $v_{t_{i+1}}$.
  - For $1 \leq i < a-1$, it is $v_{t_{i-1}}$. Thus, v_{t_i} is right point of $v_{t_{i-1}}$.
- Analogously, v_{t_i} is left point of $v_{t_{i+1}}$ for $i \geq a$.
- Edges (v_{t_i}, v_k), $1 \leq i < a-1$, are right edges.
- Similarly, (v_{t_i}, v_k), for $a+1 \leq i \leq l$, are left edges.

Refined Canonical Order $\to$ REL



Coloring.

- Color right (left) edges in red (blue).
- Color a base edge (v_{t_i}, v_k) red if $i = 1$ and blue if $i = l$ and otherwise arbitrarily.

Let T_r be the red edges and T_b the blue edges.

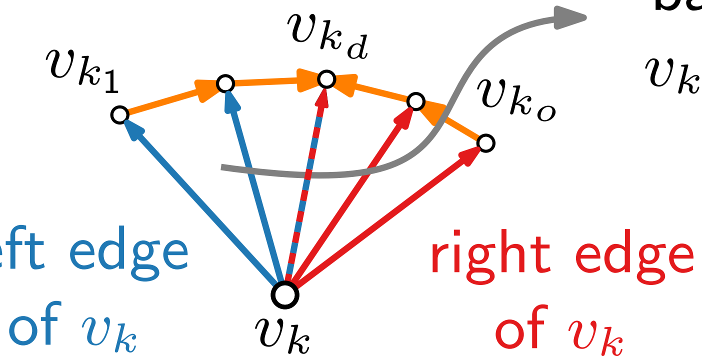
> **Lemma 3.**
> $\{T_r, T_b\}$ is a regular edge labeling.

Proof.

$k_o \geq 2$

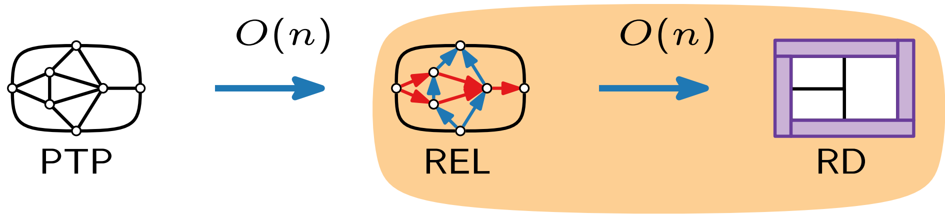
$k_d = \max\{k_1, \ldots, k_o\}$

base edges of $v_{k_2}, \ldots, v_{k_{o-1}}$



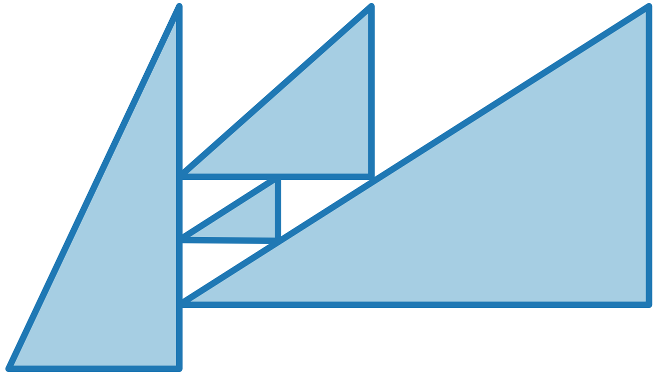
- $k_1 < k_2 < \ldots < k_d$ and $k_d > k_{d+1} > \ldots > k_o$
- (v_k, v_{k_i}), $2 \leq i \leq d-1$ are blue
- (v_k, v_{k_i}), $d+1 \leq i \leq o-1$ are red
- (v_k, v_{k_d}) is either red or blue

$\Rightarrow$ Circular order of outgoing edges at v_k correct.

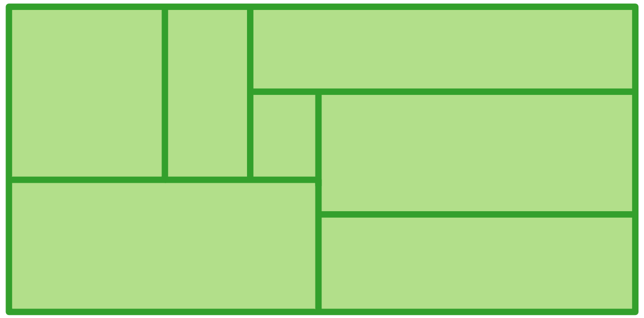
[Kant, He '94]: 

Visualization of Graphs

Lecture 8:
Contact Representations of Planar Graphs: Triangle Contacts and Rectangular Duals

