Business Cycles

- Exercise 4 -

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Production and Labor Supply

Question:
Suppose that the household only lives for one period. The household's optimization problem is:

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

In this problem, the household receives no dividend from the firm.

a) Solve for the optimality condition characterizing the household problem.

b) From this optimality condition, what can you say about the effect of w_t on N_t? What is your explanation for this finding?

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

Lagrange:

$$\mathcal{L} =$$

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

Lagrange:

$$\mathcal{L} = \ln C_t + \theta_t \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma} + \lambda(C_t - w_t N_t)$$

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1 - N_t)^{1-\gamma}}{1 - \gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

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$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

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$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} + \lambda = 0 \Rightarrow -\frac{1}{C_t} = \lambda$$

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Lagrange:

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$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} + \lambda = 0 \Rightarrow -\frac{1}{C_t} = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = -\theta_t(1-\gamma)\frac{(1-N_t)^{1-\gamma-1}}{1-\gamma} - \lambda w_t = 0 \Rightarrow -\frac{\theta_t}{w_t(1-N_t)^{\gamma}} = \lambda$$

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

Lagrange:

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$$-\frac{1}{C_t} = -\frac{\theta_t}{w_t(1-N_t)^{\gamma}}$$

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

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$$-\frac{1}{C_t} = -\frac{\theta_t}{w_t(1-N_t)^{\gamma}}$$

$$w_t = \frac{\theta_t C_t}{(1-N_t)^{\gamma}}$$

$$\max_{C_t, N_t} U = \ln C_t + \theta_t \frac{(1-N_t)^{1-\gamma}}{1-\gamma} \qquad \text{s.t.} \quad C_t = w_t N_t$$

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$$\mathcal{L} = \ln C_t + \theta_t \frac{(1-N_t)^{1-\gamma}}{1-\gamma} + \lambda(C_t - w_t N_t)$$

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$$-\frac{1}{C_t} = -\frac{\theta_t}{w_t(1-N_t)^{\gamma}}$$

$$w_t = \frac{\theta_t C_t}{(1-N_t)^{\gamma}} = \frac{u_L}{u_C}$$

1. Labor Supply

$$w_t = \frac{\theta_t C_t}{(1 - N_t)^\gamma} = \theta_t C_t (1 - N_t)^{-\gamma}$$

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$$\left(\frac{w_t}{\theta_t C_t}\right)^{-\frac{1}{\gamma}} = \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} = 1 - N_t$$

$$
\begin{aligned}
w_t &= \frac{\theta_t C_t}{(1-N_t)^\gamma} = \theta_t C_t (1-N_t)^{-\gamma} \\
\left(\frac{w_t}{\theta_t C_t}\right)^{-\frac{1}{\gamma}} &= \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} = 1 - N_t \\
1 - \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} &= N_t = N^S(C_t, w_t, \theta_t)
\end{aligned}
$$

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w_t &= \frac{\theta_t C_t}{(1-N_t)^\gamma} = \theta_t C_t (1-N_t)^{-\gamma} \\
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1 - \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} &= N_t = N^S(\underbrace{C_t}_{-}, \underbrace{w_t}_{+}, \underbrace{\theta_t}_{-})
\end{aligned}
$$

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$$1 - \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} = N_t = N^S(\underbrace{C_t}_{-}, \underbrace{w_t}_{+}, \underbrace{\theta_t}_{-})$$

- Substitution effect dominates income effect

$$w_t = \frac{\theta_t C_t}{(1-N_t)^\gamma} = \theta_t C_t (1-N_t)^{-\gamma}$$

$$\left(\frac{w_t}{\theta_t C_t}\right)^{-\frac{1}{\gamma}} = \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} = 1 - N_t$$

$$1 - \left(\frac{\theta_t C_t}{w_t}\right)^{\frac{1}{\gamma}} = N_t = N^S(\underbrace{C_t}_{-}, \underbrace{w_t}_{+}, \underbrace{\theta_t}_{-})$$

- Substitution effect dominates income effect
- Upward-sloping labor supply curve

Question:
Suppose that you have a firm with a Cobb-Douglas production function for production in period t:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

