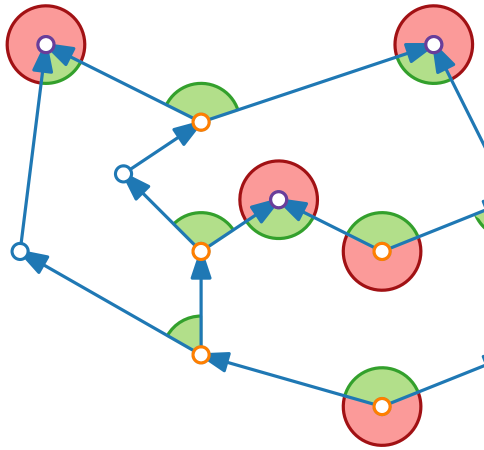
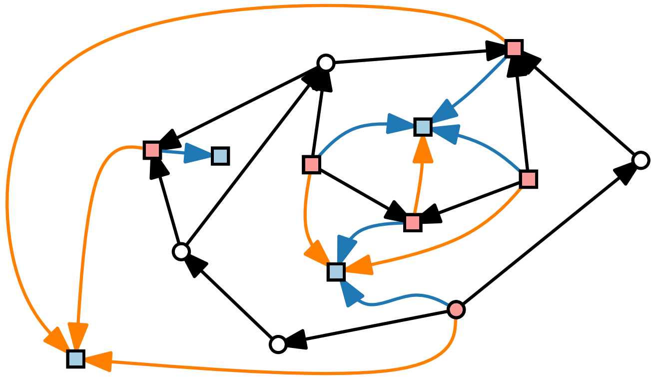
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 6:
Upward Planar Drawings

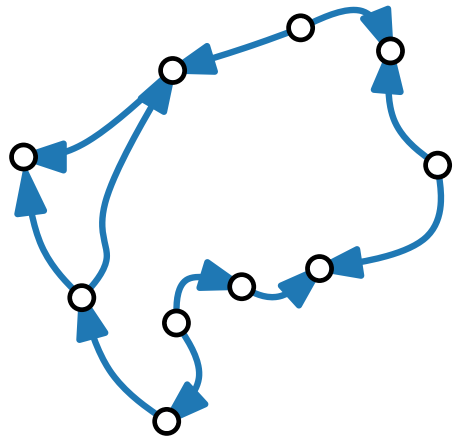


Part I:
Characterization



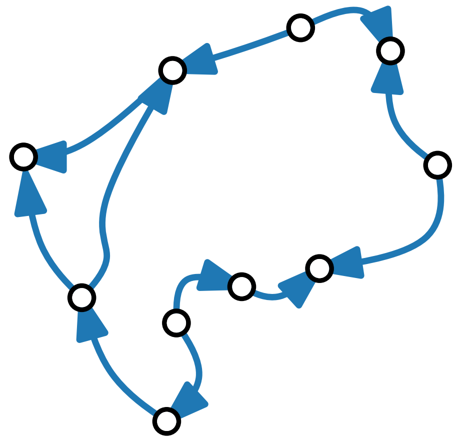
Alexander Wolff

Upward Planar Drawings – Motivation



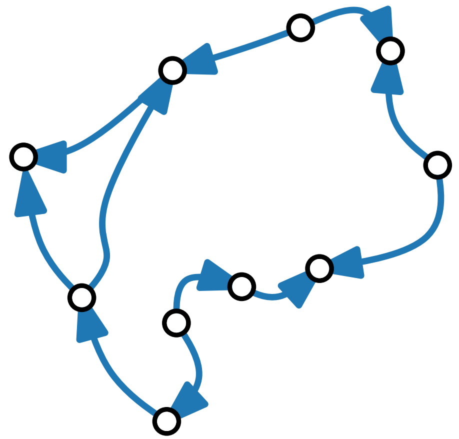
Upward Planar Drawings – Motivation

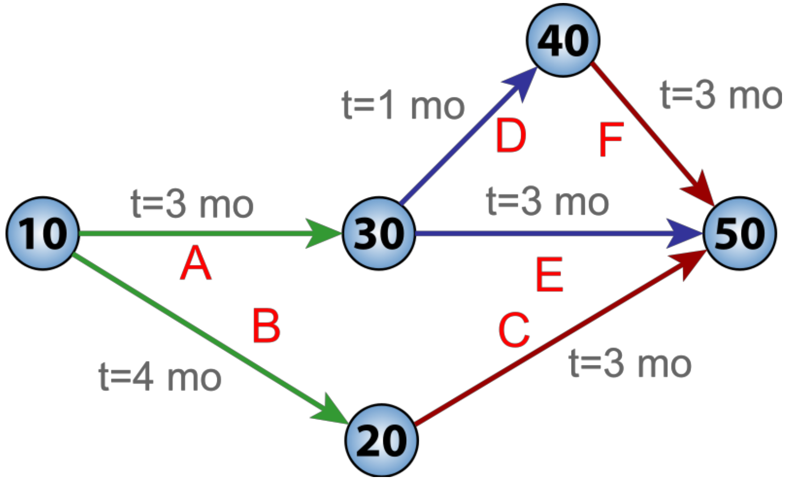
- What may the direction of edges in a digraph represent?



Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time

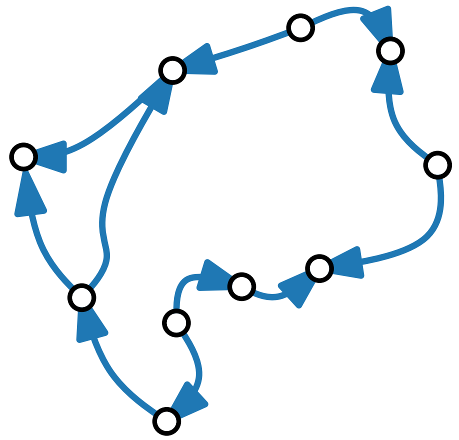




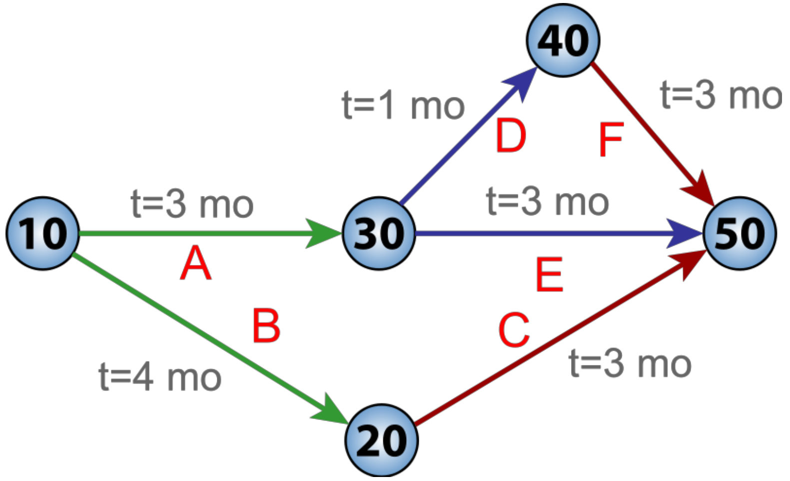
PERT diagram

Program Evaluation and Review Technique
(Project management)

Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
  - Flow



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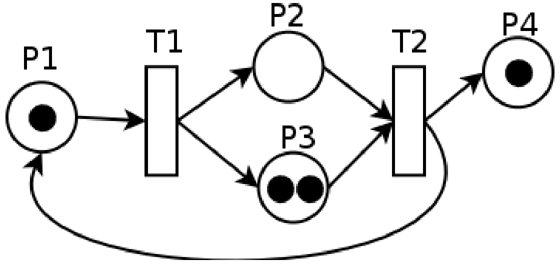
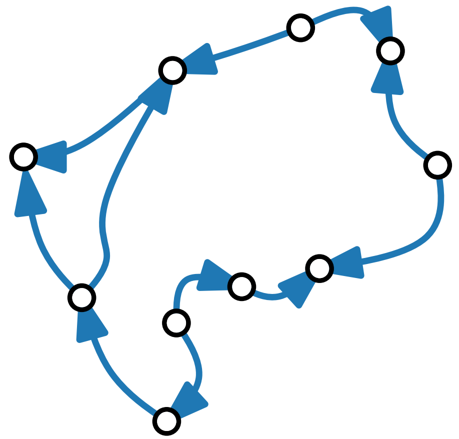


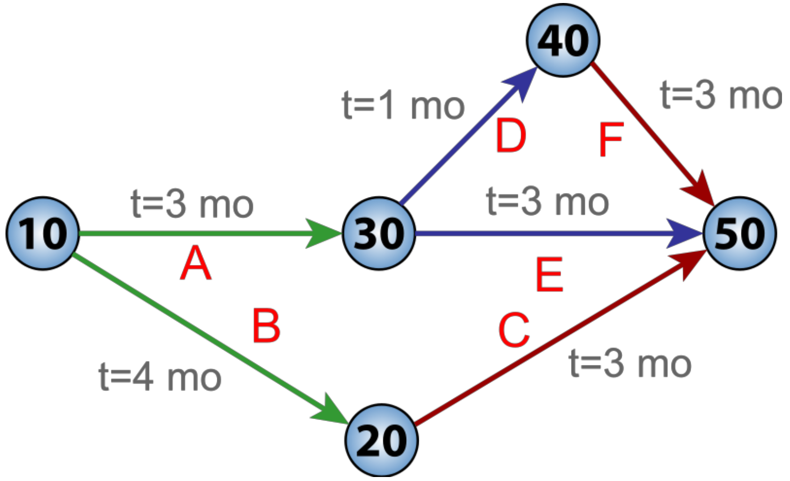
Petri net

Place/Transition net
(Modeling languages for distributed systems)

Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy





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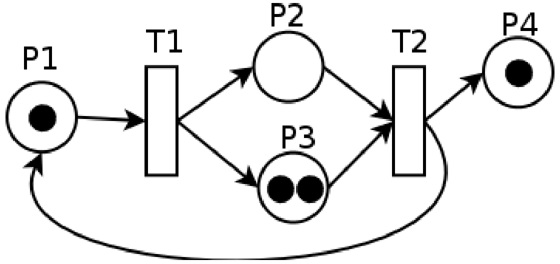

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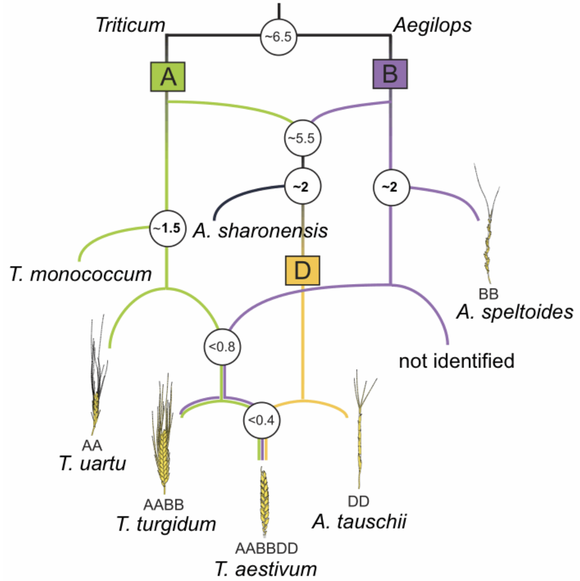
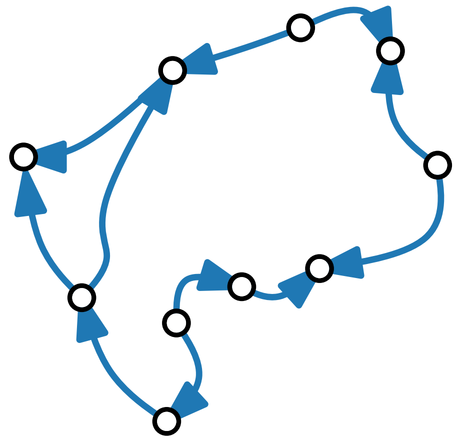


Phylogenetic network

Ancestral trees / networks
(Biology)

Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
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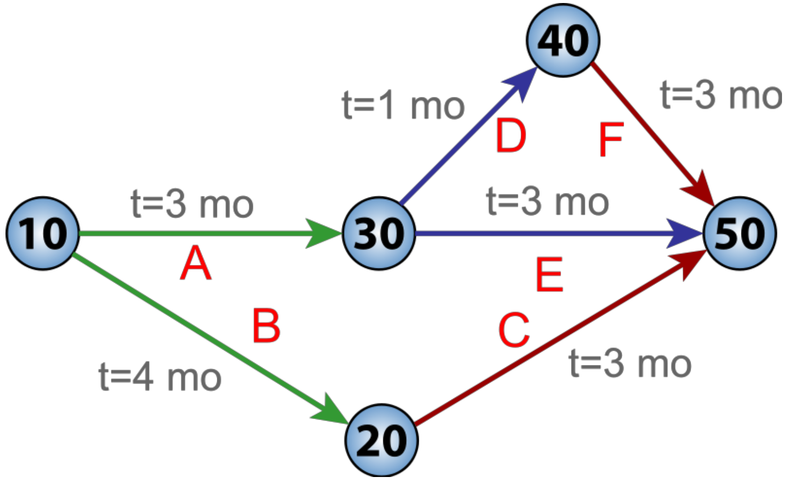
  - . . .



PERT diagram

Program Evaluation and Review Technique
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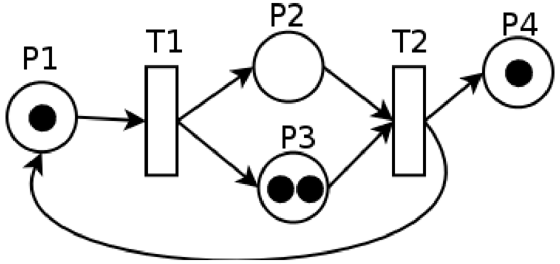

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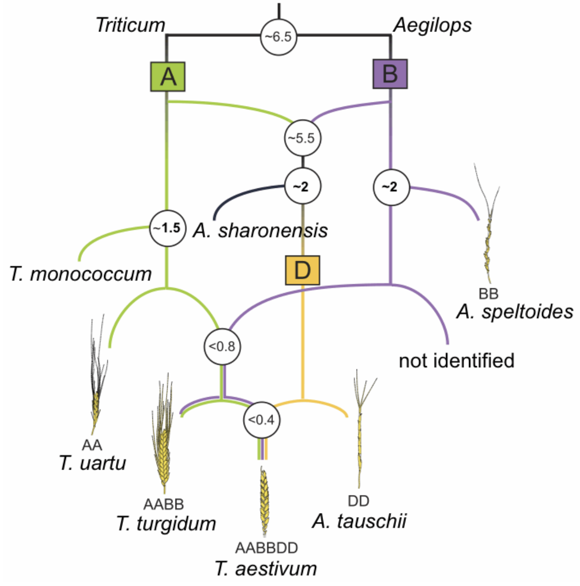

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Ancestral trees / networks
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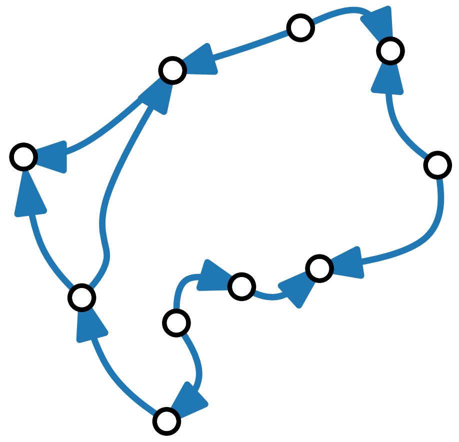
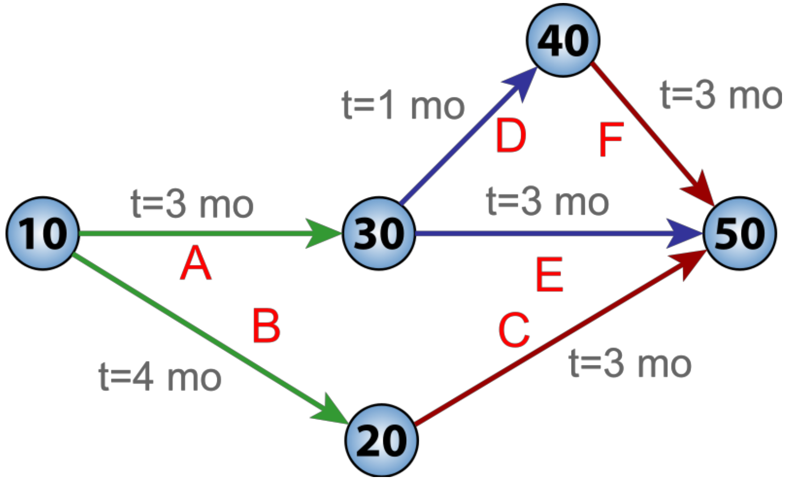
Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - . . .
- Would be nice to have general direction preserved in drawing.



PERT diagram

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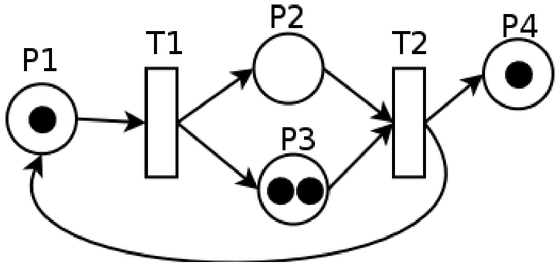

Petri net

Place/Transition net
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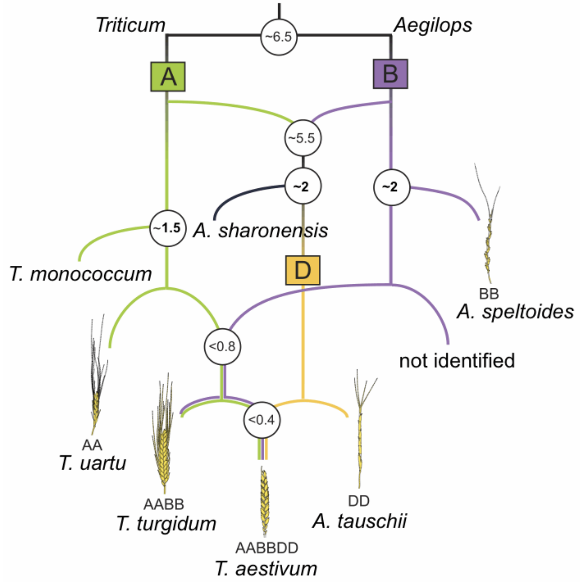

Phylogenetic network

Ancestral trees / networks
(Biology)

Upward Planar Drawings – Definition

A directed graph is **upward planar** when it admits a drawing that is

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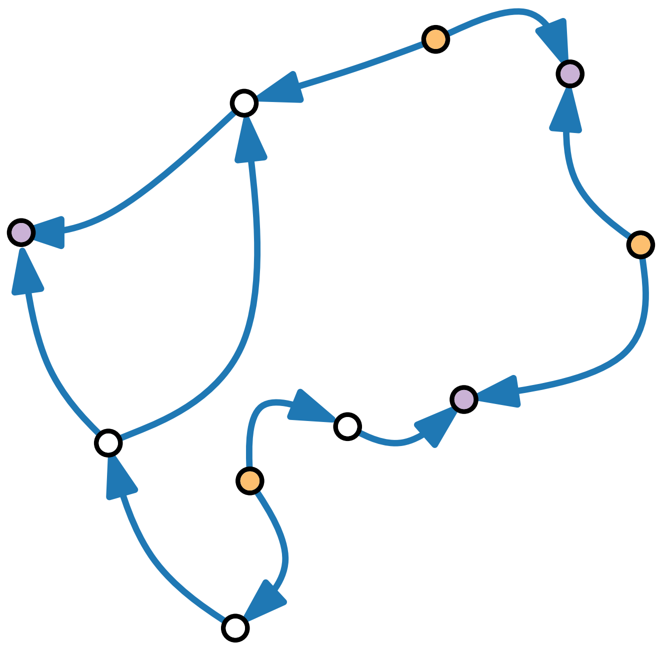
-



Upward Planar Drawings – Definition

A directed graph is **upward planar** when it admits a drawing that is
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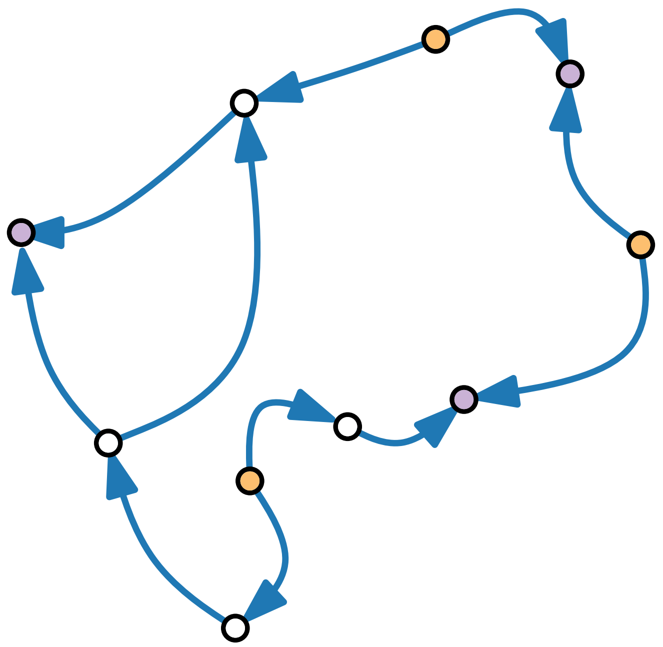
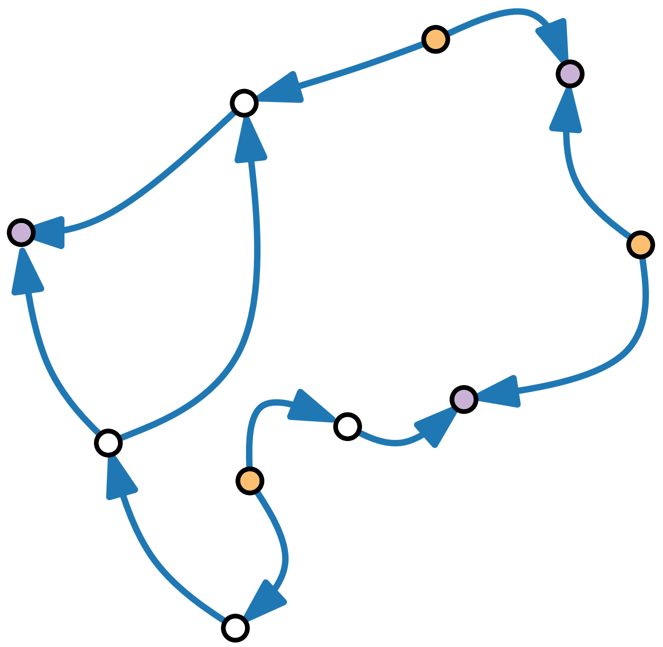
-



Upward Planar Drawings – Definition

A directed graph is **upward planar** when it admits a drawing that is

- planar and
- where each edge is drawn as an upward, y-monotone curve.



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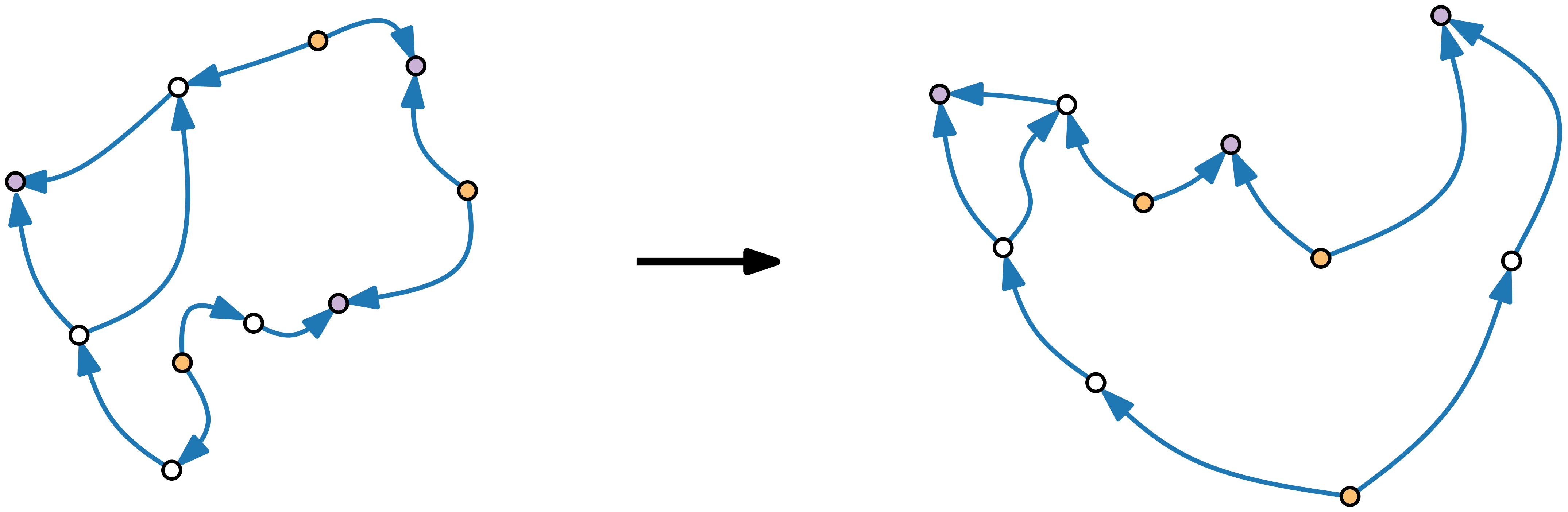
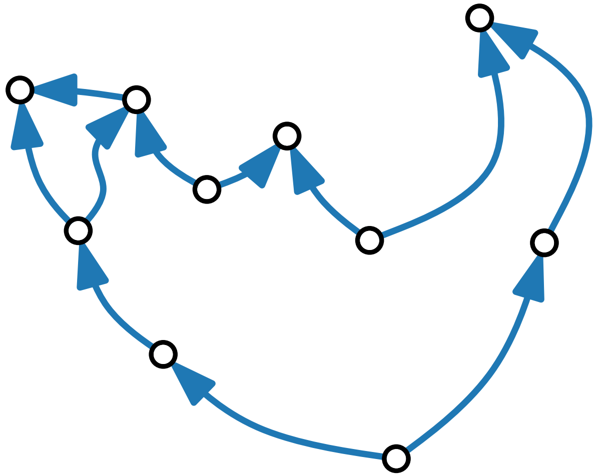


Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar



Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
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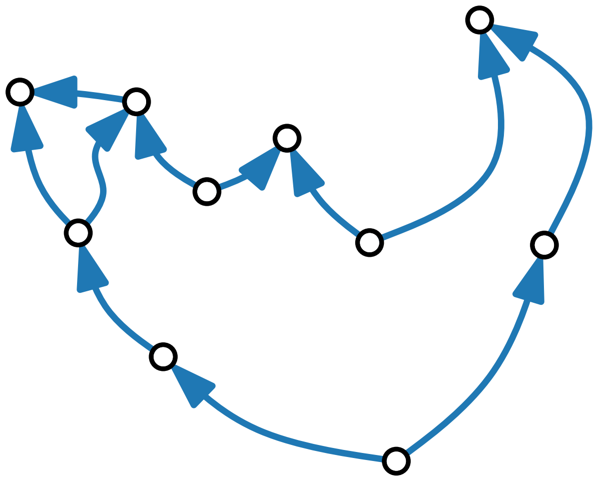

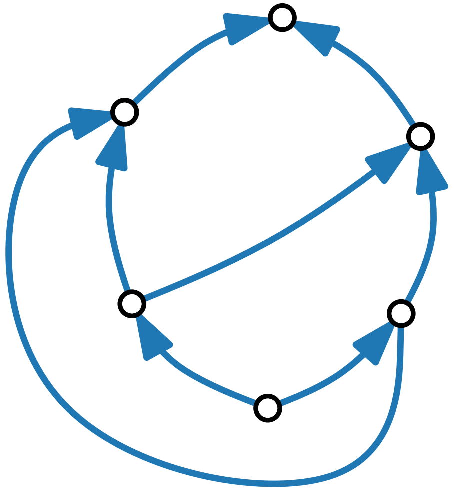

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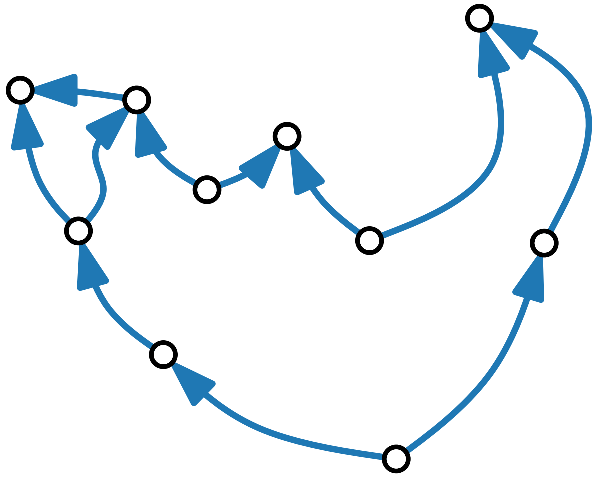

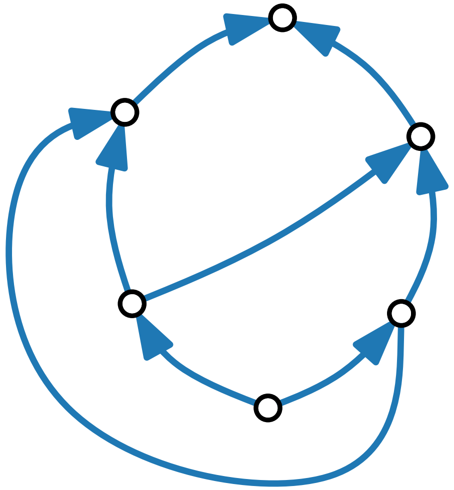

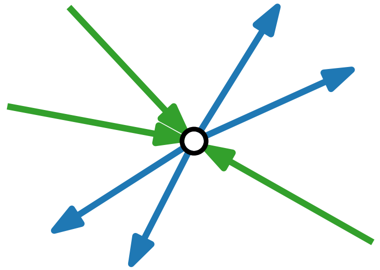

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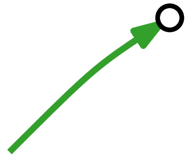

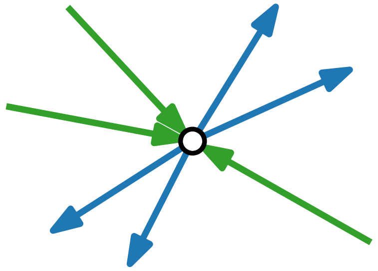

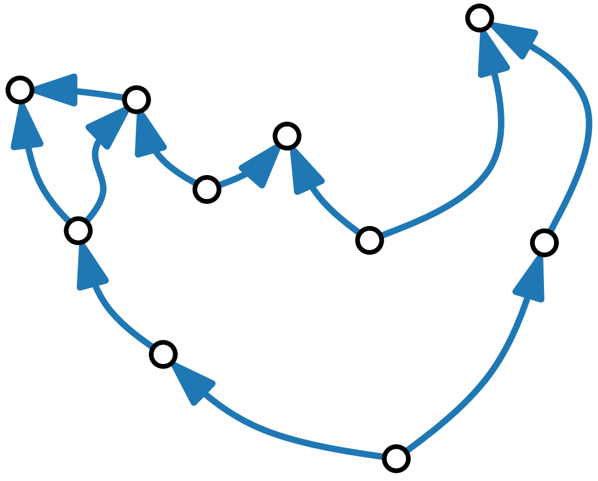

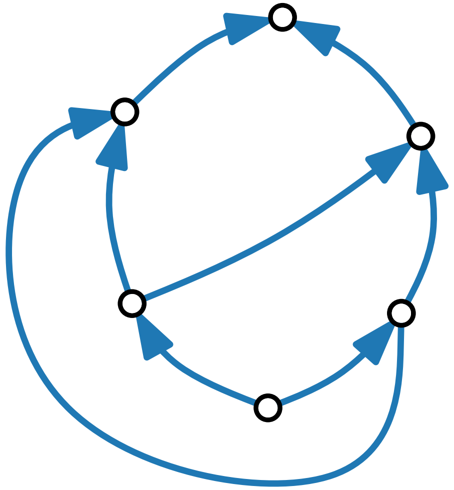

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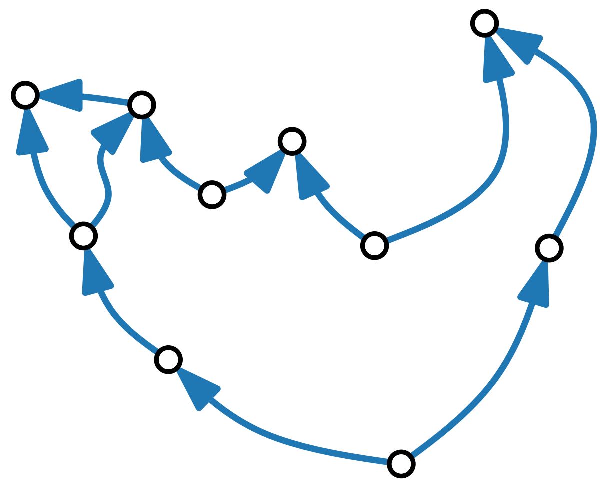

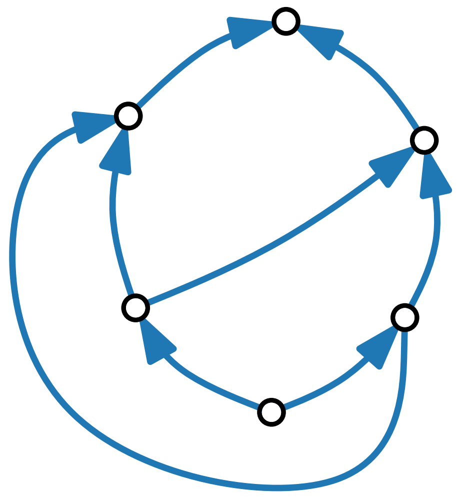

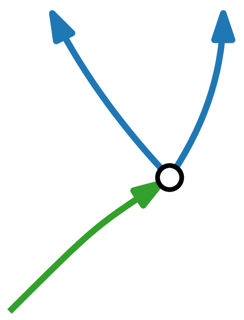

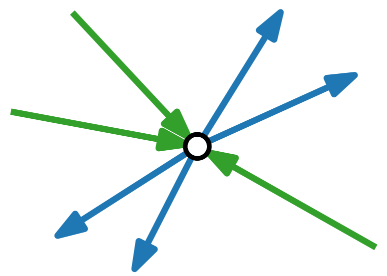

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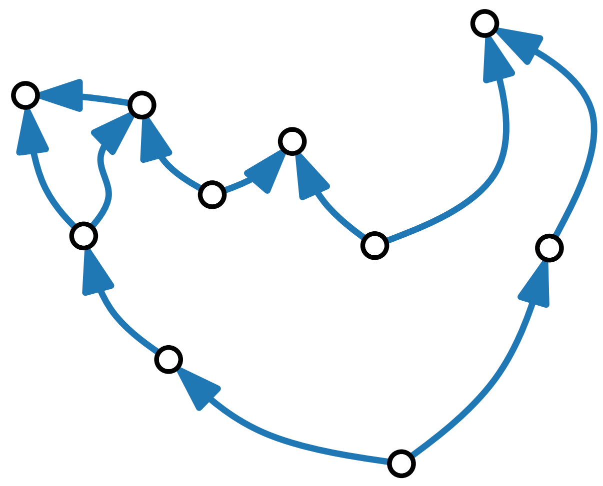

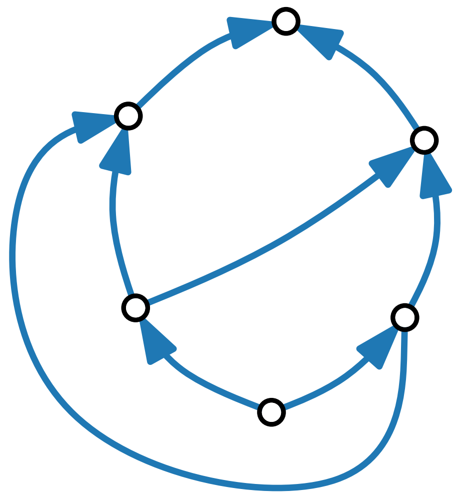

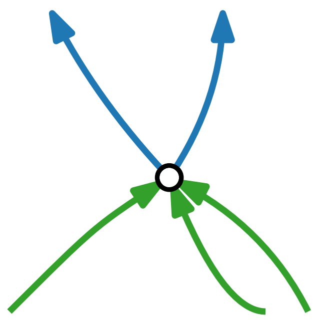

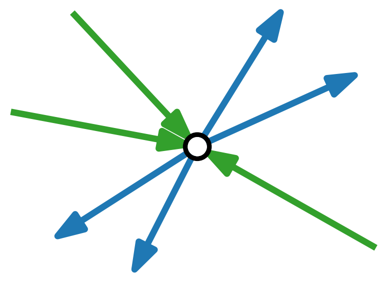

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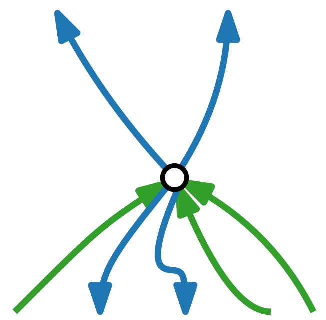

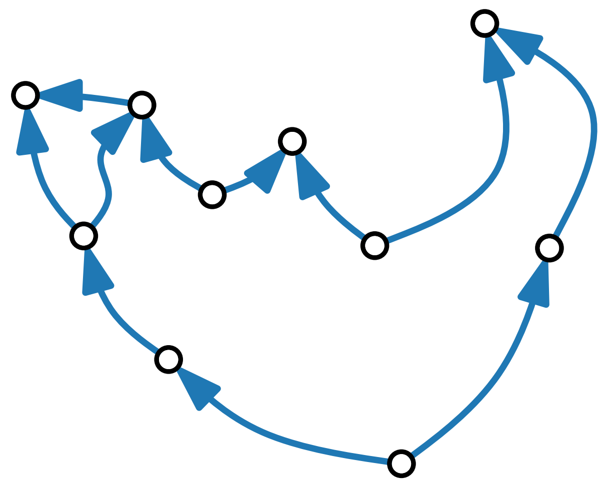

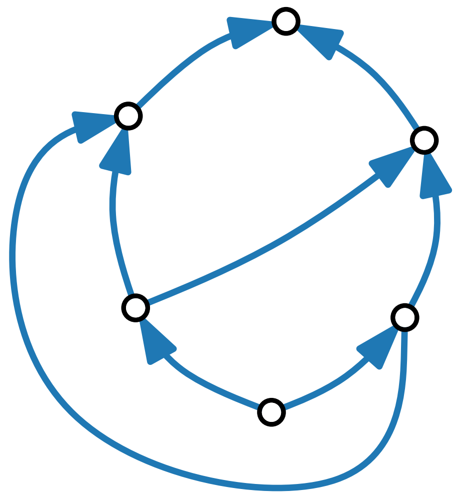

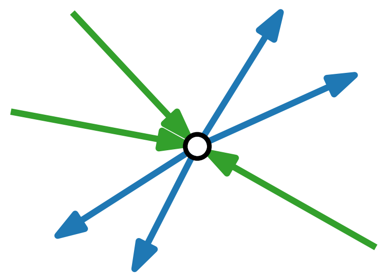

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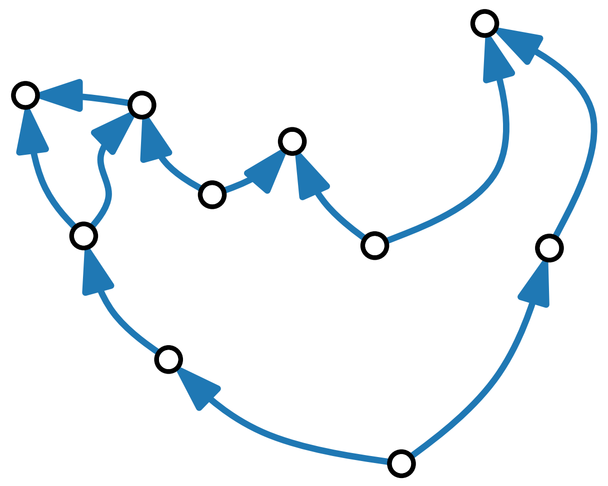

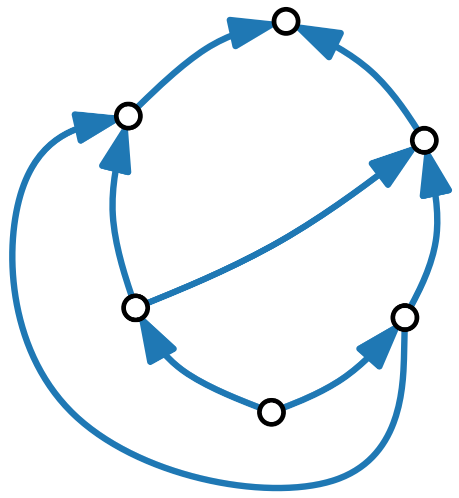

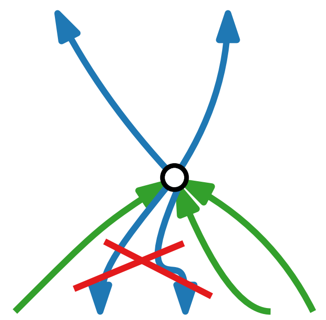

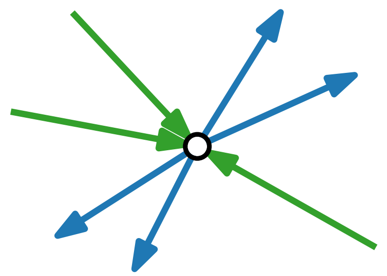

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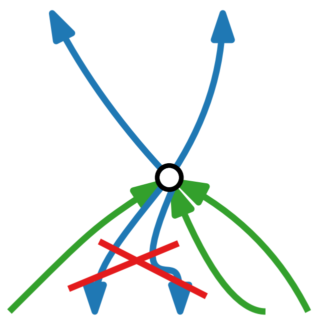

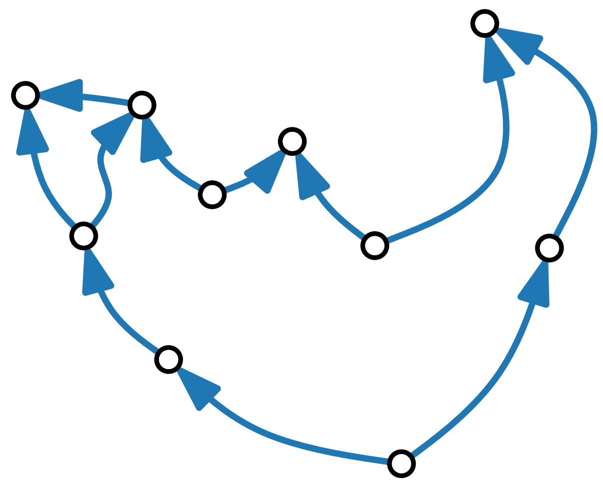

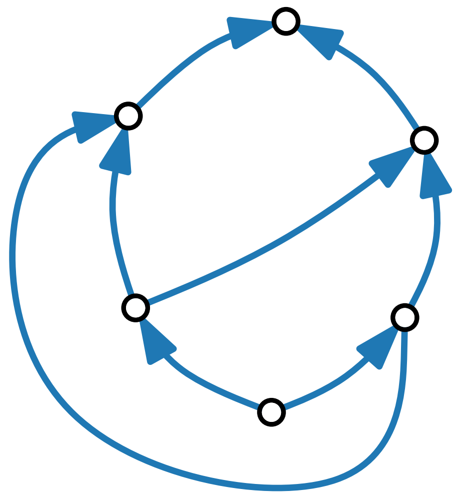

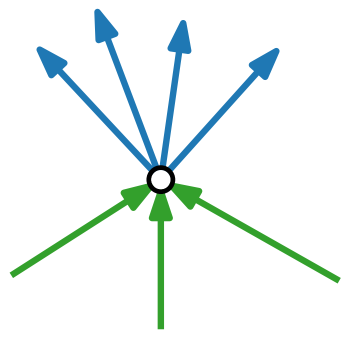

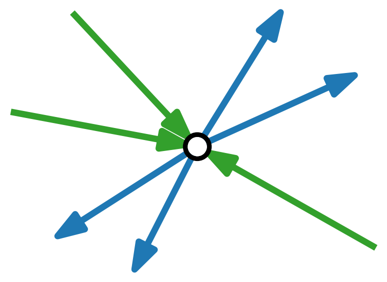

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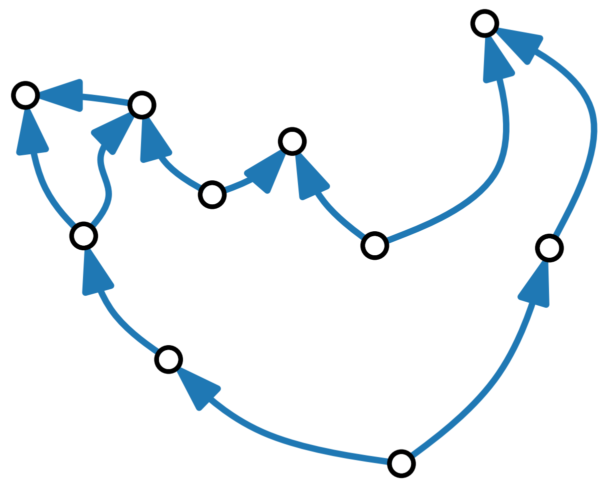
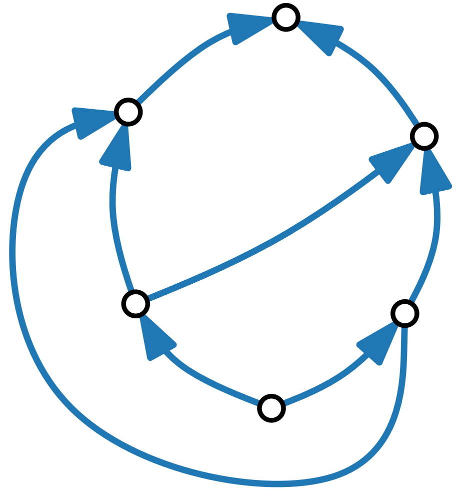
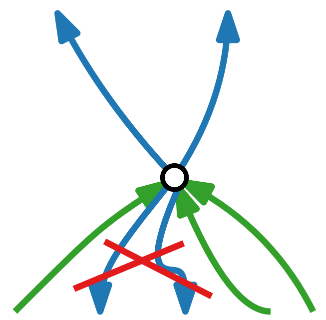
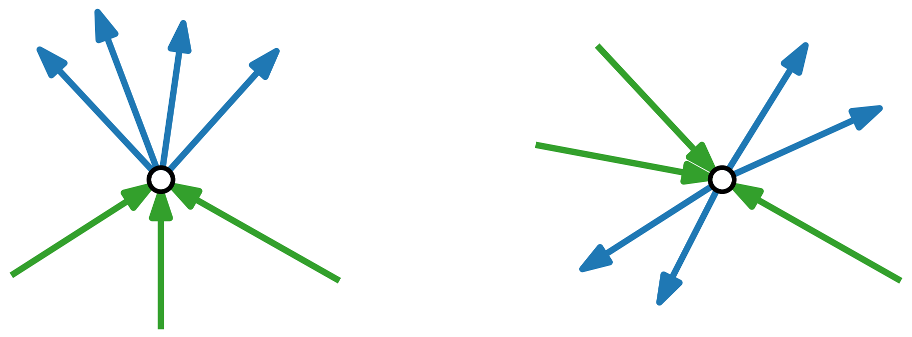