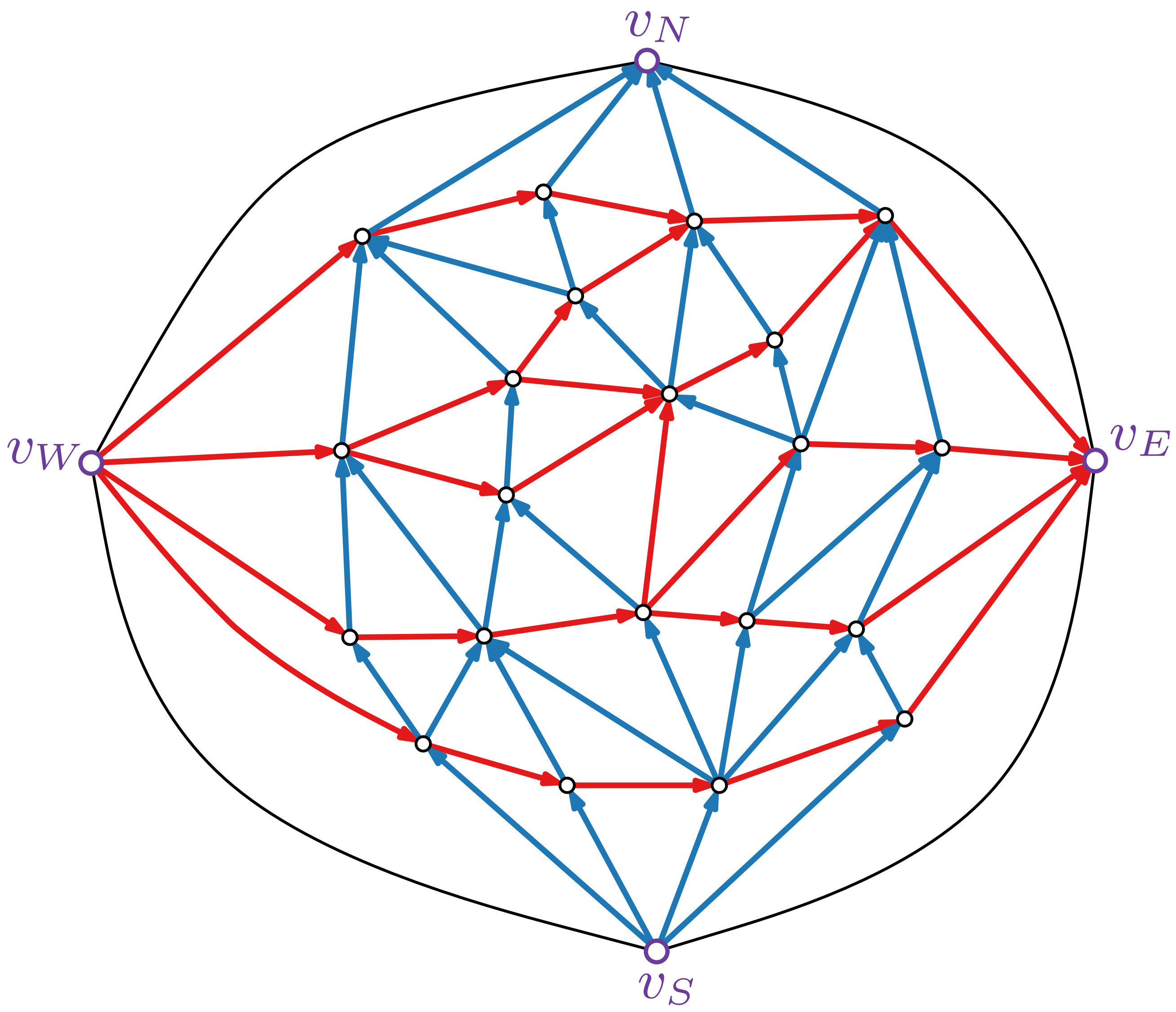


Part V:
Computing the Coordinates

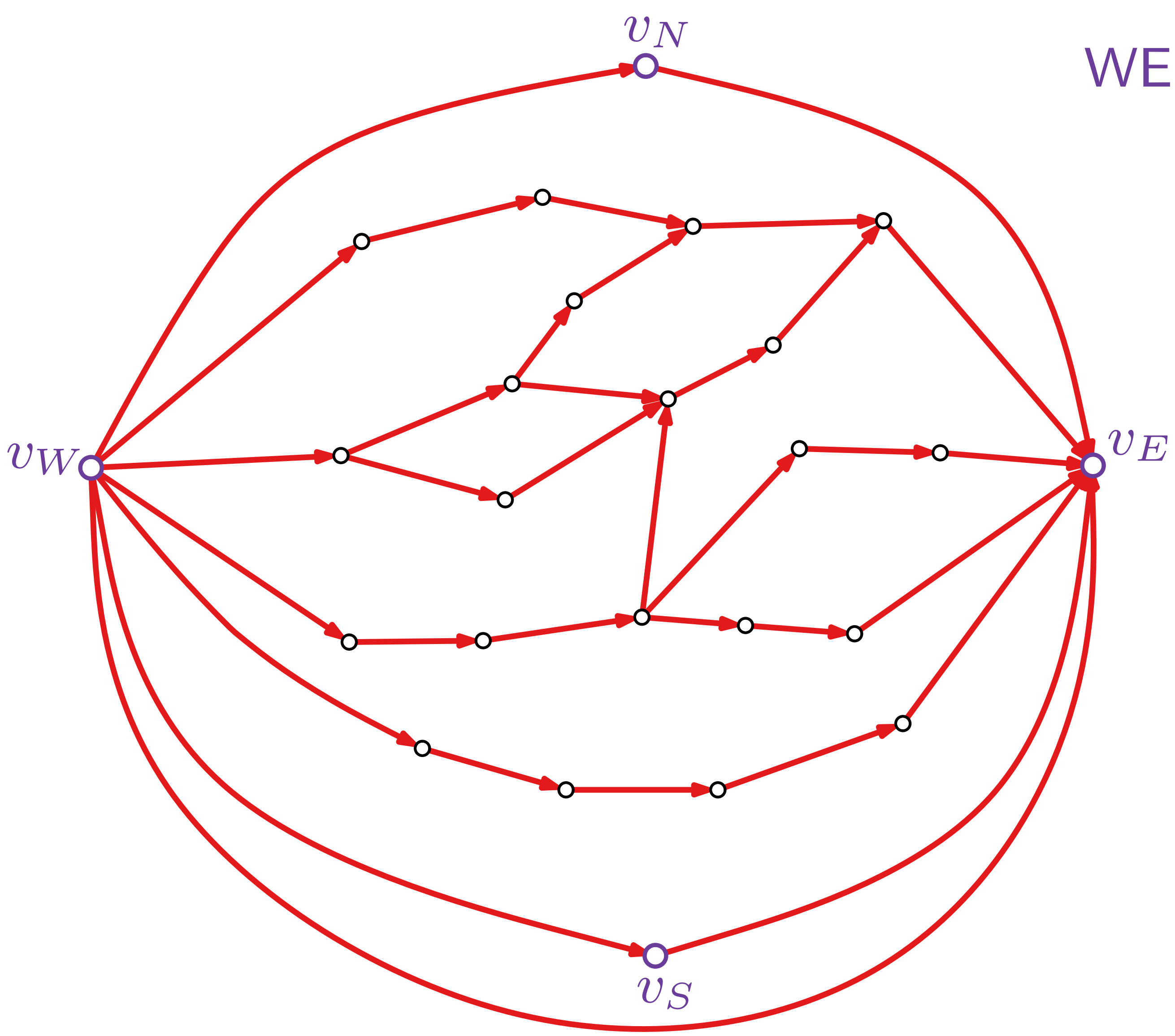


Alexander Wolff

From REL to st-Digraphs to Coordinates

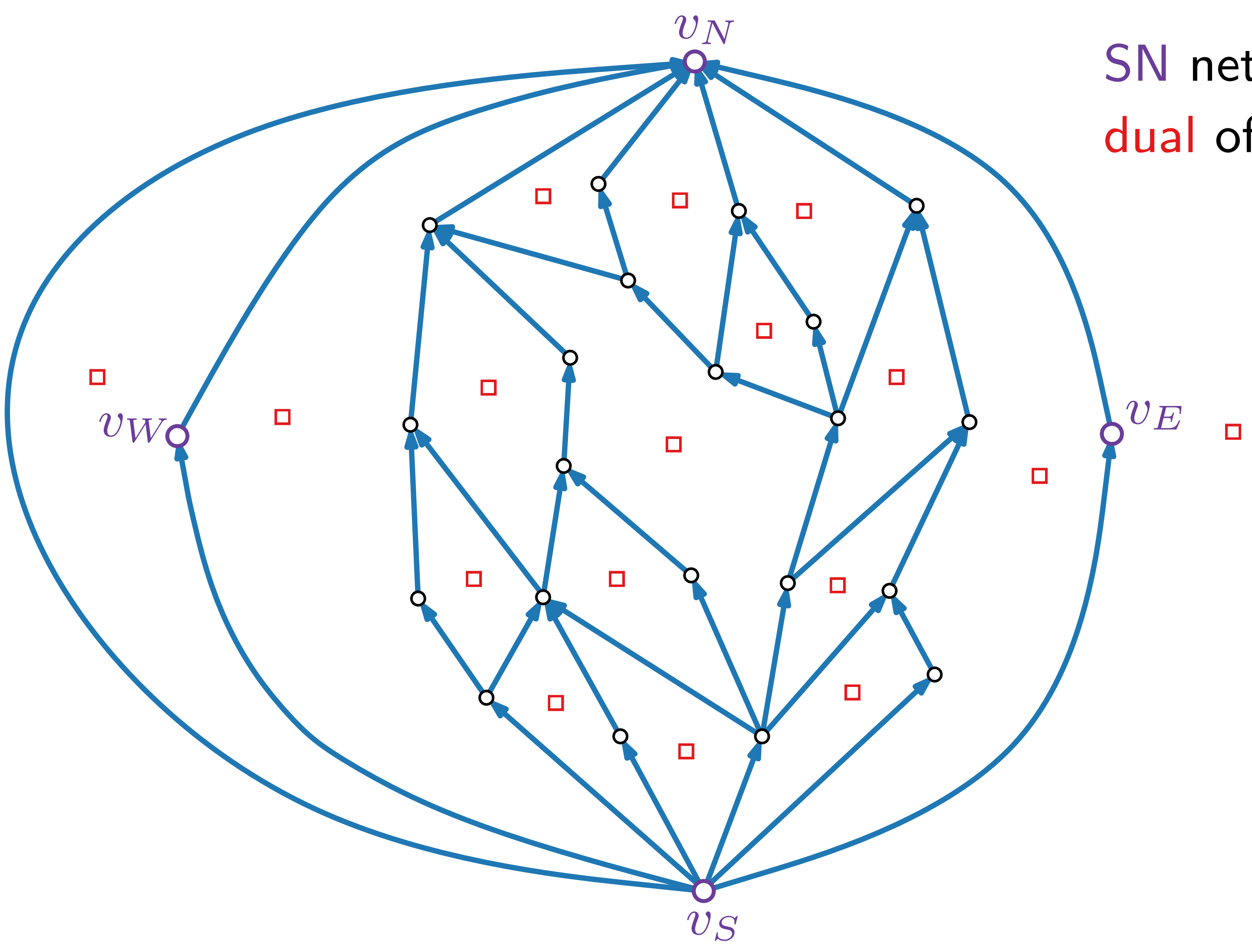


From REL to st-Digraphs to Coordinates



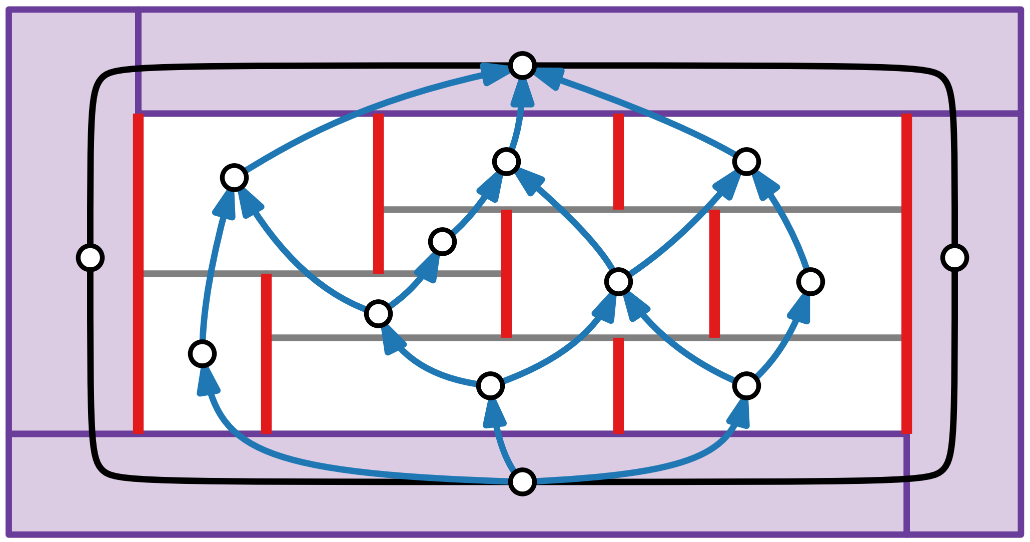