The only twist relative to our setup in the main text is that the firm does not use labor to produce output in period $t+1$. The production function in that period is:

$$Y_{t+1} = A_{t+1} K_{t+1}^{\alpha}$$

a) Write down the optimization problem for the firm in this setup. It has to pay labor in period t w_t, and it discounts future dividends by $\frac{1}{1+r_t}$. It must borrow to finance its investment at r_t. The capital accumulation equation is standard.

b) What is the terminal condition for the firm? Explain the economic logic.

c) Using this specification of production, derive the first order optimality conditions for the optimal choices of N_t and K_{t+1}.

$$\max V_t = D_t + \frac{D_{t+1}}{1 + r_t}$$

$$\max V_t = D_t + \frac{D_{t+1}}{1+r_t}$$
$$= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - w_{t+1}N_{t+1} - I_{t+1} - (1+r_t)B_t^I\right]$$

$$\begin{aligned}\max V_t &= D_t + \frac{D_{t+1}}{1+r_t} \\ &= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - I_{t+1} - (1+r_t)B_t^I\right]\end{aligned}$$

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$$\max V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1} K_{t+1}^{\alpha} - I_{t+1} - (1+r_t)B_t^I\right]$$

$$
\begin{aligned}
\max V_t &= D_t + \frac{D_{t+1}}{1+r_t} \\
&= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - I_{t+1} - (1+r_t)B_t^I\right] \\
\max V_t &= A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1} K_{t+1}^{\alpha} - I_{t+1} - (1+r_t)B_t^I\right] \\
\text{s.t.} \quad K_{t+1} &= I_t + (1-\delta)K_t \\
B_t^I &= I_t
\end{aligned}
$$

$$\max V_t = D_t + \frac{D_{t+1}}{1+r_t}$$
$$= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - I_{t+1} - (1+r_t)B_t^I\right]$$
$$\max V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1}K_{t+1}^{\alpha} - I_{t+1} - (1+r_t)B_t^I\right]$$

s.t.
$$K_{t+1} = I_t + (1-\delta)K_t$$
$$B_t^I = I_t$$
$$K_{t+2} = I_{t+1} + (1-\delta)K_{t+1}$$

Economic Intuition?

$$\begin{aligned}
\max V_t &= D_t + \frac{D_{t+1}}{1+r_t} \\
&= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - I_{t+1} - (1+r_t)B_t^I\right]
\end{aligned}$$

$$\begin{aligned}
\max V_t &= A_t K_t^\alpha N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1} K_{t+1}^\alpha - I_{t+1} - (1+r_t)B_t^I\right] \\
\text{s.t.} \quad K_{t+1} &= I_t + (1-\delta)K_t \\
B_t^I &= I_t \\
K_{t+2} &= I_{t+1} + (1-\delta)K_{t+1} = 0
\end{aligned}$$

Economic Intuition?

$$
\begin{aligned}
\max V_t &= D_t + \frac{D_{t+1}}{1+r_t} \\
&= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - I_{t+1} - (1+r_t)B_t^I\right]
\end{aligned}
$$

$$
\max V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1} K_{t+1}^{\alpha} - I_{t+1} - (1+r_t)B_t^I\right]
$$

$$
\text{s.t.} \quad B_t^I = K_{t+1} - (1-\delta)K_t
$$

$$\max V_t = D_t + \frac{D_{t+1}}{1+r_t}$$
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s.t. $B_t^I = K_{t+1} - (1-\delta)K_t$

$$I_{t+1} = -(1-\delta)K_{t+1}$$

$$\begin{aligned}\max V_t &= D_t + \frac{D_{t+1}}{1+r_t} \\ &= Y_t - w_t N_t + \frac{1}{1+r_t}\left[Y_{t+1} - I_{t+1} - (1+r_t)B_t^I\right] \\ \max V_t &= A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1}K_{t+1}^{\alpha} - I_{t+1} - (1+r_t)B_t^I\right]\end{aligned}$$

$$\begin{aligned}\text{s.t.} \quad B_t^I &= K_{t+1} - (1-\delta)K_t \\ I_{t+1} &= -(1-\delta)K_{t+1}\end{aligned}$$