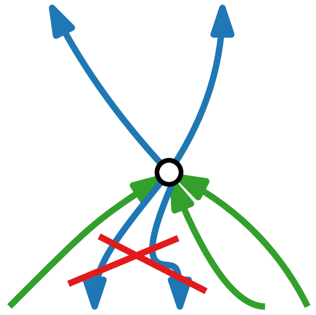


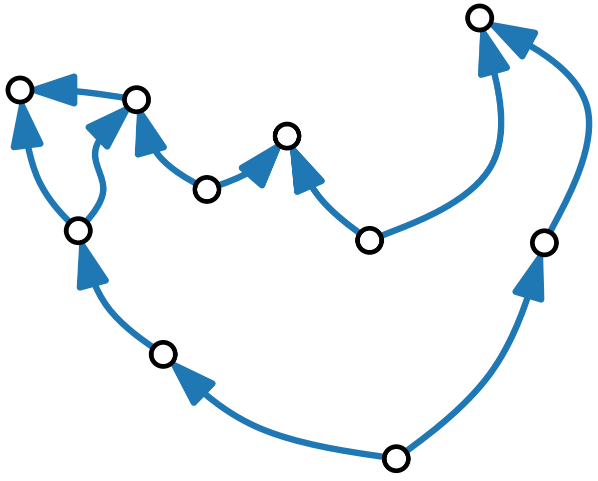
bimodal vertex *not* bimodal

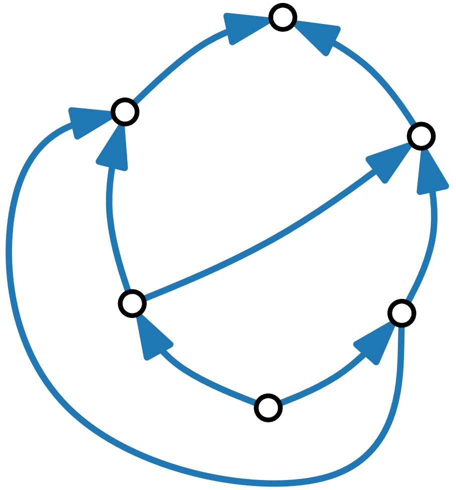


Upward Planarity – Necessary Conditions

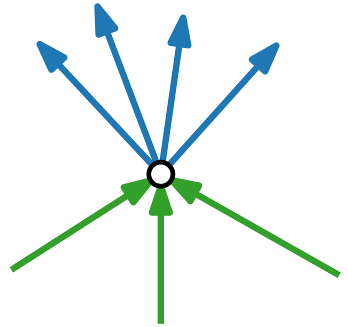
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal



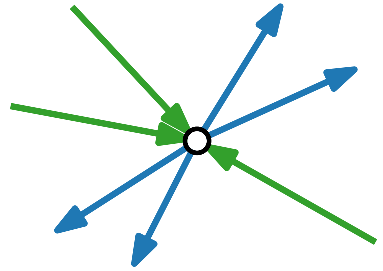




bimodal vertex



not bimodal



Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
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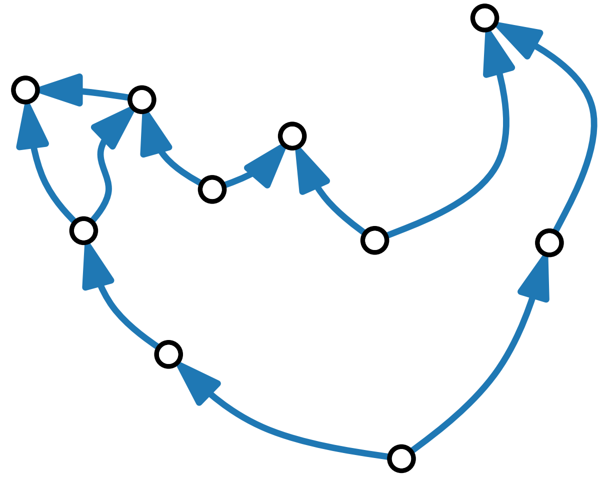
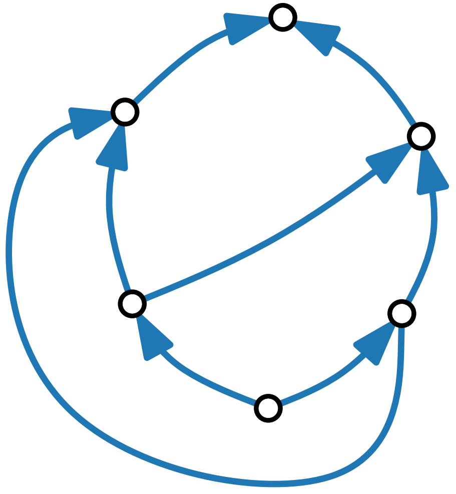
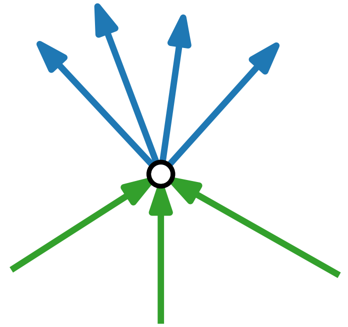
- . . . but these conditions are *not sufficient*.

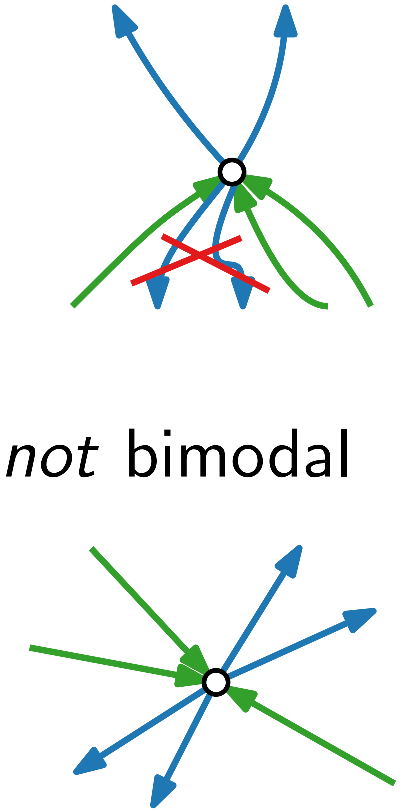




bimodal vertex



not bimodal



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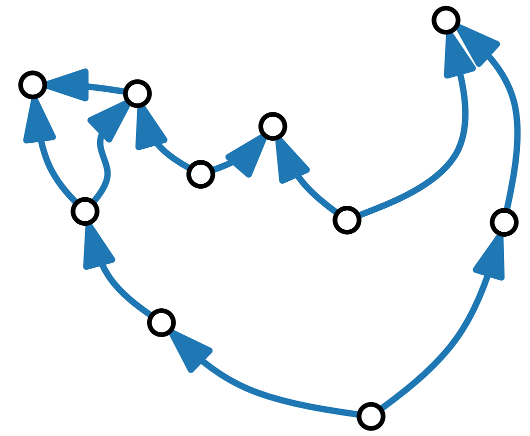

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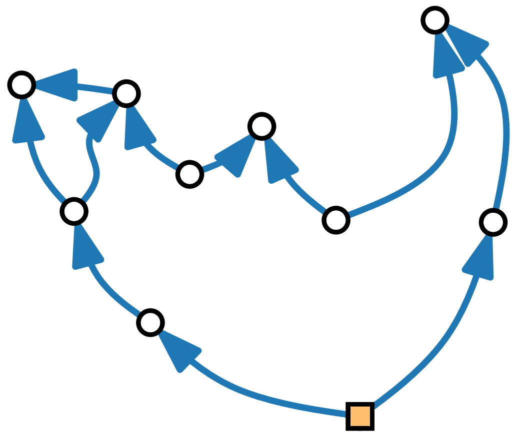

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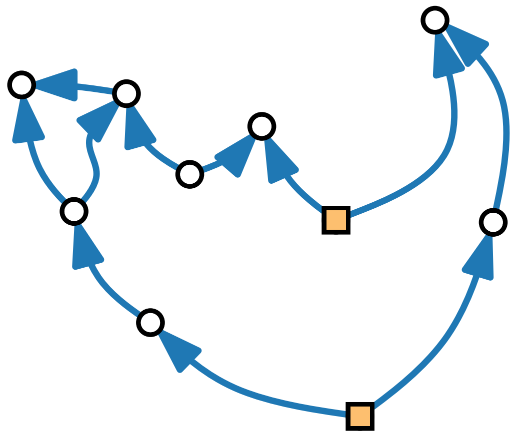

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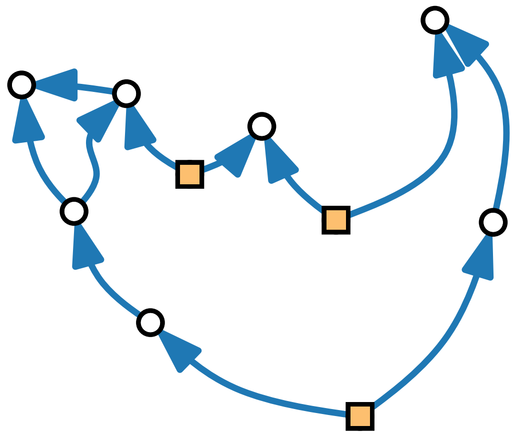

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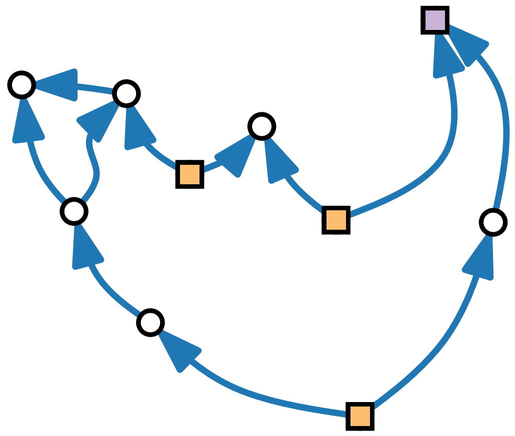

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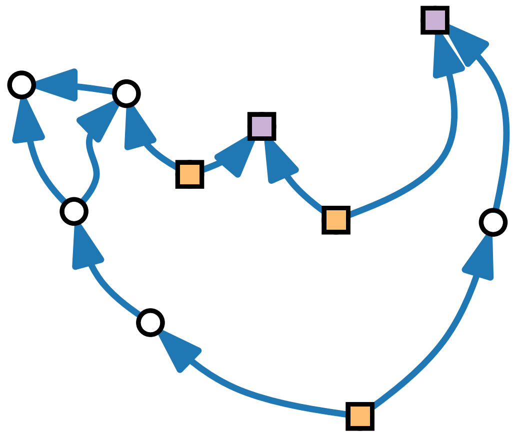

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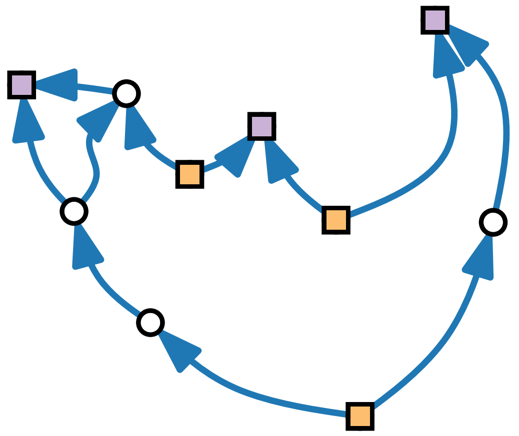

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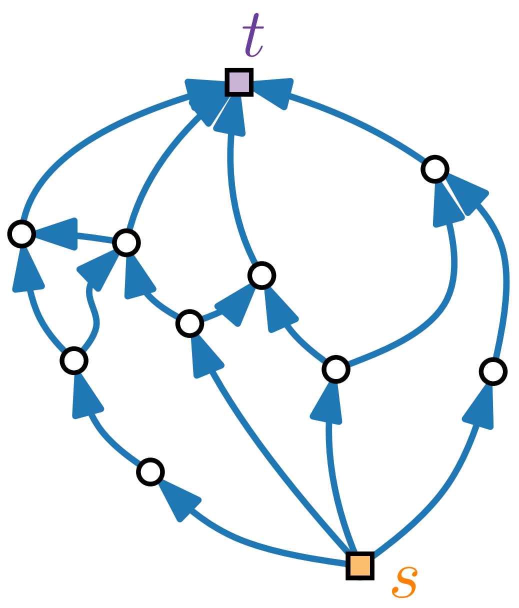

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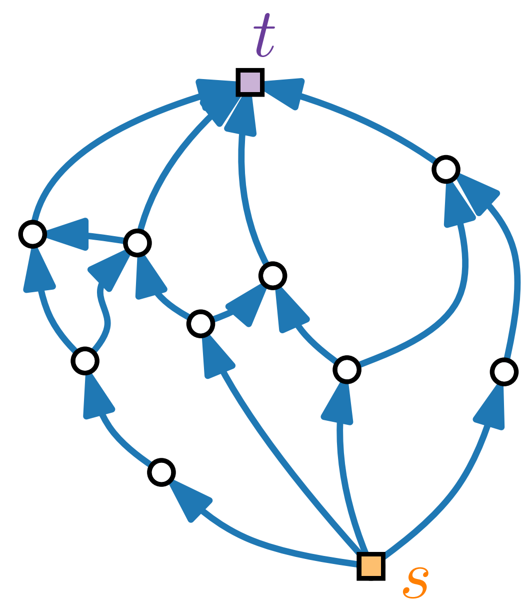

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Additionally:
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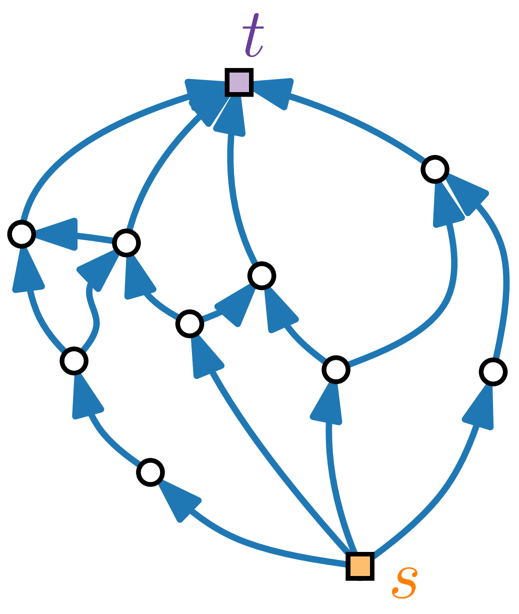

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Additionally:
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or:
Edge (s, t) exists.

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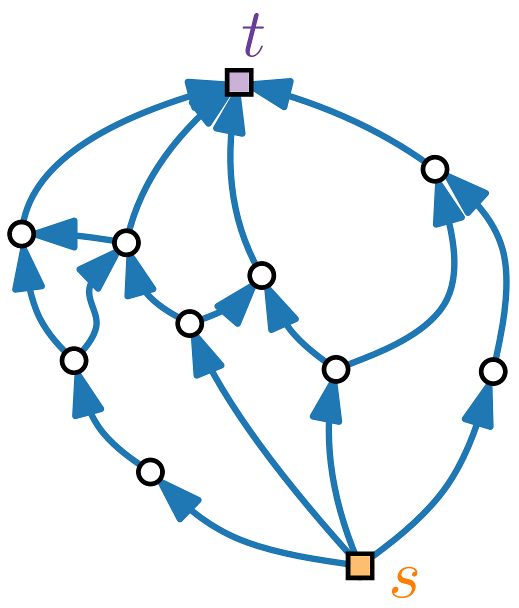

Proof.

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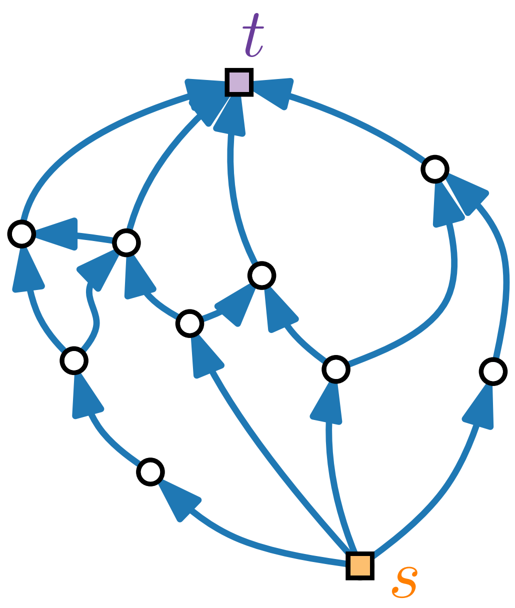

Proof.

(2) $\Rightarrow$ **(1)** By definition.

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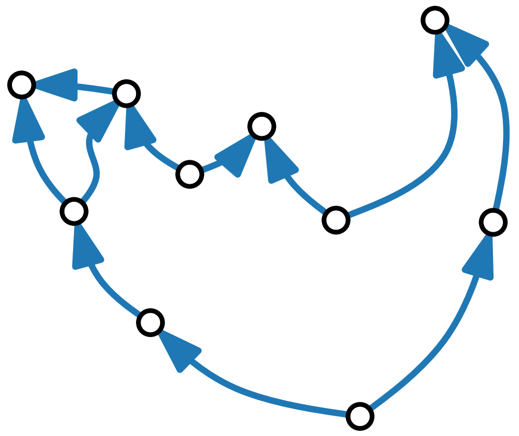

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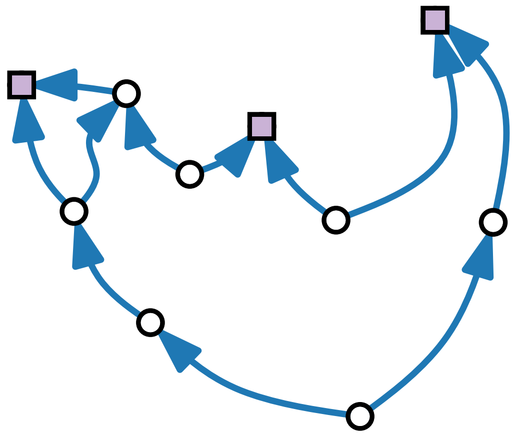

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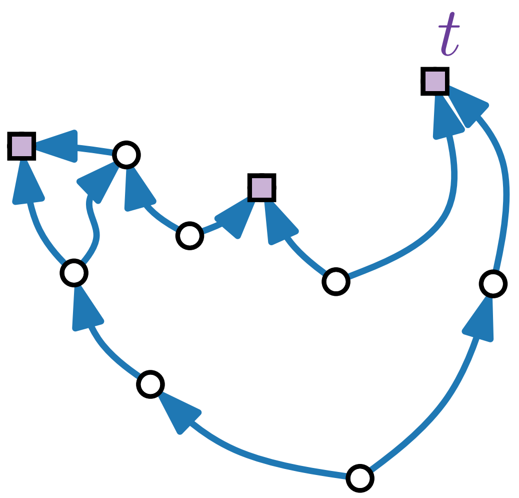

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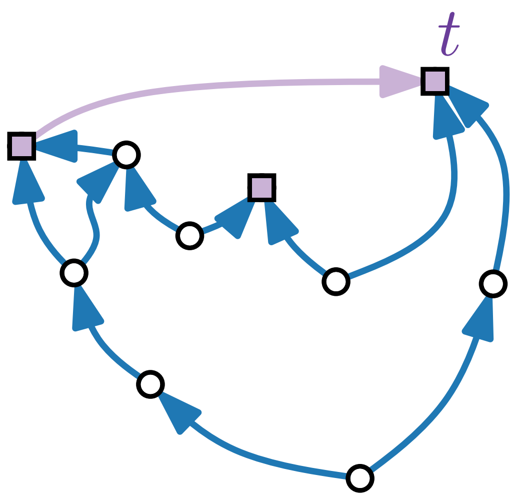

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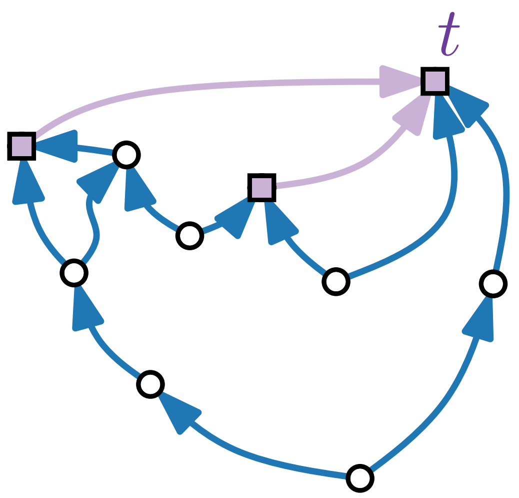

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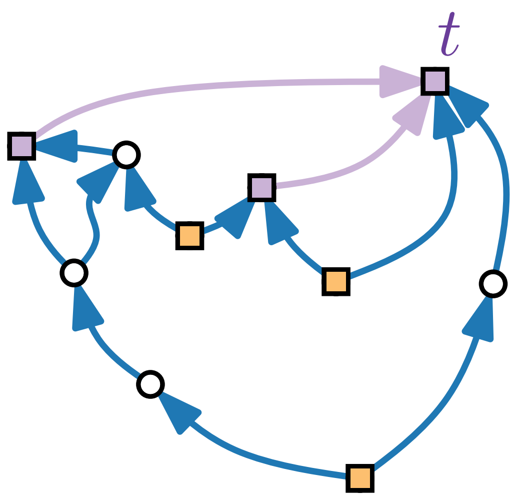

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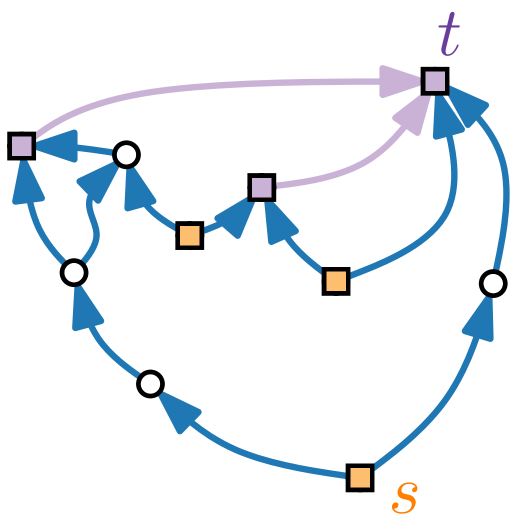

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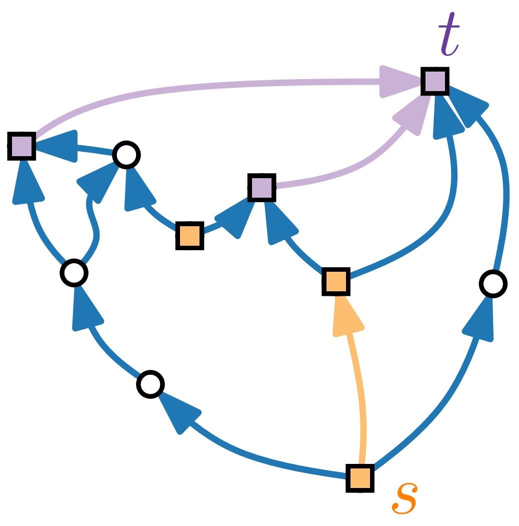

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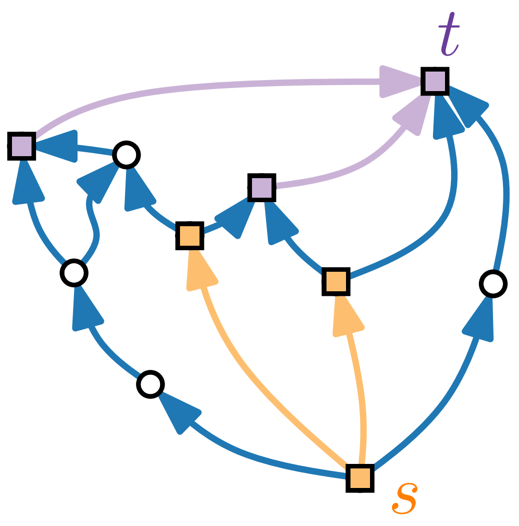

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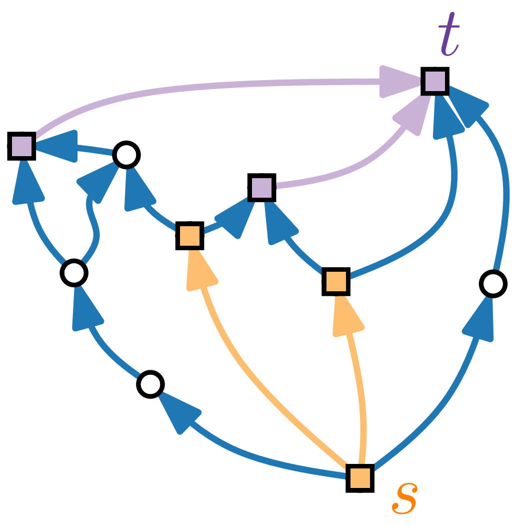

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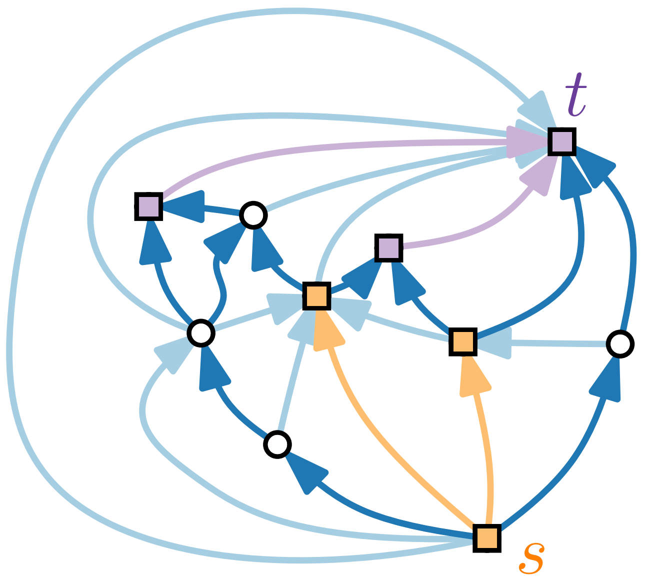

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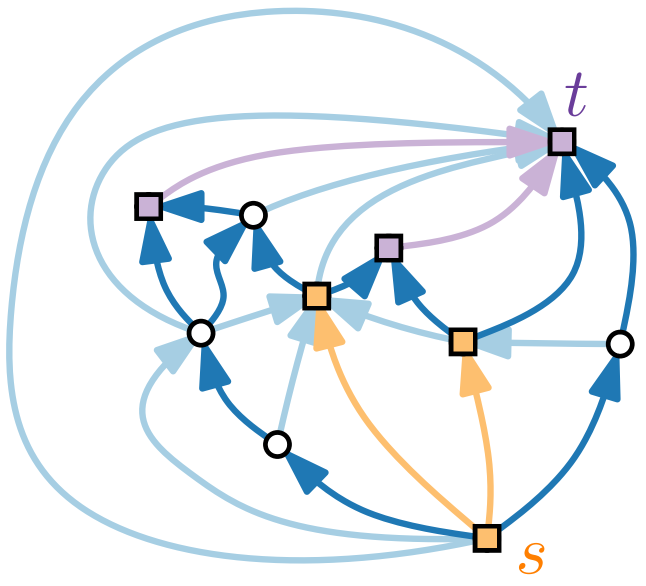

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Claim.
Can draw in prespecified triangle.

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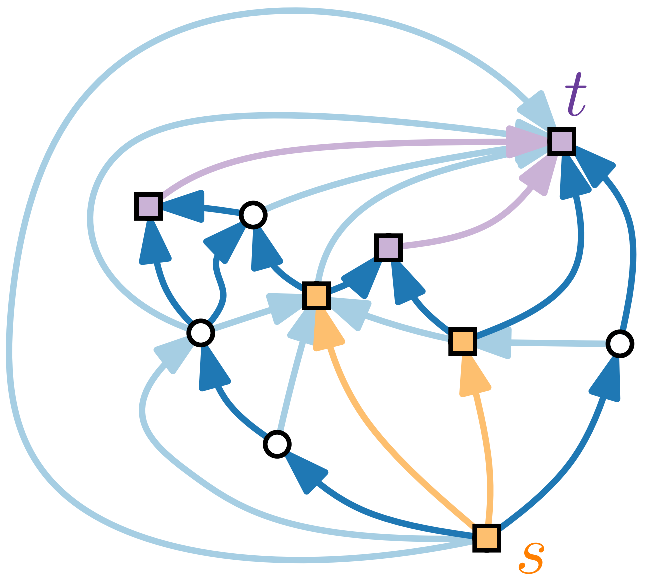

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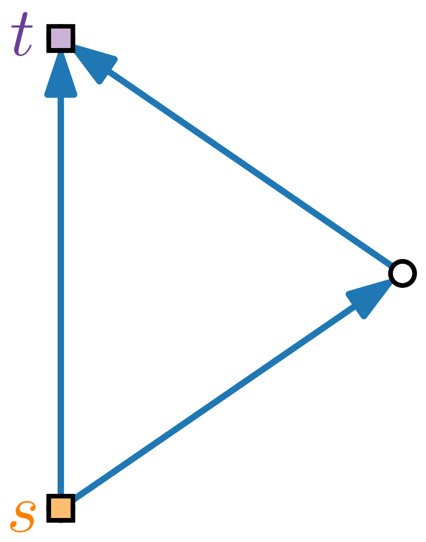

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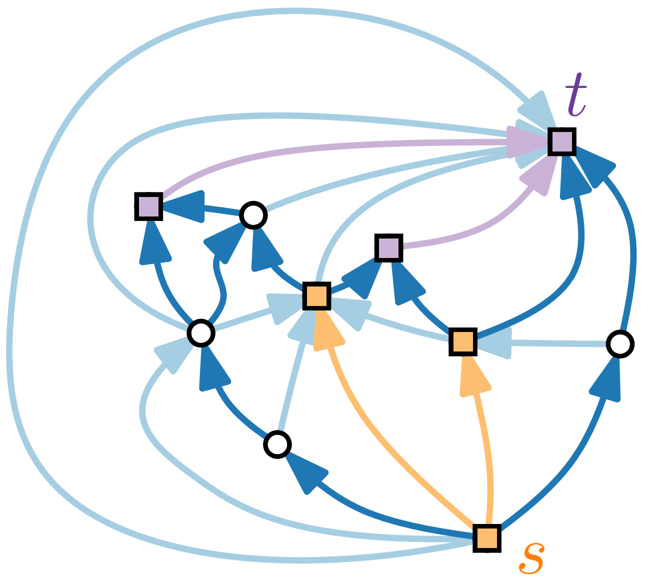

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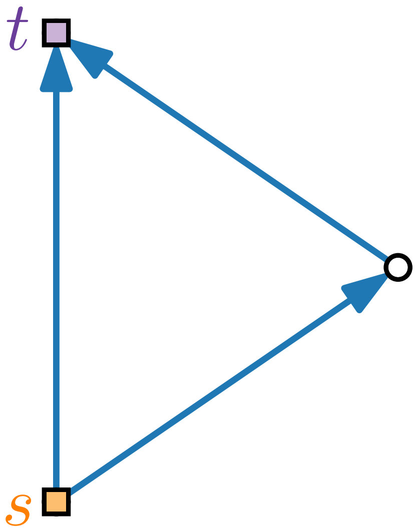
Induction on n.



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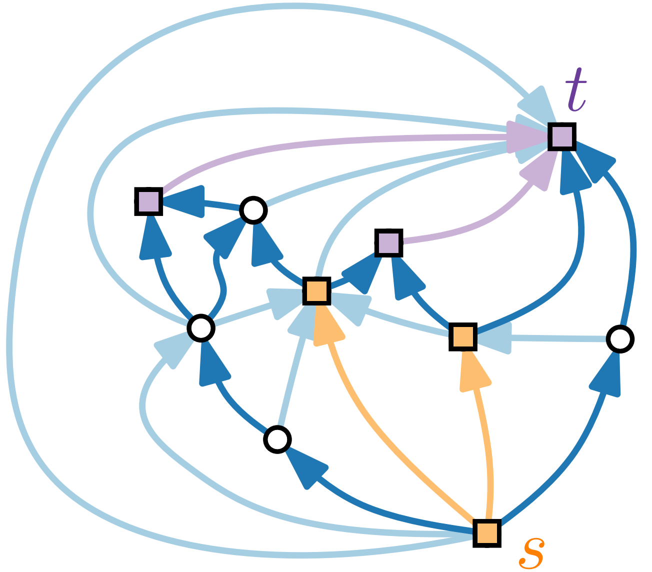

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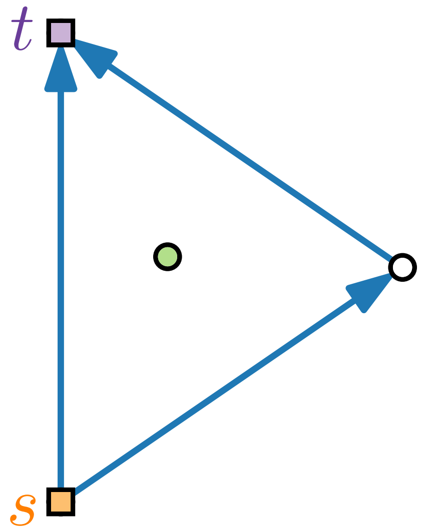

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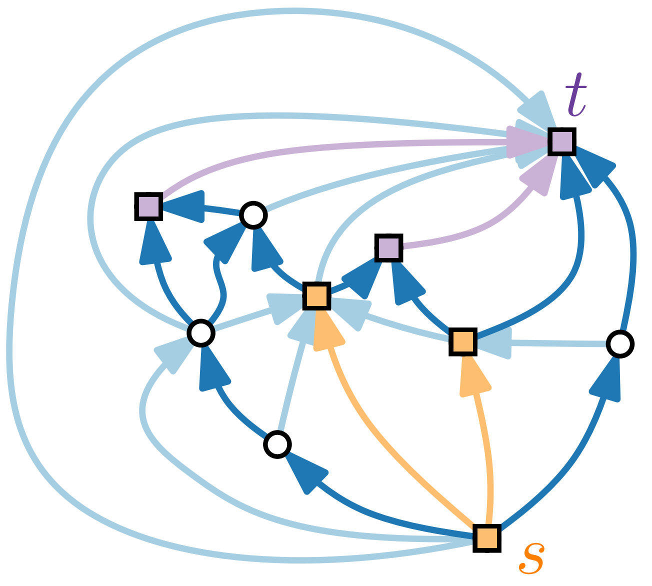

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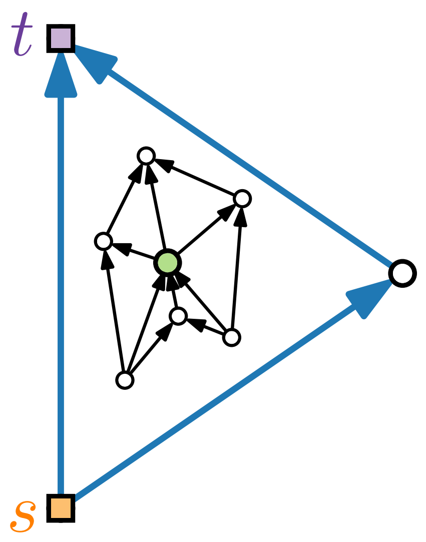

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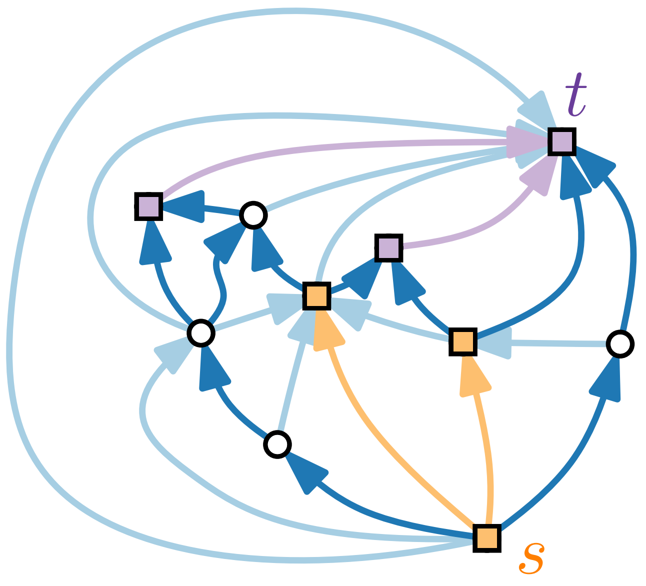

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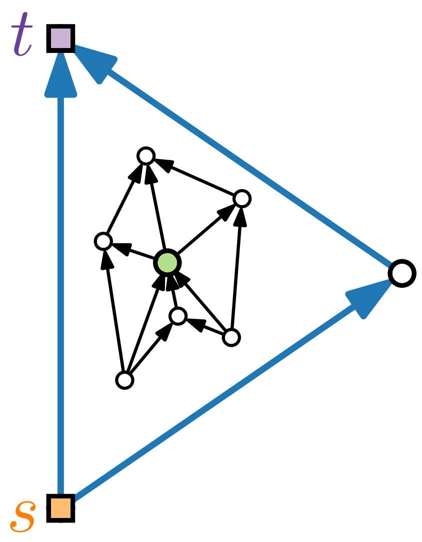
Case 1: chord



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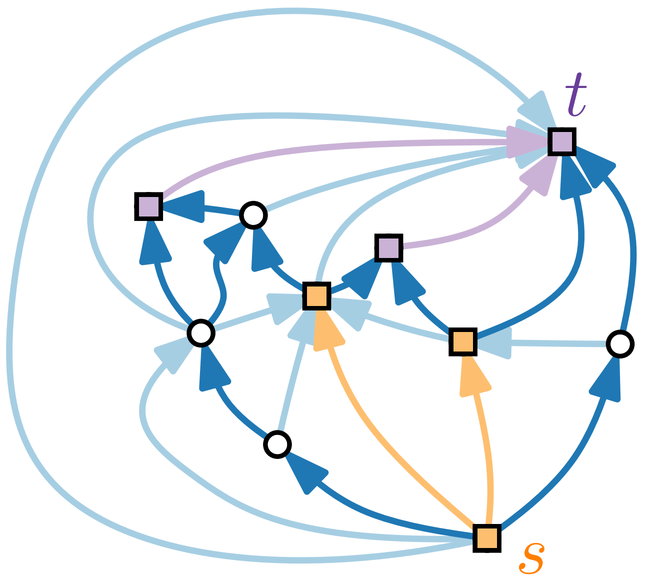

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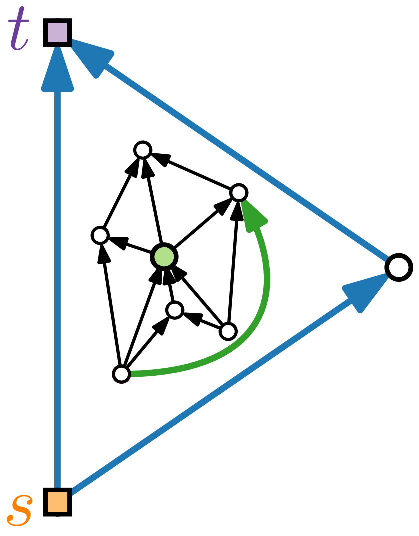

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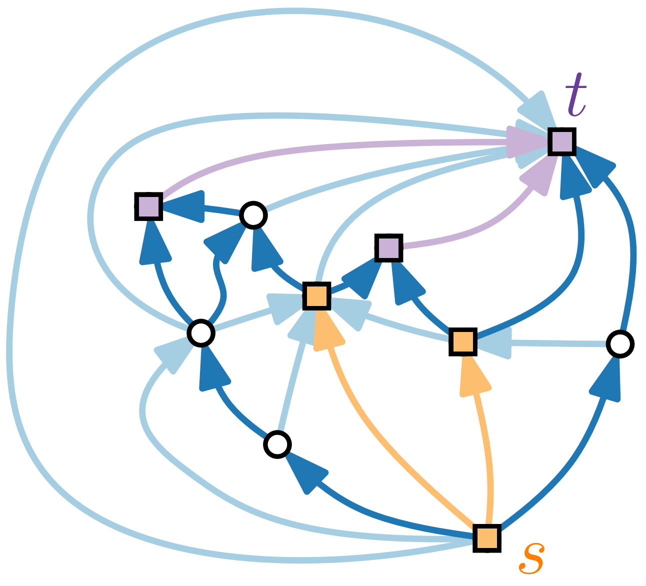

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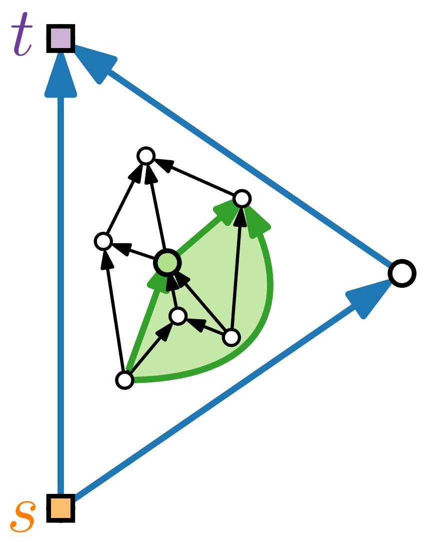

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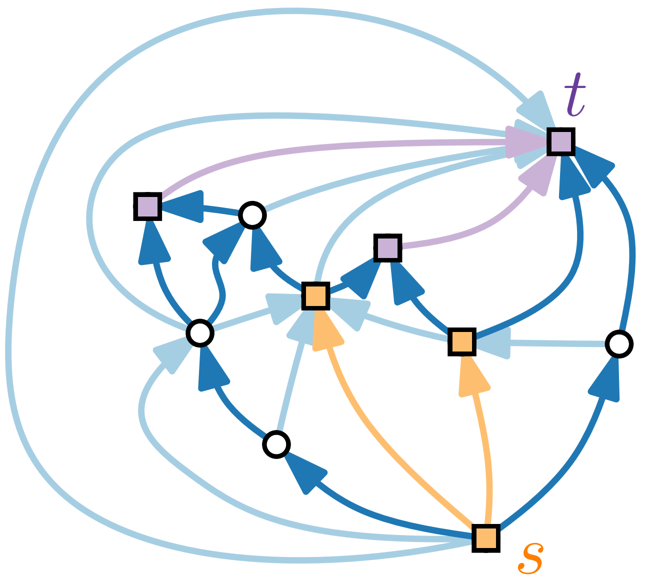

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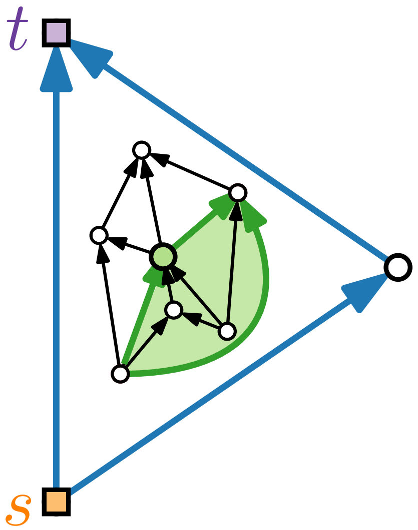

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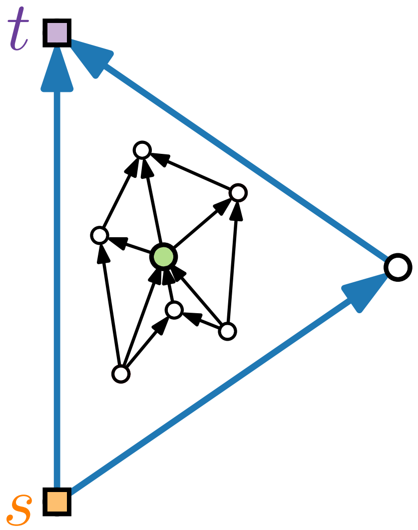