WE network G_{hor}

From REL to st-Digraphs to Coordinates

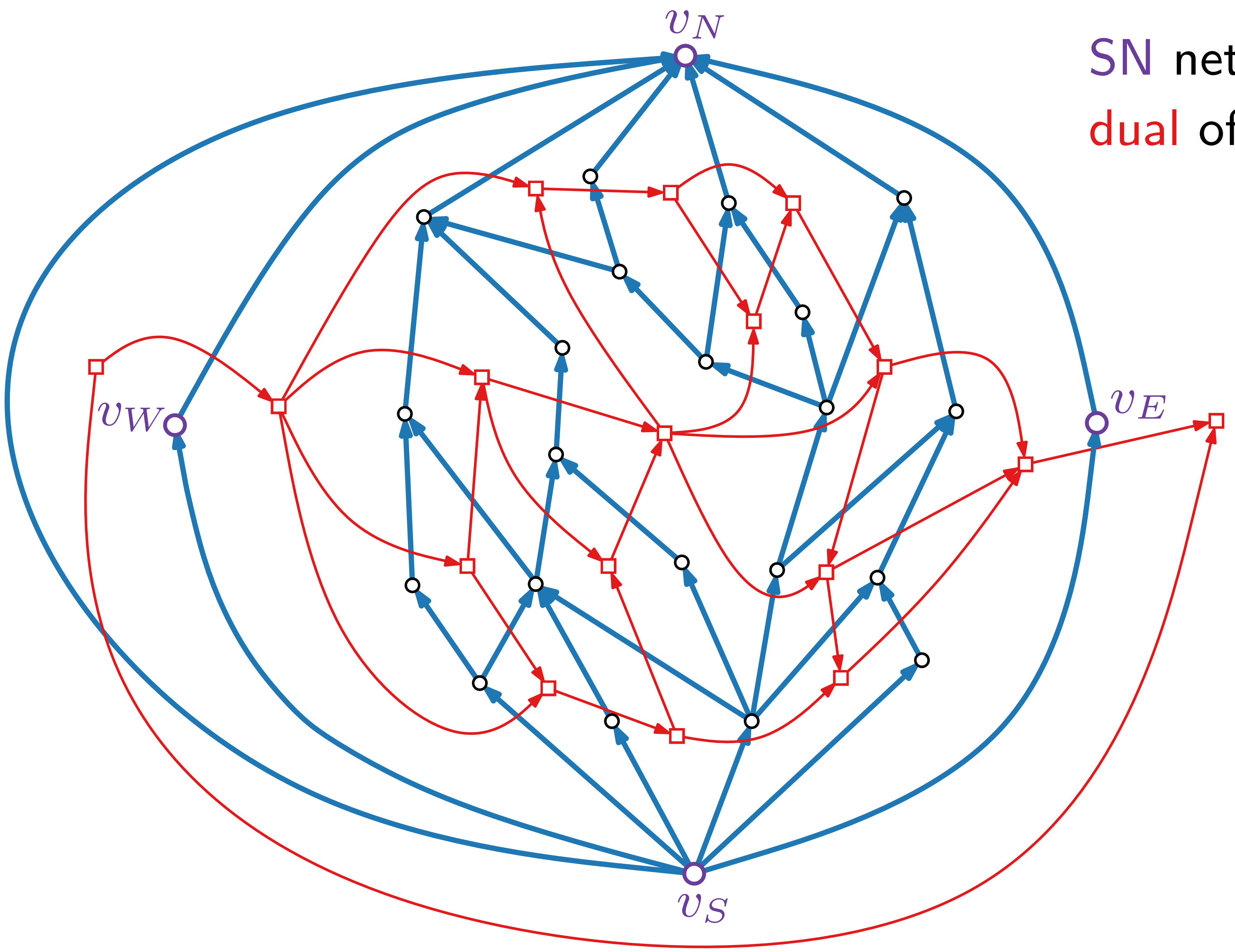


SN network G_{ver}

dual of G_{ver}



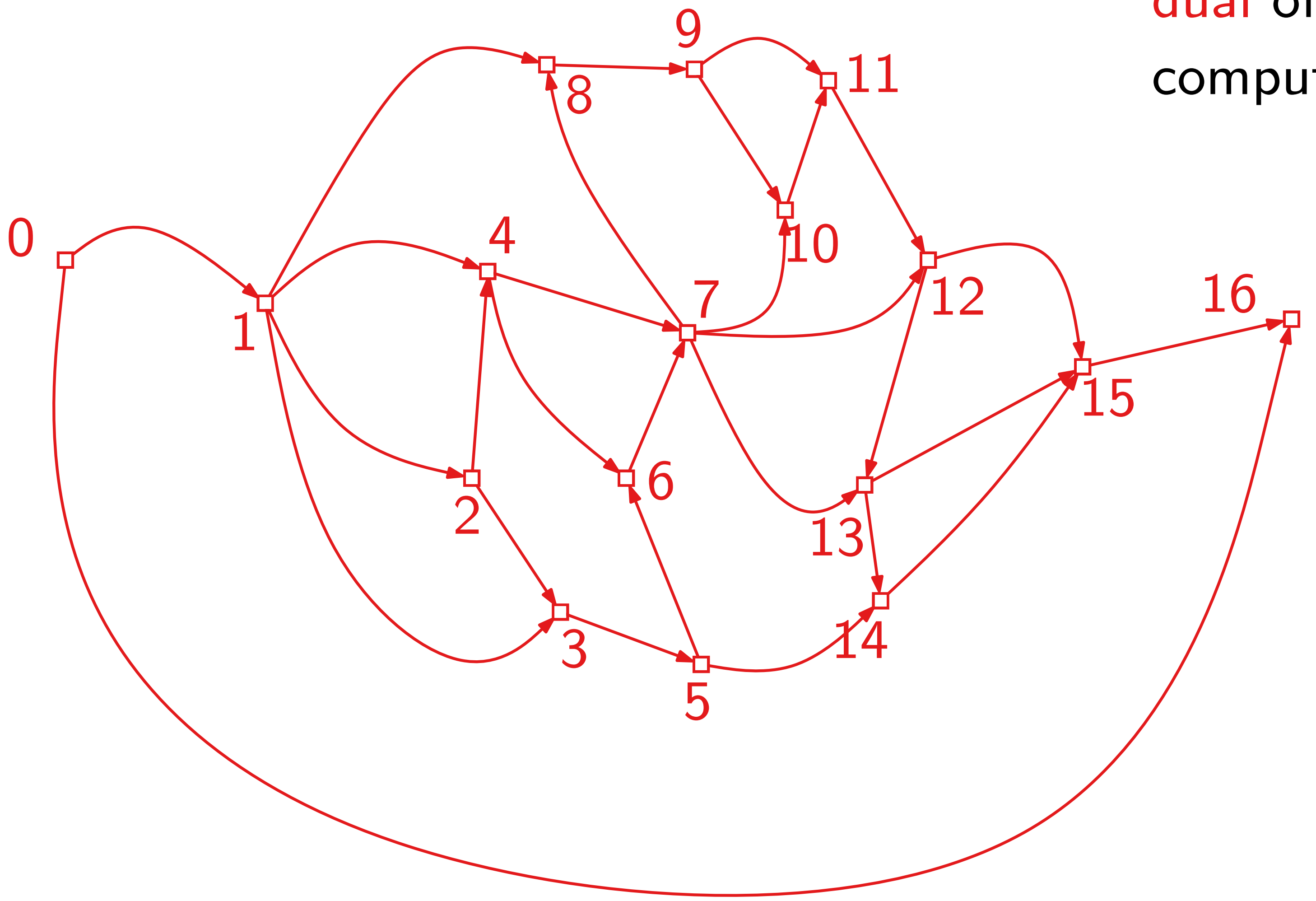
From REL to st-Digraphs to Coordinates



SN network G_{ver}

dual of G_{ver}

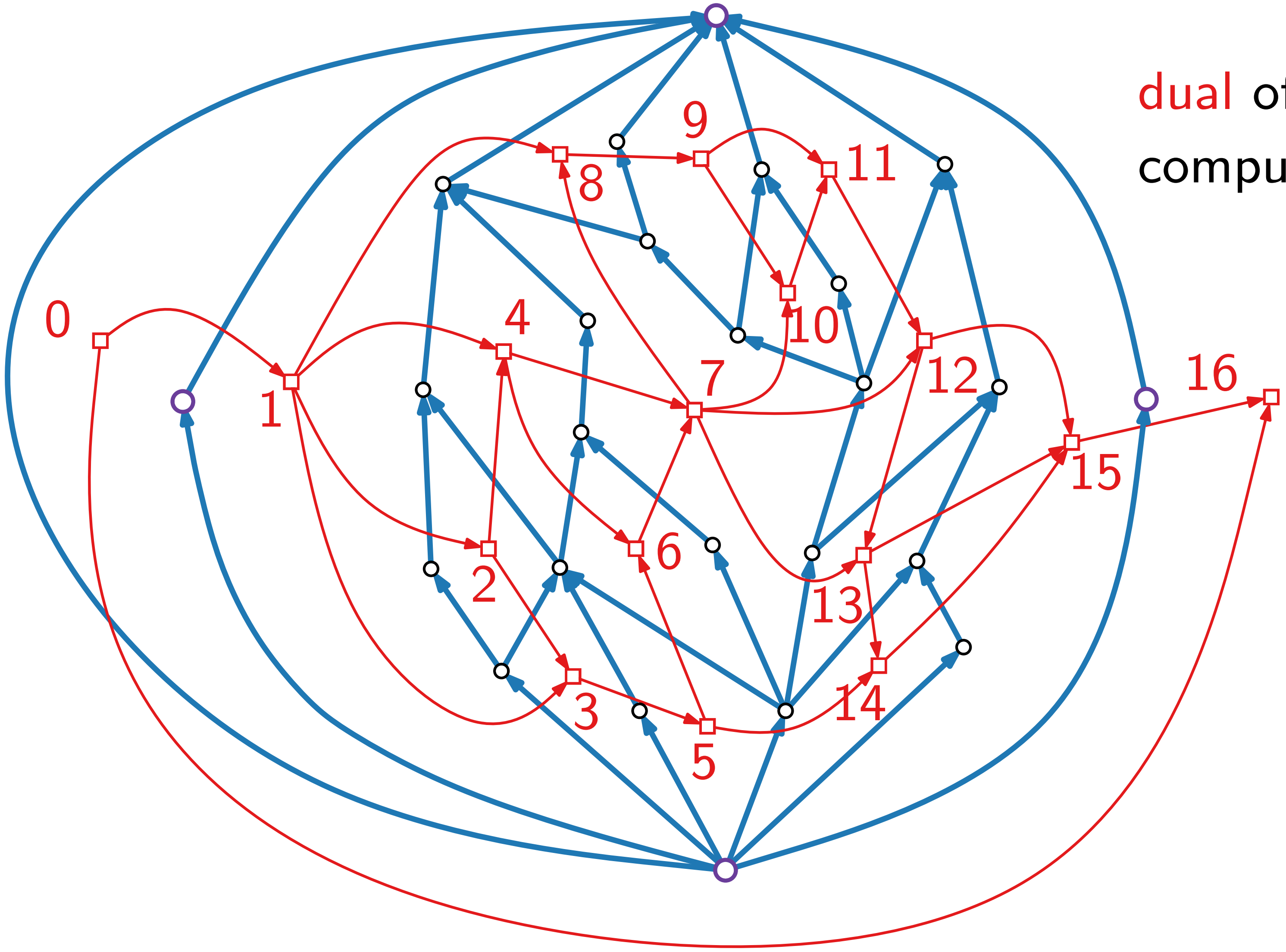