$$\begin{aligned}\max_{N_t, K_{t+1}} V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\Big[A_{t+1}K_{t+1}^{\alpha} \ldots \\ \ldots + (1-\delta)K_{t+1} - (1+r_t)K_{t+1} + (1+r_t)(1-\delta)K_t\Big]\end{aligned}$$

$$\max_{N_t, K_{t+1}} V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t} \left[A_{t+1} K_{t+1}^{\alpha} \ldots \right.$$
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$$\frac{\partial V_t}{\partial N_t} =$$

$$\max_{N_t, K_{t+1}} V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t} \left[A_{t+1} K_{t+1}^{\alpha} \ldots \right.$$

$$\left. \ldots + (1-\delta) K_{t+1} - (1+r_t) K_{t+1} + (1+r_t)(1-\delta) K_t \right]$$

$$\frac{\partial V_t}{\partial N_t} = (1-\alpha) A_t \left(\frac{K_t}{N_t} \right)^{\alpha} - w_t = 0 \Rightarrow w_t = (1-\alpha) A_t \left(\frac{K_t}{N_t} \right)^{\alpha}$$

$$\max_{N_t, K_{t+1}} V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t} \left[A_{t+1} K_{t+1}^{\alpha} \ldots \right.$$
$$\ldots \left. + (1-\delta) K_{t+1} - (1+r_t) K_{t+1} + (1+r_t)(1-\delta) K_t \right]$$

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$$\left. \ldots + (1-\delta) K_{t+1} - (1+r_t) K_{t+1} + (1+r_t)(1-\delta) K_t\right]$$

$$\frac{\partial V_t}{\partial N_t} = (1-\alpha) A_t \left(\frac{K_t}{N_t}\right)^{\alpha} - w_t = 0 \Rightarrow w_t = (1-\alpha) A_t \left(\frac{K_t}{N_t}\right)^{\alpha}$$
$$\frac{\partial V_t}{\partial K_{t+1}} = \frac{1}{1+r_t}\left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) - (1+r_t)\right] = 0$$

$$
\begin{aligned}
\max_{N_t, K_{t+1}} V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t} \big[A_{t+1} K_{t+1}^{\alpha} \ldots \\
\ldots + (1-\delta) K_{t+1} - (1+r_t) K_{t+1} + (1+r_t)(1-\delta) K_t \big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_t}{\partial N_t} =& (1-\alpha) A_t \left(\frac{K_t}{N_t} \right)^{\alpha} - w_t = 0 \Rightarrow w_t = (1-\alpha) A_t \left(\frac{K_t}{N_t} \right)^{\alpha} \\
\frac{\partial V_t}{\partial K_{t+1}} =& \frac{1}{1+r_t} \left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) - (1+r_t) \right] = 0 \\
& \frac{1}{1+r_t} \left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right] = 1
\end{aligned}
$$

$$\max_{N_t, K_{t+1}} V_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t + \frac{1}{1+r_t}\left[A_{t+1} K_{t+1}^{\alpha} \dots\right.$$
$$\left.\dots + (1-\delta)K_{t+1} - (1+r_t)K_{t+1} + (1+r_t)(1-\delta)K_t\right]$$

$$\frac{\partial V_t}{\partial N_t} = (1-\alpha)A_t\left(\frac{K_t}{N_t}\right)^{\alpha} - w_t = 0 \Rightarrow w_t = (1-\alpha)A_t\left(\frac{K_t}{N_t}\right)^{\alpha}$$
$$\frac{\partial V_t}{\partial K_{t+1}} = \frac{1}{1+r_t}\left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) - (1+r_t)\right] = 0$$
$$\frac{1}{1+r_t}\left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta)\right] = 1$$
$$\left[\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta)\right] = 1 + r_t$$

d) Re-arrange the first order optimality conditions to derive explicit expression for the demand for N_t and the demand for K_{t+1}.