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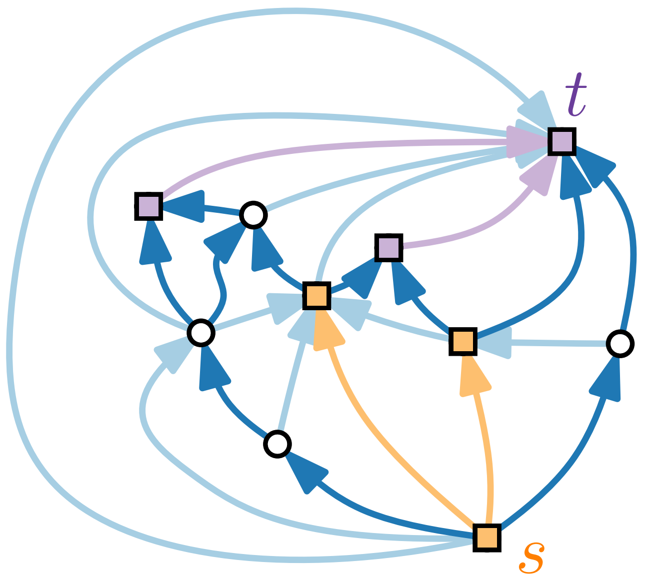

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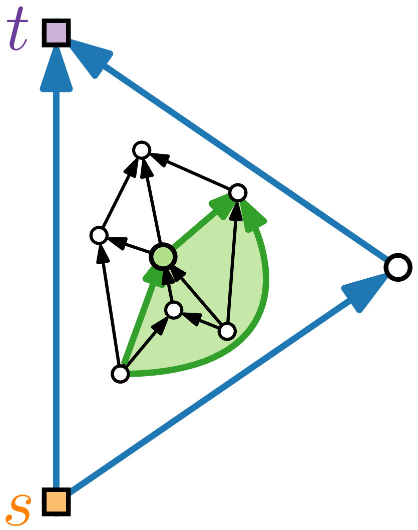

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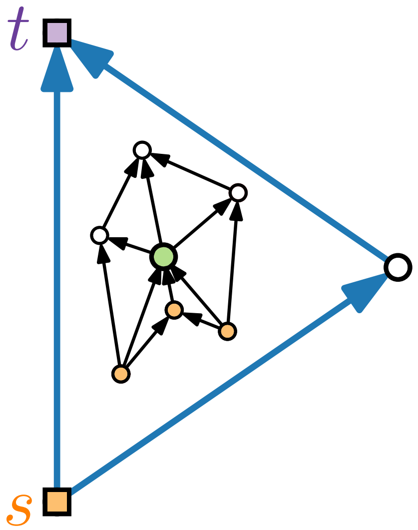

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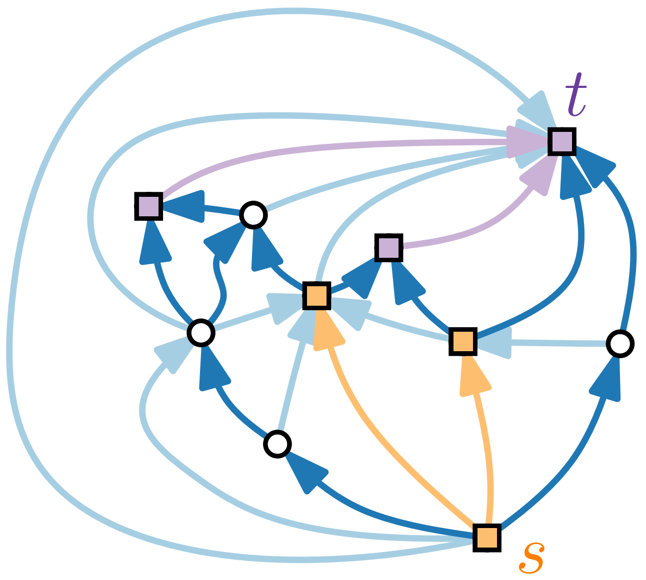

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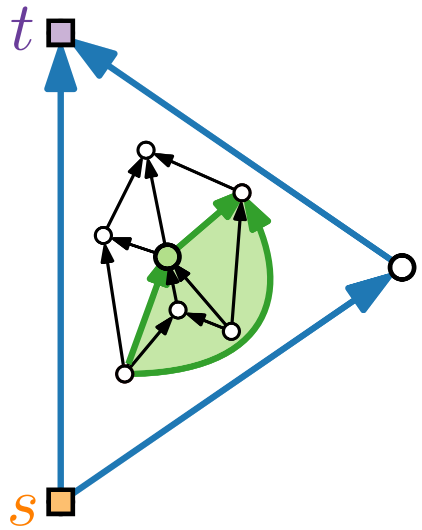

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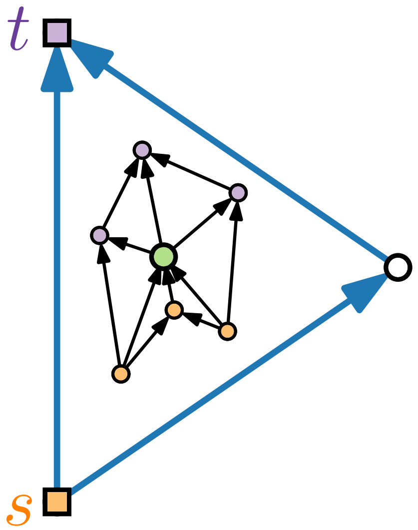

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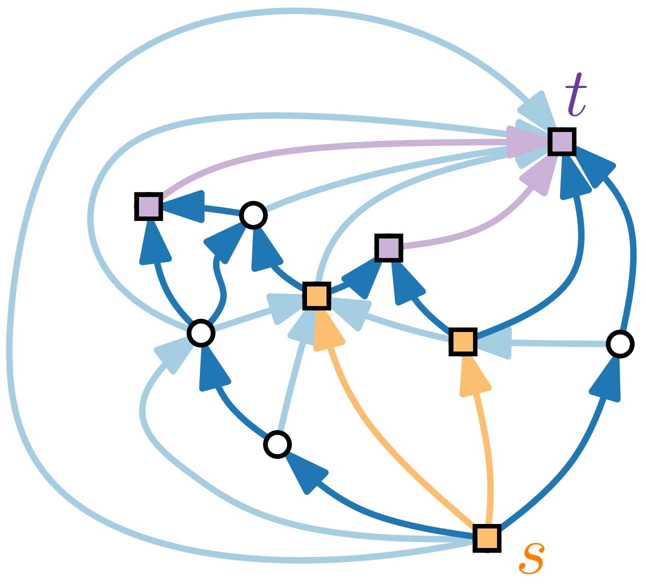

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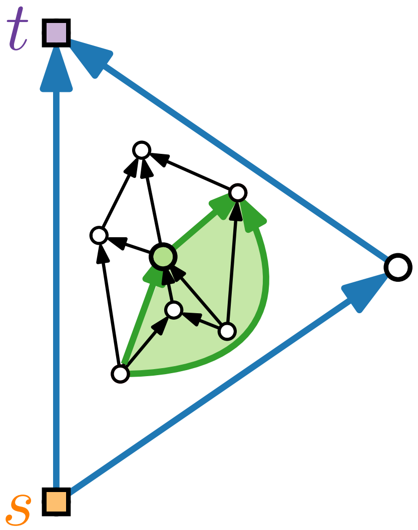

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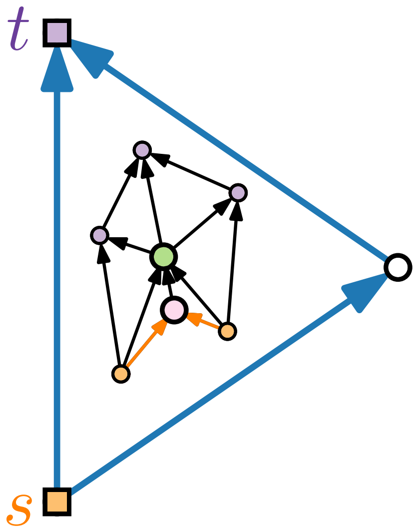

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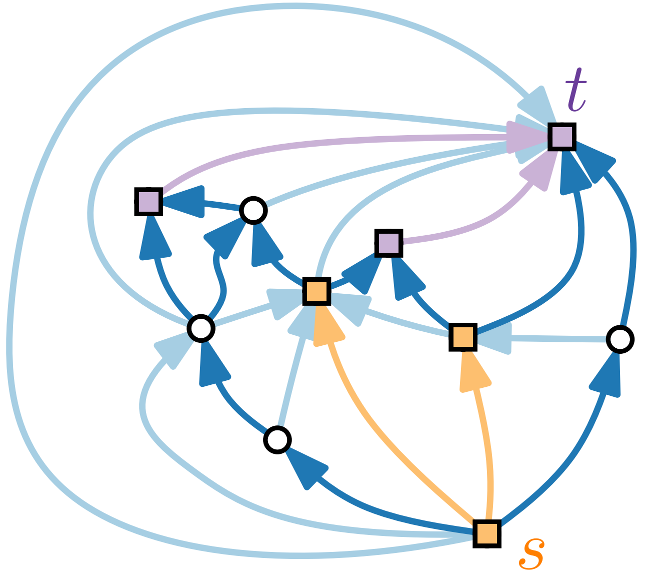

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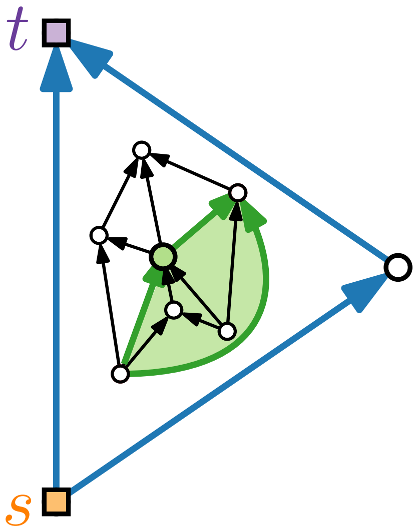

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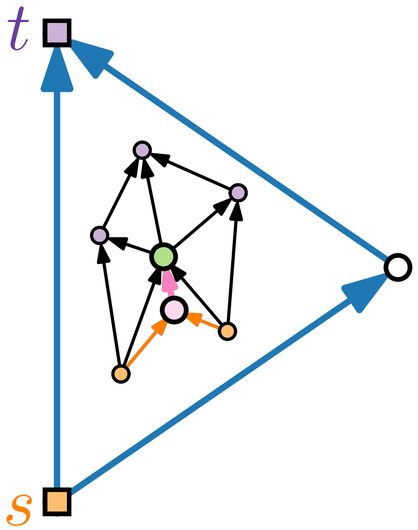

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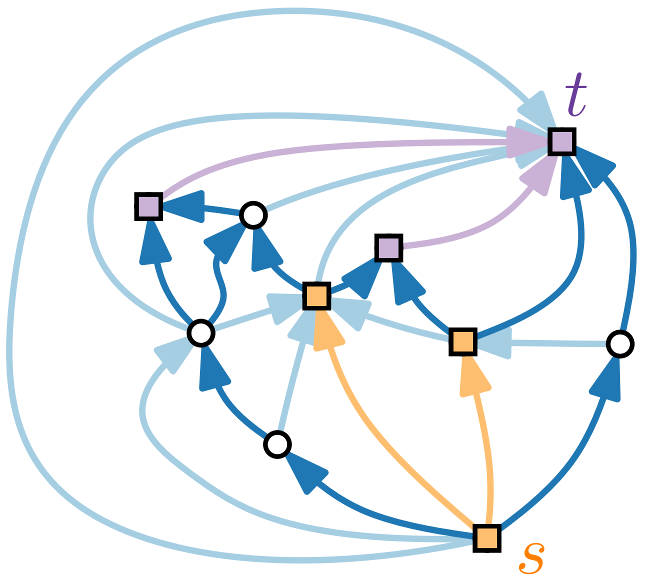

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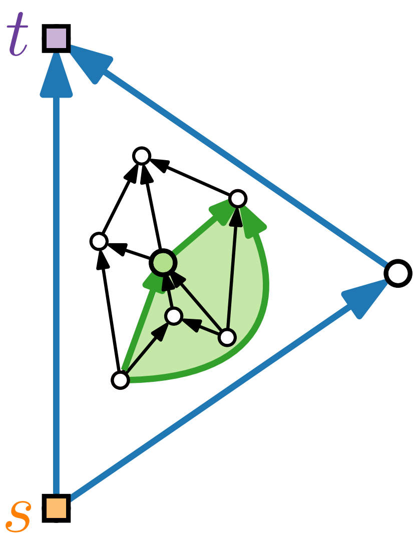

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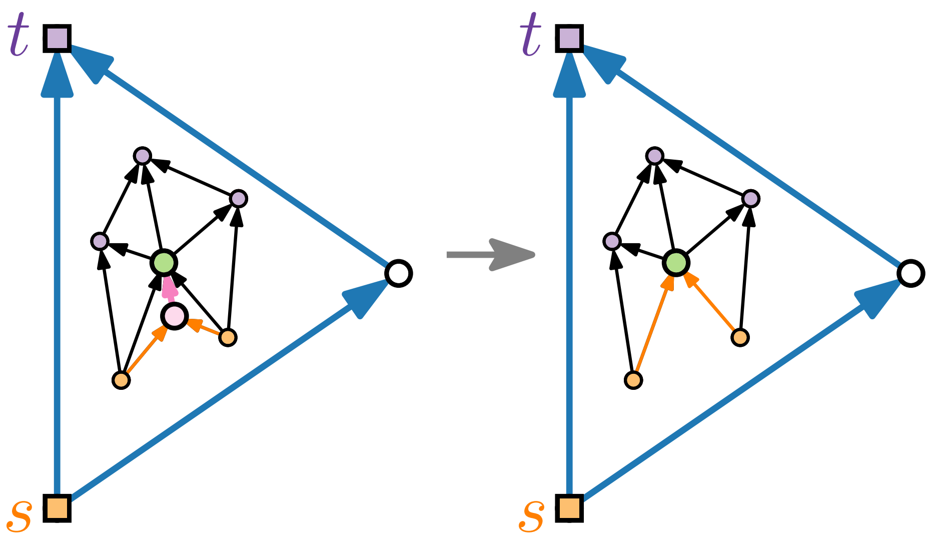

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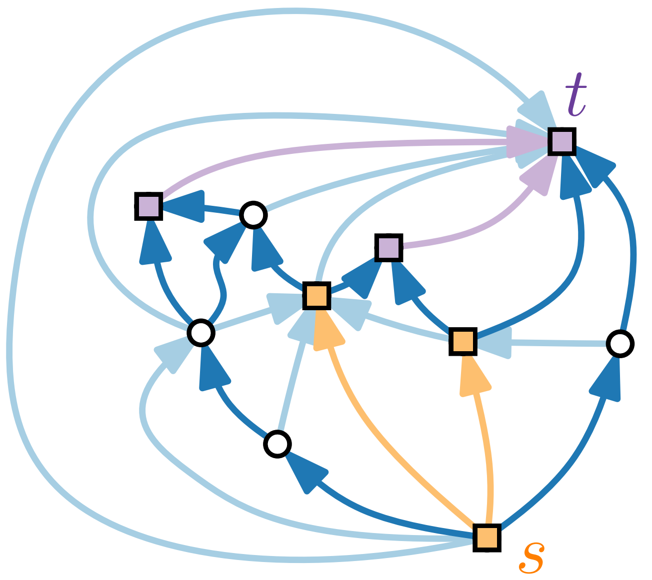

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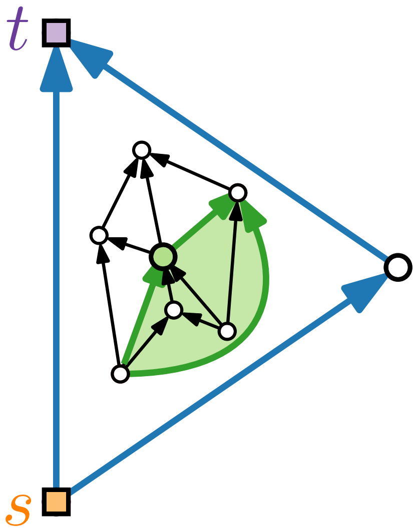

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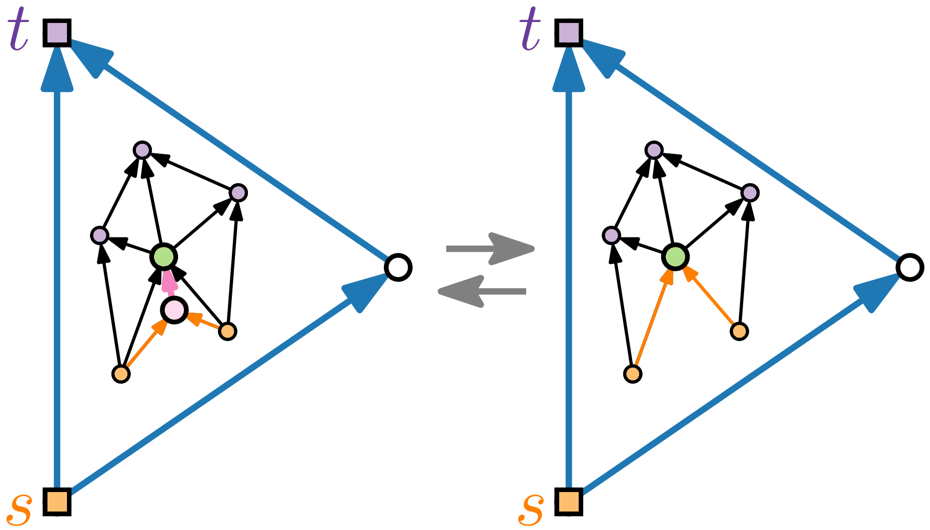

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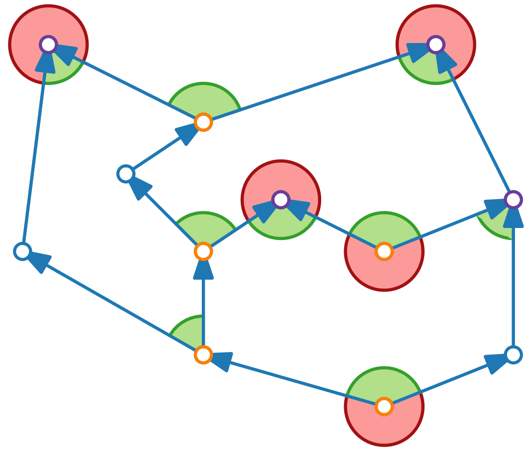
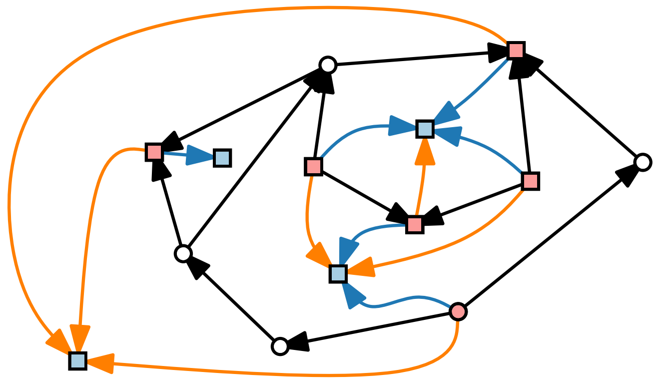
WÜRZBURG

Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part II:
Assignment Problem



Alexander Wolff

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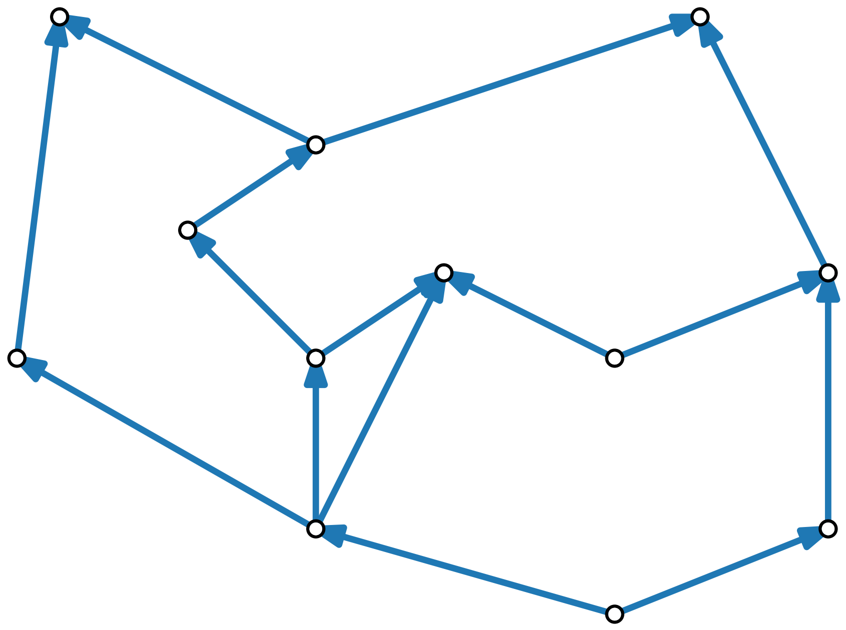
Plan.

- Find property that any upward planar drawing of G satisfies.
- Formalize property.
- Find algorithm to test property.

Angles, Local Sources & Sinks

Definitions.



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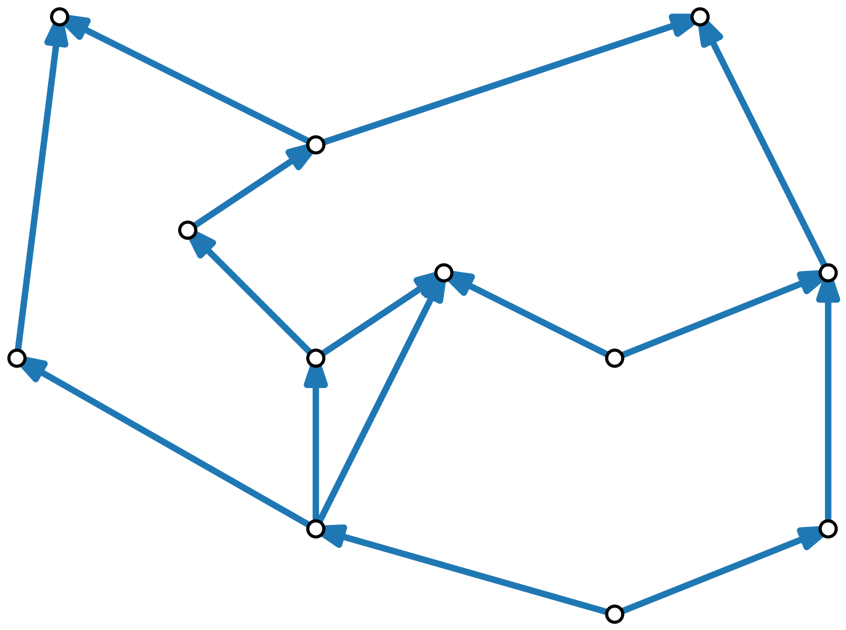

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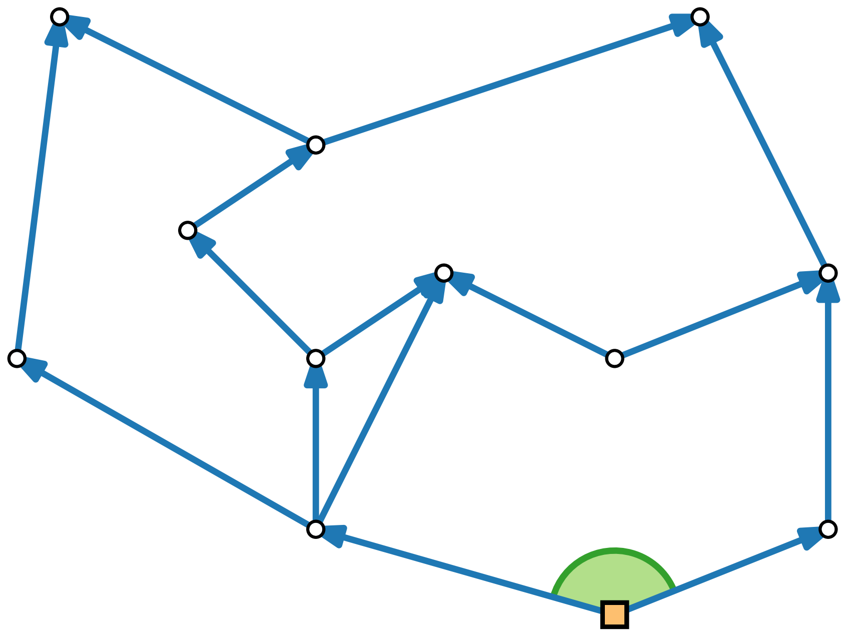

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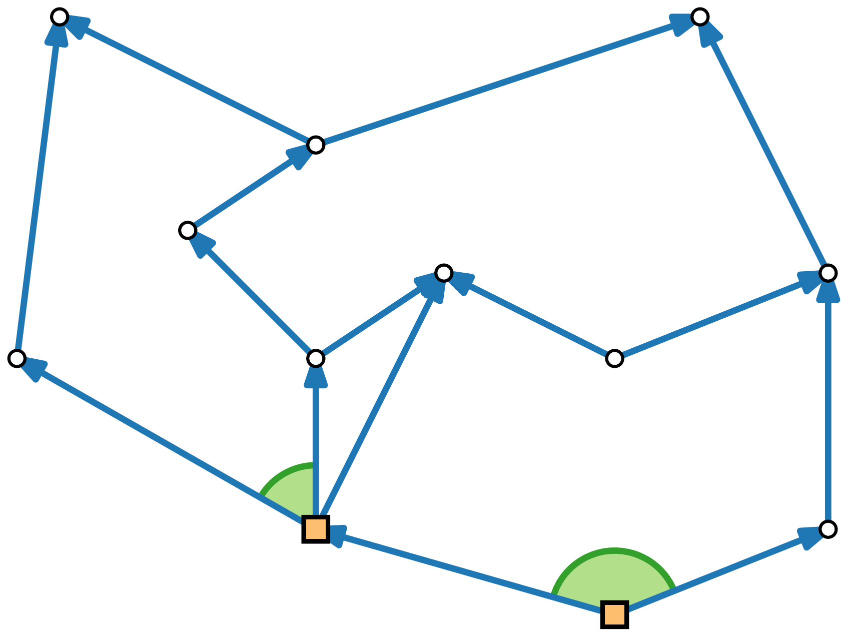

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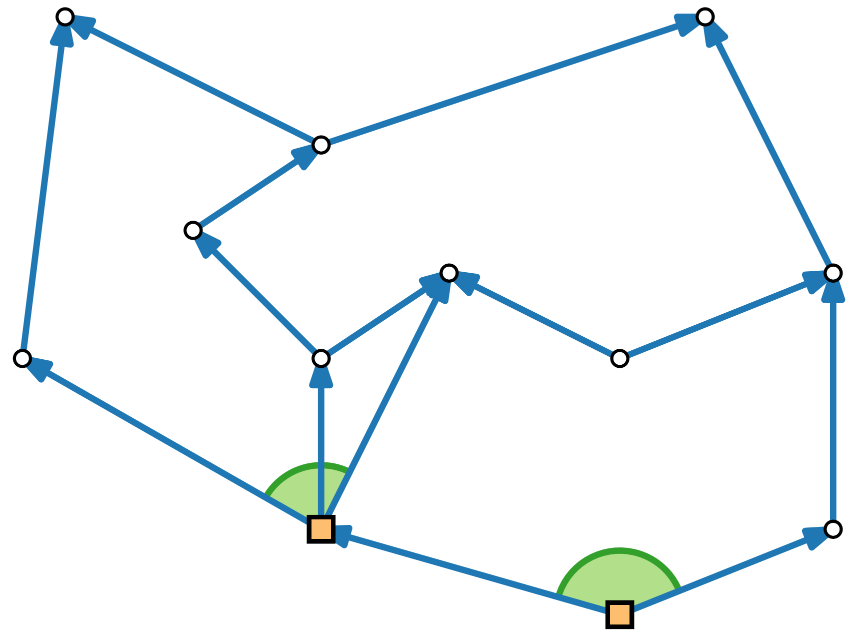

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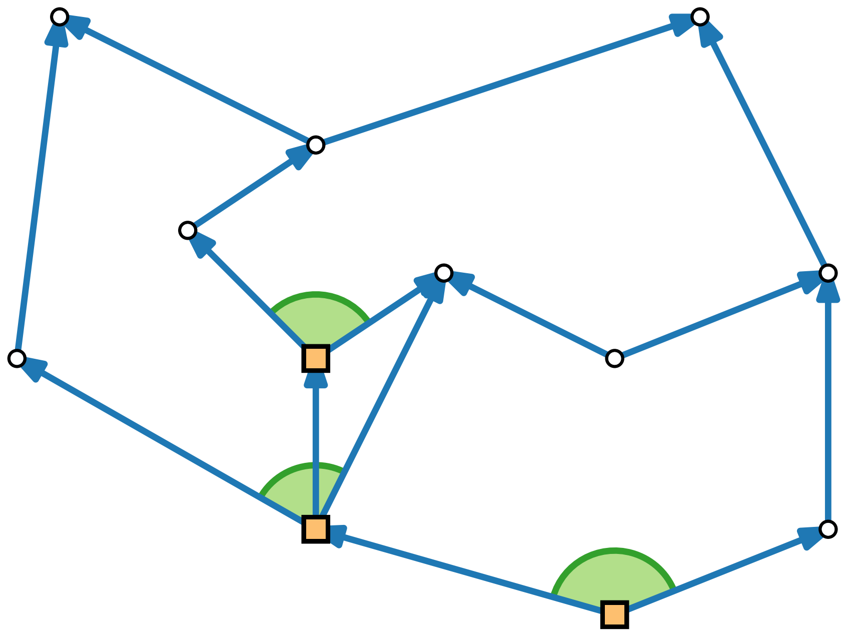

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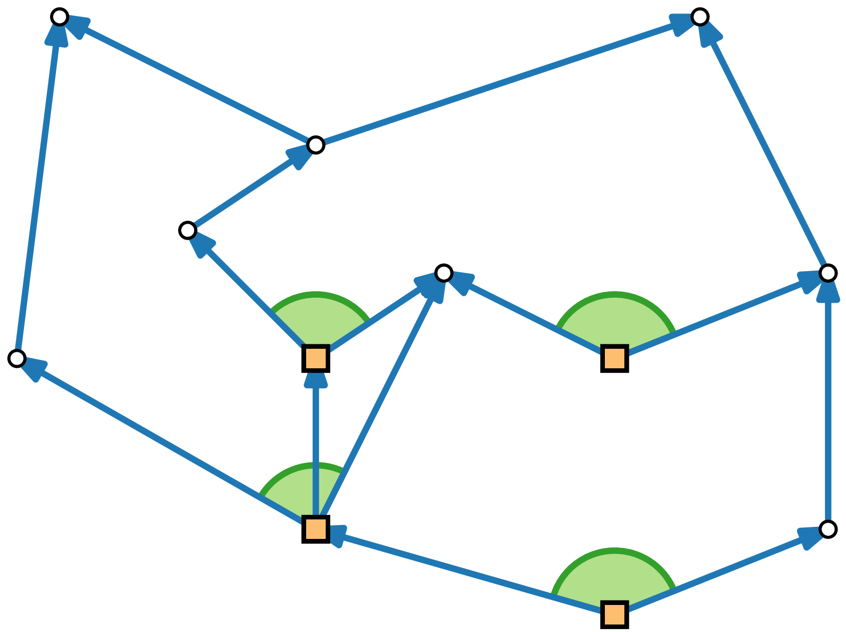

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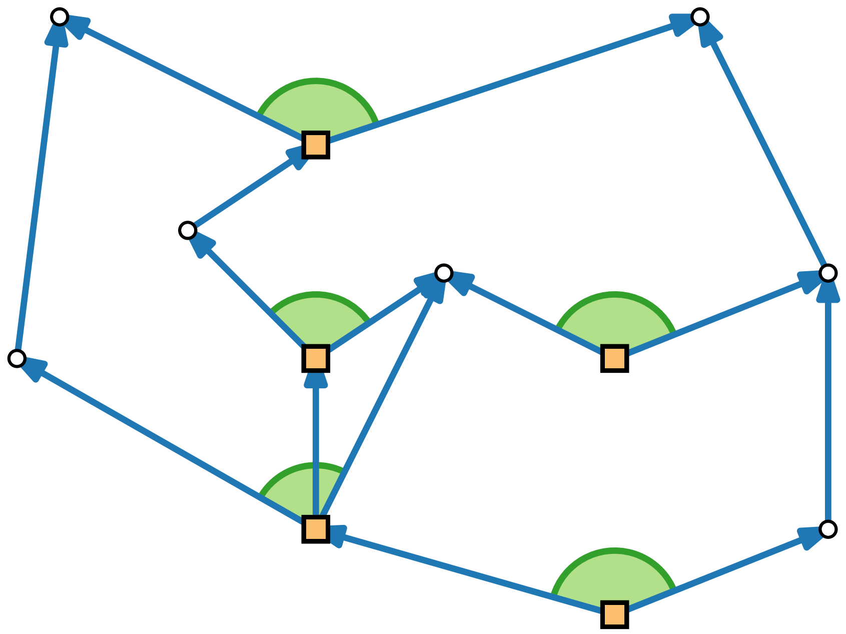

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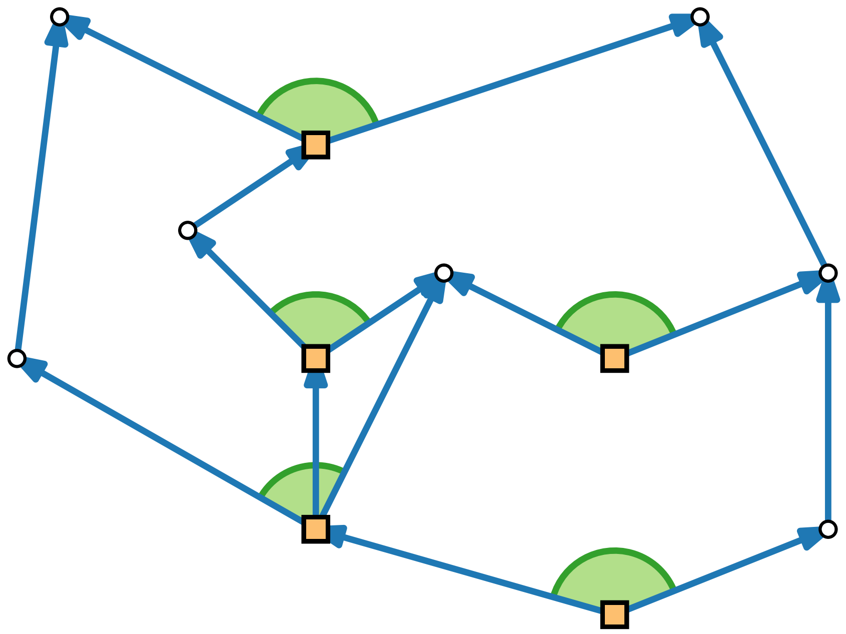

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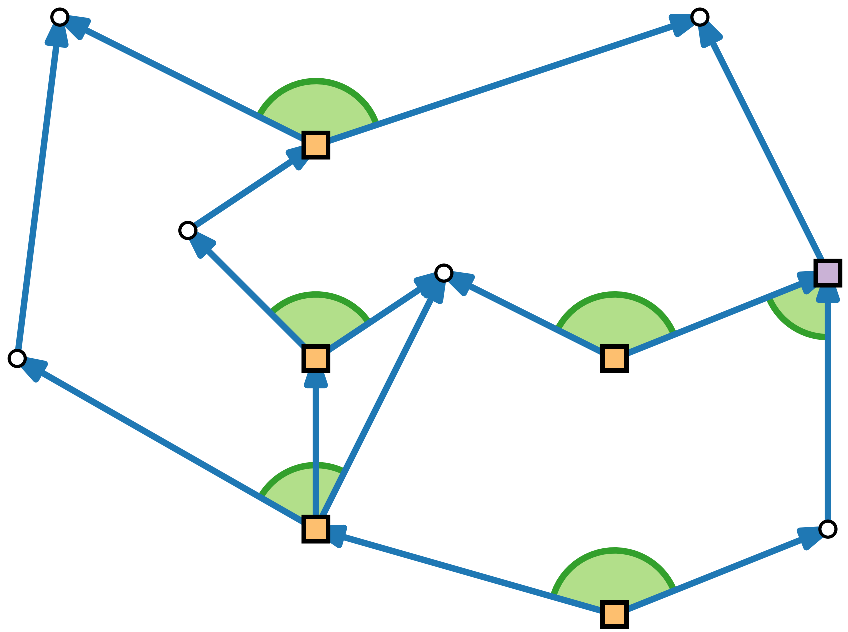

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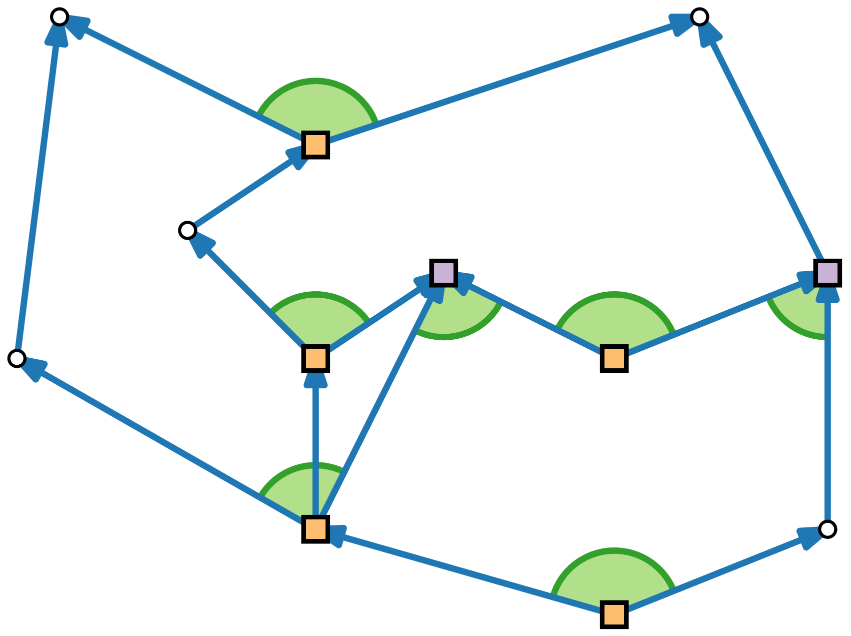

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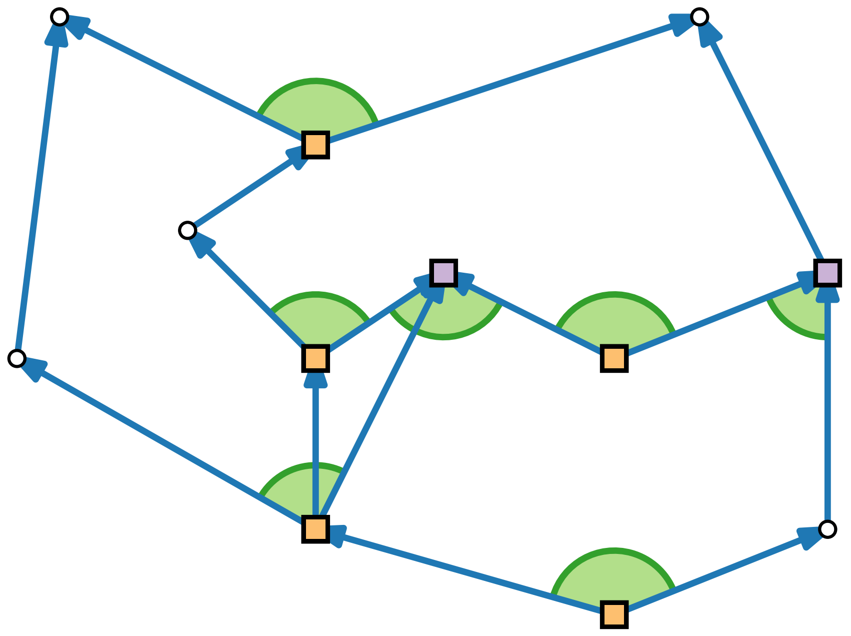

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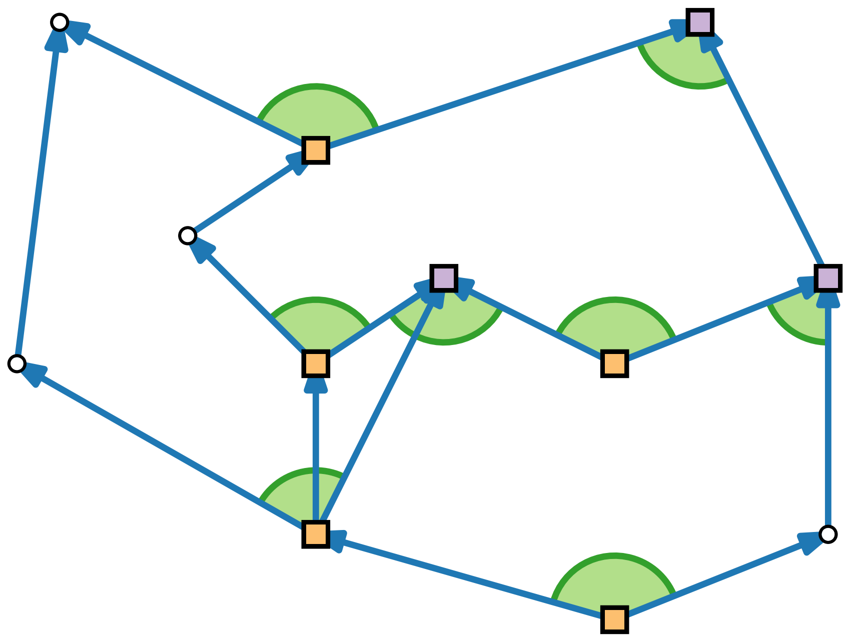

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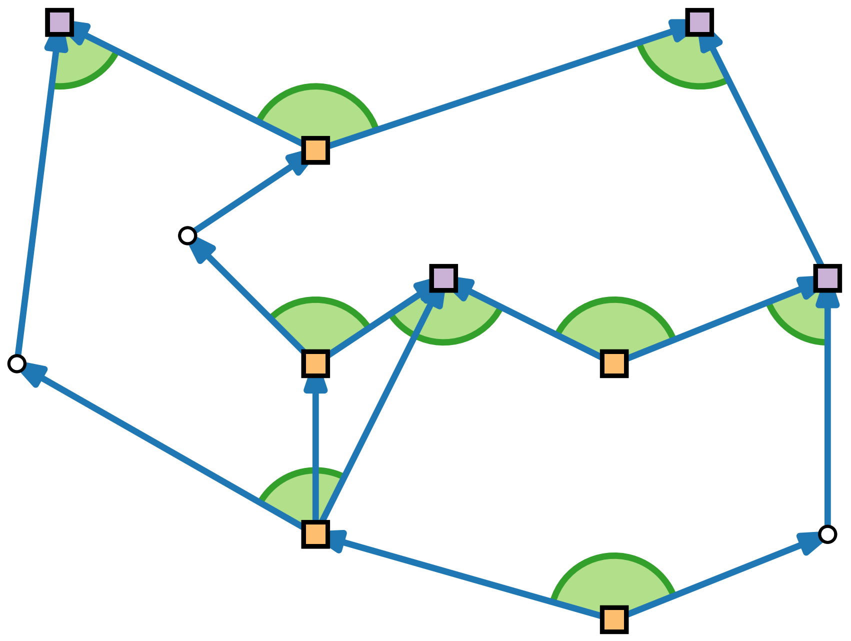

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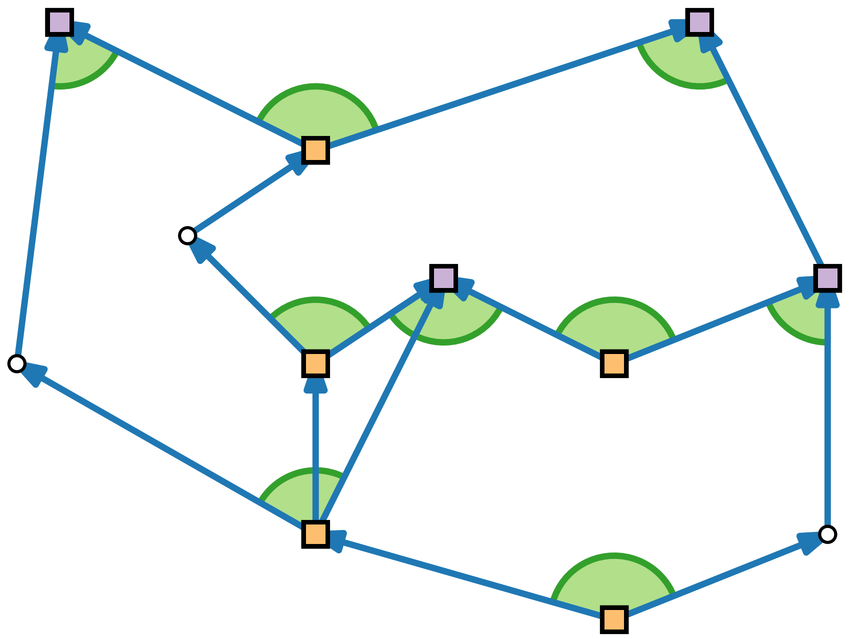