From REL to st-Digraphs to Coordinates



dual of G_{ver}

compute topological order

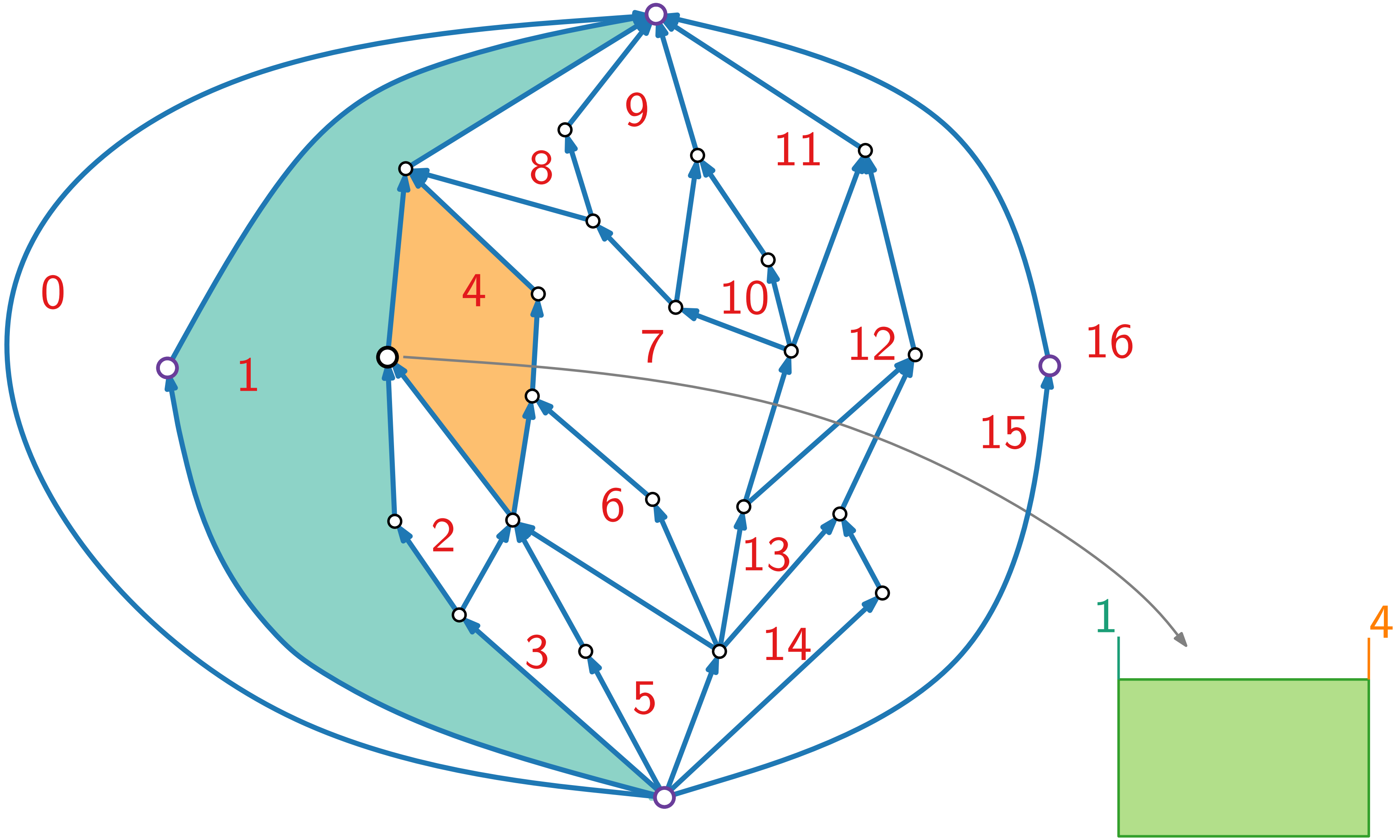
From REL to st-Digraphs to Coordinates



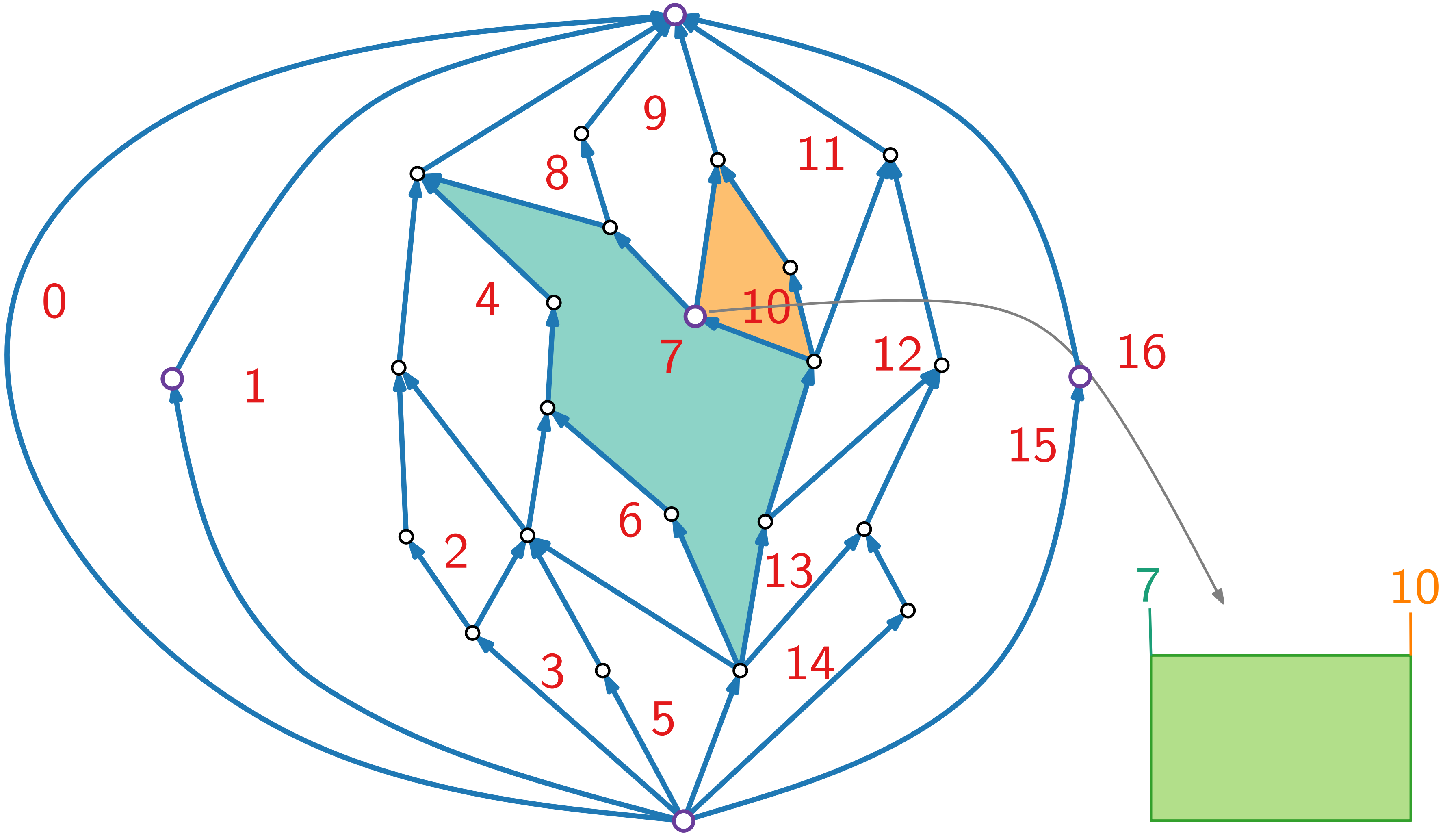
dual of G_{ver}

compute topological order

From REL to st-Digraphs to Coordinates