$$\frac{\partial V_t}{\partial N_t}: \quad w_t = (1-\alpha)A_t\left(\frac{K_t}{N_t}\right)^{\alpha}$$

d) Re-arrange the first order optimality conditions to derive explicit expression for the demand for N_t and the demand for K_{t+1}.

$$\frac{\partial V_t}{\partial N_t}: \quad w_t = (1-\alpha)A_t\left(\frac{K_t}{N_t}\right)^{\alpha}$$

$$N_t = \left(\frac{1-\alpha}{w_t}A_t K_t^{\alpha}\right)^{\frac{1}{\alpha}} = \left(\frac{1-\alpha}{w_t}A_t\right)^{\frac{1}{\alpha}} K_t$$

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$$N_t = \left(\frac{1-\alpha}{w_t}A_tK_t^{\alpha}\right)^{\frac{1}{\alpha}} = \left(\frac{1-\alpha}{w_t}A_t\right)^{\frac{1}{\alpha}}K_t$$

$$\frac{\partial V_t}{\partial K_{t+1}}: \quad 1 + r_t = \alpha A_{t+1}K_{t+1}^{\alpha-1} + (1-\delta)$$

d) Re-arrange the first order optimality conditions to derive explicit expression for the demand for N_t and the demand for K_{t+1}.

$$\frac{\partial V_t}{\partial N_t}: \quad w_t = (1-\alpha) A_t \left(\frac{K_t}{N_t}\right)^{\alpha}$$

$$N_t = \left(\frac{1-\alpha}{w_t} A_t K_t^{\alpha}\right)^{\frac{1}{\alpha}} = \left(\frac{1-\alpha}{w_t} A_t\right)^{\frac{1}{\alpha}} K_t$$

$$\frac{\partial V_t}{\partial K_{t+1}}: \quad 1 + r_t = \alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta)$$

$$K_{t+1}^{\alpha-1} = \frac{(1+r_t) - (1-\delta)}{\alpha A_{t+1}} = \frac{r_t + \delta}{\alpha A_{t+1}}$$

d) Re-arrange the first order optimality conditions to derive explicit expression for the demand for N_t and the demand for K_{t+1}.

$$\frac{\partial V_t}{\partial N_t}: \quad w_t = (1-\alpha)A_t\left(\frac{K_t}{N_t}\right)^{\alpha}$$

$$N_t = \left(\frac{1-\alpha}{w_t}A_tK_t^{\alpha}\right)^{\frac{1}{\alpha}} = \left(\frac{1-\alpha}{w_t}A_t\right)^{\frac{1}{\alpha}}K_t$$

$$\frac{\partial V_t}{\partial K_{t+1}}: \quad 1+r_t = \alpha A_{t+1}K_{t+1}^{\alpha-1} + (1-\delta)$$

$$K_{t+1}^{\alpha-1} = \frac{(1+r_t)-(1-\delta)}{\alpha A_{t+1}} = \frac{r_t+\delta}{\alpha A_{t+1}}$$

$$K_{t+1} = \left(\frac{r_t+\delta}{\alpha A_{t+1}}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha A_{t+1}}{r_t+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$K_{t+1} = \left(\frac{\alpha A_{t+1}}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

e) Why is investment increasing in future productivity but is not affected by current productivity?

$$K_{t+1} = \left(\frac{\alpha A_{t+1}}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

e) Why is investment increasing in future productivity but is not affected by current productivity?

$$K_{t+1} = K^d(??)$$

$$K_{t+1} = \left(\frac{\alpha A_{t+1}}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

e) Why is investment increasing in future productivity but is not affected by current productivity?

$$K_{t+1} = K^d(r_t, A_{t+1})$$

$$K_{t+1} = \left(\frac{\alpha A_{t+1}}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

e) Why is investment increasing in future productivity but is not affected by current productivity?

$$K_{t+1} = K^d(\underbrace{r_t}_{-}, \underbrace{A_{t+1}}_{+})$$

$$K_{t+1} = \left(\frac{\alpha A_{t+1}}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

e) Why is investment increasing in future productivity but is not affected by current productivity?