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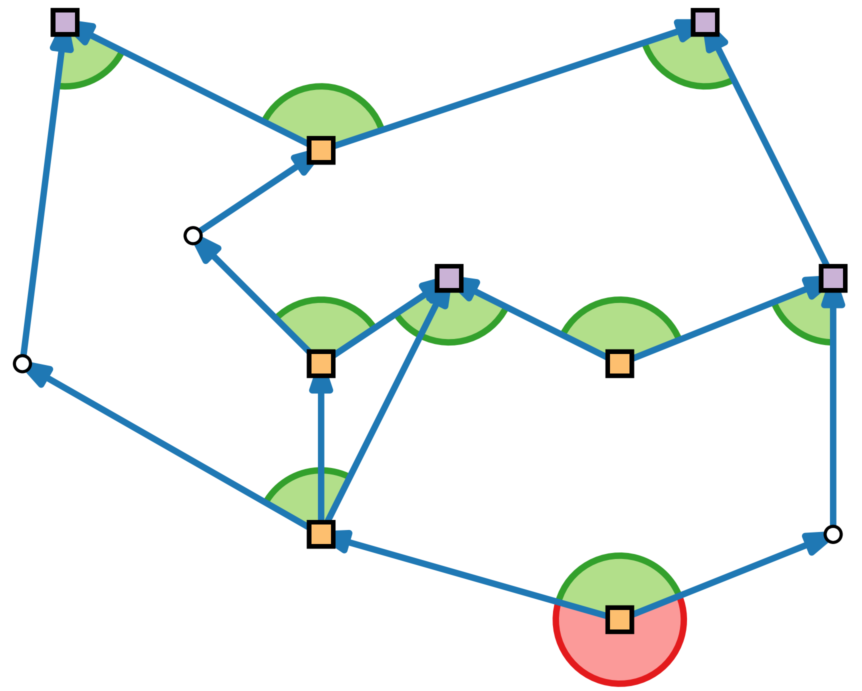

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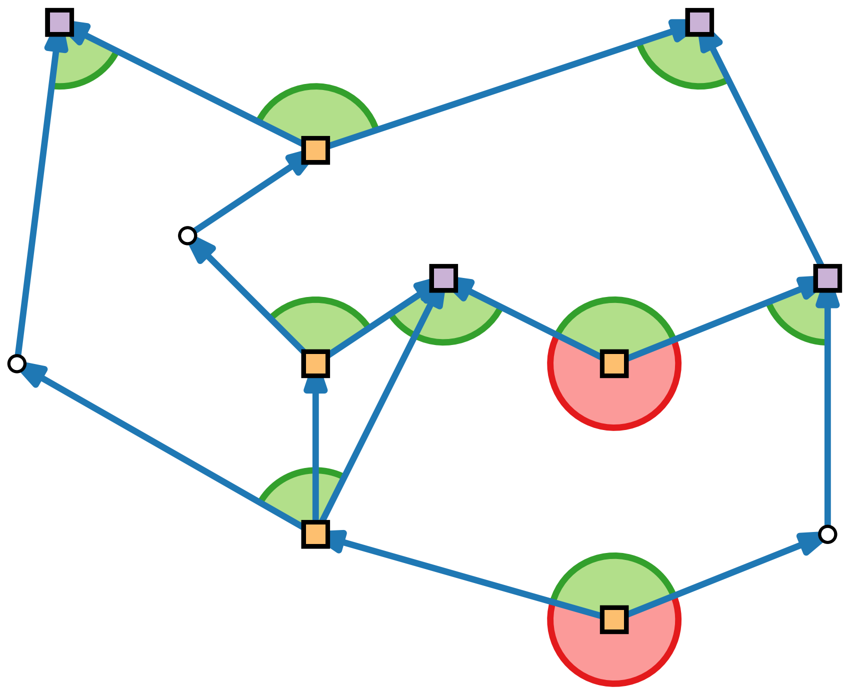

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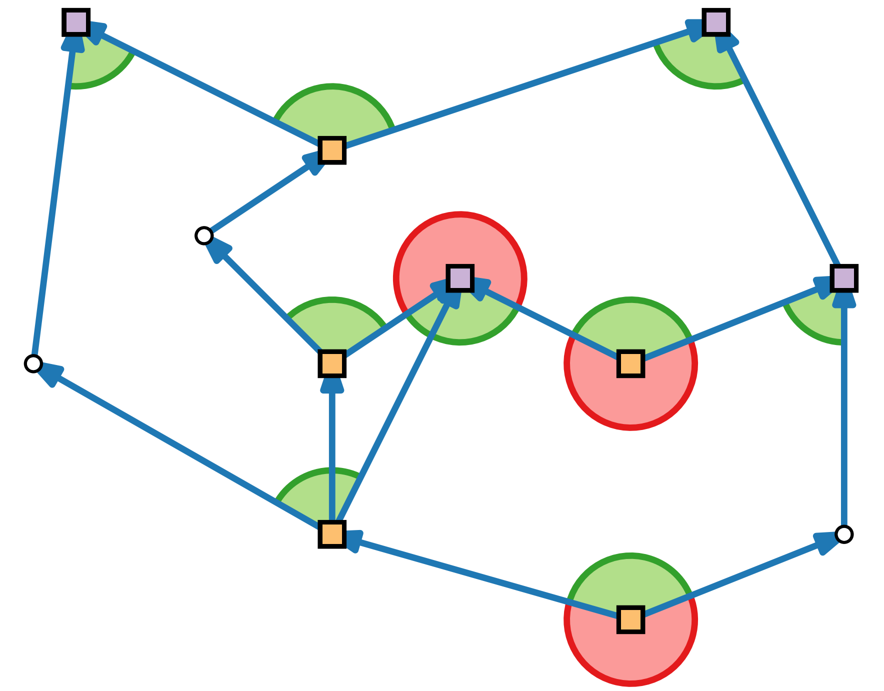

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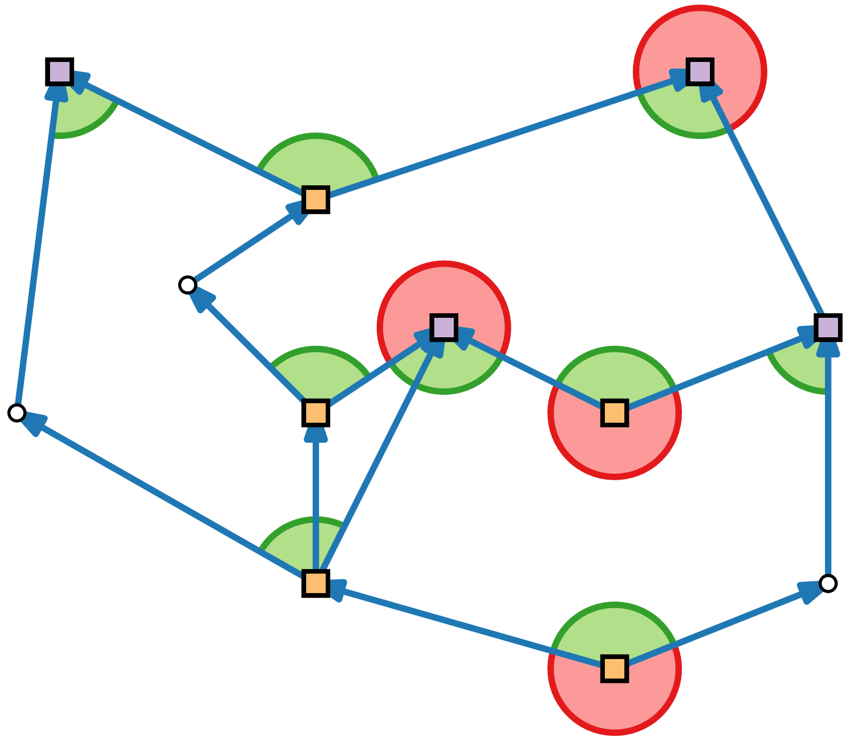

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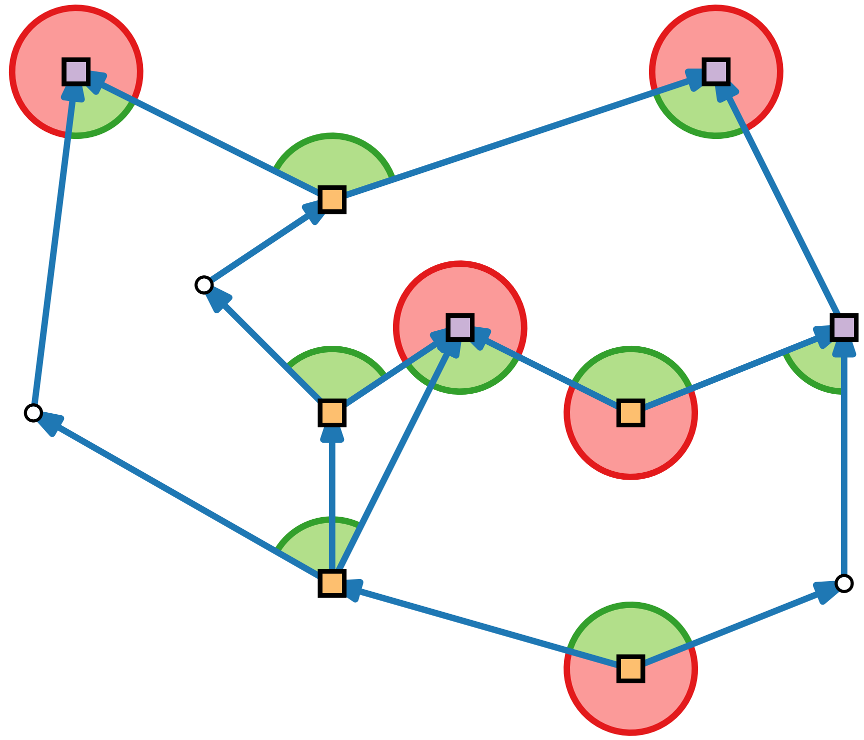

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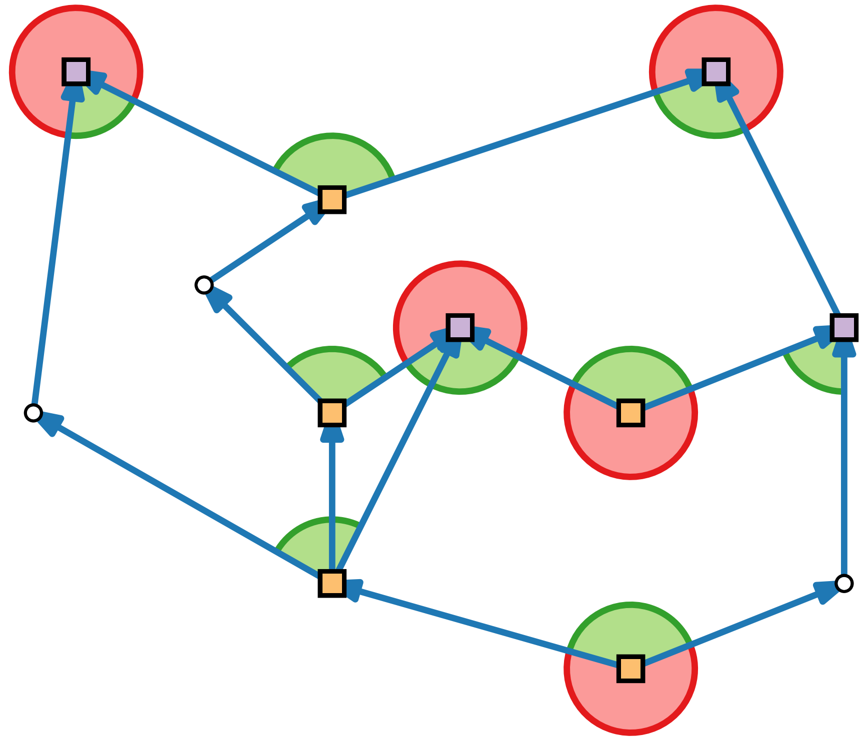

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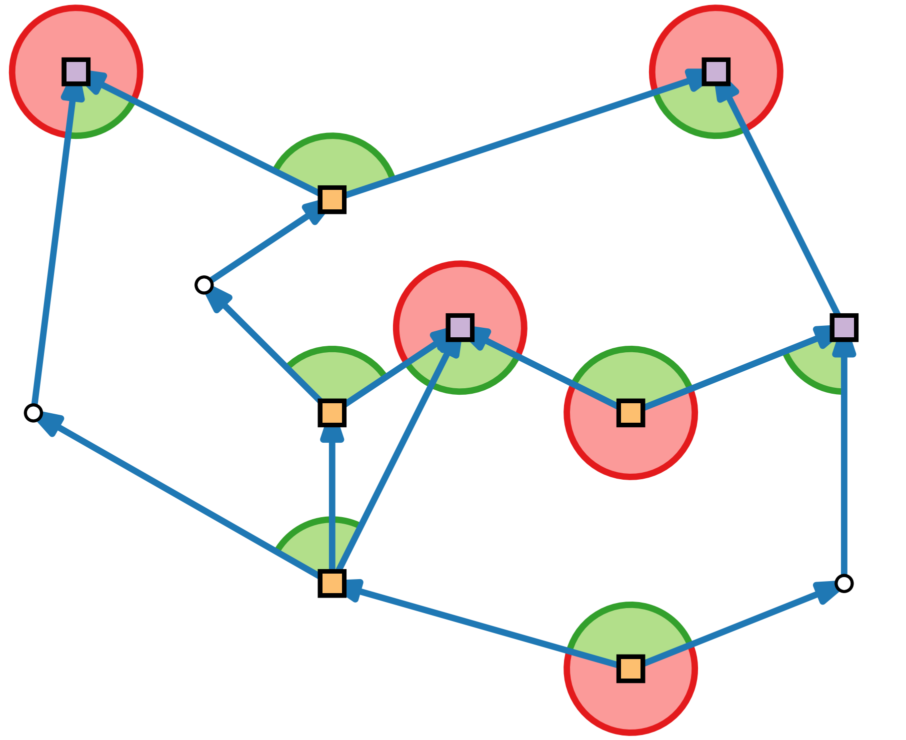

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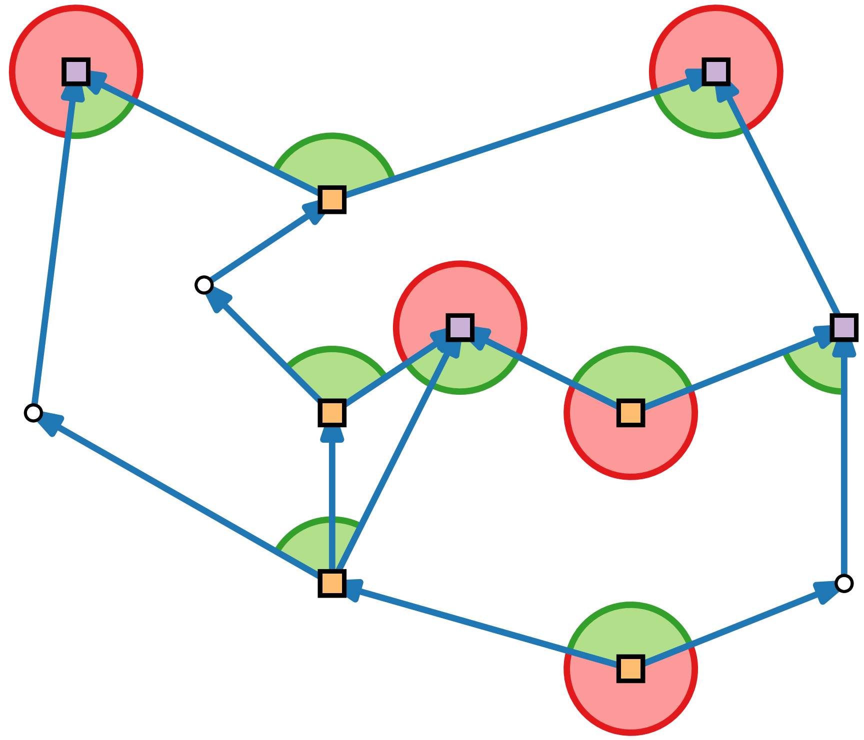

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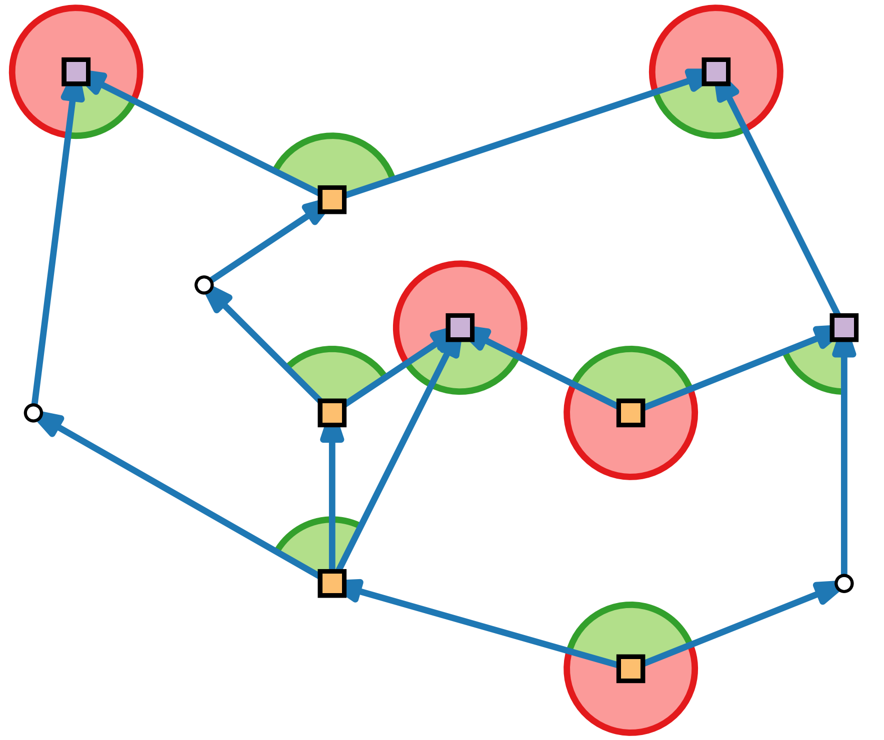

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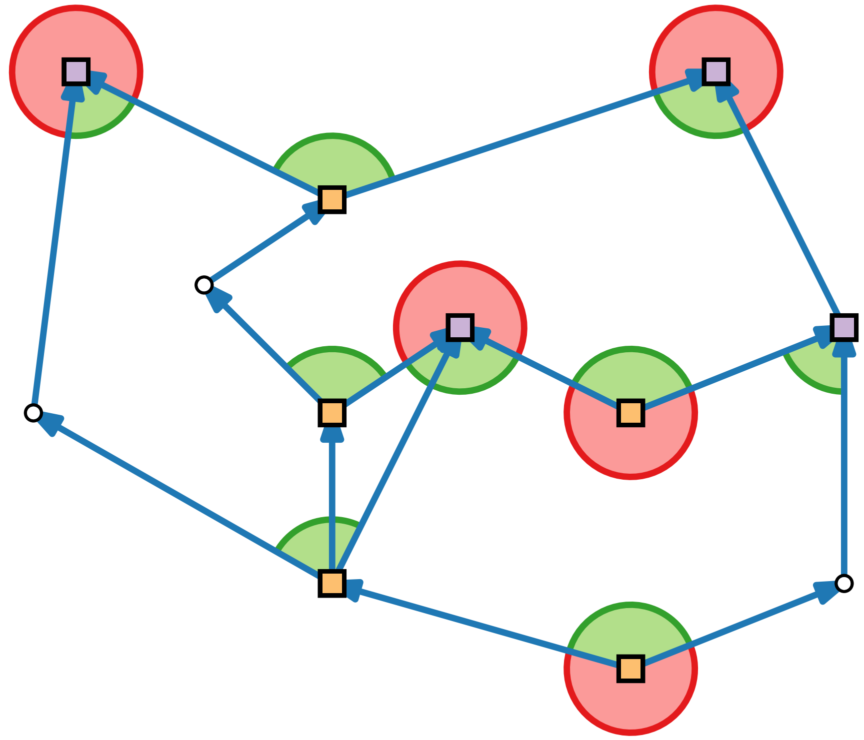

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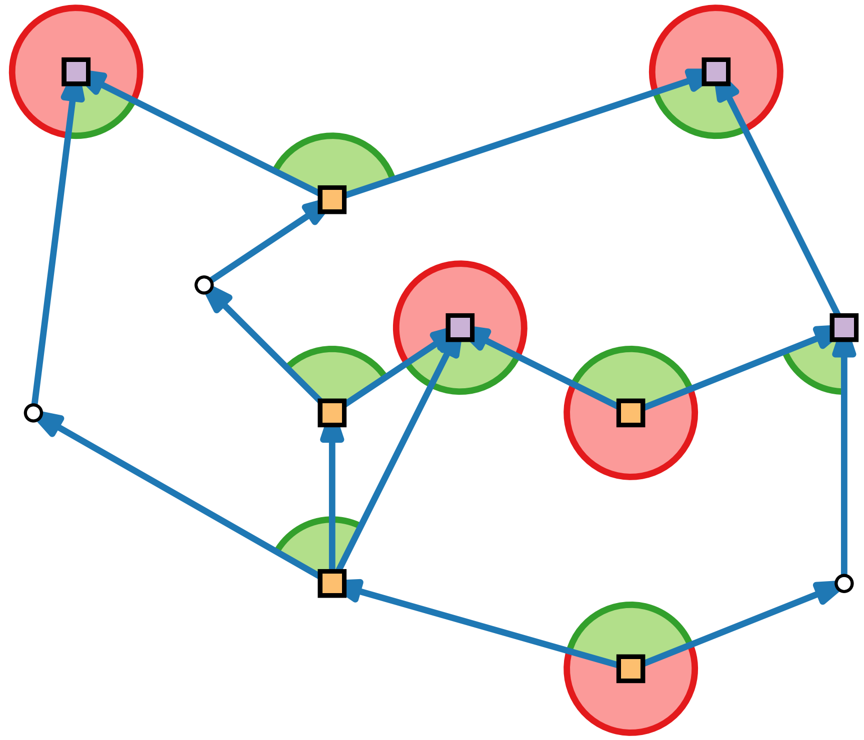

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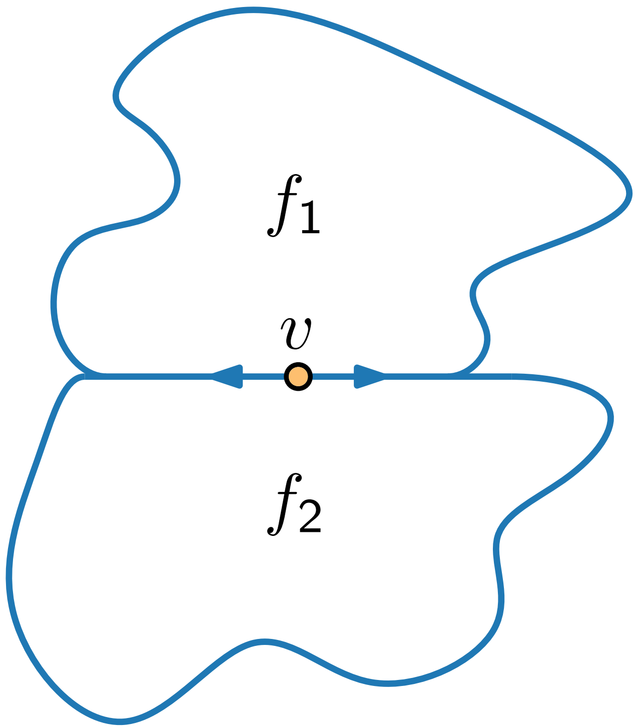
Lemma 1.
$L(f) + S(f) = 2A(f)$

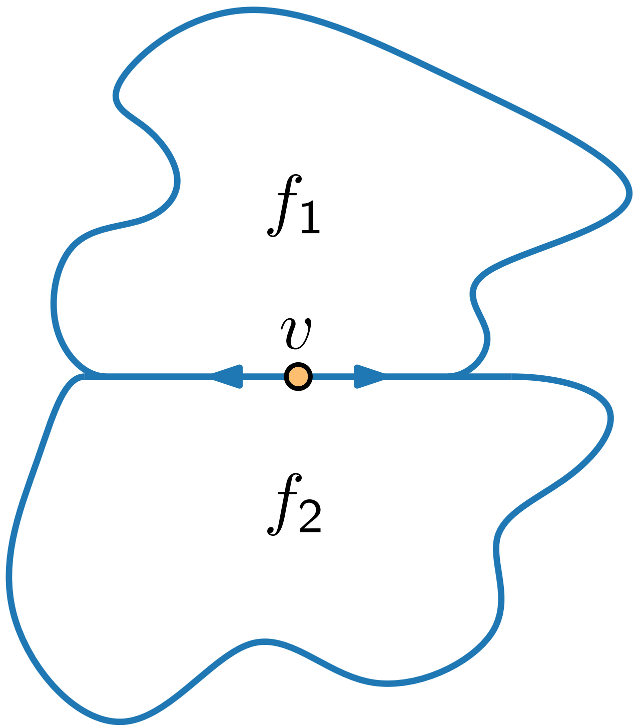
Assignment Problem

- Vertex v is a global source at faces f_1 and f_2.



Assignment Problem

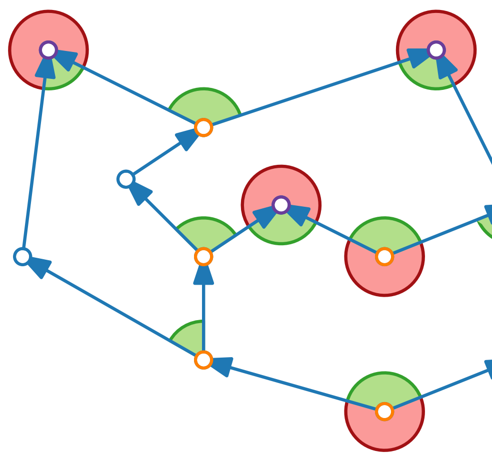
- Vertex v is a global source at faces f_1 and f_2.
- Does v have a **large** angle in f_1 or f_2?



Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

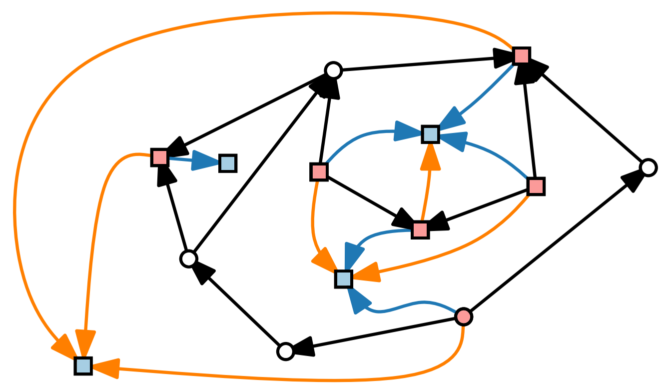
Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part III:
Angle Relations

Alexander Wolff



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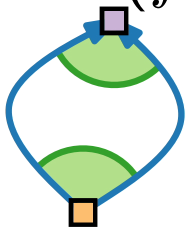

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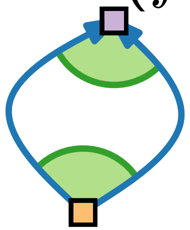
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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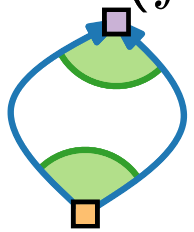
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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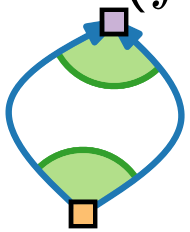

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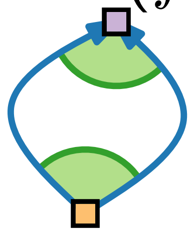
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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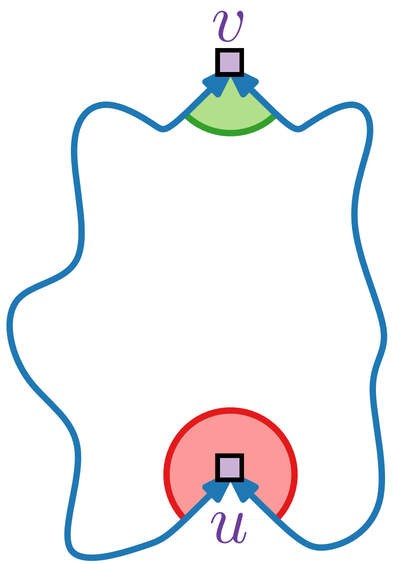

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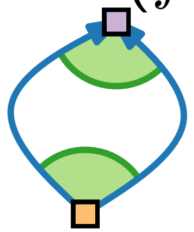
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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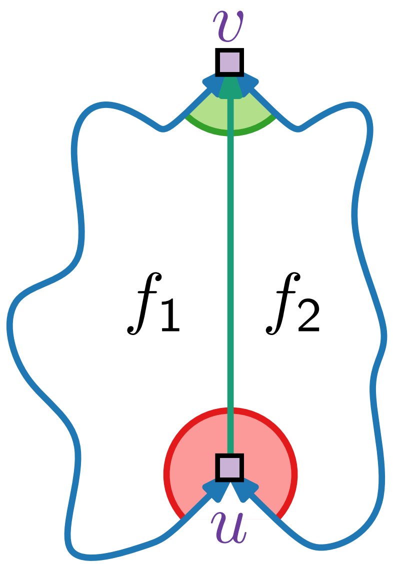

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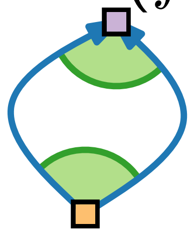
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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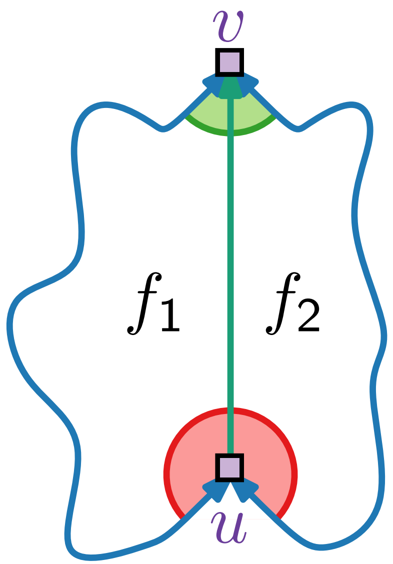

$L(f) - S(f)$

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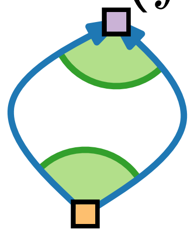
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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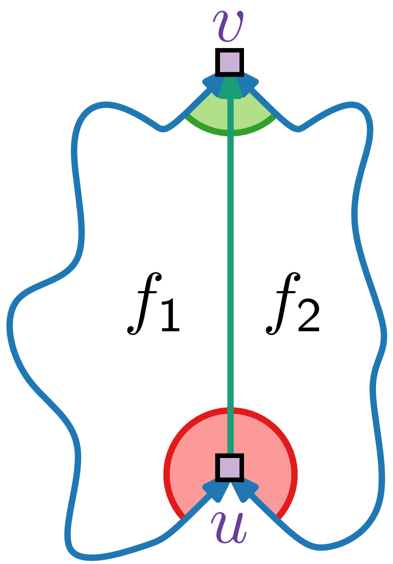

$$L(f) - S(f) = L(f_1) + L(f_2) + 1$$

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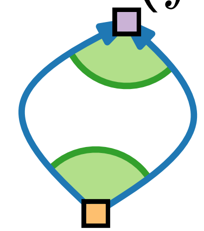

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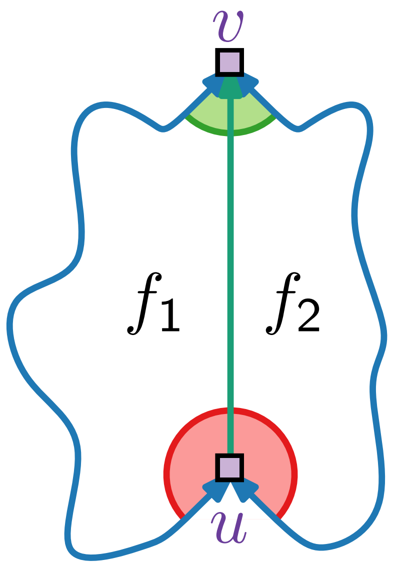

$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

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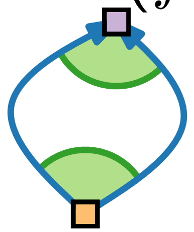

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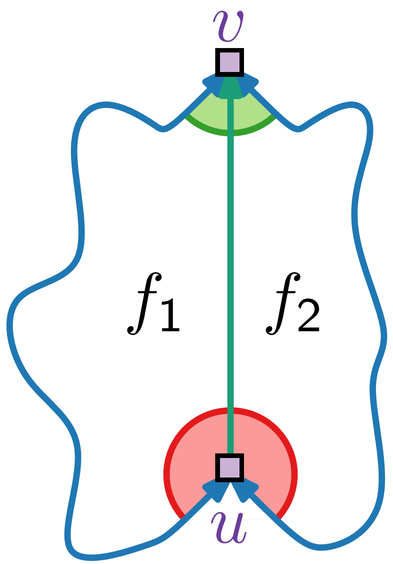

$-2 \qquad -2$

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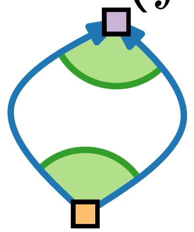

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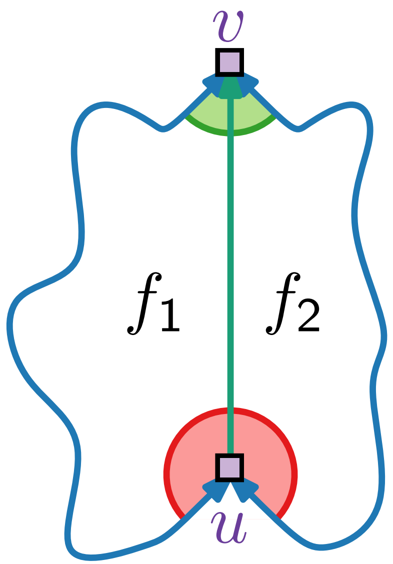

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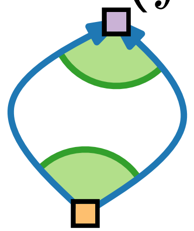

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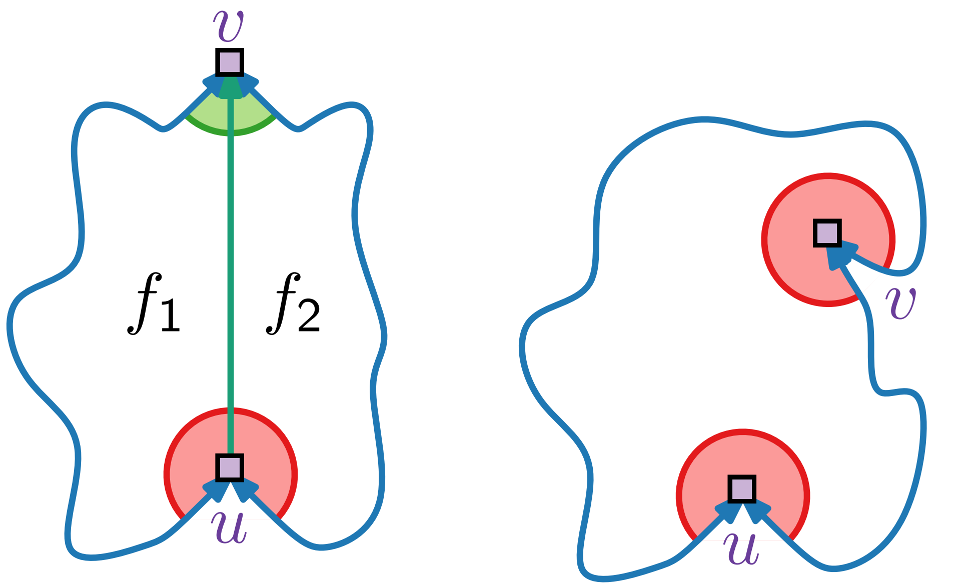

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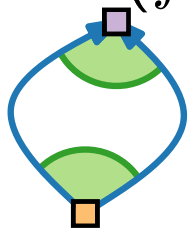
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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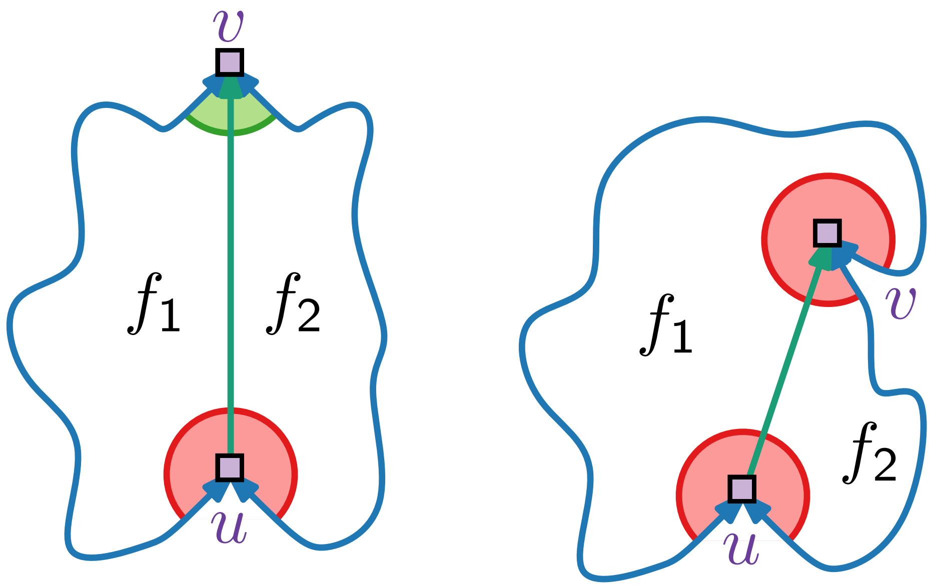

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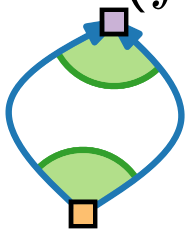

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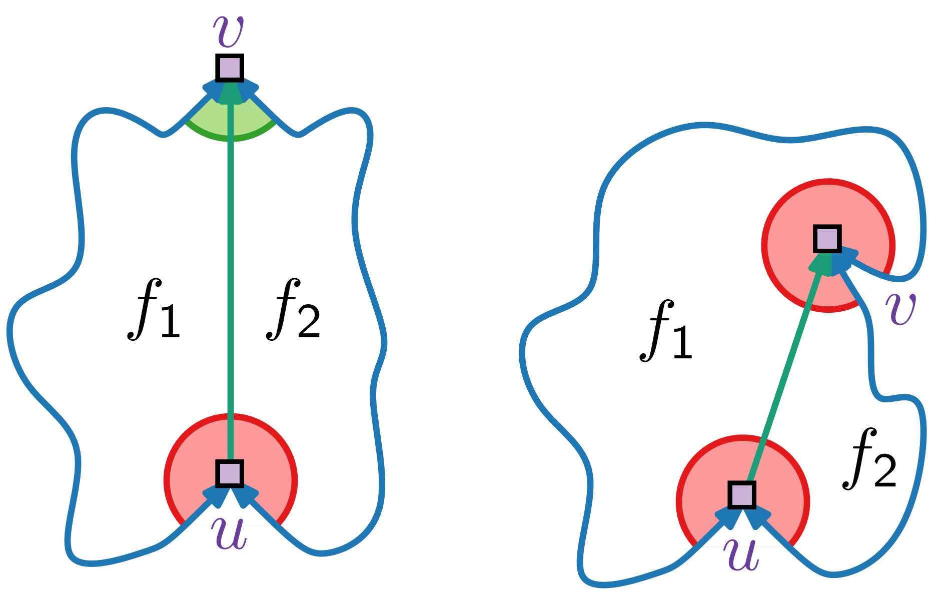

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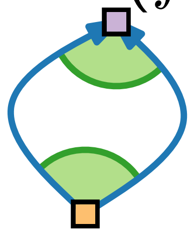
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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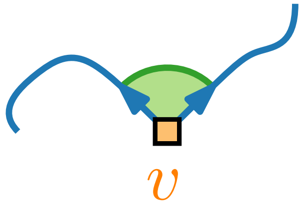

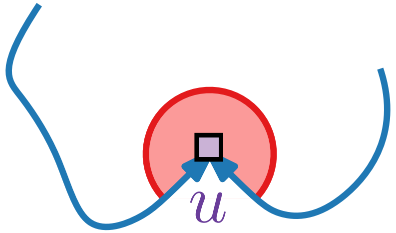

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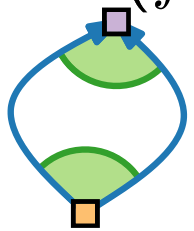
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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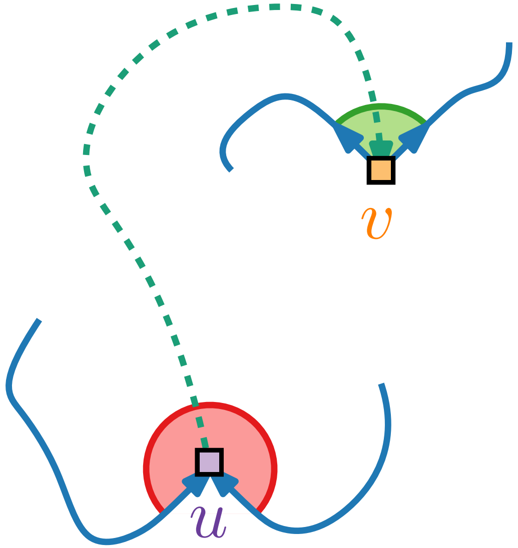

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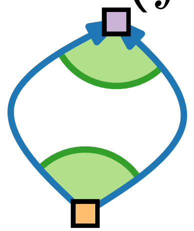
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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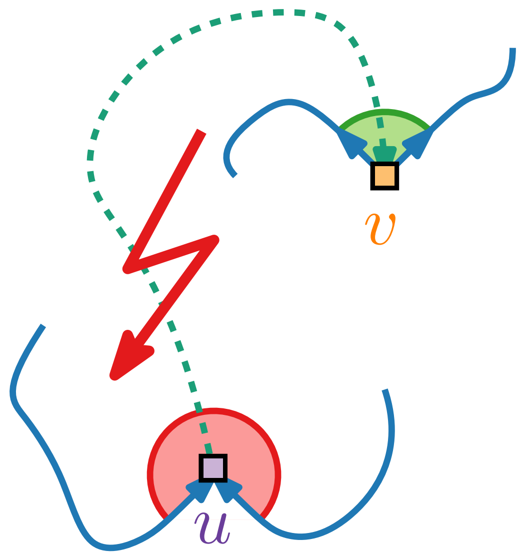

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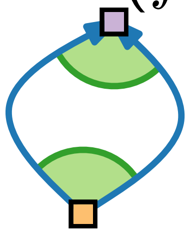
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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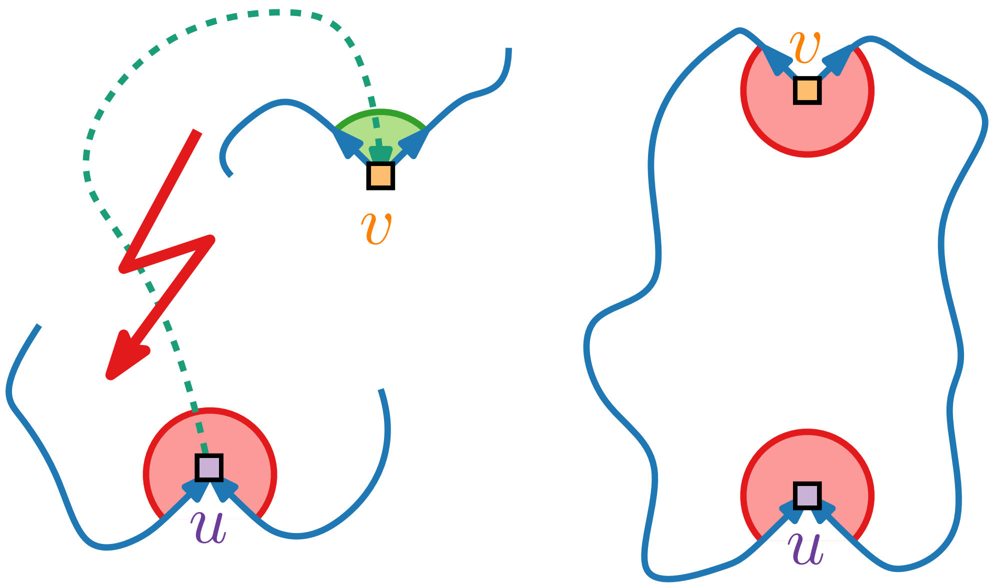

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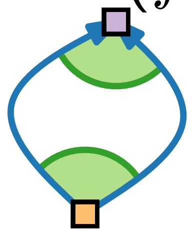
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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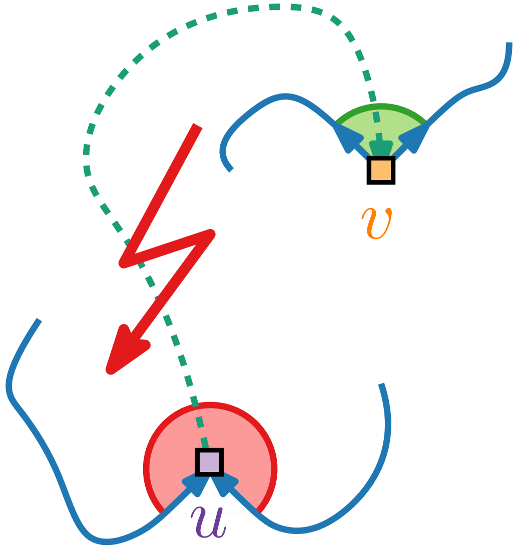