From REL to st-Digraphs to Coordinates

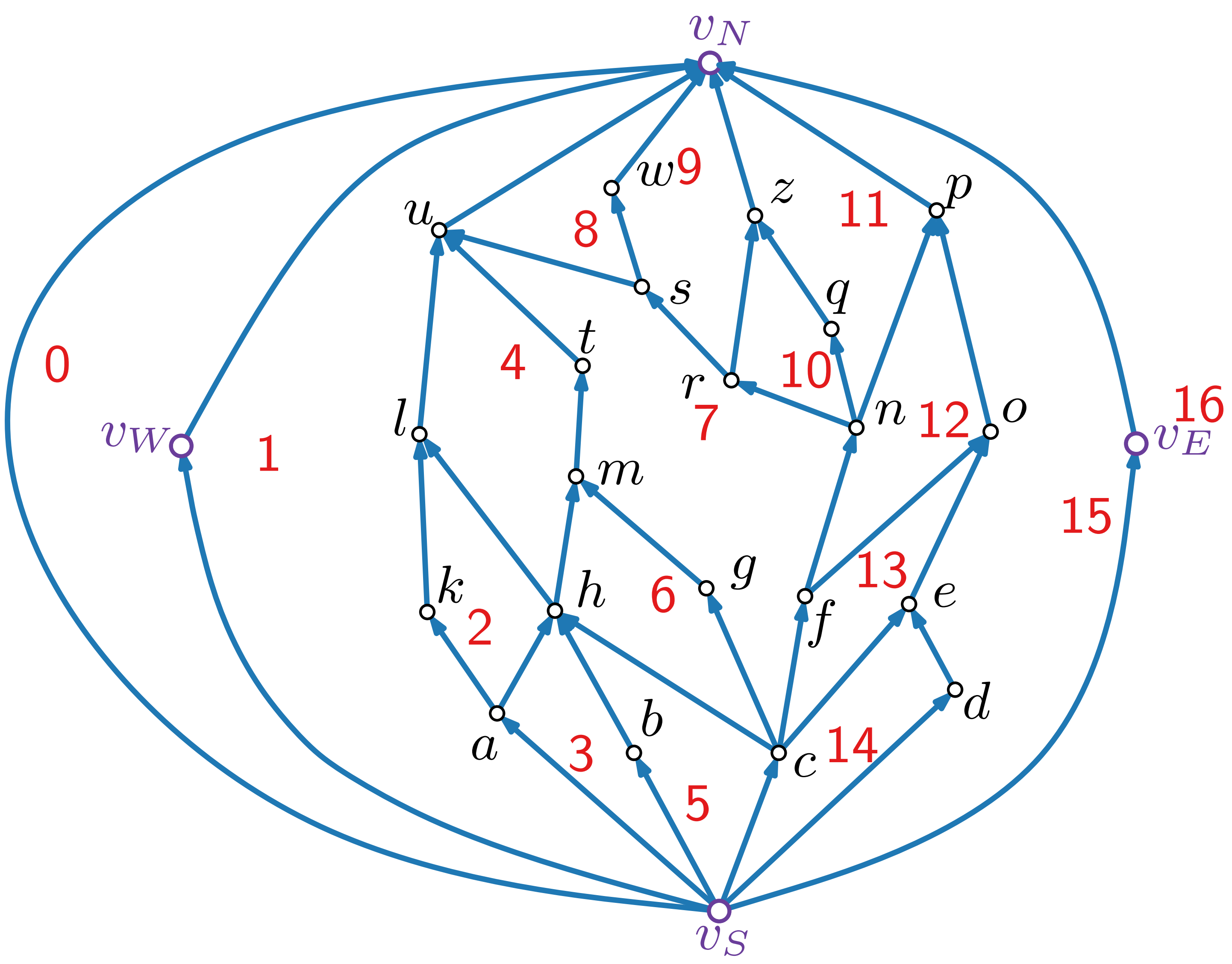


Rectangular Dual Algorithm

For a PTP graph $G = (V, E)$:

- Find a REL $\{T_r, T_b\}$ of G;
- Construct a SN network G_{ver} of G (consists of T_b plus outer edges)
- Construct the dual $G^{\star}_{\text{ver}}$ of G_{ver} and compute