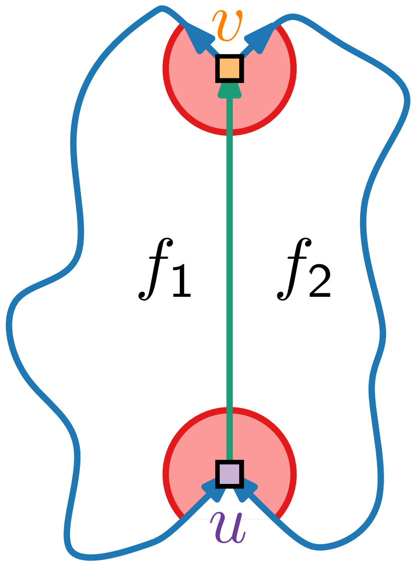

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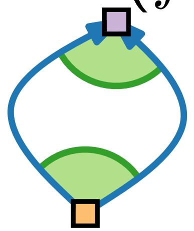
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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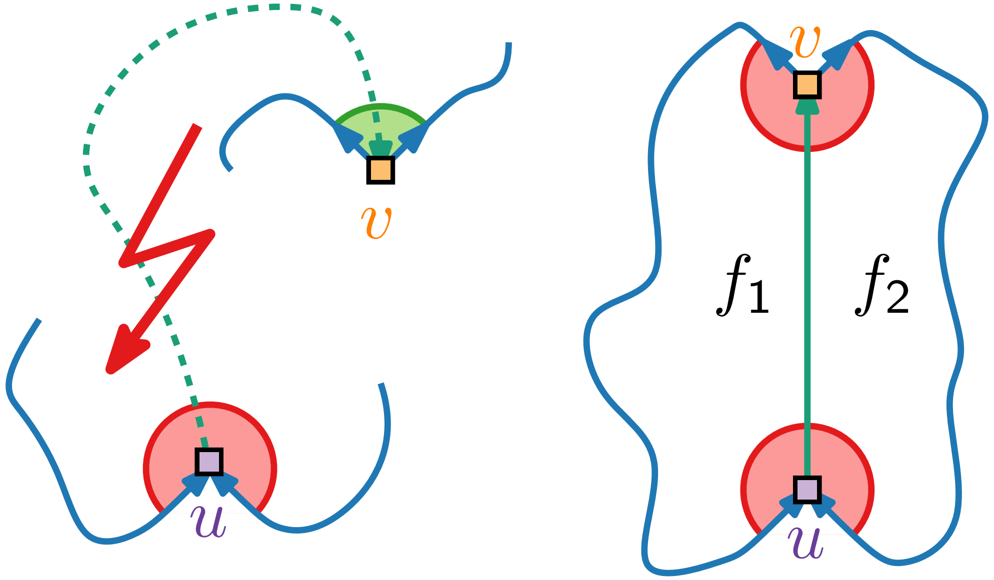

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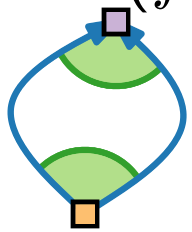
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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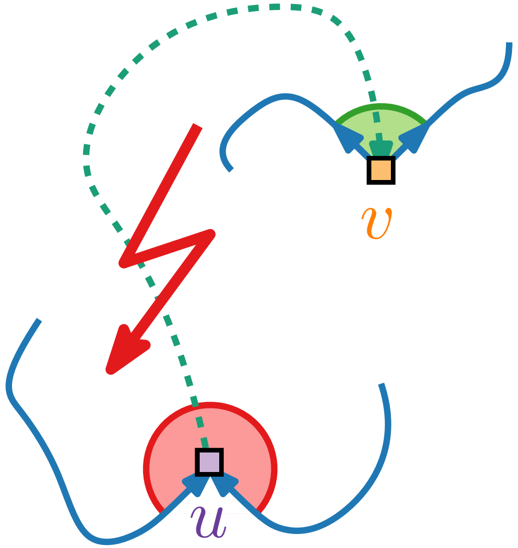

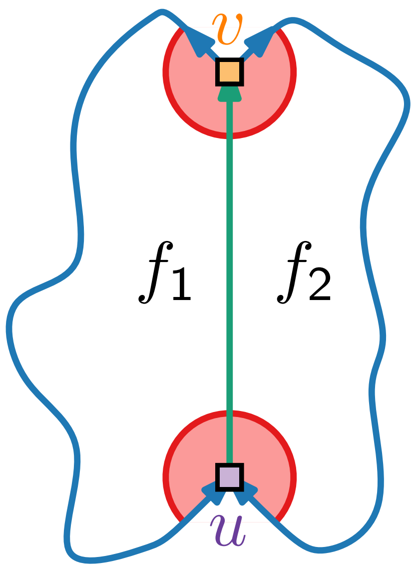

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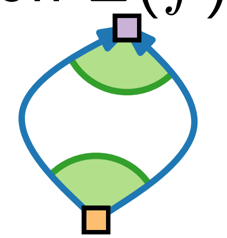

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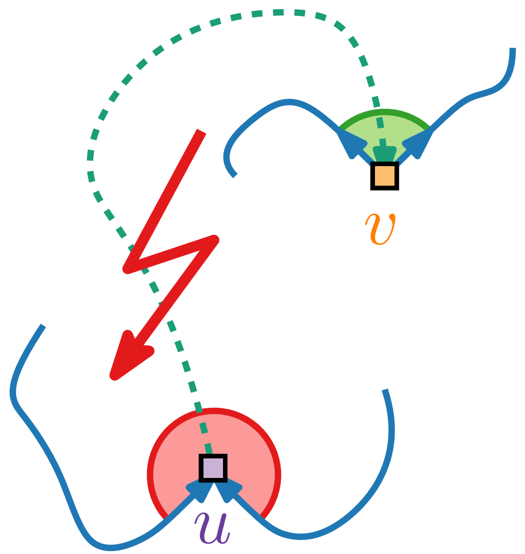

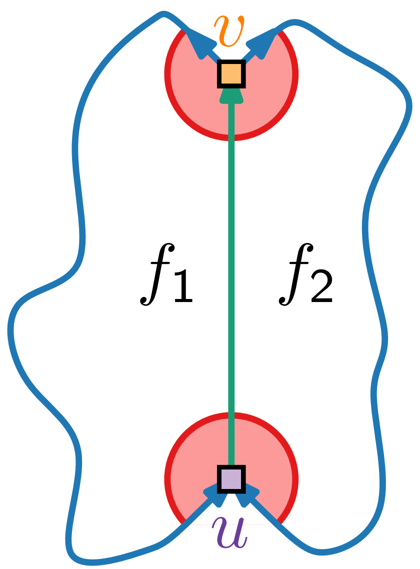

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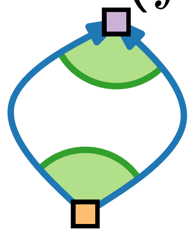
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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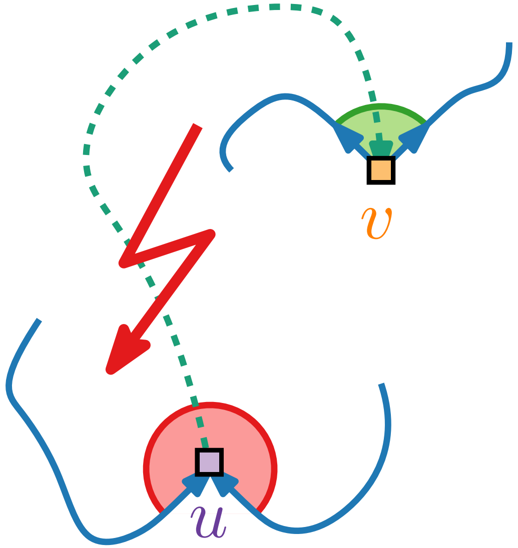

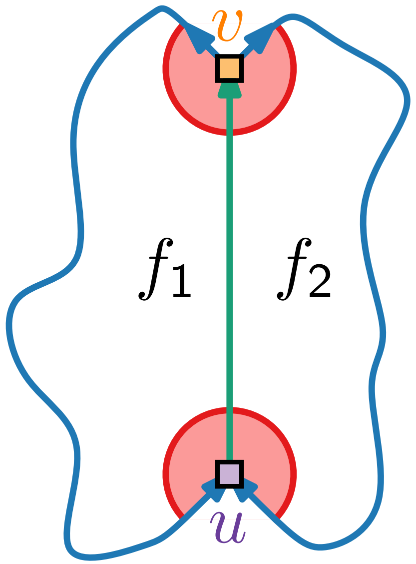

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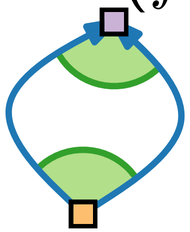
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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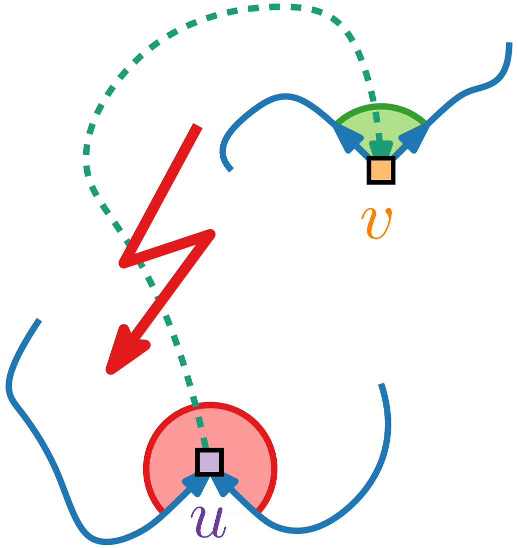

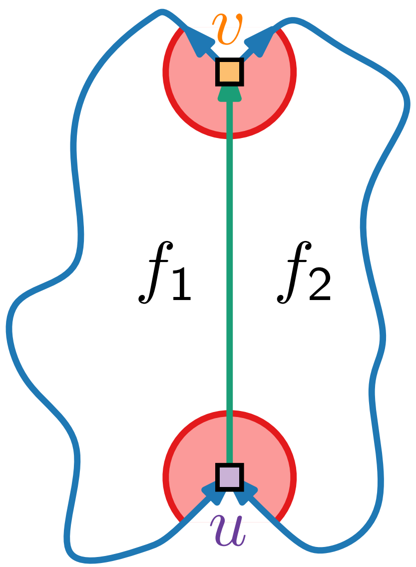

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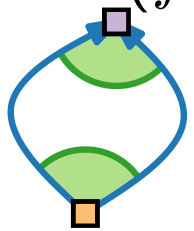

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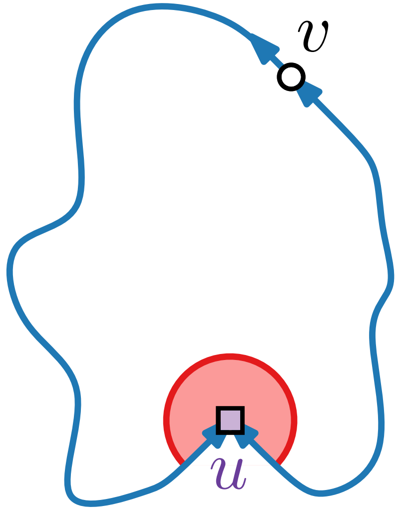

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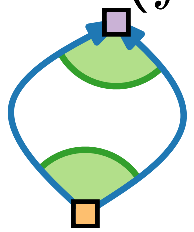
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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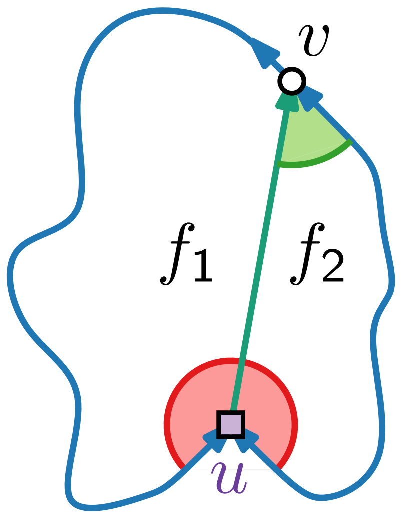

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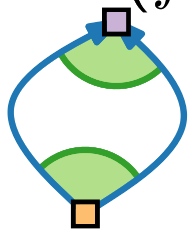
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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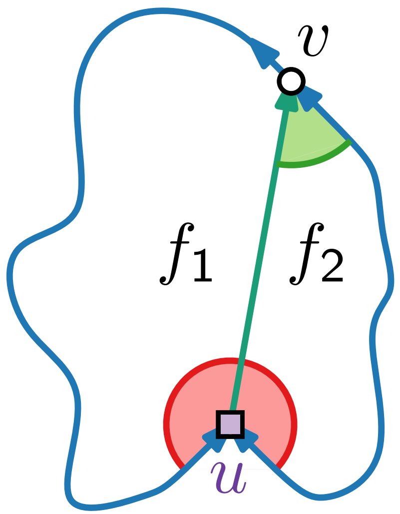

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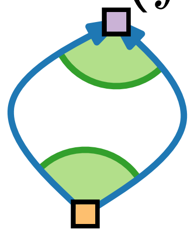

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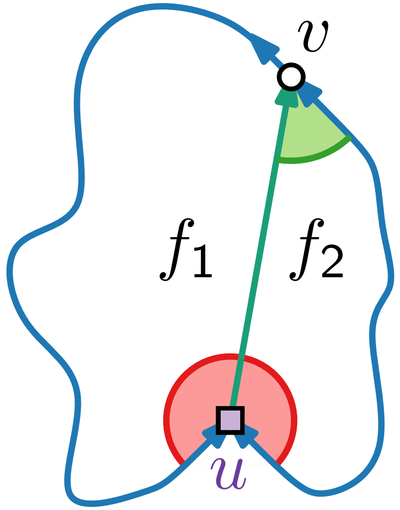

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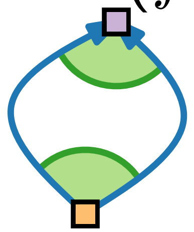
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓
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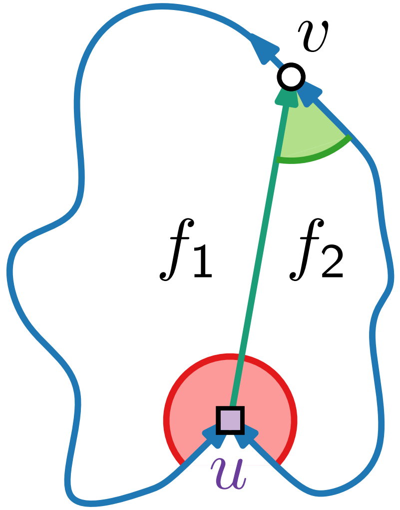

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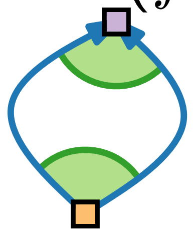

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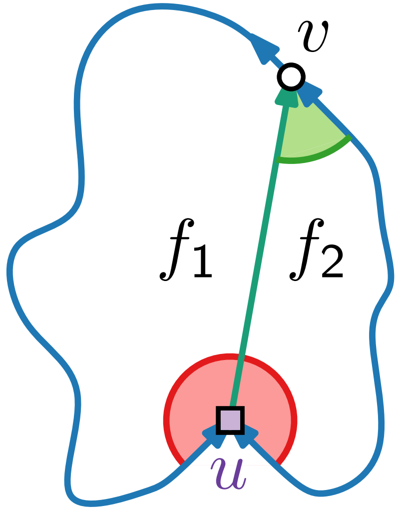

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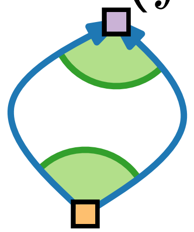
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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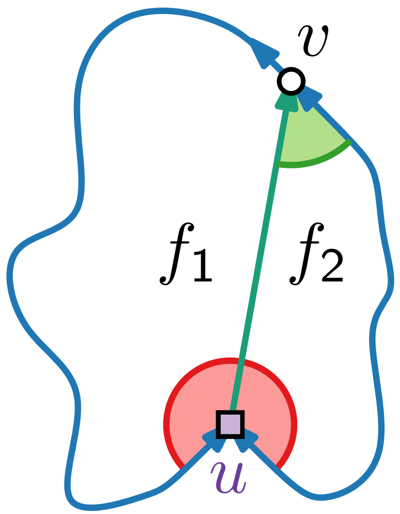

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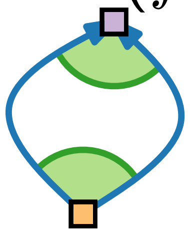

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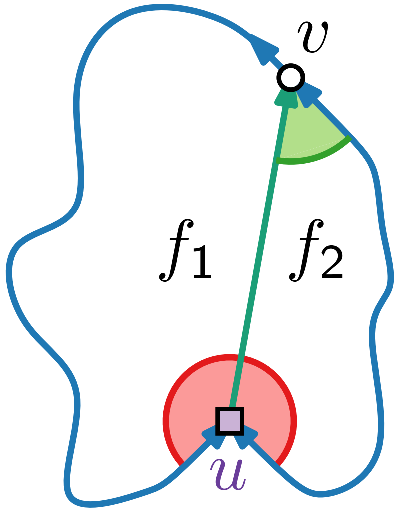

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- Otherwise "high" source u exists.

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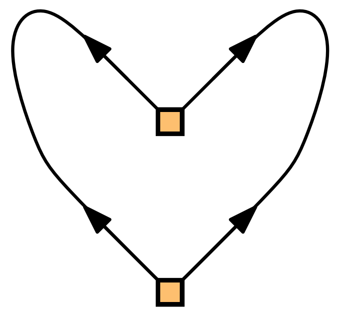

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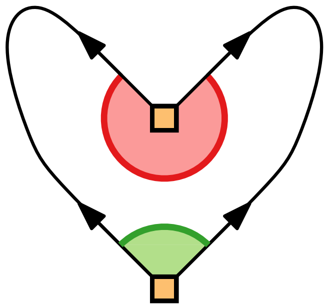

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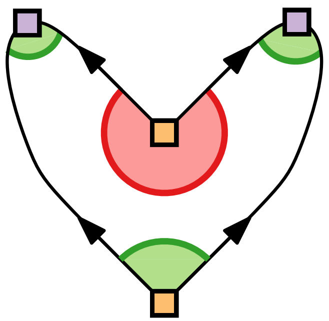

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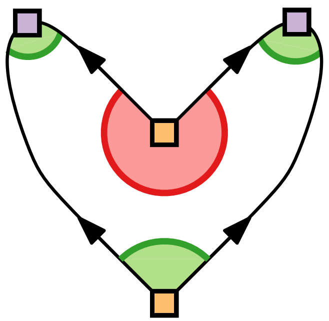

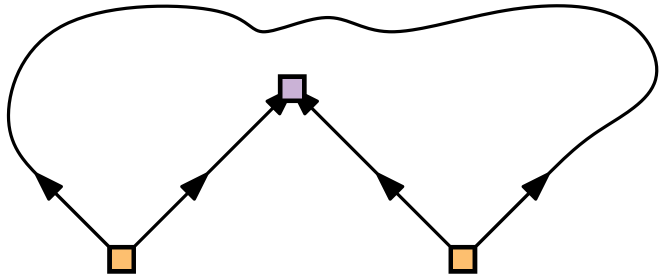

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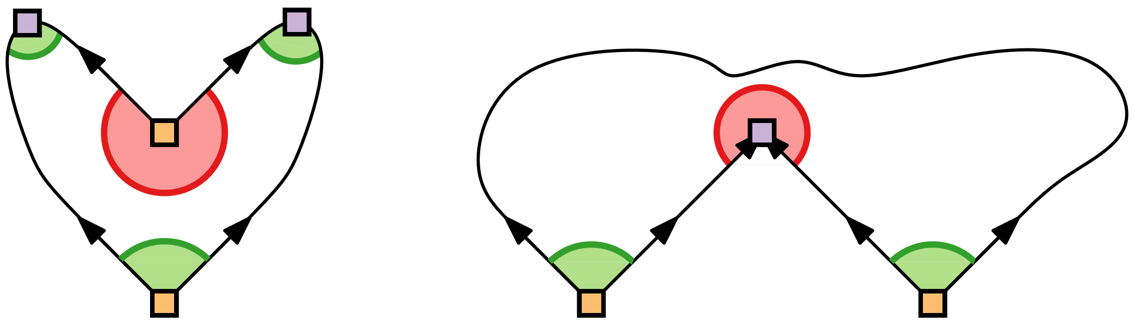

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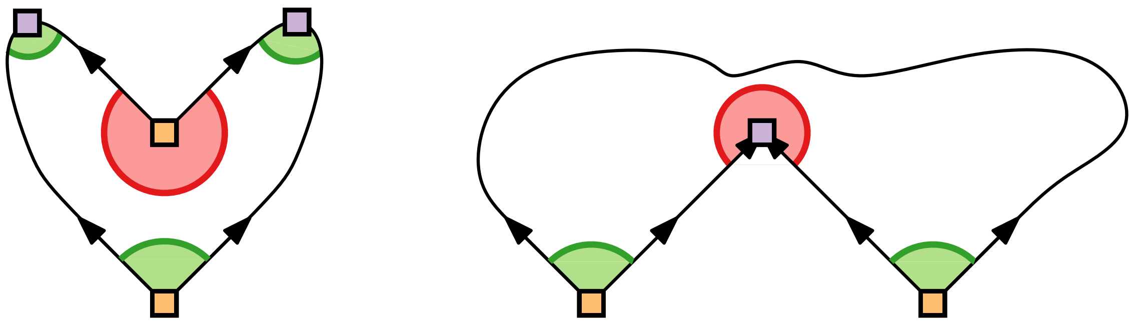

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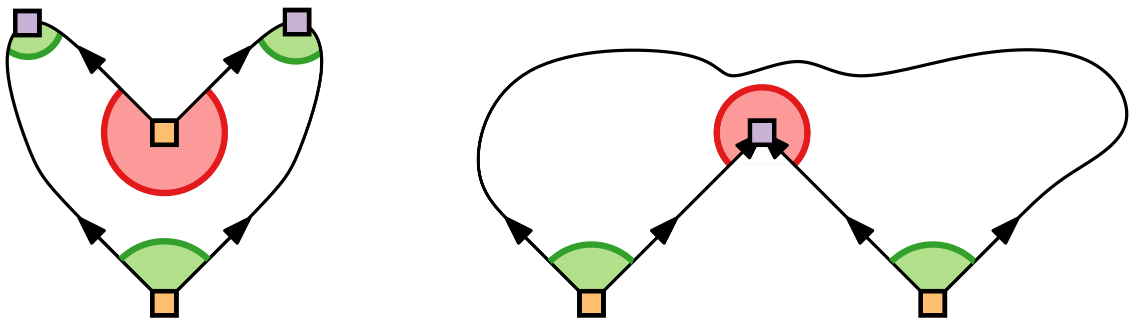

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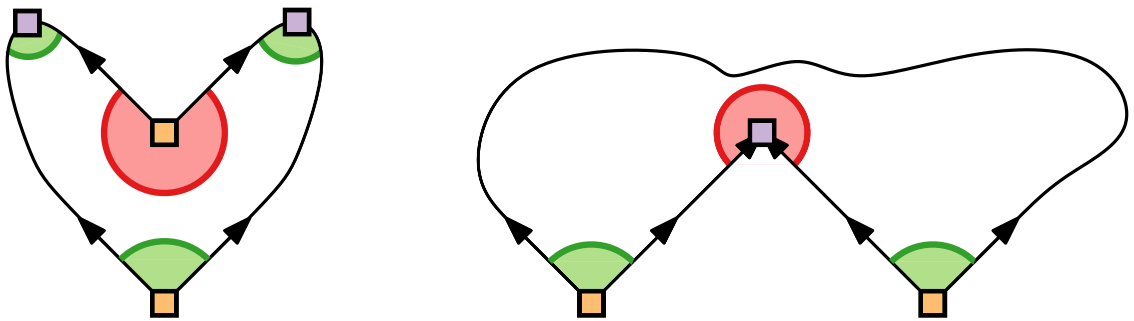

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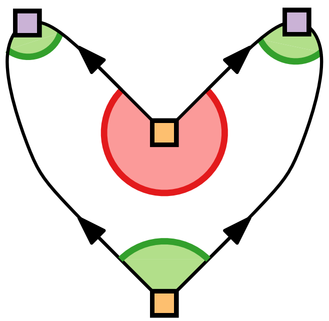

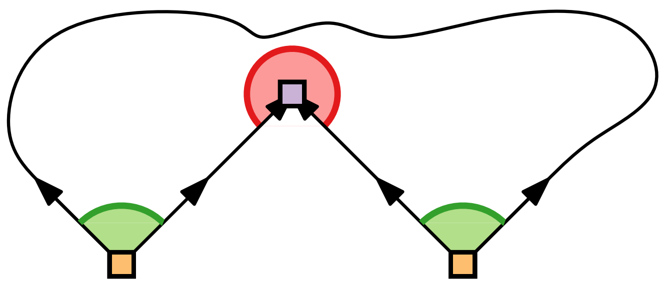

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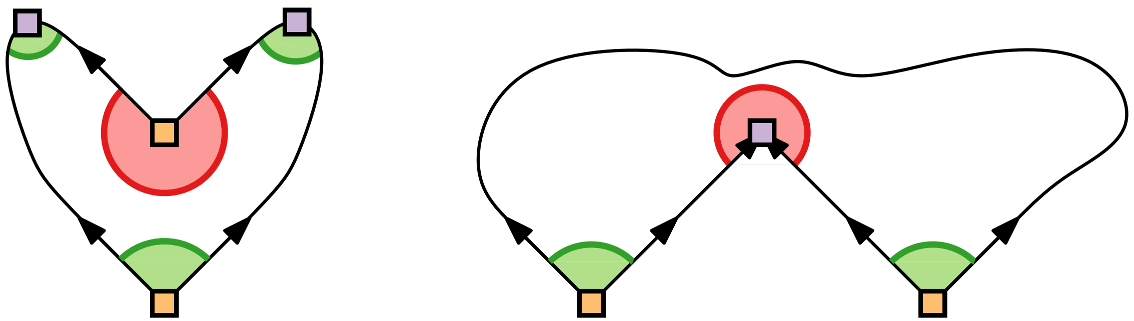

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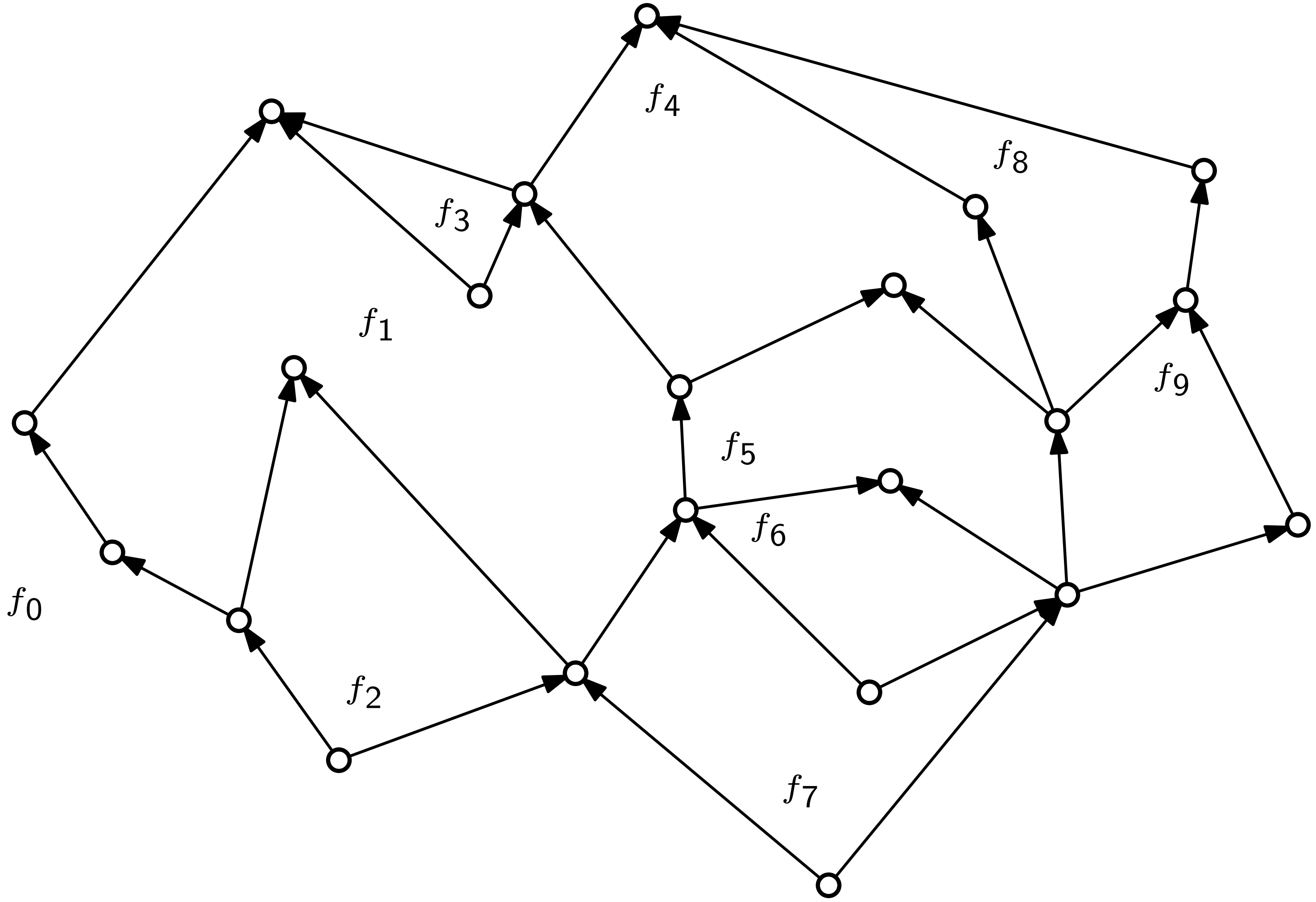

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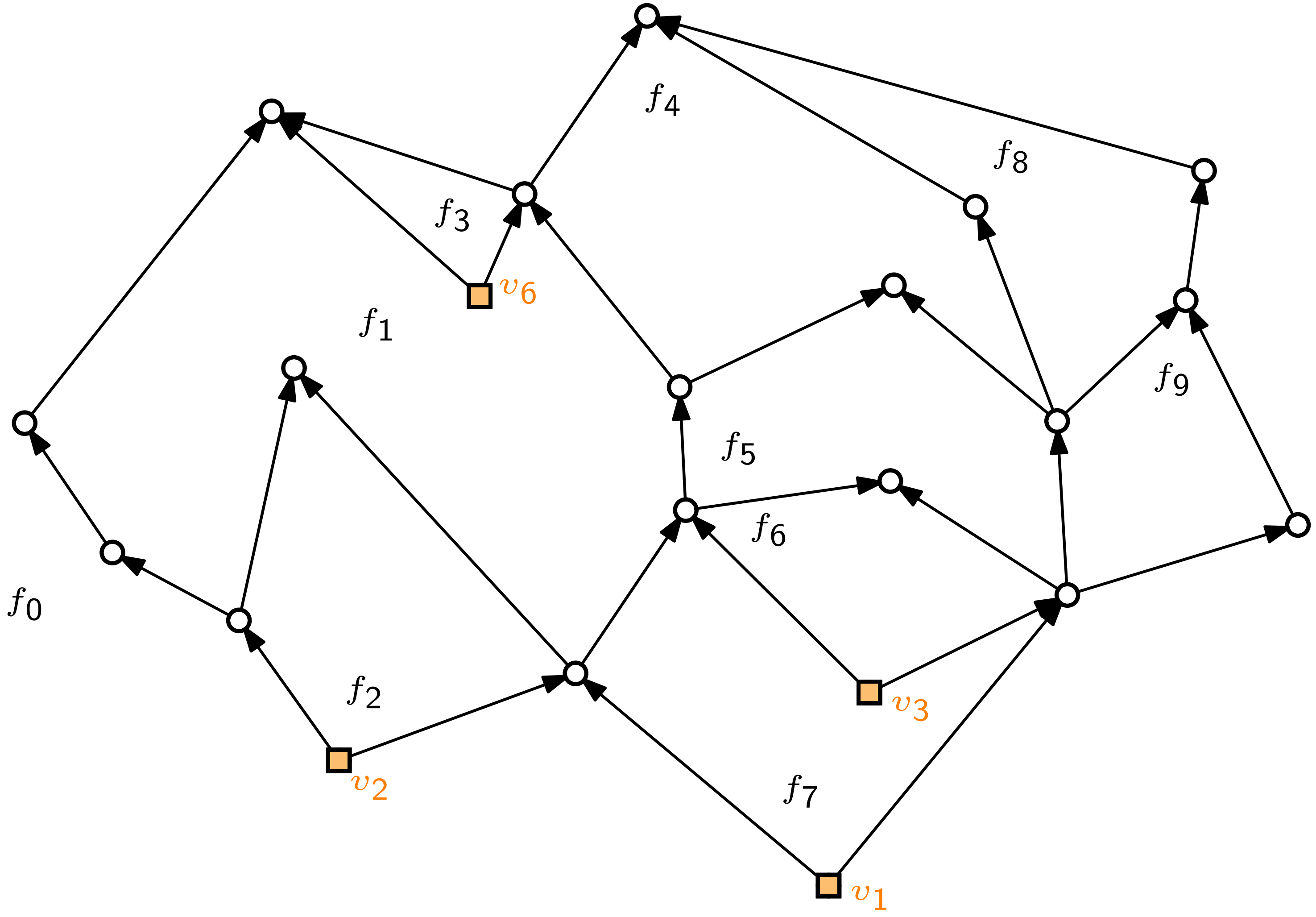

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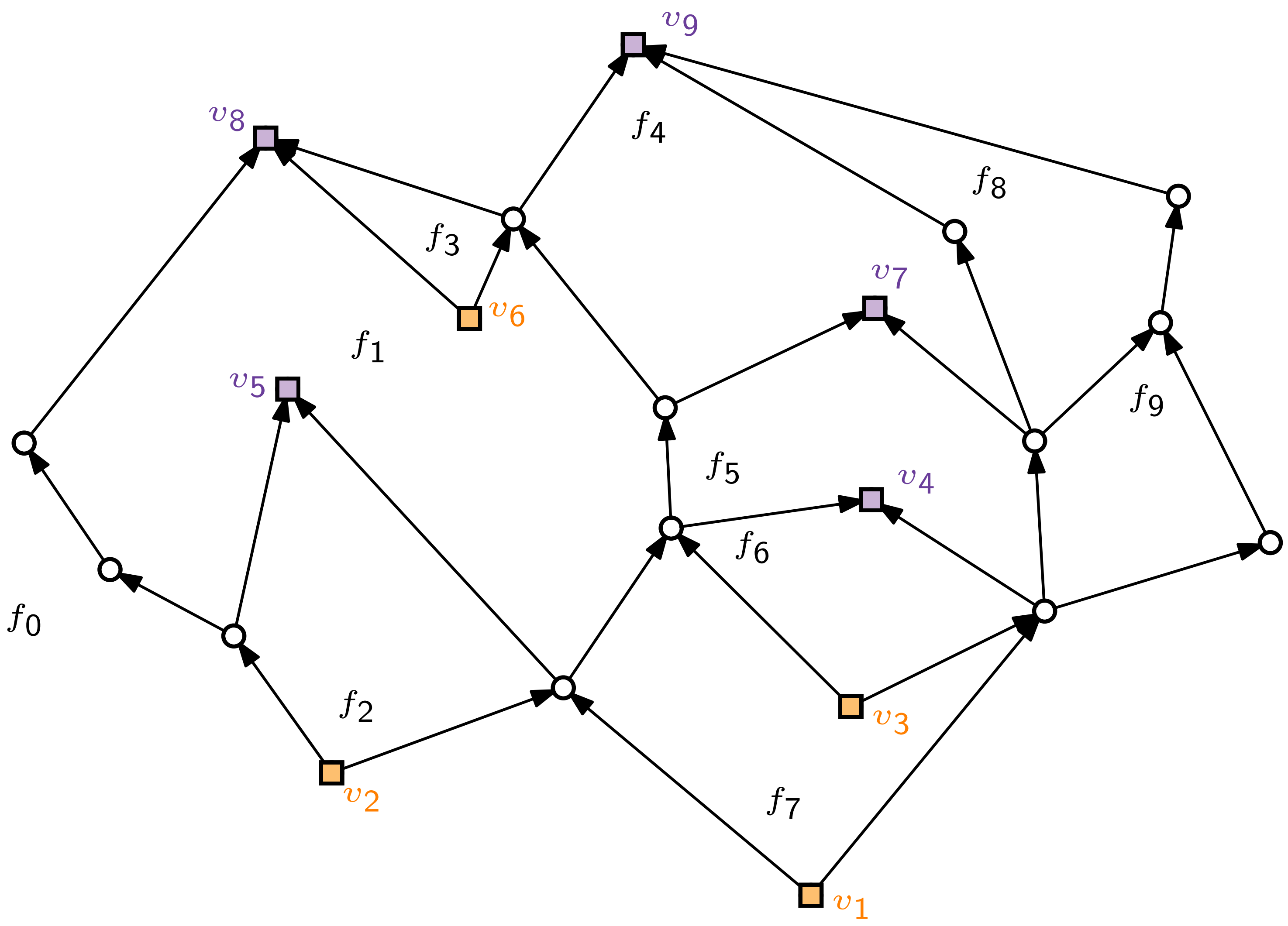

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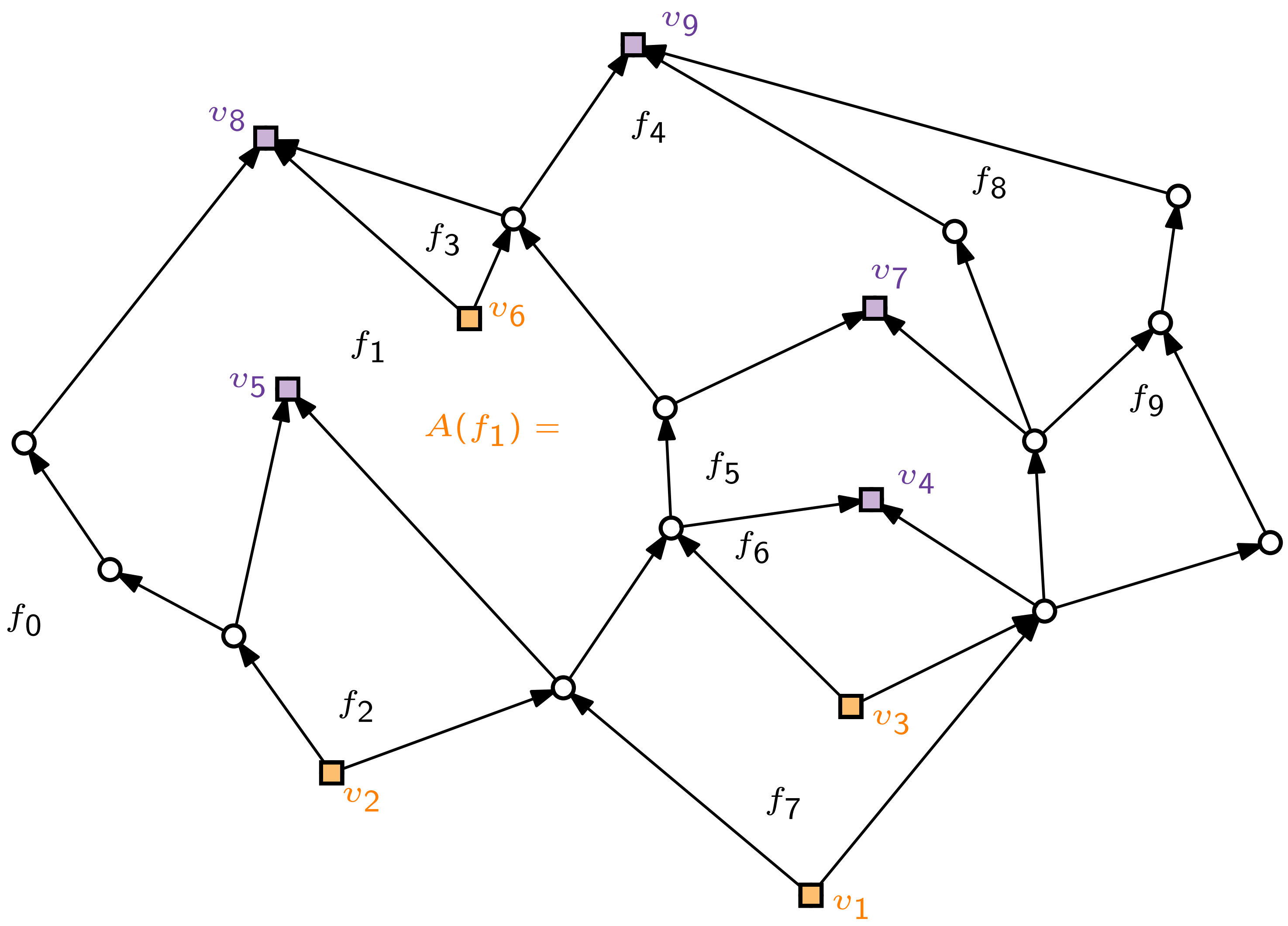

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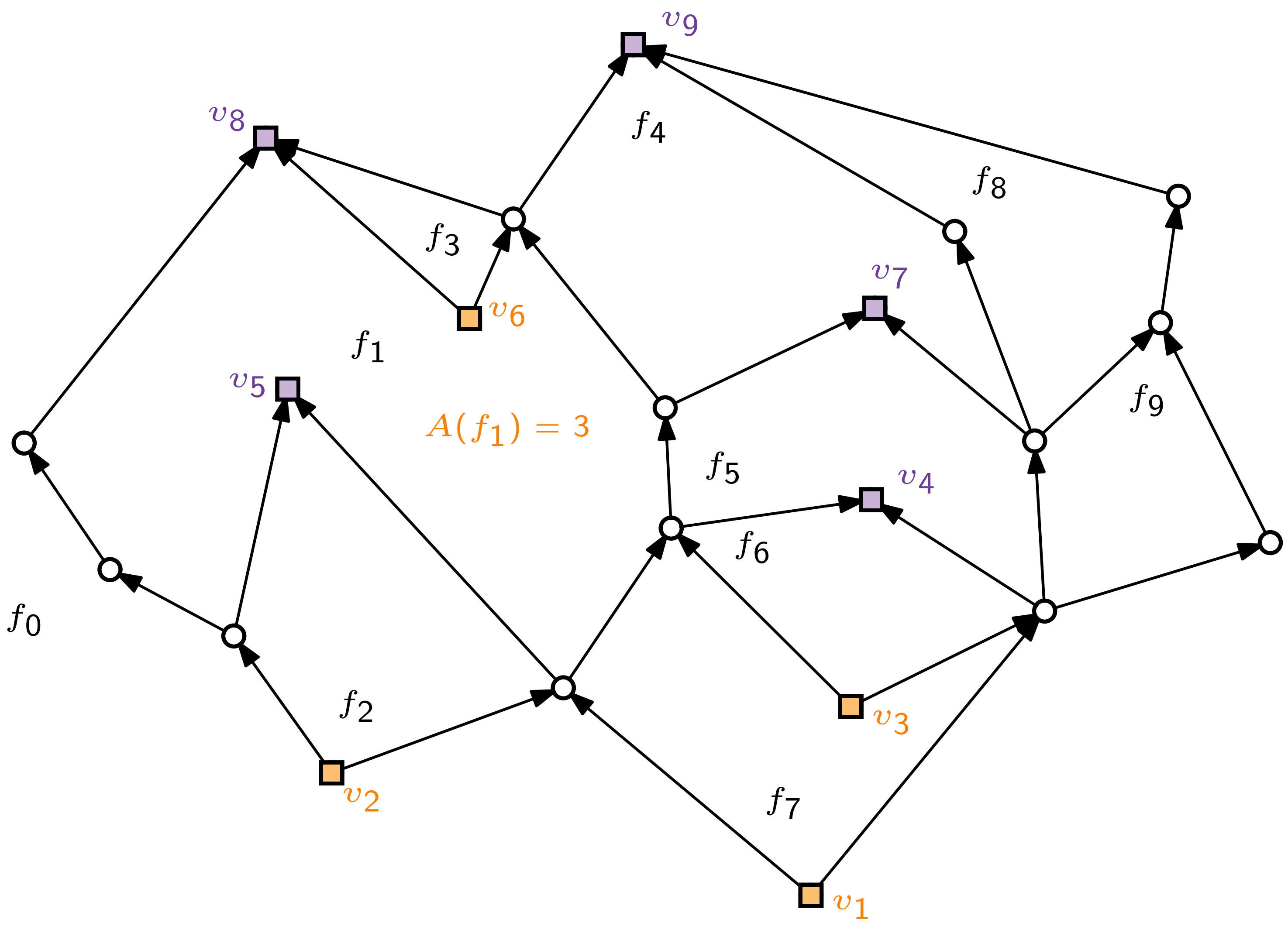

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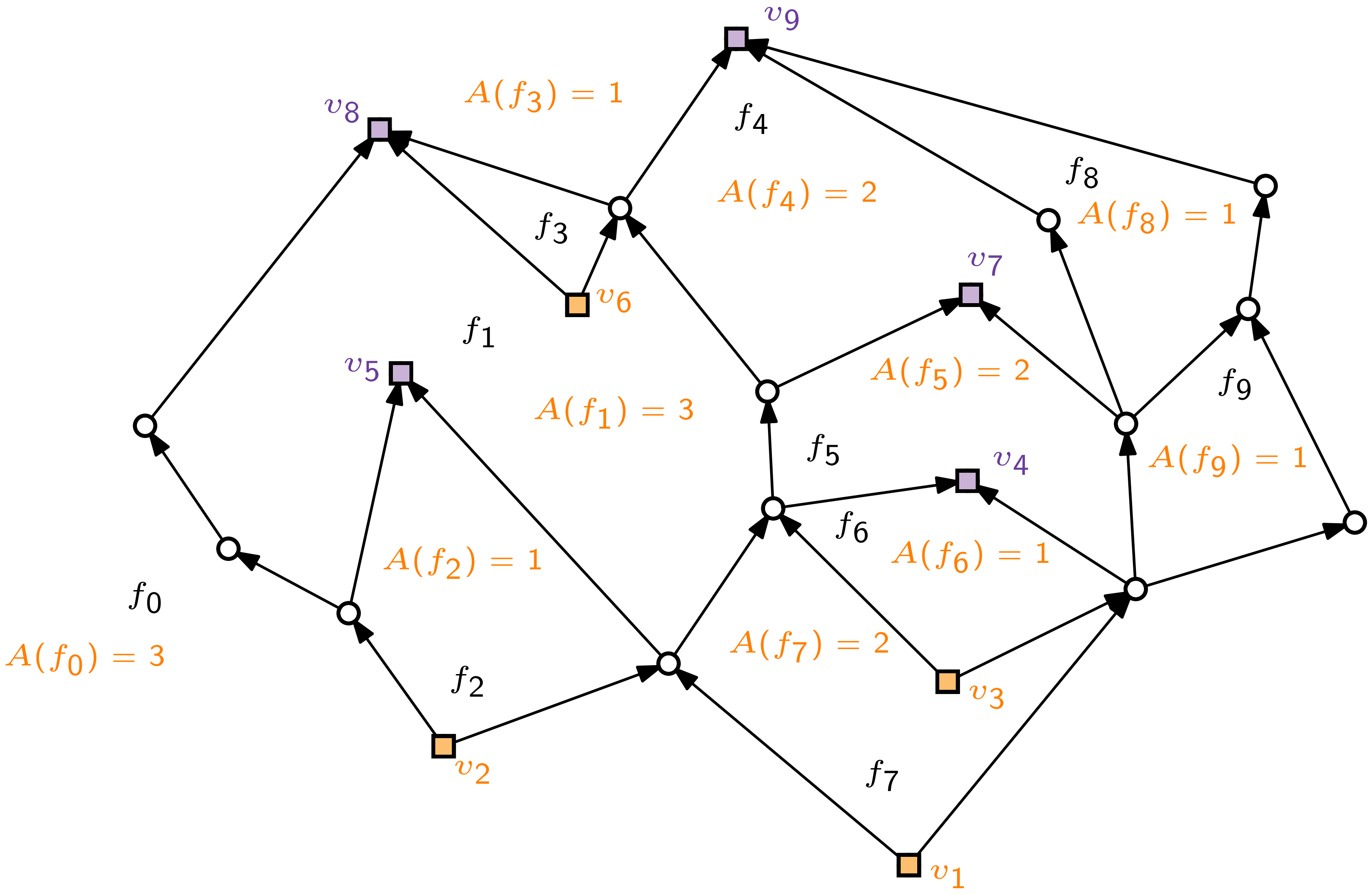

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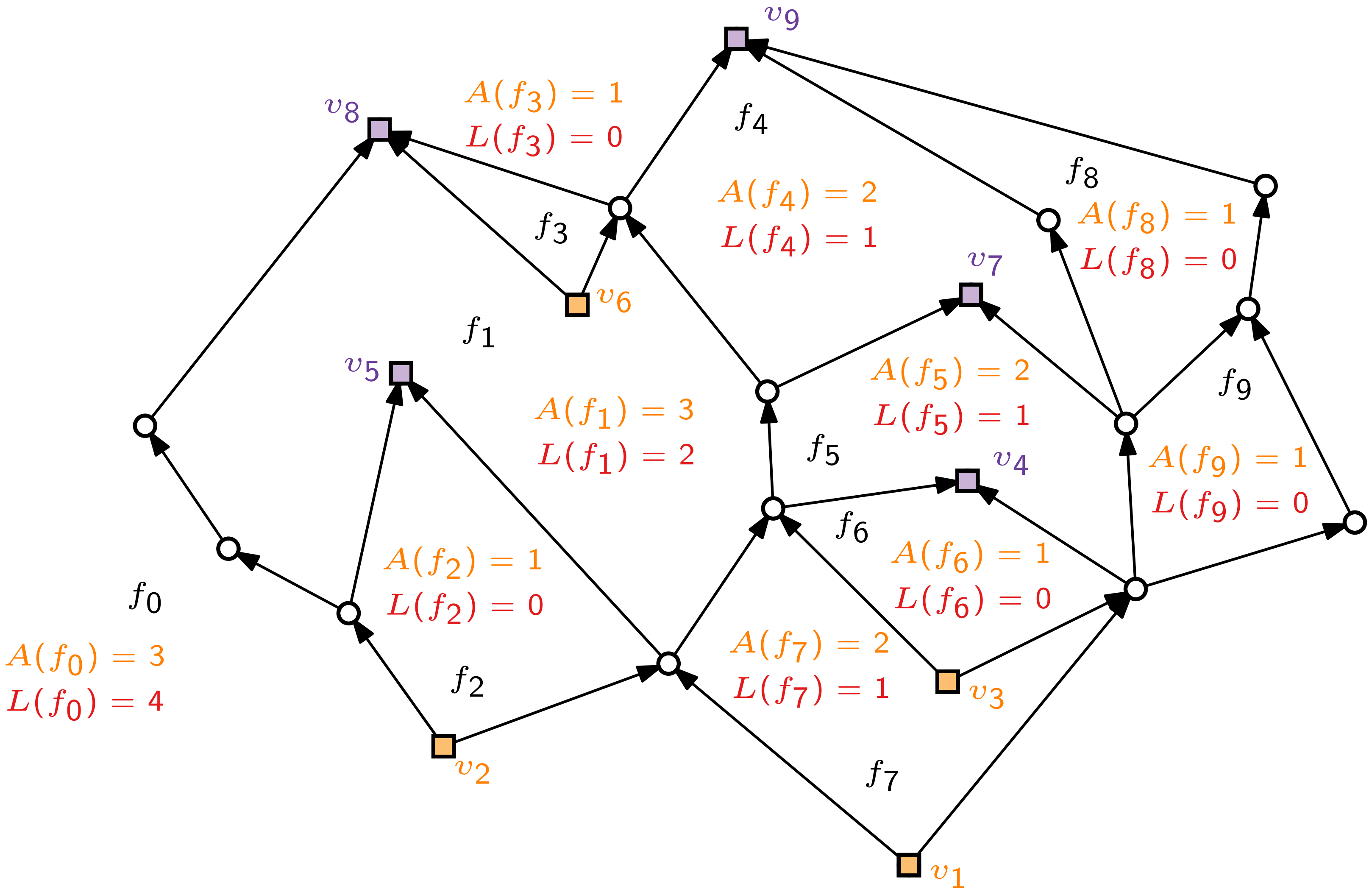

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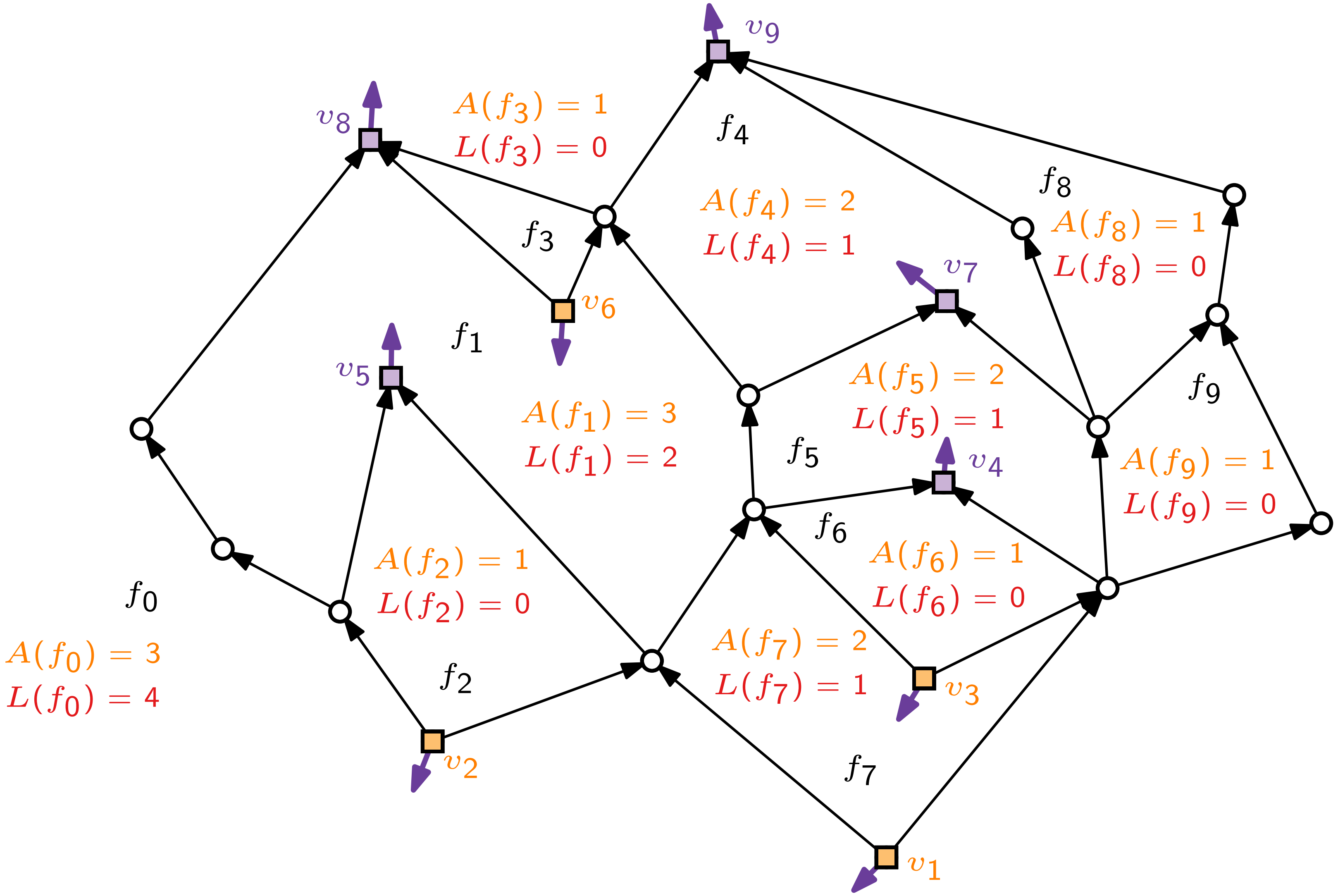

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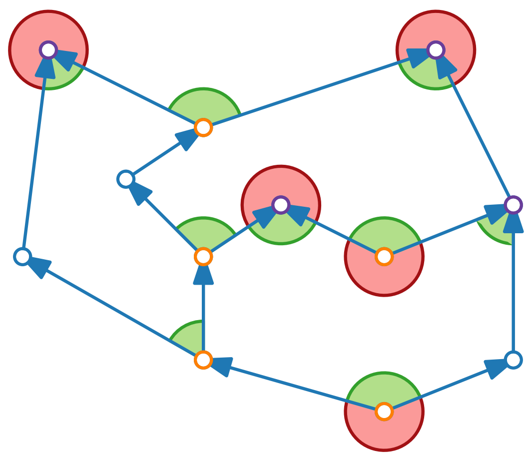
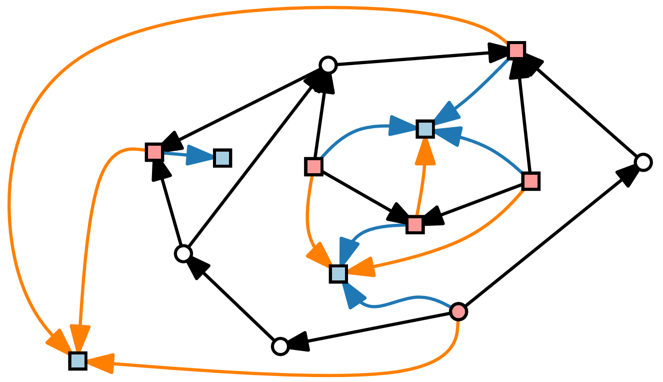
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part IV:
Refinement Algorithm



Alexander Wolff

Result Characterization

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- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

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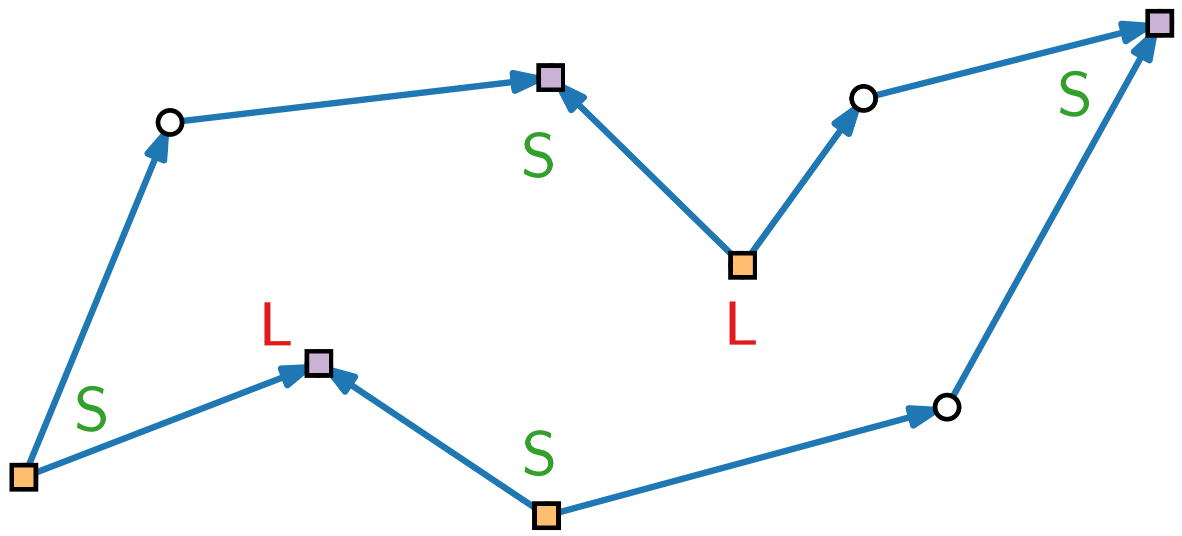

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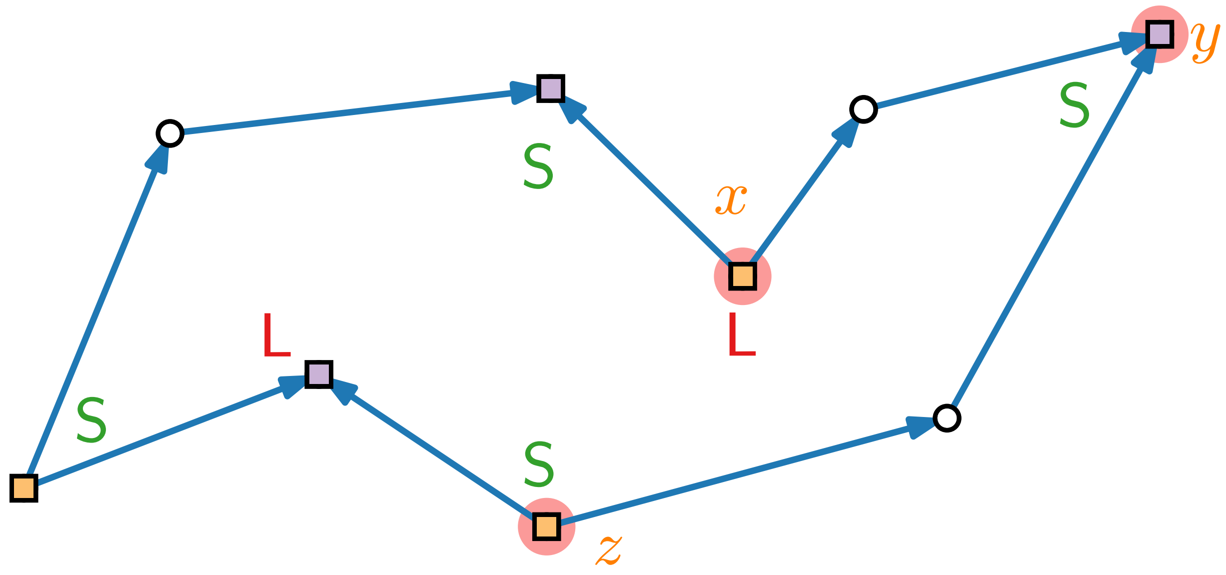

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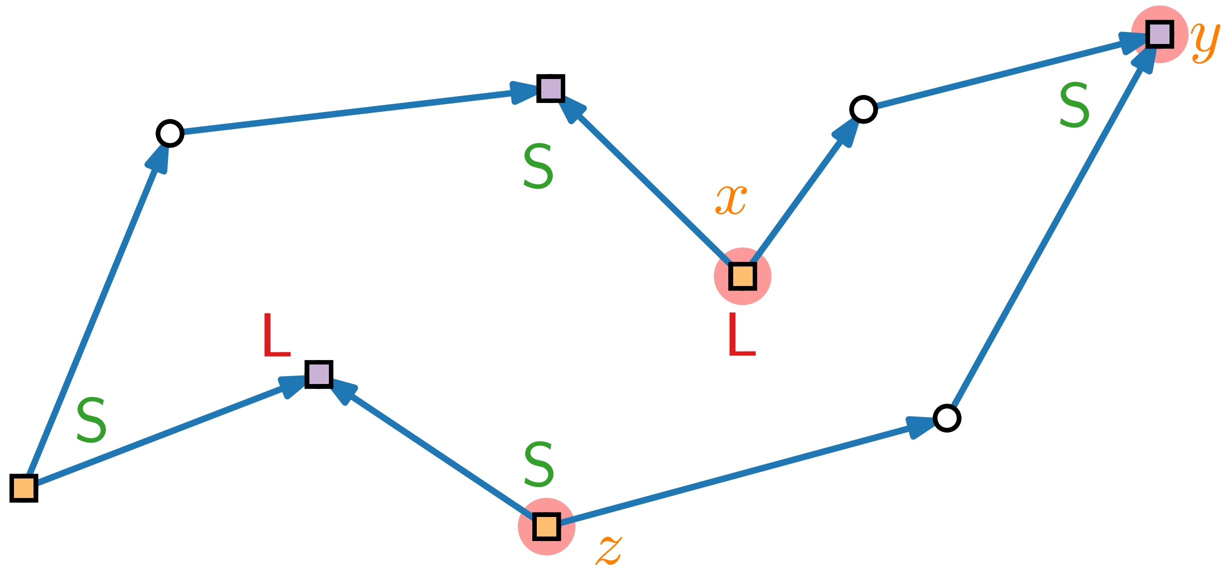

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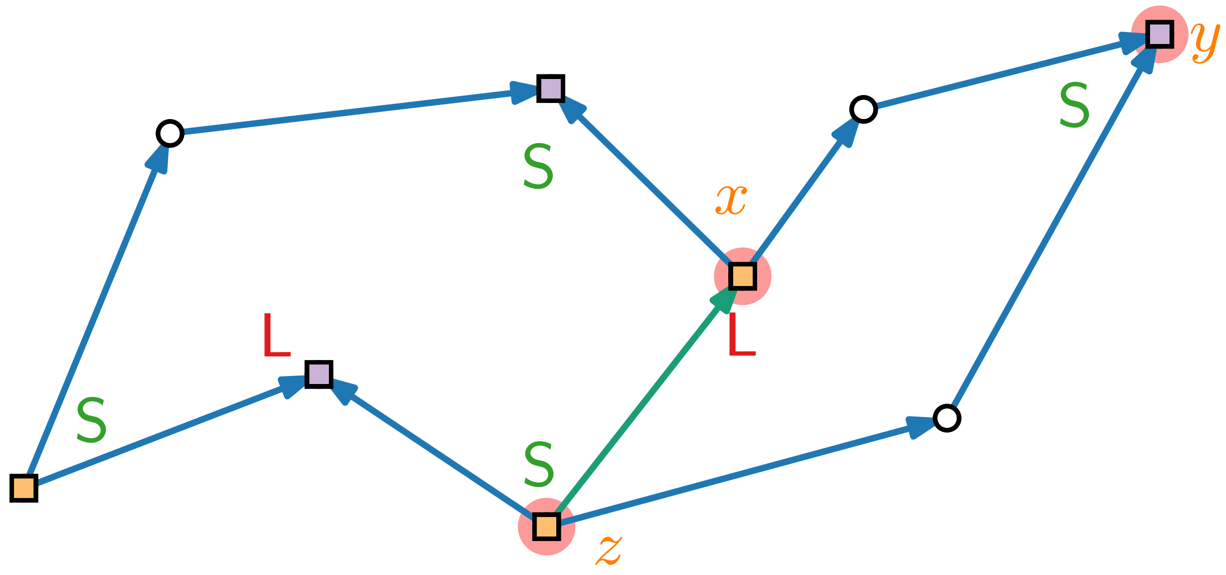

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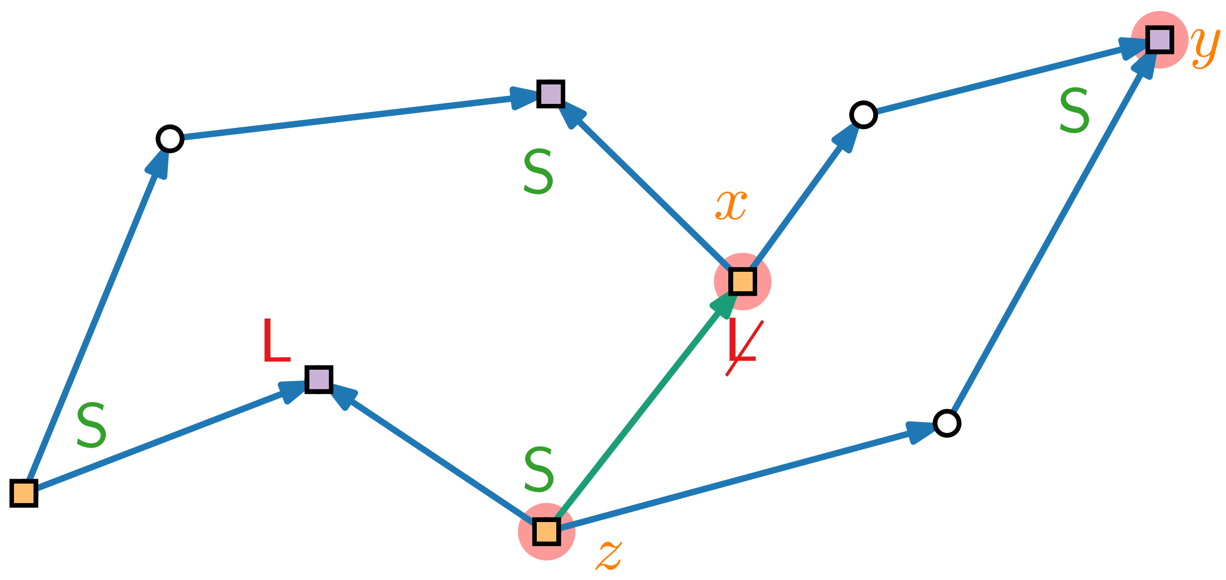

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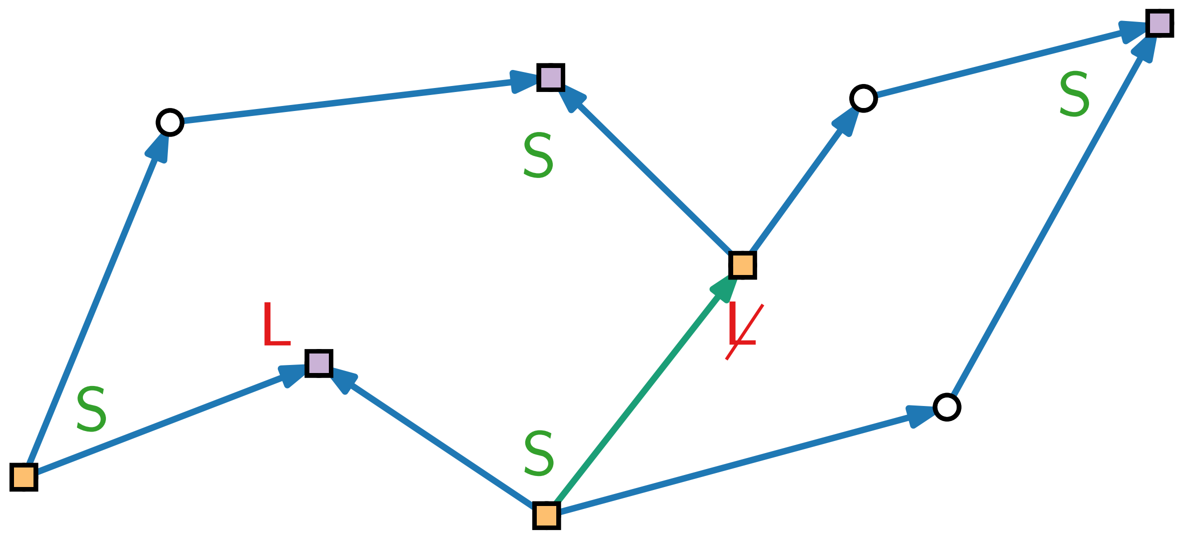

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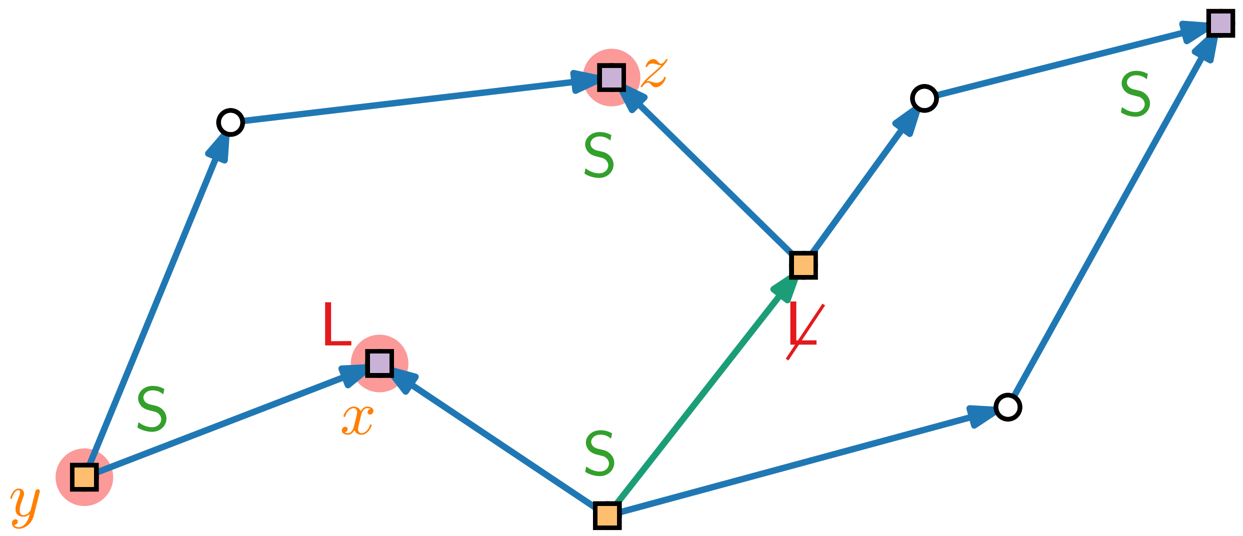

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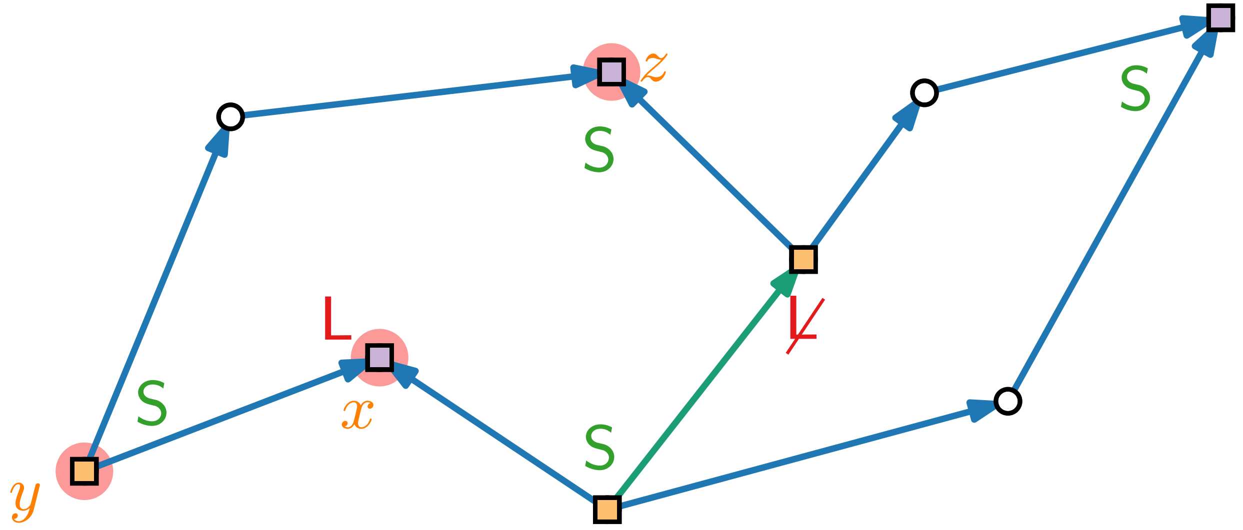

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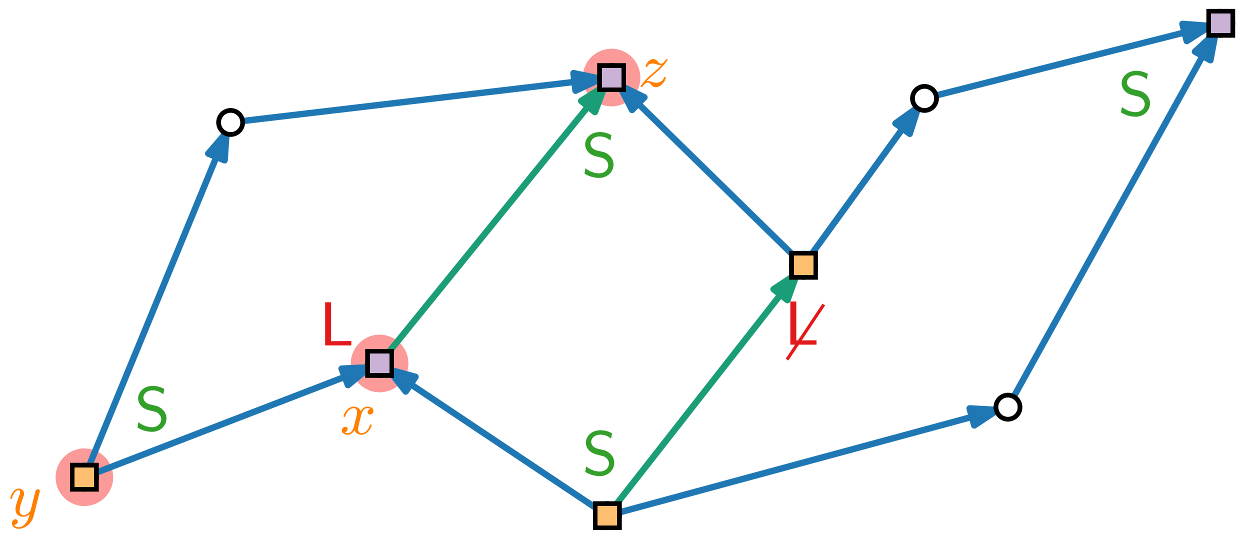

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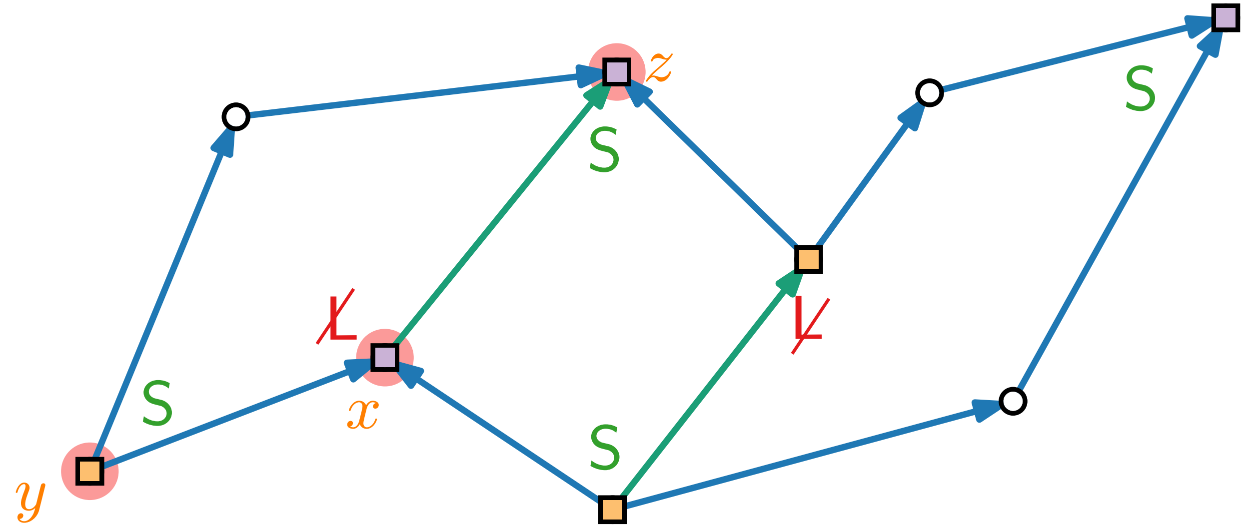

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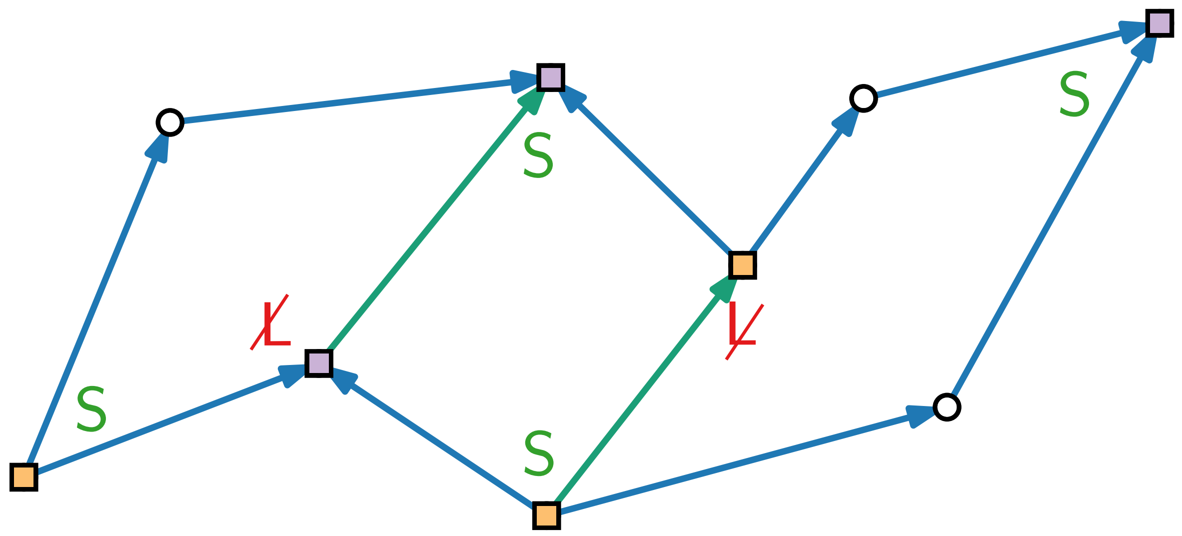

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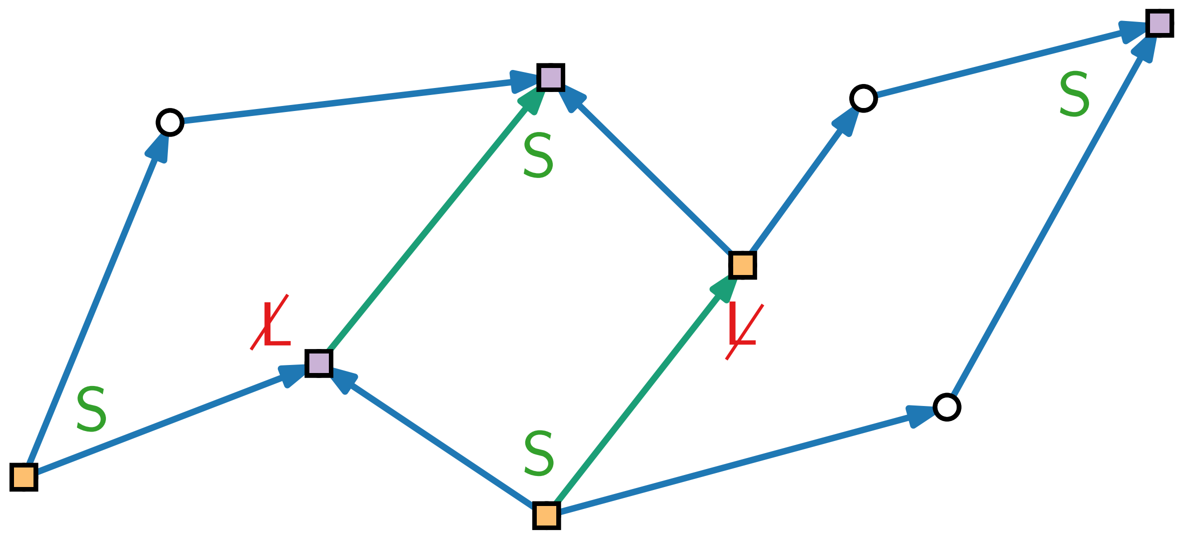

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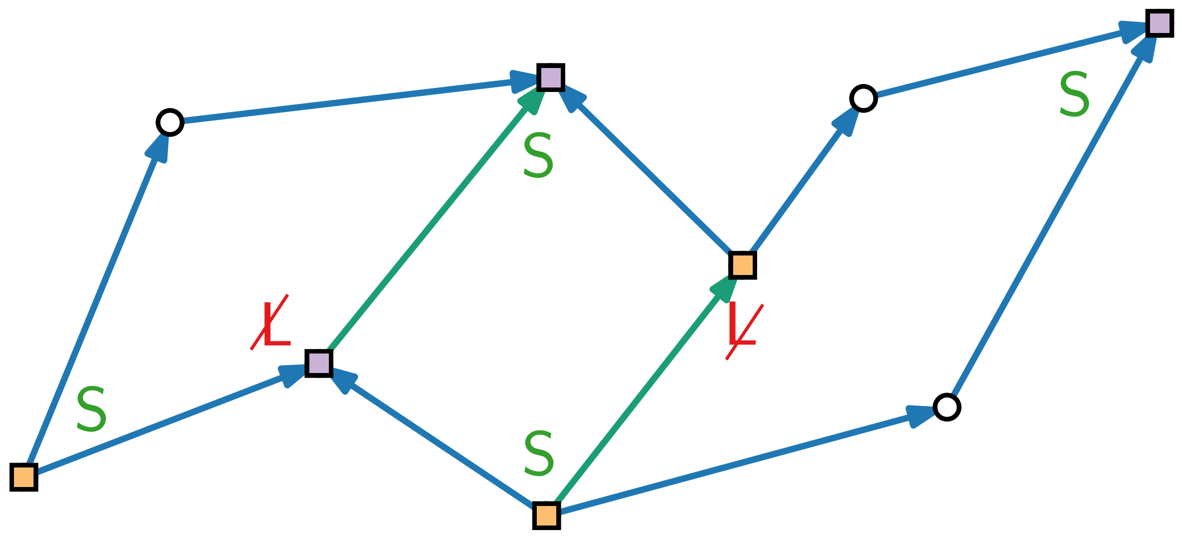

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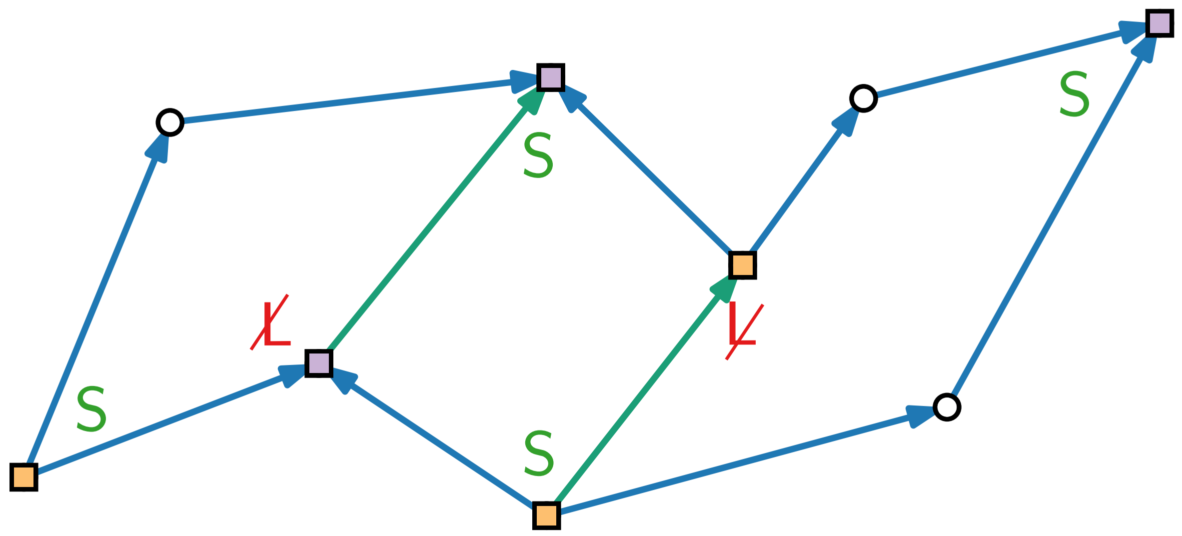

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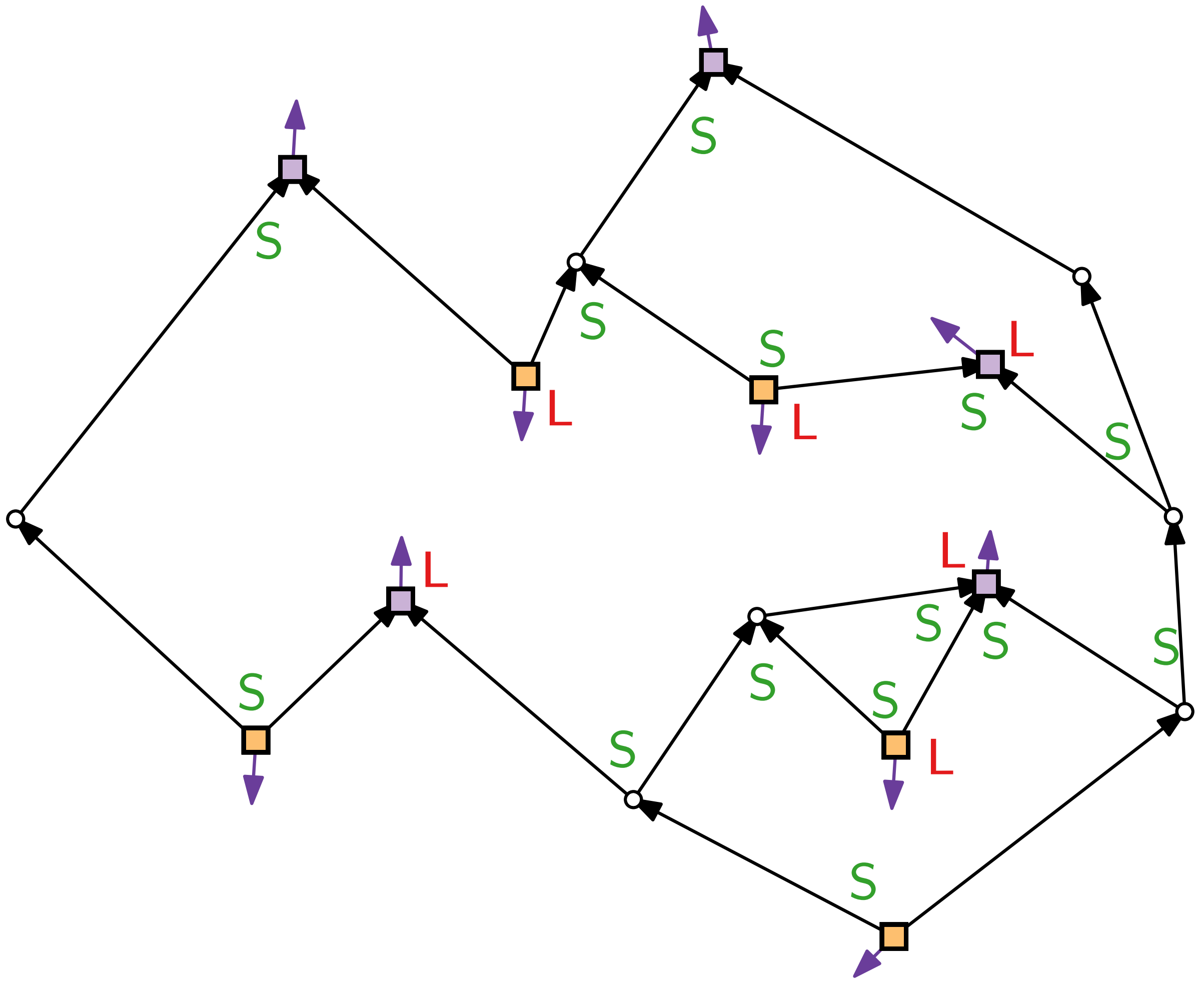
- Planarity, acyclicity, bimodality are invariants under construction.