a topological ordering f_{ver} of $G^{\star}_{\text{ver}}$
- For each vertex $v \in V$, let g and h be the face on the left and face on the right of v. Set $x_1(v) = f_{\text{ver}}(g)$ and $x_2(v) = f_{\text{ver}}(h)$.
- Define $x_1(v_N) = 1, x_1(v_S) = 2$ and $x_2(v_N) = \max f_{\text{ver}} - 1, x_2(v_S) = \max f_{\text{ver}}$
- Analogously compute y_1 and y_2 with G_{hor}.
- For each $v \in V$, let $R(v) = [x_1(v), x_2(v)] \times [y_1(v), y_2(v)]$.

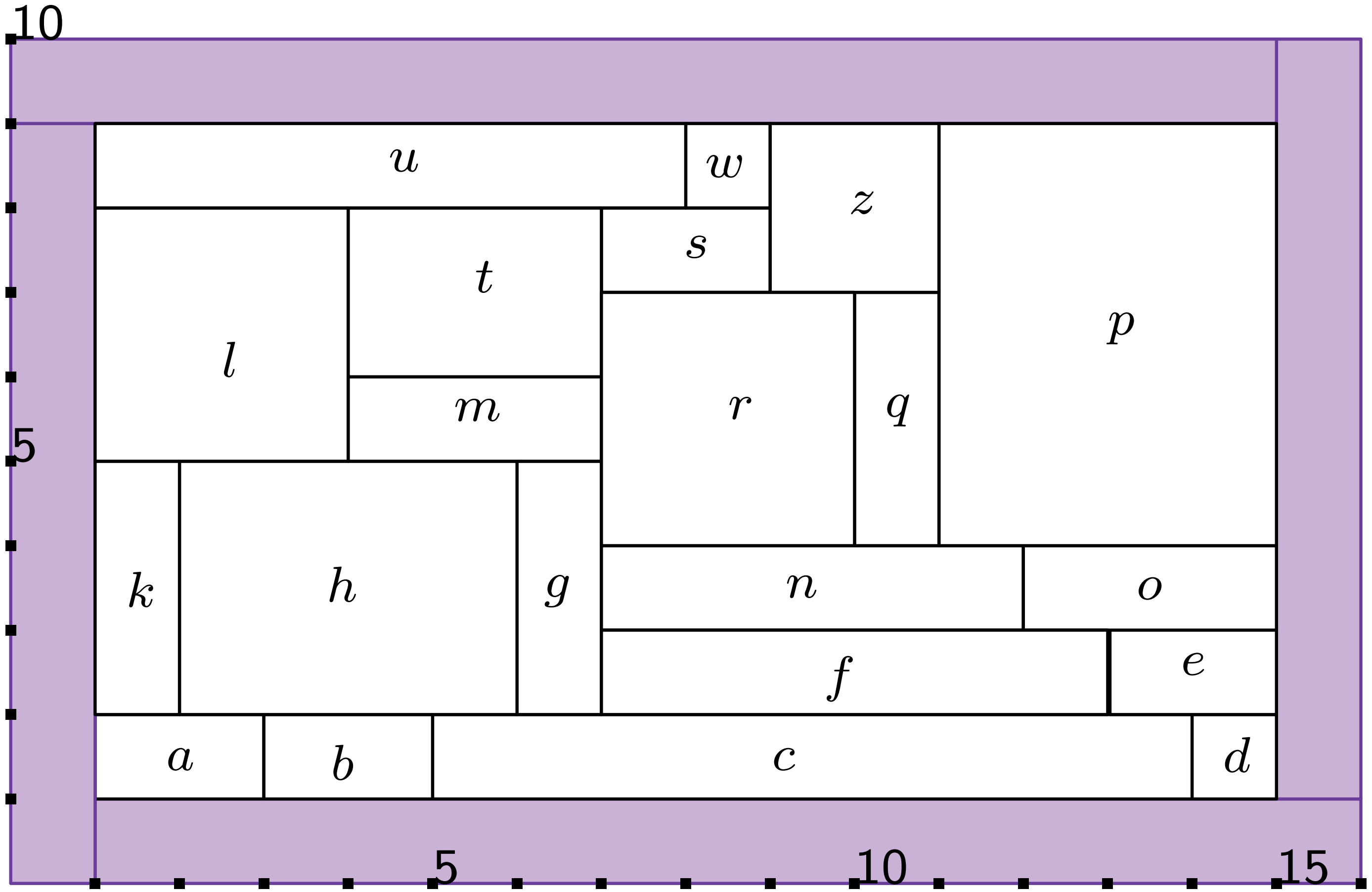
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 2,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
...