Refinement Example



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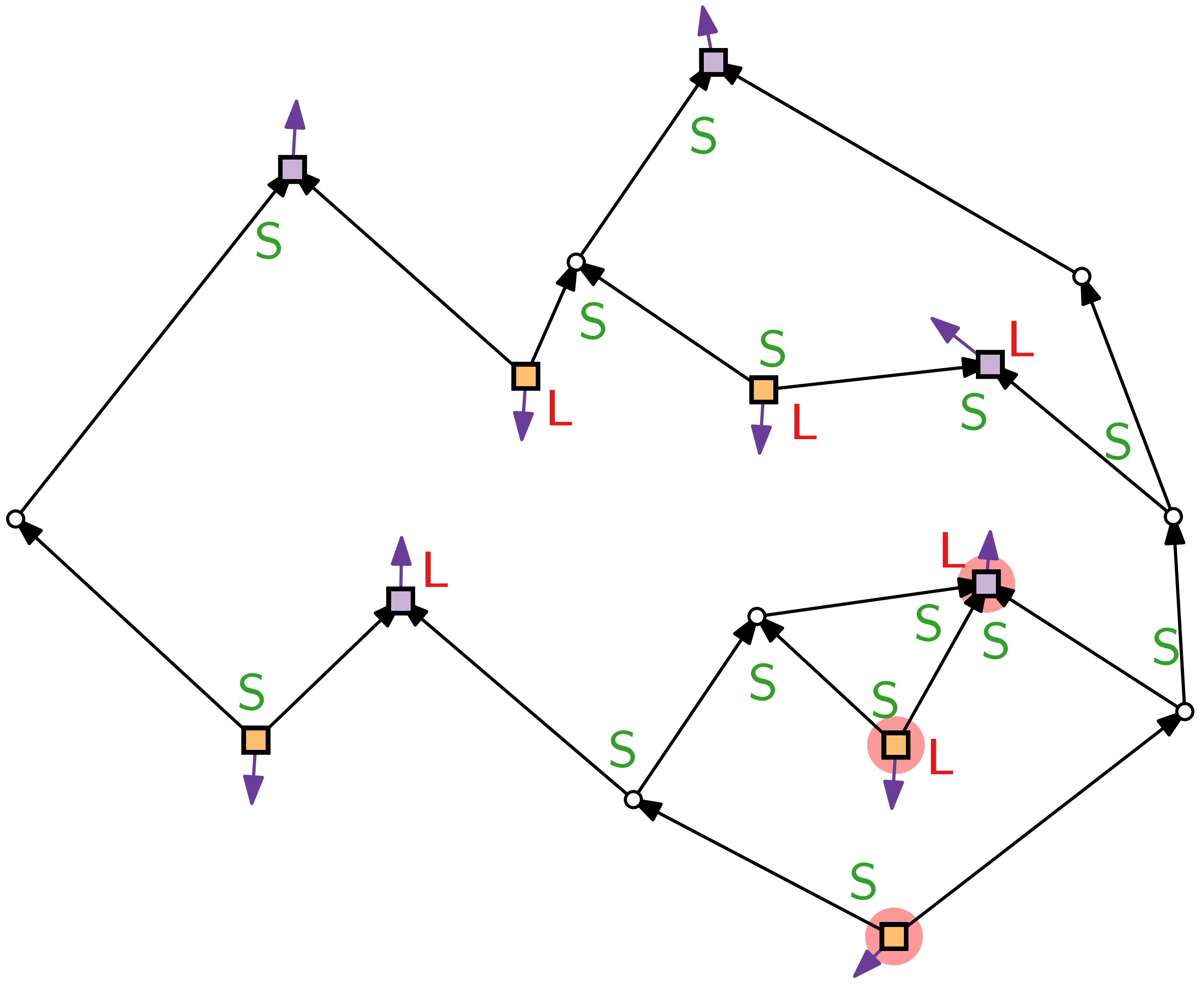

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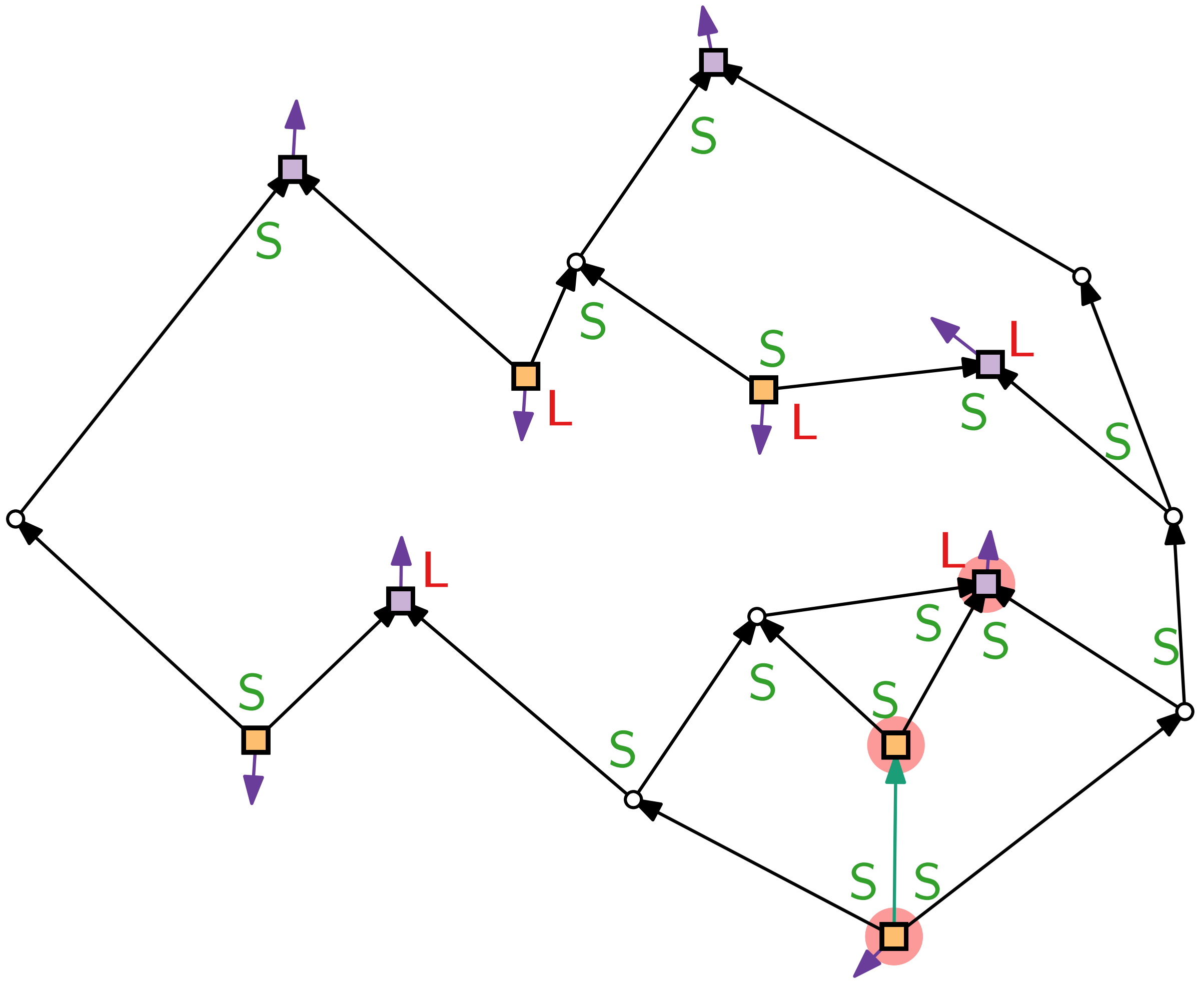

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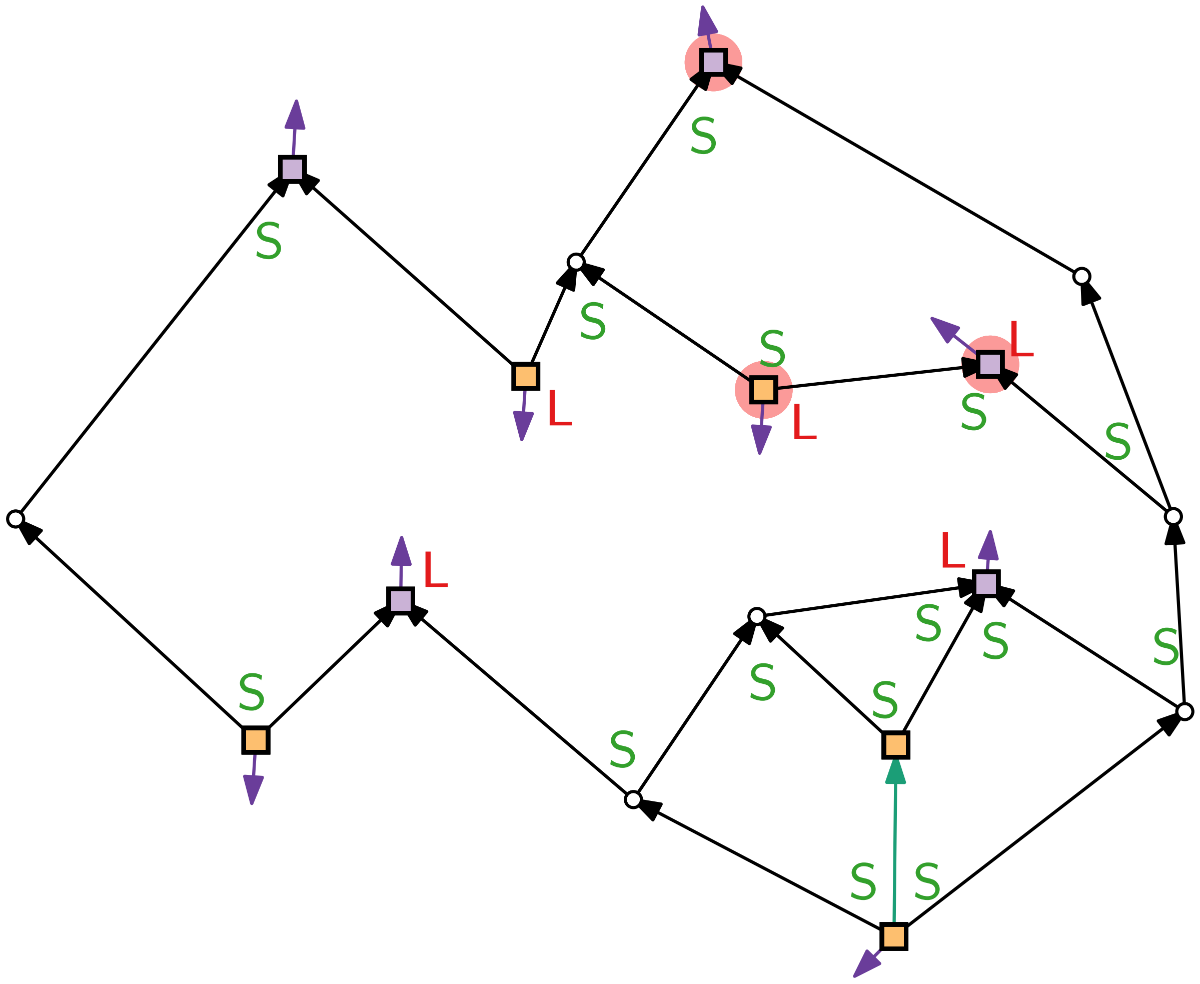

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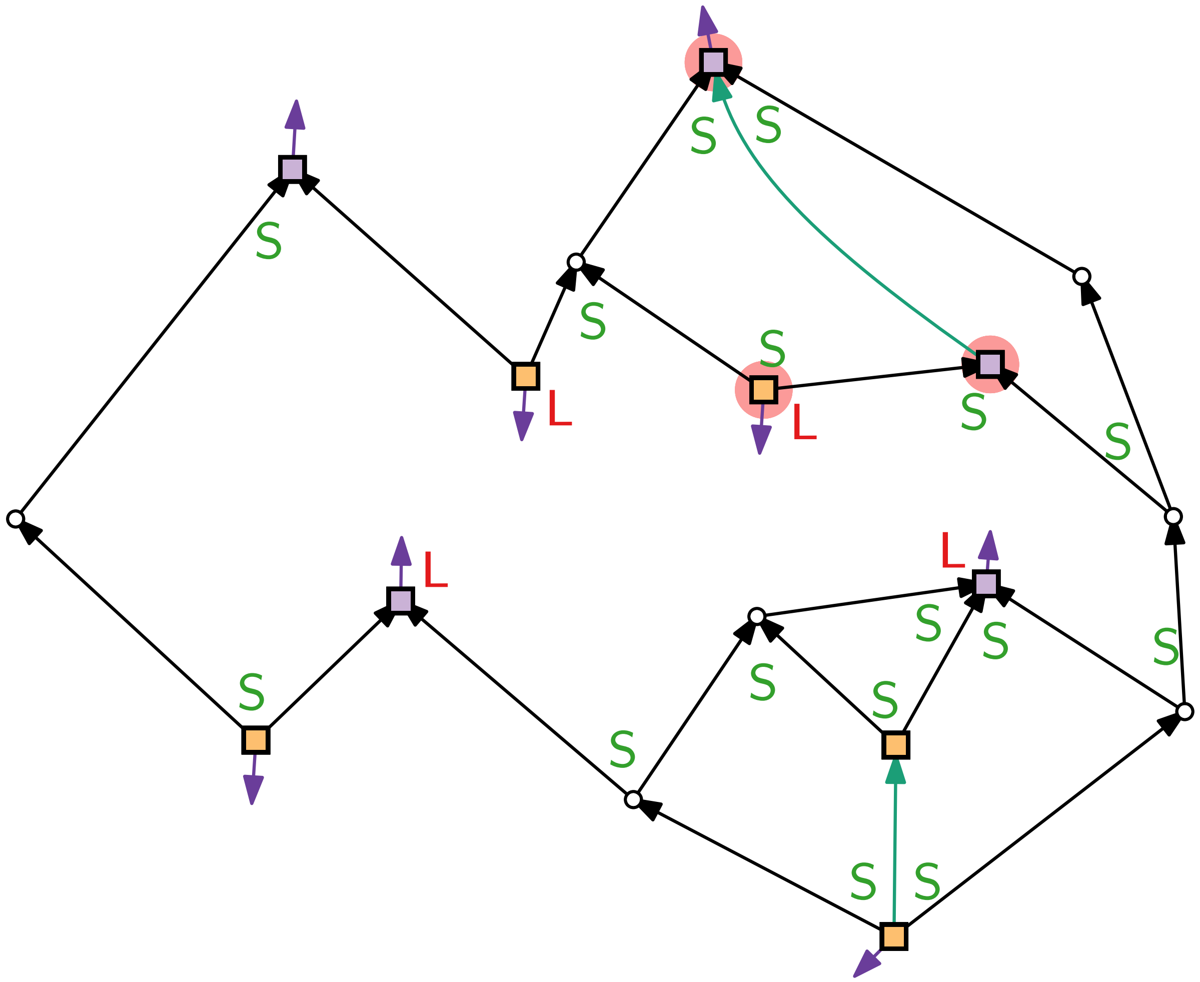

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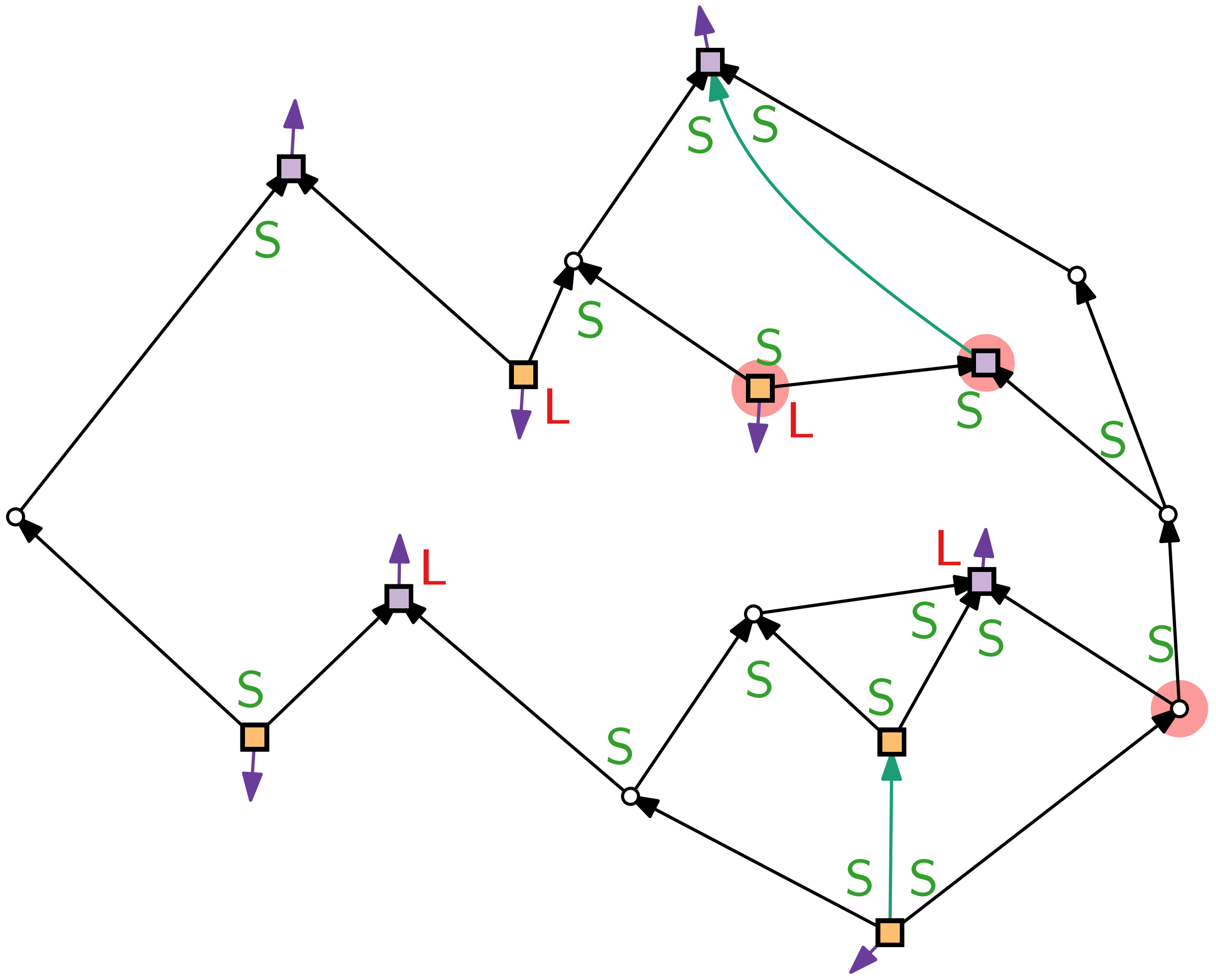

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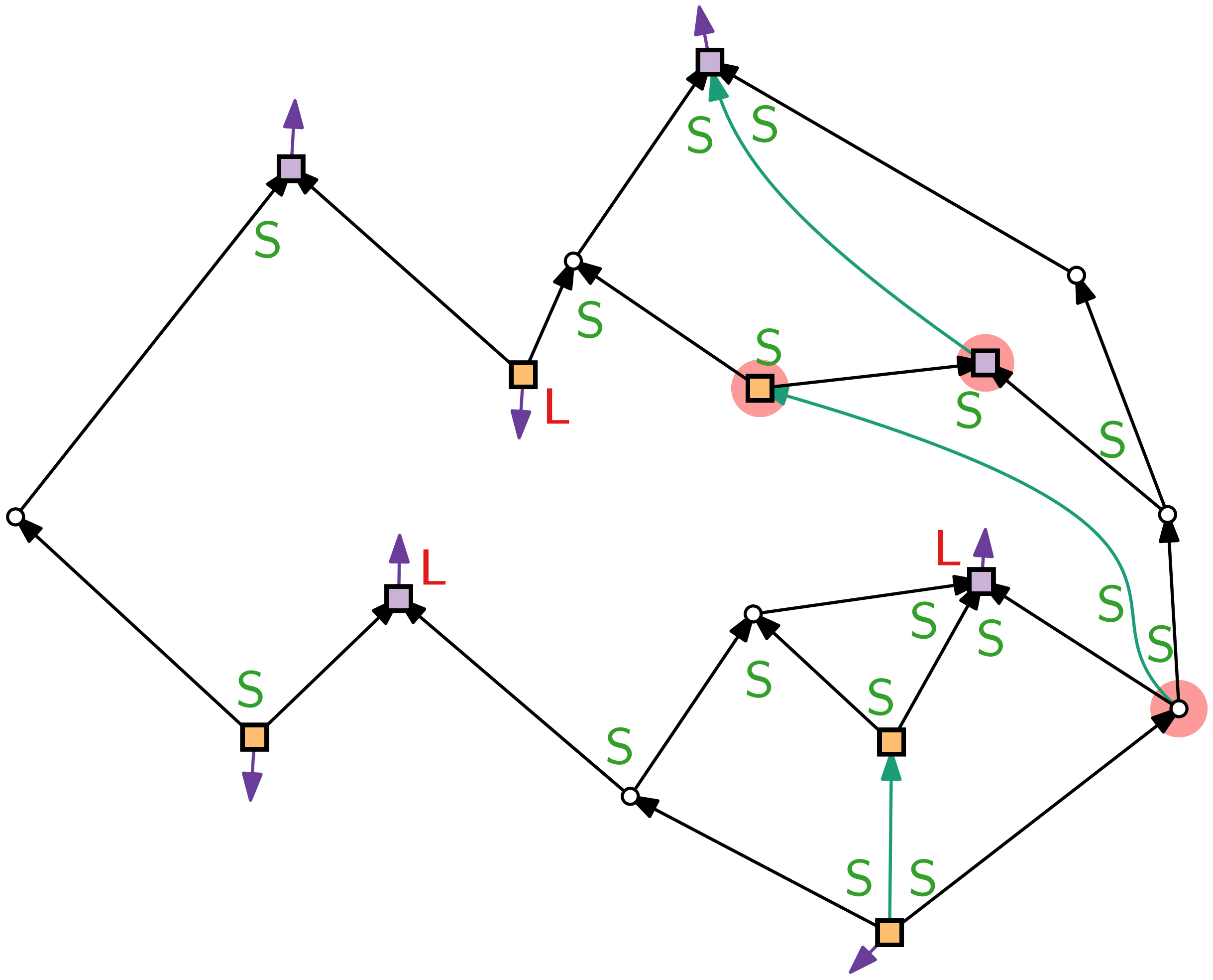

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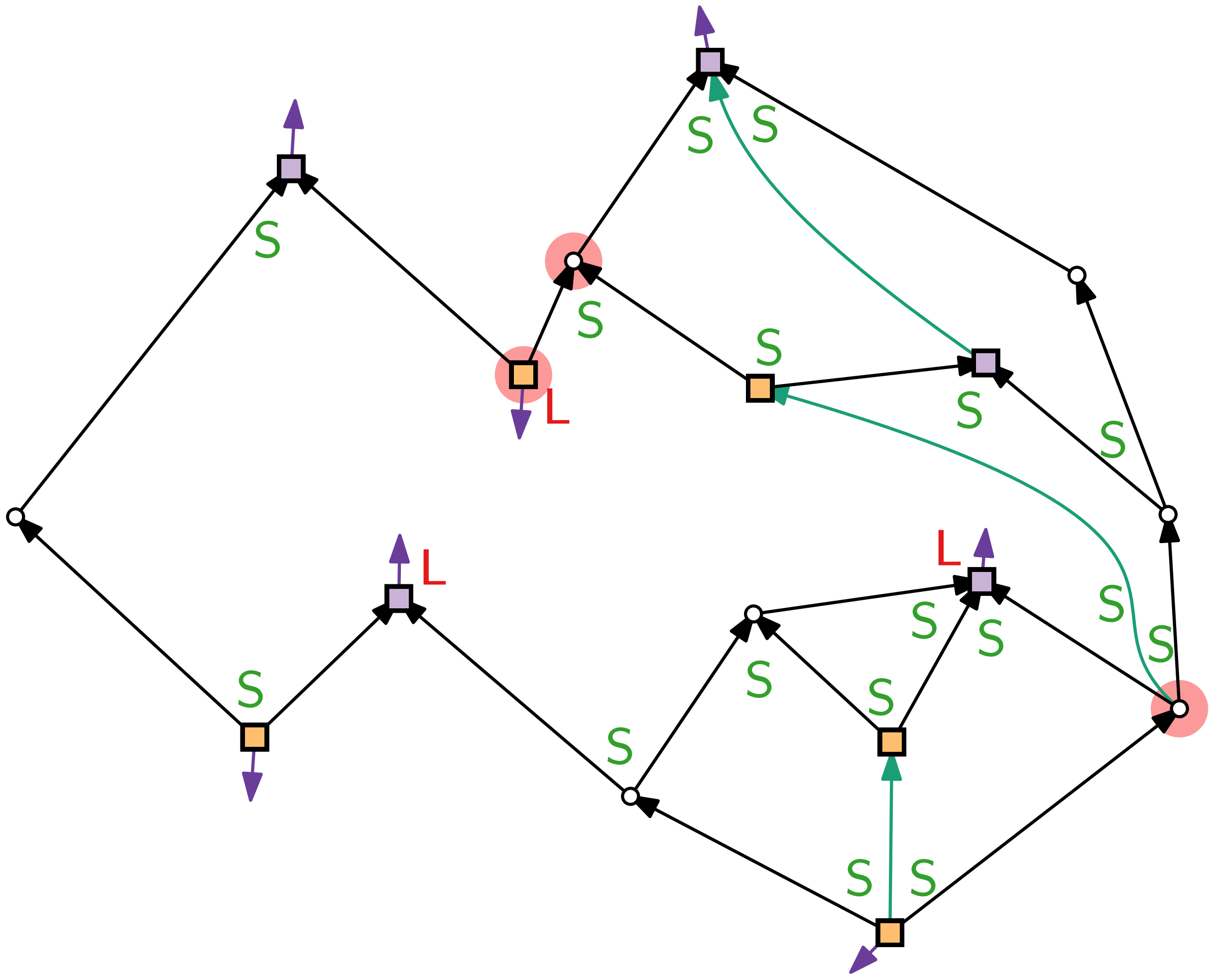

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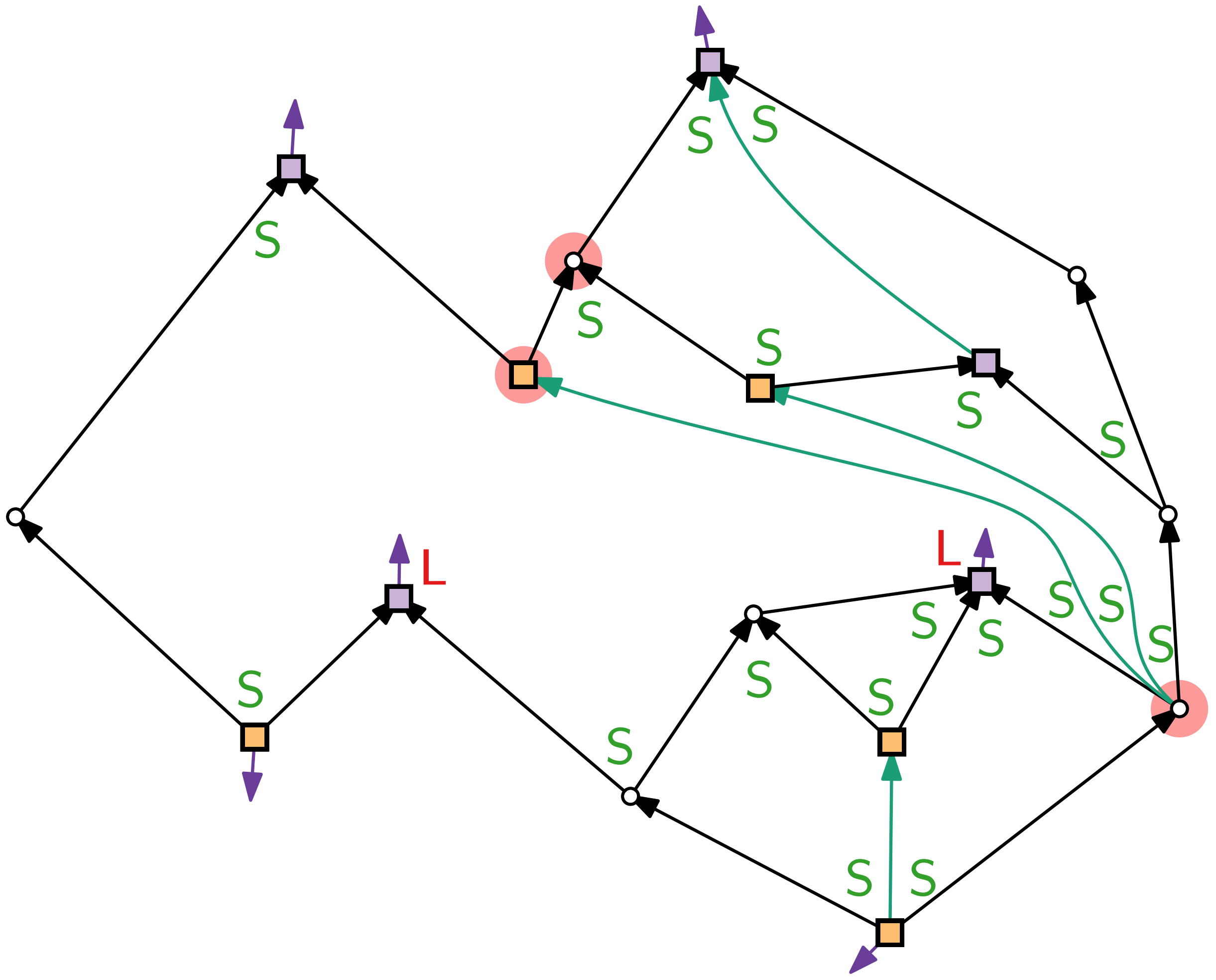

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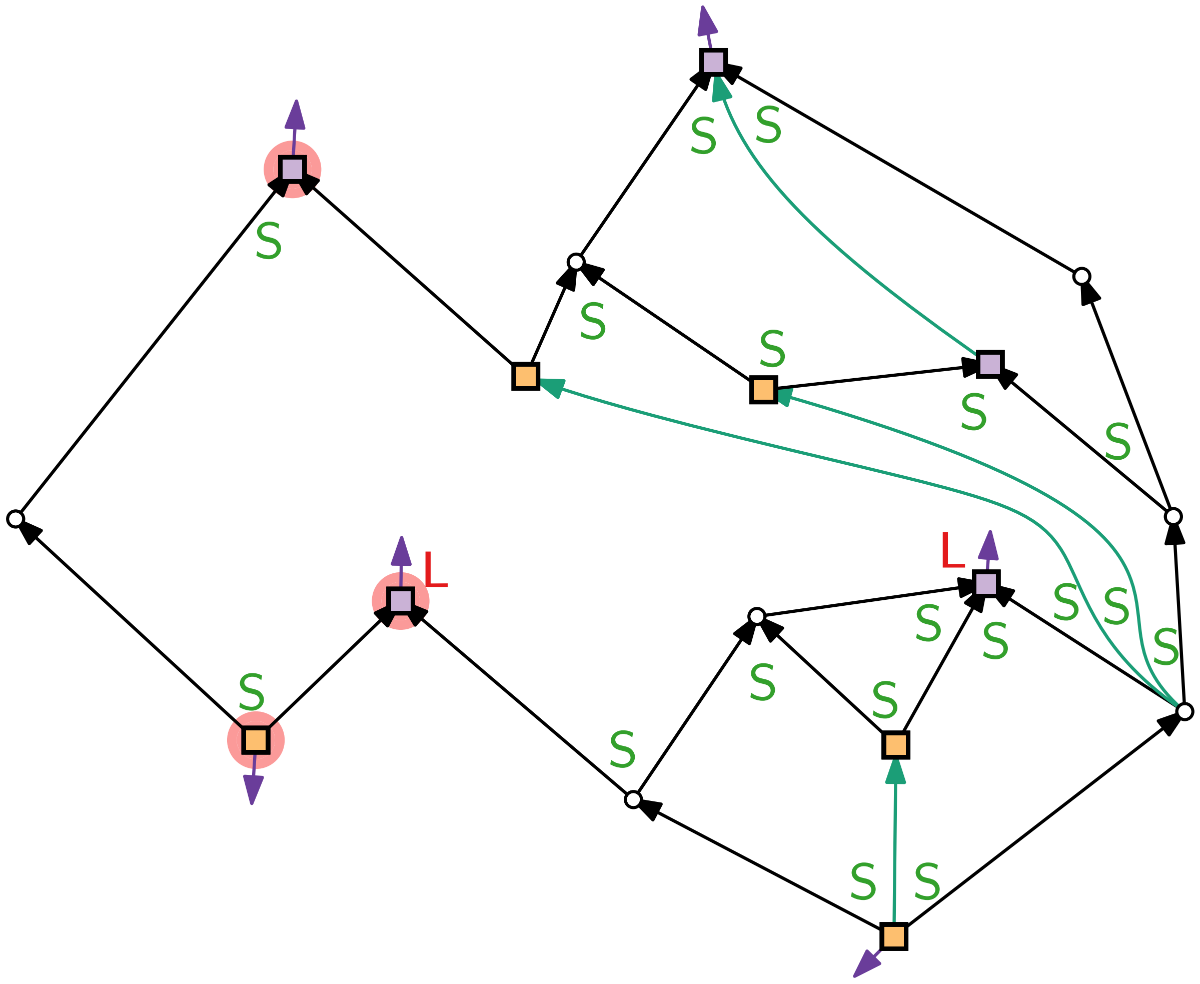

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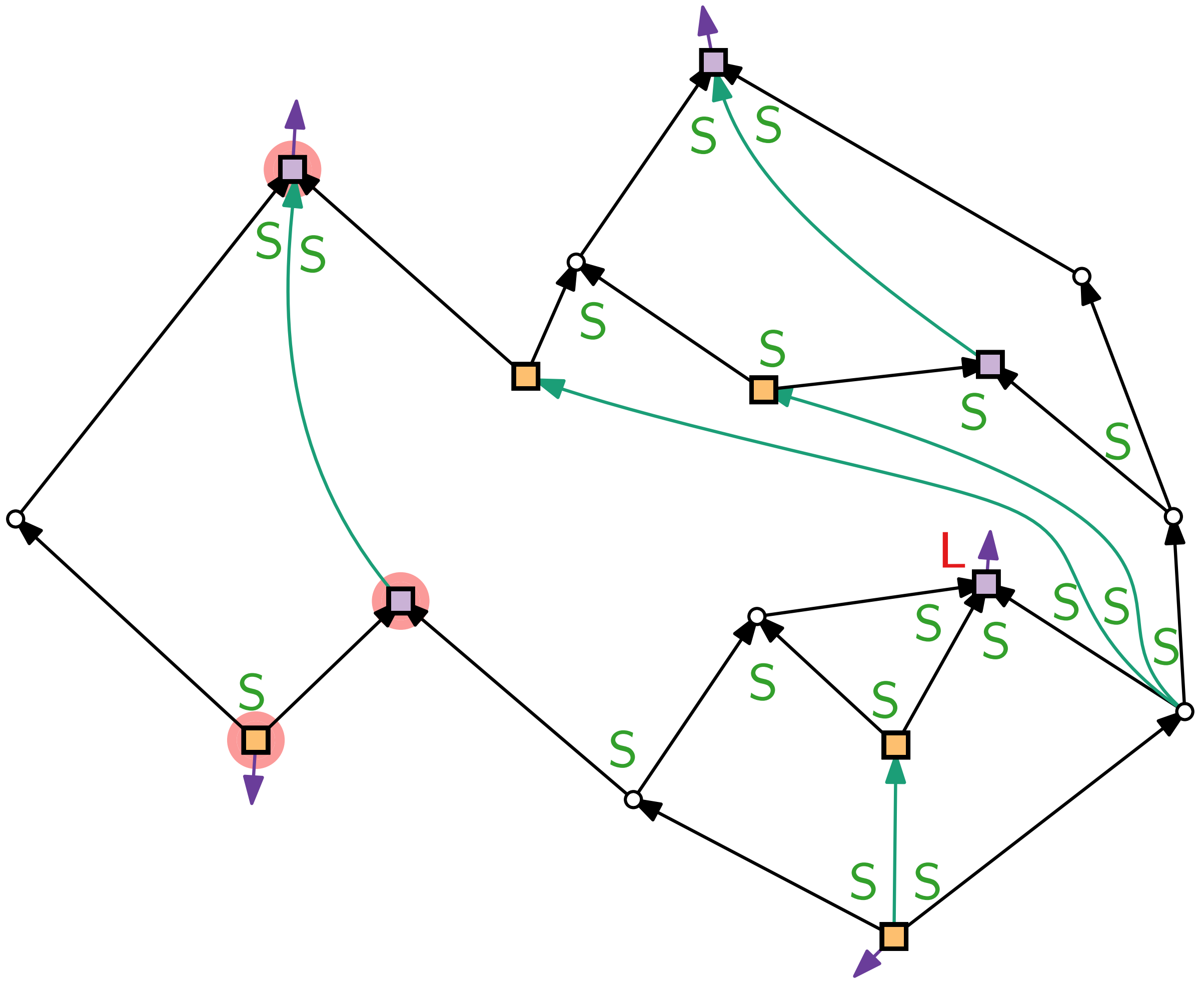

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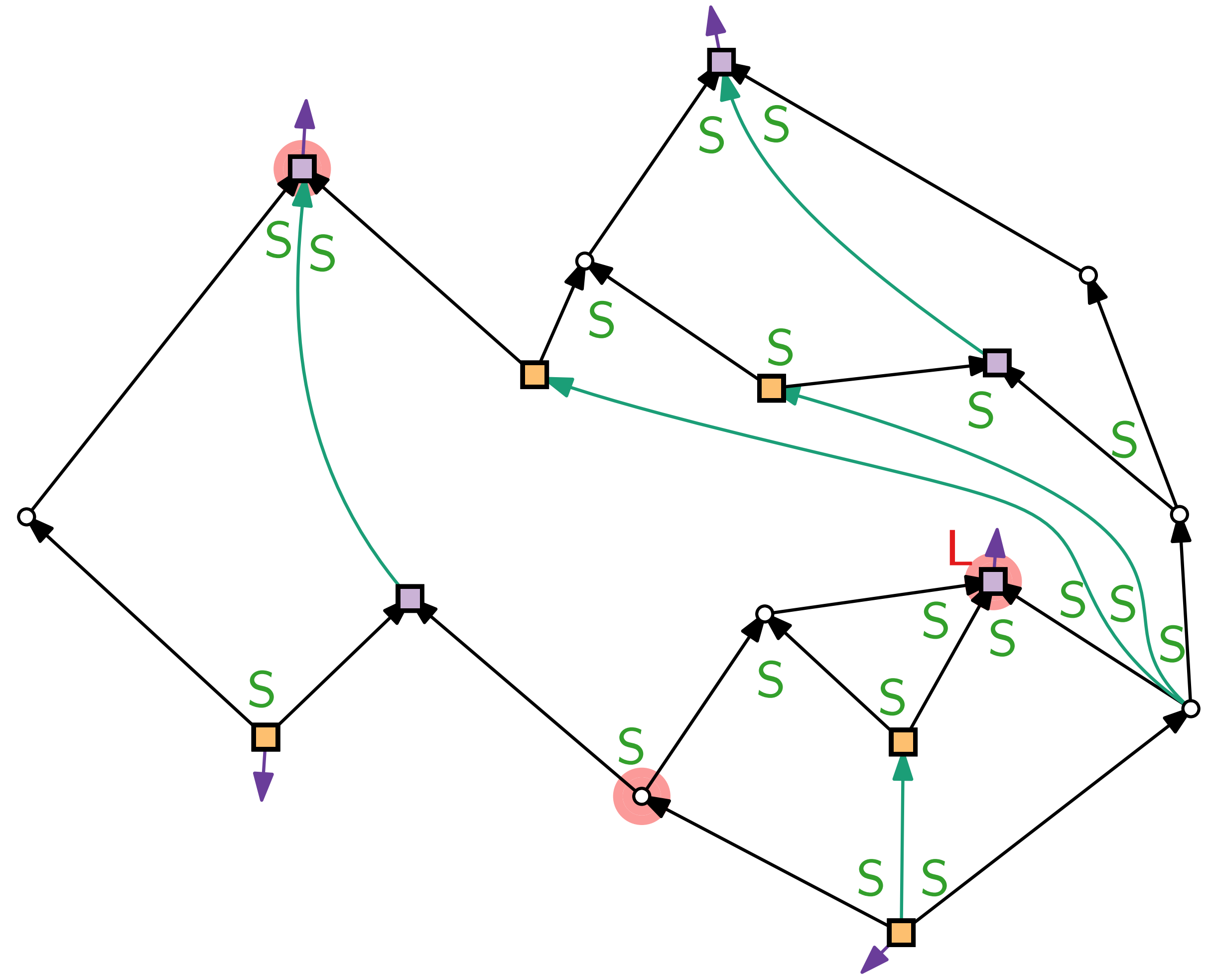

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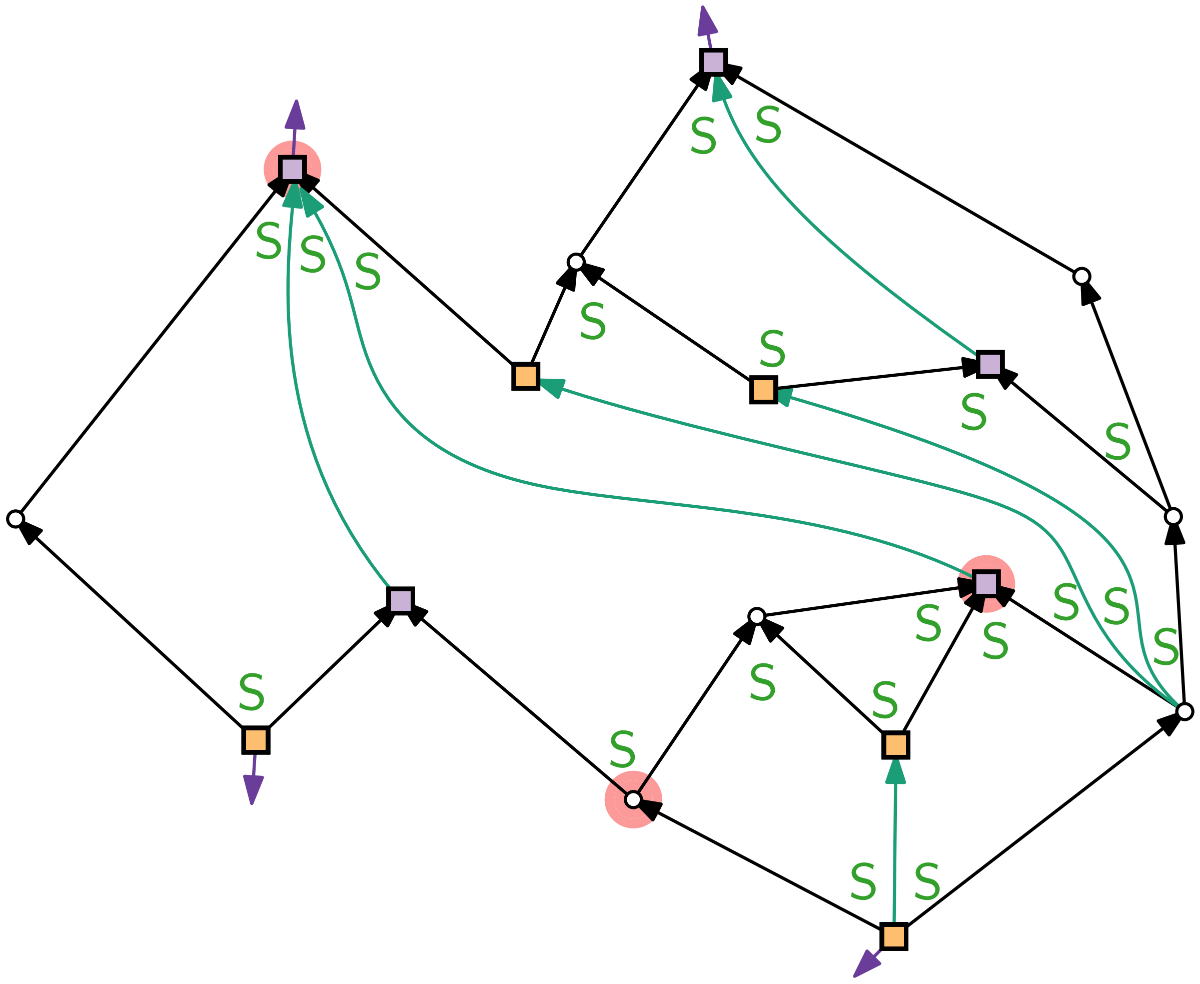