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
...

Reading off Coordinates to Get Rectangular Dual

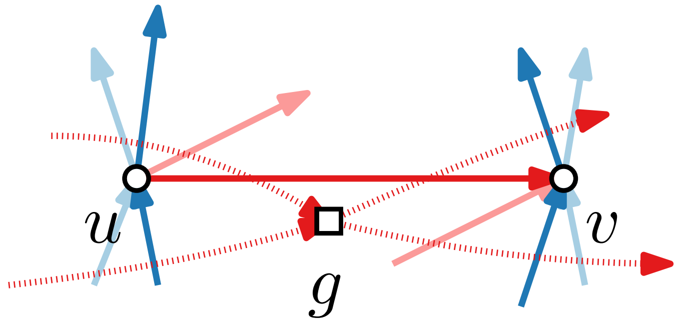


$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 2,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
...

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
...

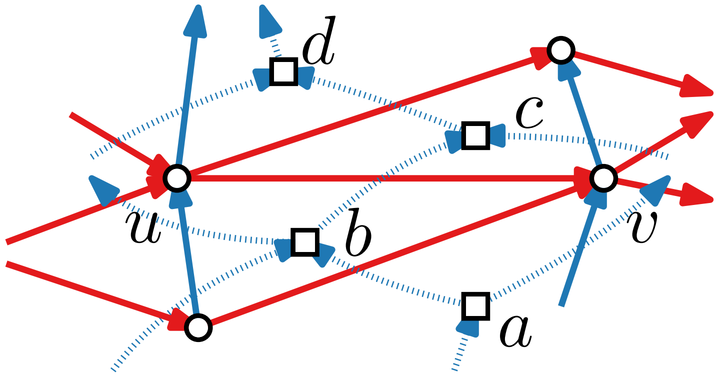
Correctness of Algorithm (Sketch)

- If edge (u, v) exists, then $x_2(u) = x_1(v)$



$$x_2(u) = f_{\mathsf{ver}}(g) = x_1(v)$$

- and the vertical segments of their rectangles overlap



$$y_1(v) = f_{\mathsf{hor}}(a) \leq y_1(u) = f_{\mathsf{hor}}(b)$$
$$< y_2(v) = f_{\mathsf{hor}}(c) \leq y_2(u) = f_{\mathsf{hor}}(d)$$

- If path from u to v in red at least two edges long, then $x_2(u) < x_1(v)$.
- No two boxes overlap.
- For details, see He's paper [He '93].

Rectangular Dual Result

Theorem.
Every PTP graph G has a rectangular dual, which can be computed in linear time.

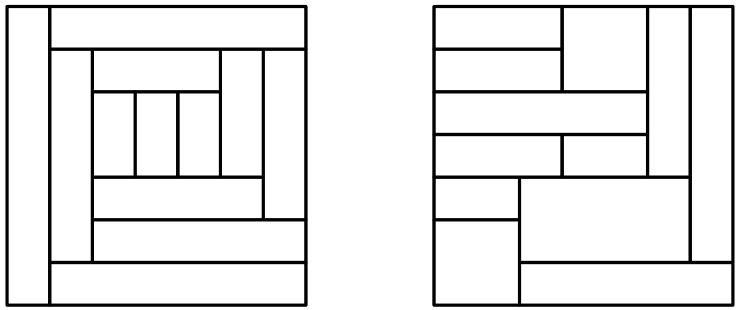
Proof.

- Compute a planar embedding of G.
- Compute a refined canonical ordering of G.
- Traverse the graph and color the edges.
- Construct G_{ver} and G_{hor}.
- Construct their duals $G^{\star}_{\text{ver}}$ and $G^{\star}_{\text{hor}}$.
- Compute a topological ordering for vertices of $G^{\star}_{\text{ver}}$ and $G^{\star}_{\text{hor}}$.
- Assing coordinates to the rectangles representing vertices.

Discussion

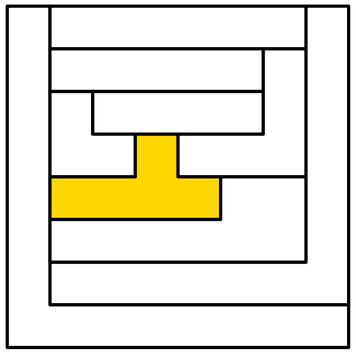
- A layout is **area-universal** if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.
- A rectangular layout is **area-universal** if and only if it is **one-sided**. [Eppstein et al. SIAM J. Comp. 2012]

one-sided



not one-sided

- Area-universal **rectlinear** representation: possible for all planar graphs.
- [Alam et al. 2013]: 8 sides (matches the lower bound)



Literature

Construction of triangle contact representations based on

- [de Fraysseix, Ossona de Mendez, Rosenstiehl '94] On Triangle Contact Graphs

Construction of rectangular dual based on

- [He '93] On Finding the Rectangular Duals of Planar Triangulated Graphs
- [Kant, He '94] Two algorithms for finding rectangular duals of planar graphs

and originally from

- [Koźmiński, Kinnen '85] Rectangular Duals of Planar Graphs