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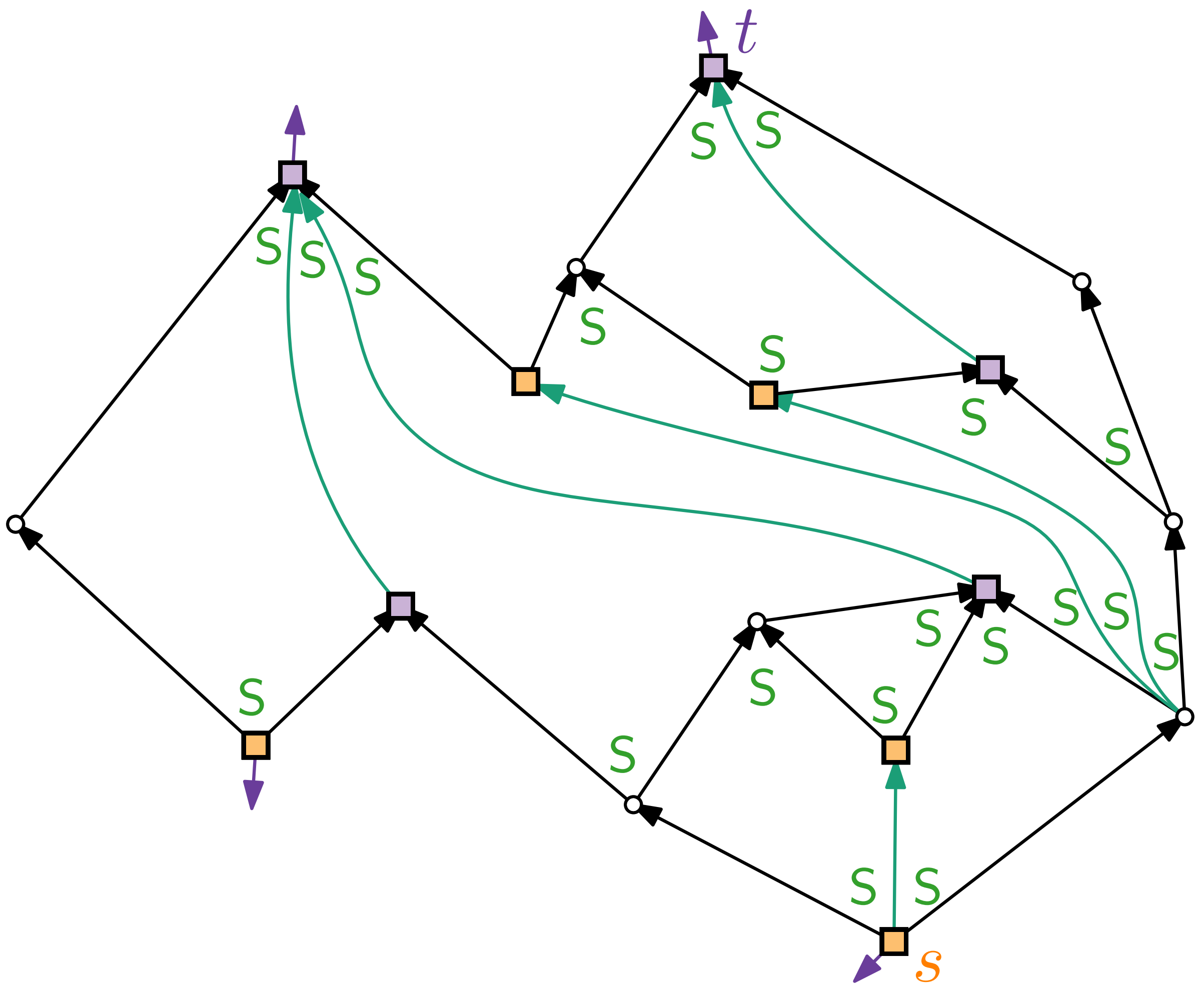

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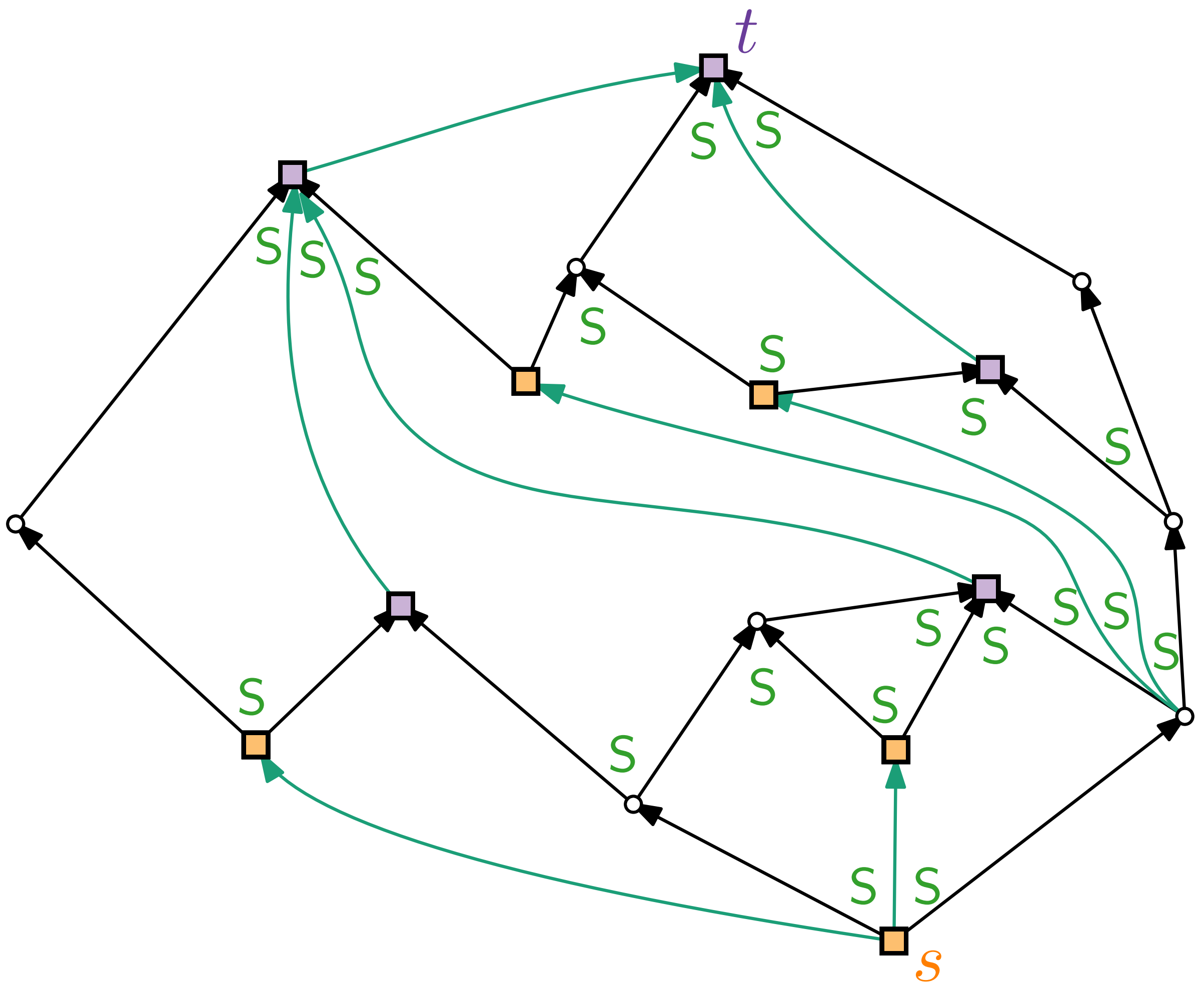

Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
Given a *combinatorially embedded* planar digraph G,
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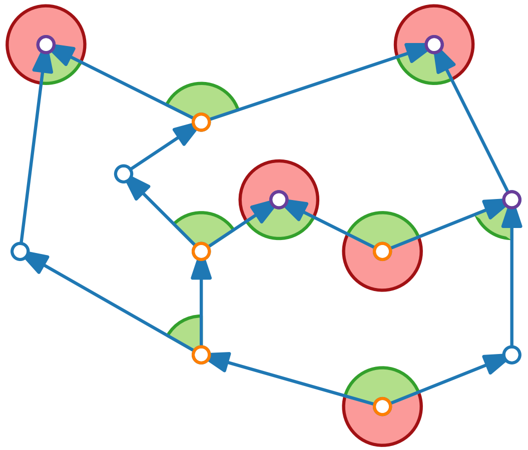
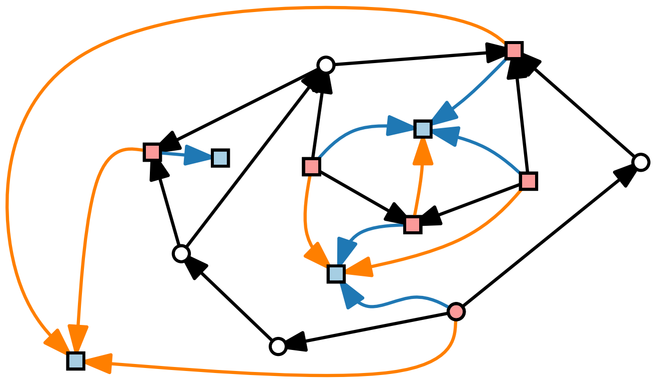
- Deleted edges added in refinement step.

Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part V:
Finding a Consistent Assignment



Alexander Wolff

Finding a Consistent Assignment

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Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

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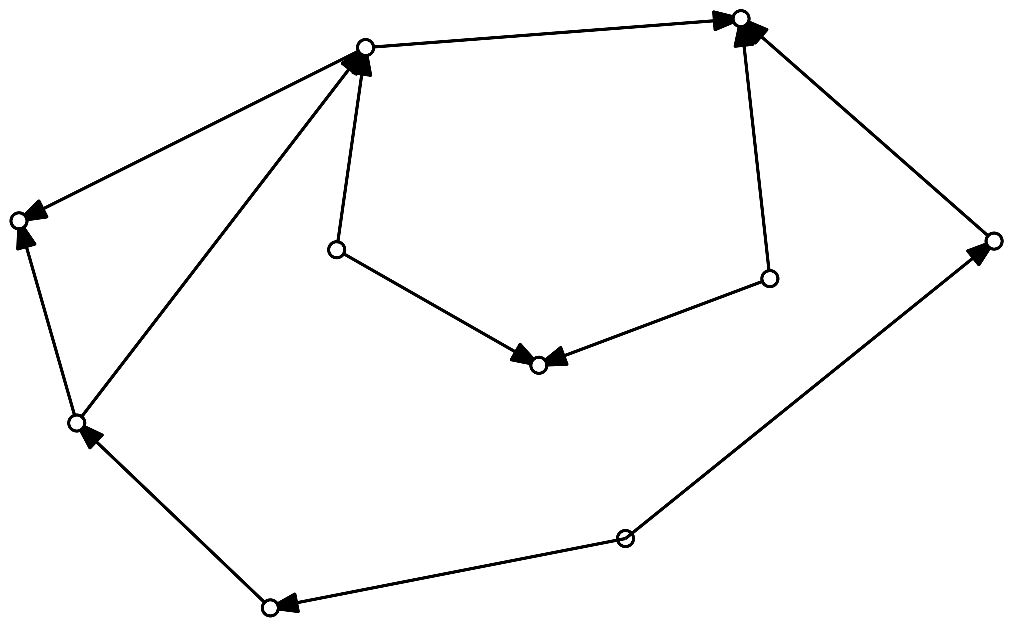

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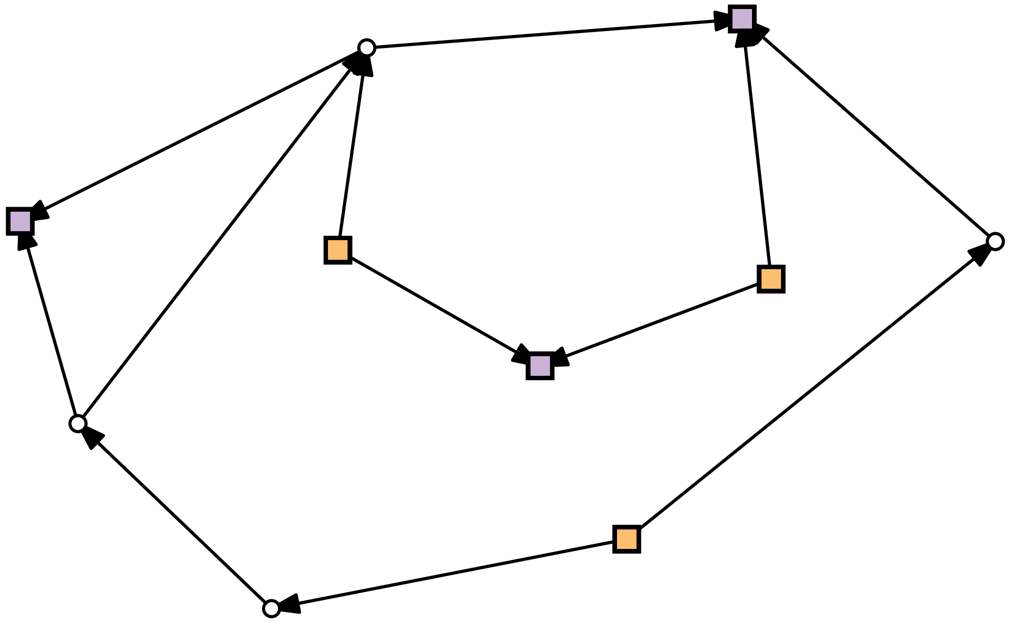

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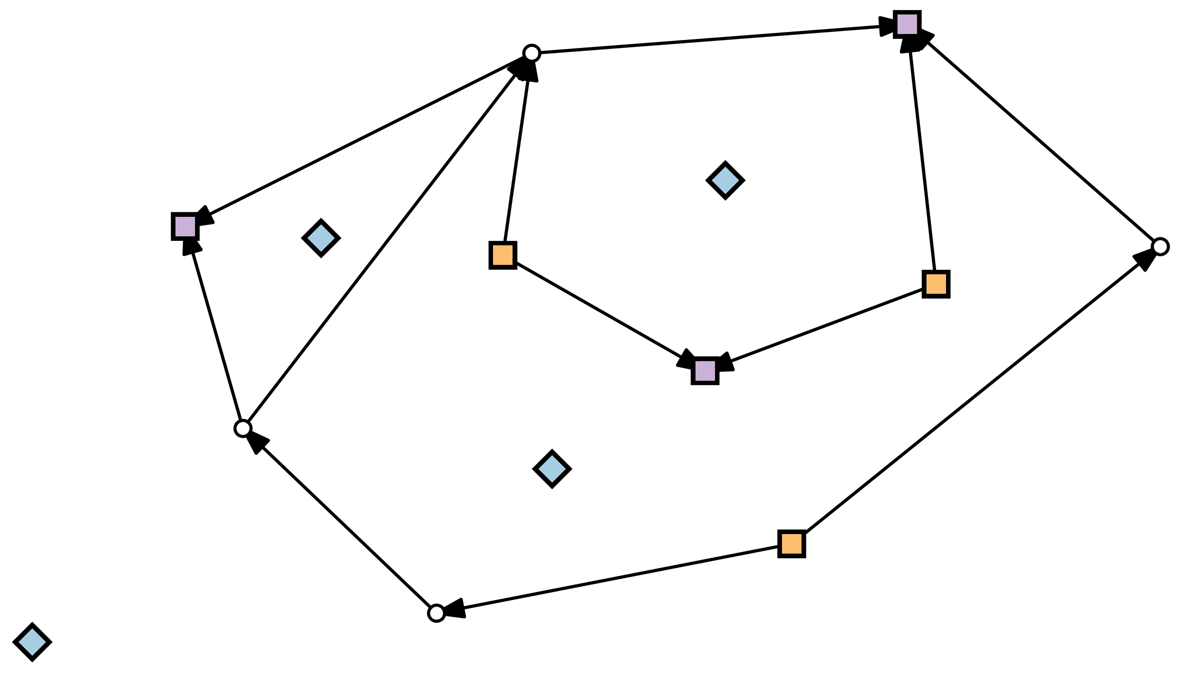

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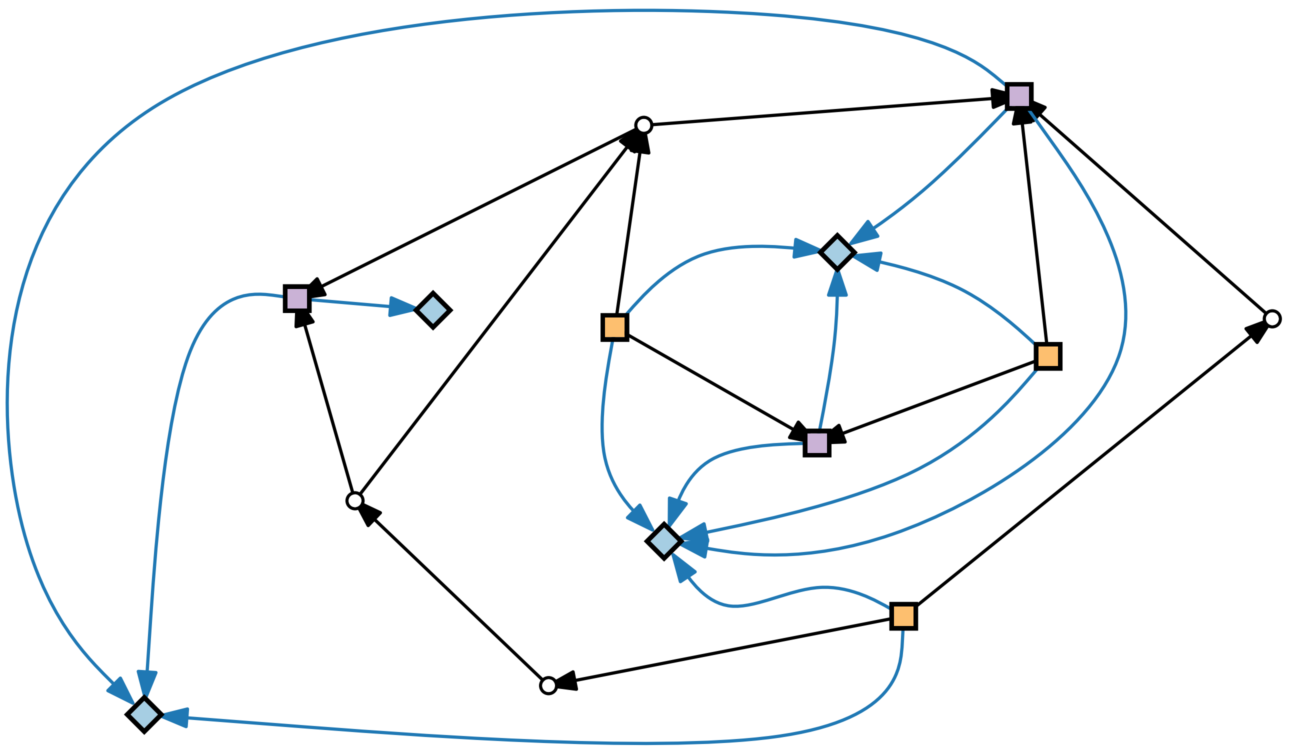

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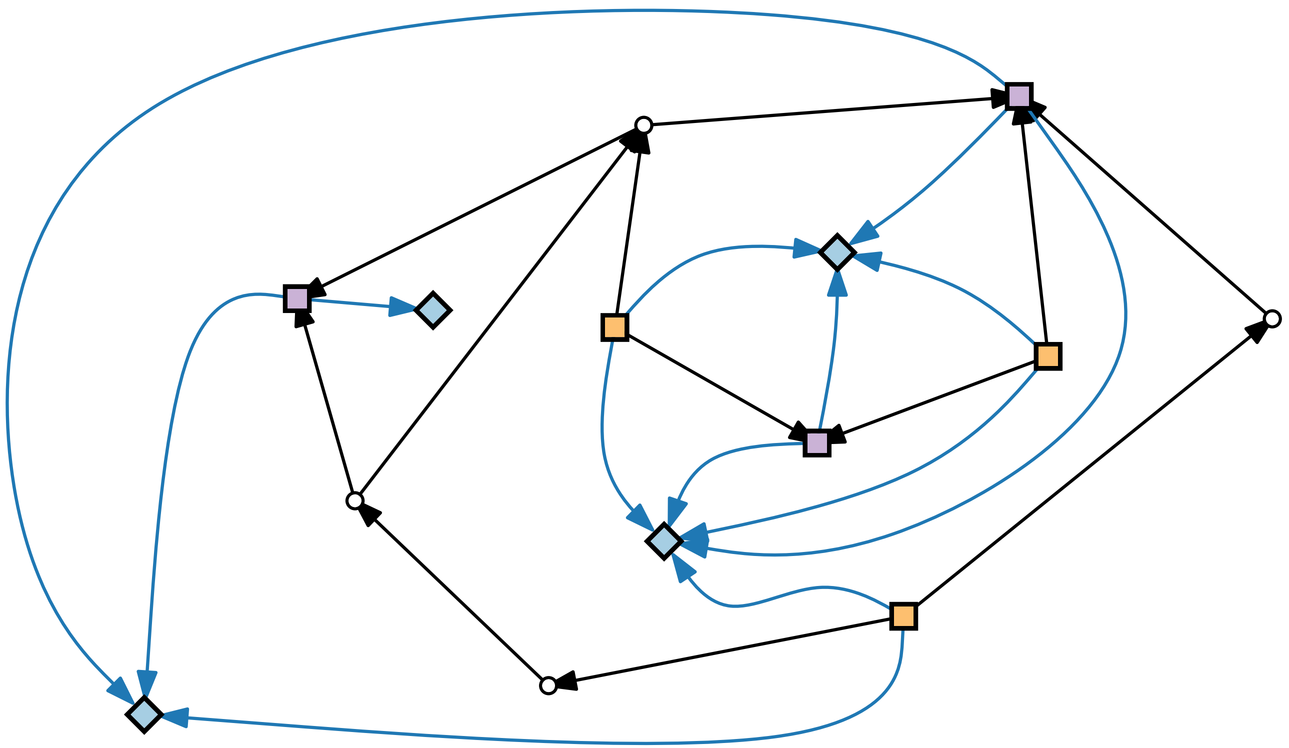

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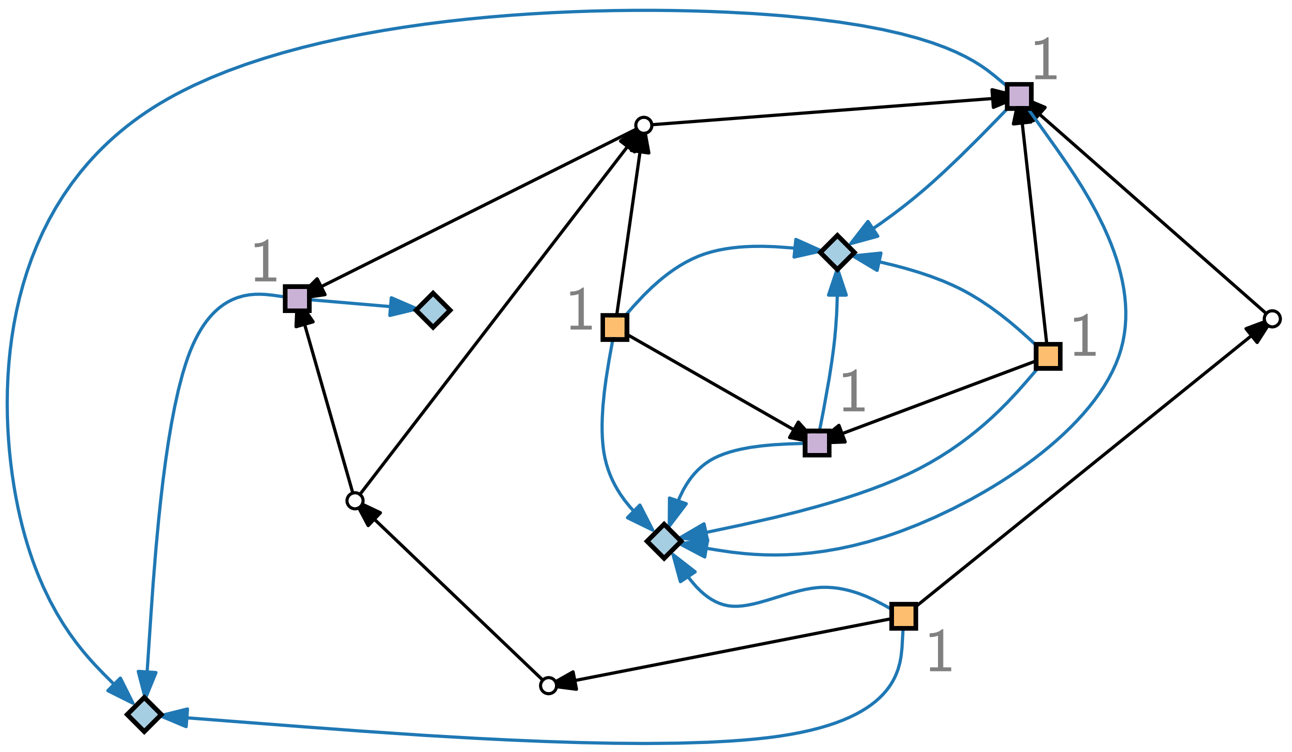

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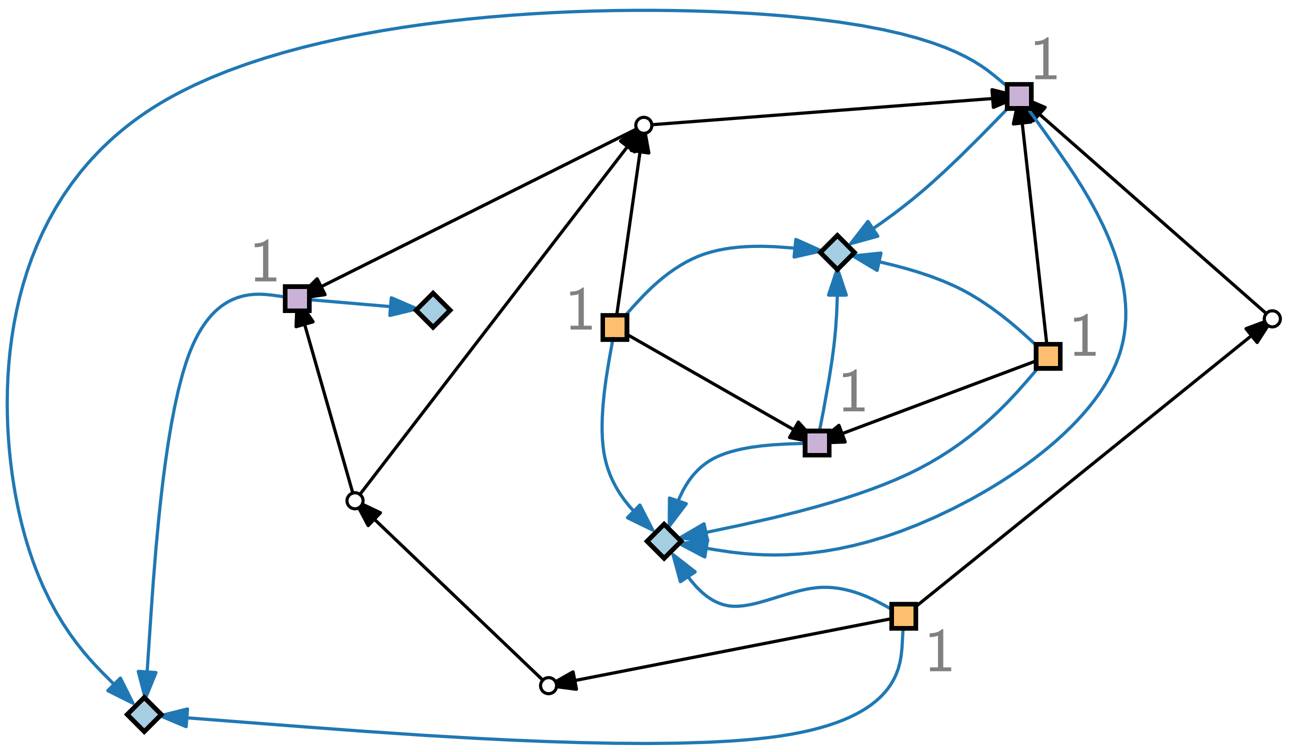

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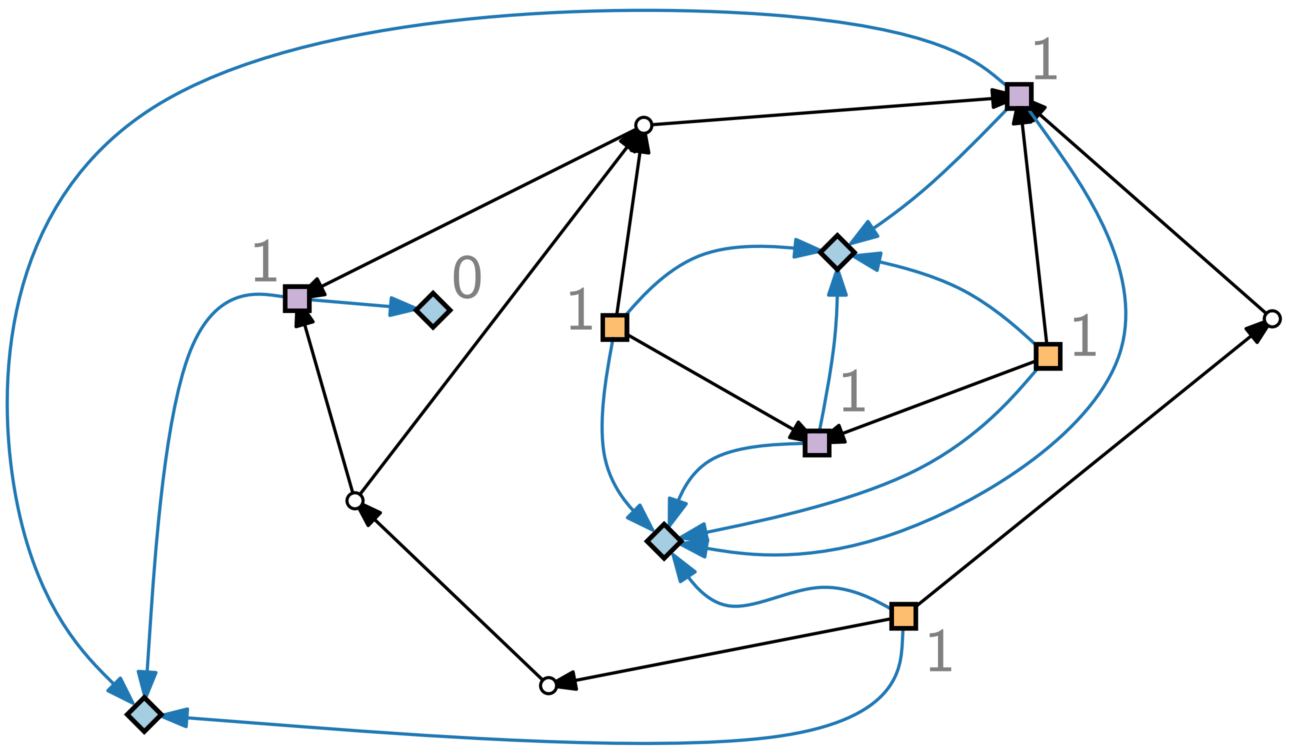

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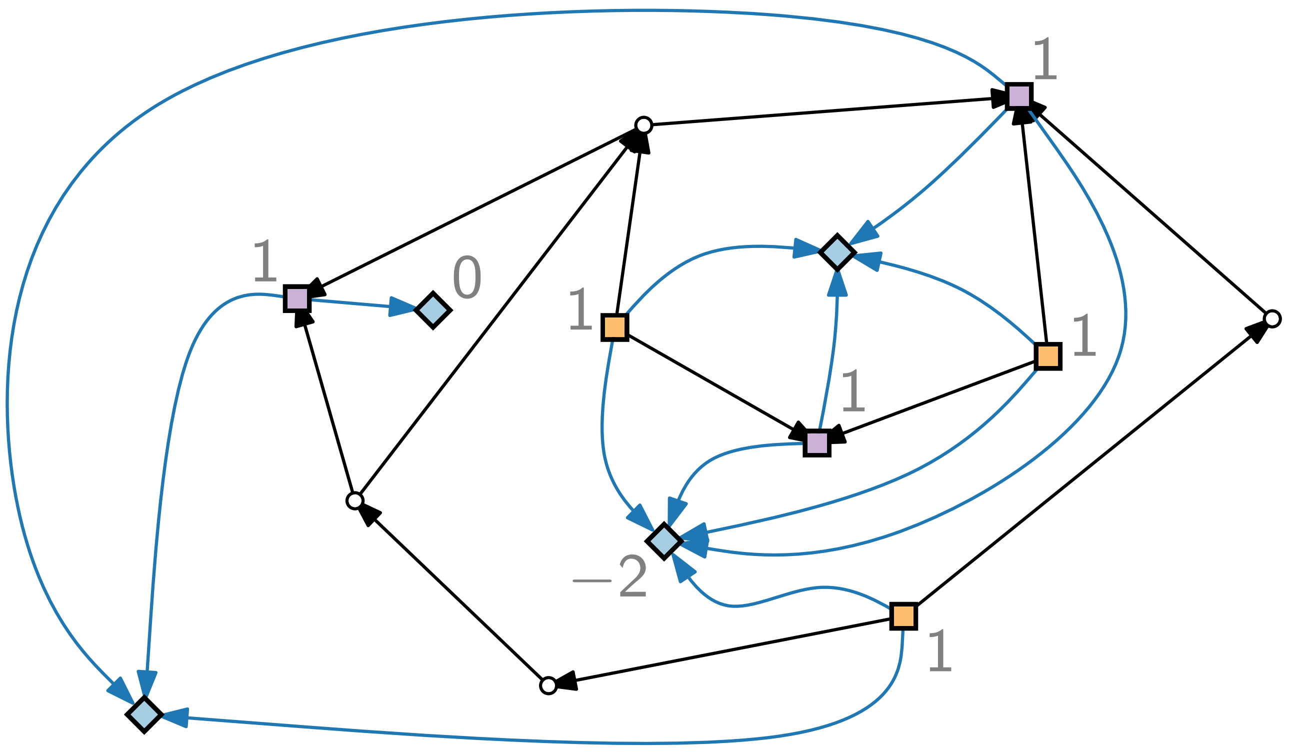

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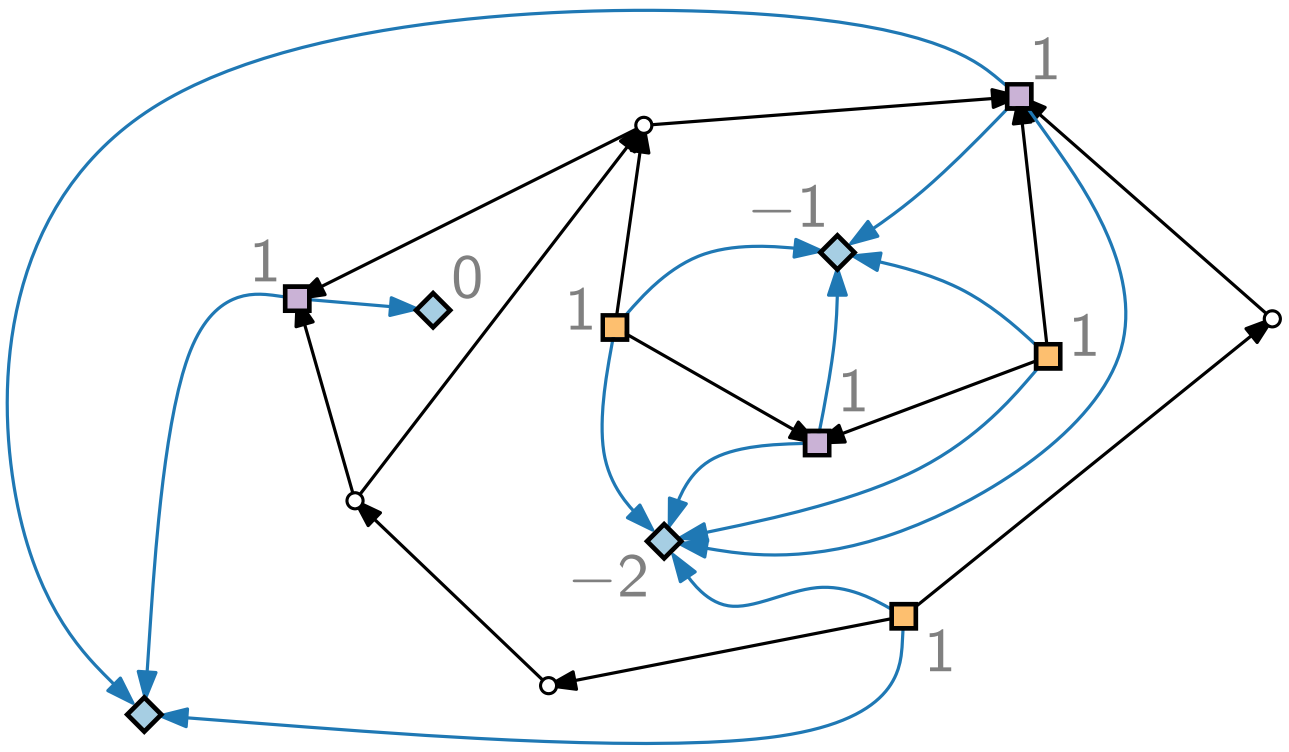

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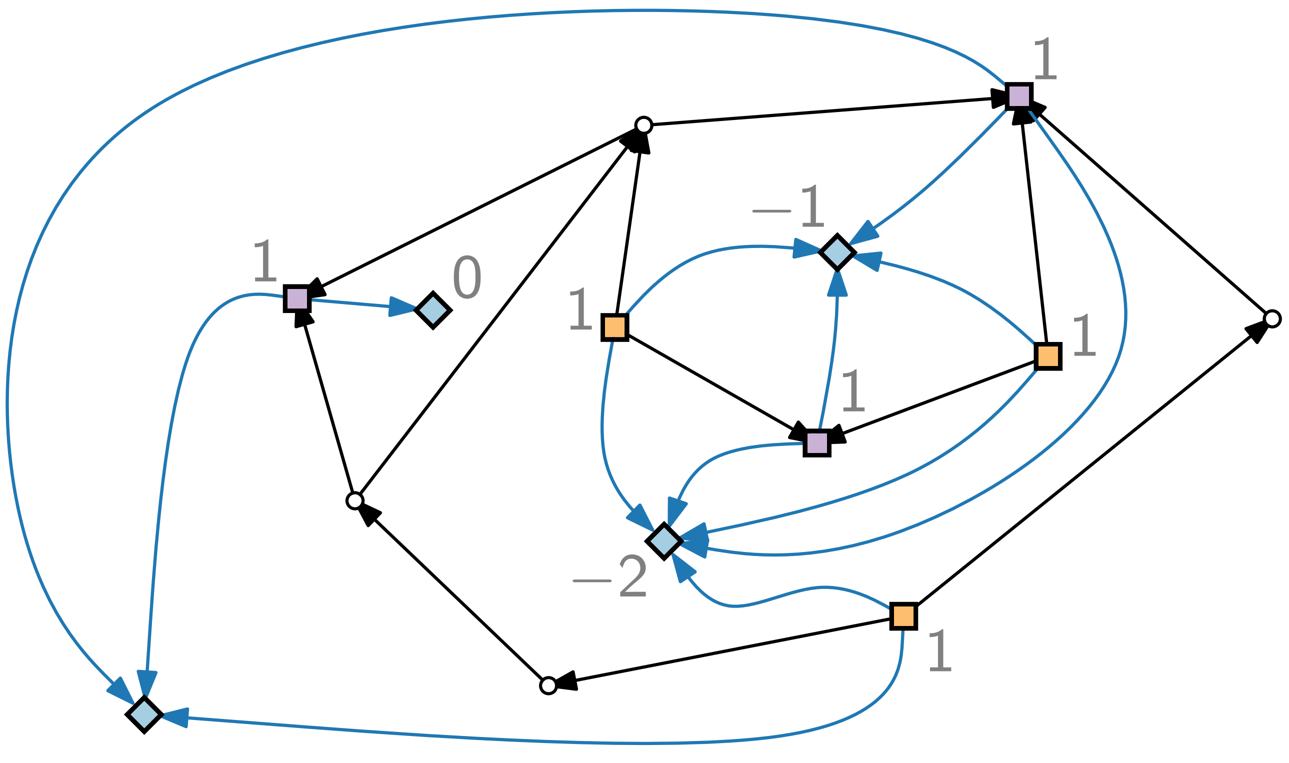

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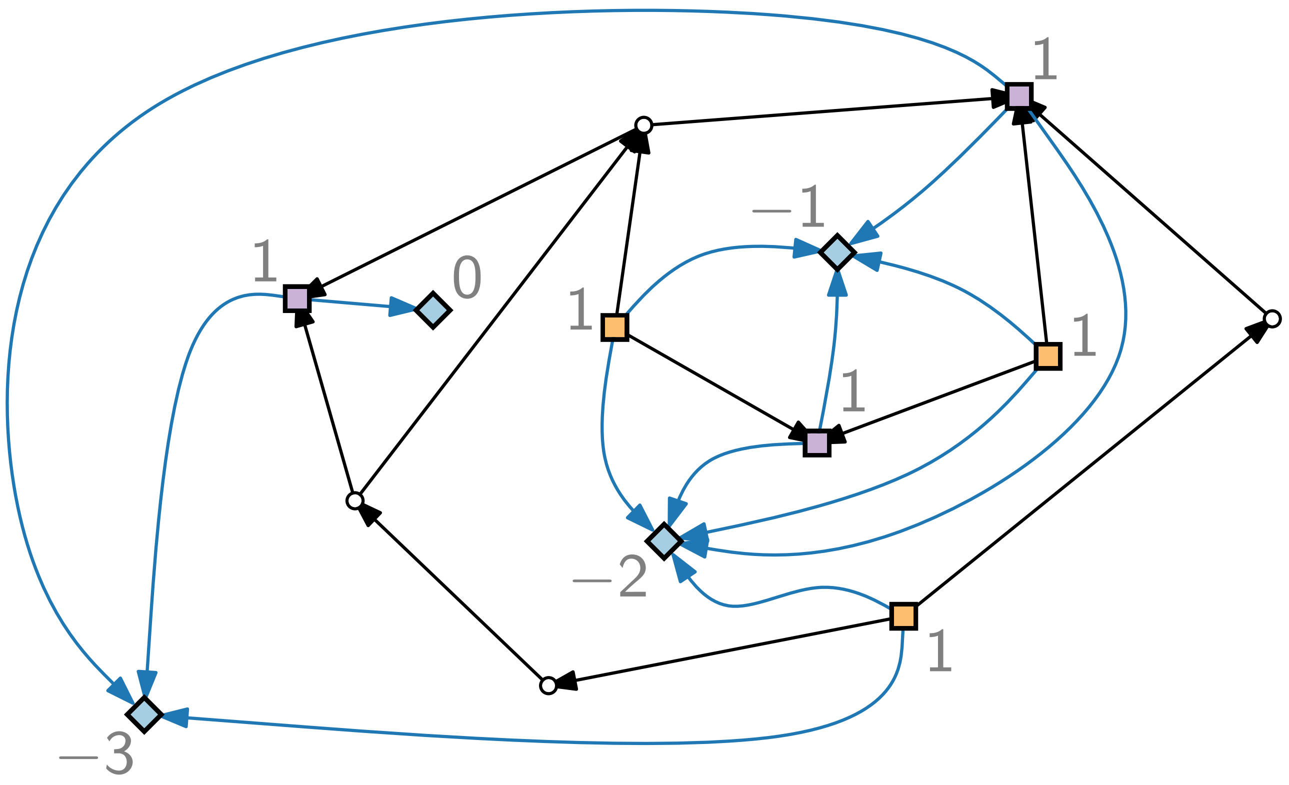

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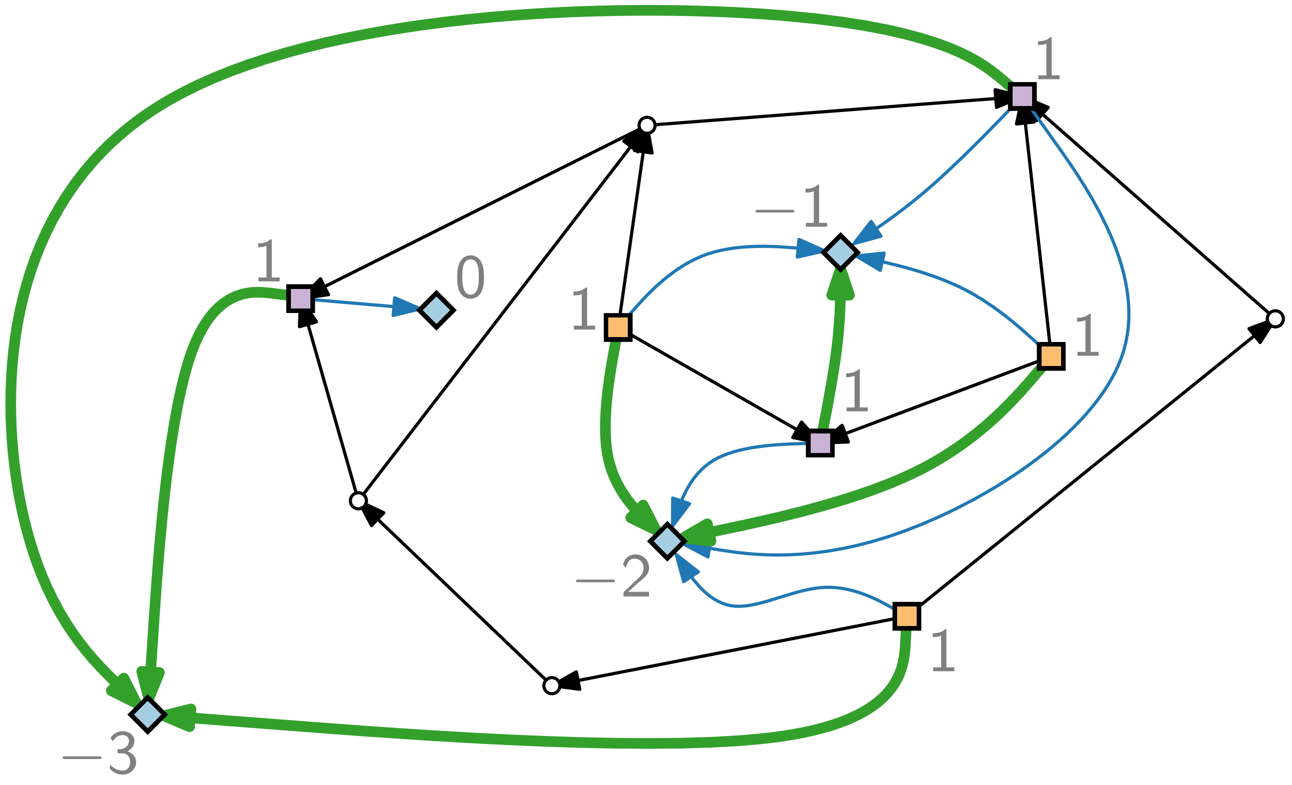

Discussion

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- Many related concepts have been studied:
  quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, . . .

Literature

- See [GD Ch. 6] for detailed explanation!

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista &Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg &Tamassia '95]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton & Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94]
  Upward Drawings of Triconnected Digraphs
- [Healy & Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10]
  Improving the running time of embedded upward planarity testing