$$K_{t+1} = K^d(\underbrace{r_t}_{-}, \underbrace{A_{t+1}}_{+})$$

$$I_t = I^d(r_t, A_{t+1}, K_t)$$

$$K_{t+1} = \left(\frac{\alpha A_{t+1}}{r_t + \delta} \right)^{\frac{1}{1-\alpha}}$$

e) Why is investment increasing in future productivity but is not affected by current productivity?

$$K_{t+1} = K^d(\underbrace{r_t}_{-}, \underbrace{A_{t+1}}_{+})$$

$$I_t = I^d(\underbrace{r_t}_{-}, \underbrace{A_{t+1}}_{+}, \underbrace{K_t}_{-})$$

f) Re-arrange your answer from the previous part, making use of the capital accumulation equation, to solve for an expression for I_t.

$$I_t =$$

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f) Re-arrange your answer from the previous part, making use of the capital accumulation equation, to solve for an expression for I_t.

$$I_t = K_{t+1} - (1-\delta)K_t = \left(\frac{\alpha A_{t+1}}{r_t + \delta}\right)^{\frac{1}{1-\alpha}} - (1-\delta)K_t$$

Question:
Consider a representative agent with the utility function

$$U = \ln C_t + \theta \ln(1 - N_t)$$

The budget constraint is

$$C_t + w_t(1 - N_t) = w_t + \Pi_t$$

where w_t is the wage and Π_t is non-wage income.

a) Solve for the optimal quantities of consumption and labor.

Household problem:

$$\max_{C_t, N_t} U = \ln C_t + \theta \ln(1 - N_t)$$
$$\text{s.t.} \quad C_t + w_t(1 - N_t) = w_t + \Pi_t$$

Household problem:

$$\max_{C_t, N_t} U = \ln C_t + \theta \ln(1 - N_t)$$

$$\text{s.t.} \quad C_t + w_t(1 - N_t) = w_t + \Pi_t$$

Reformulate:

$$\max_{C_t, L_t} U = \ln C_t + \theta \ln L_t$$

$$\text{s.t.} \quad C_t + w_t L_t = w_t + \Pi_t$$

Household problem:

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Reformulate:

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$$\text{s.t.} \quad C_t + w_t L_t = w_t + \Pi_t$$

Lagrange:

$$\mathcal{L} = \ln C_t + \theta \ln L_t + \lambda(C_t + w_t L_t - w_t - \Pi_t)$$

Household problem:

$$\max_{C_t, N_t} U = \ln C_t + \theta \ln(1 - N_t)$$
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$$\mathcal{L} = \ln C_t + \theta \ln L_t + \lambda(C_t + w_t L_t - w_t - \Pi_t)$$

Solution:

$$L_t = \frac{\theta}{w_t} C_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$C_t + w_t L_t = w_t + \Pi_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$C_t + w_t L_t = w_t + \Pi_t$$
$$C_t + w_t \frac{\theta}{w_t} C_t = w_t + \Pi_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned} C_t + w_t L_t &= w_t + \Pi_t \\ C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t \\ C_t &= \frac{w_t + \Pi_t}{1 + \theta} \end{aligned}$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned} C_t + w_t L_t &= w_t + \Pi_t \\ C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t \\ C_t &= \frac{w_t + \Pi_t}{1 + \theta} \end{aligned}$$

Leisure/labor supply:

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned} C_t + w_t L_t &= w_t + \Pi_t \\ C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t \\ C_t &= \frac{w_t + \Pi_t}{1+\theta} \end{aligned}$$

Leisure/labor supply:

$$L_t = \frac{\theta}{w_t} C_t = \frac{\theta}{w_t} \left(\frac{w_t + \Pi_t}{1+\theta} \right) = \frac{\theta}{1+\theta} \left(\frac{w_t + \Pi_t}{w_t} \right)$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned} C_t + w_t L_t &= w_t + \Pi_t \\ C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t \\ C_t &= \frac{w_t + \Pi_t}{1+\theta} \end{aligned}$$

Leisure/labor supply:

$$\begin{aligned} L_t &= \frac{\theta}{w_t} C_t = \frac{\theta}{w_t} \left(\frac{w_t + \Pi_t}{1+\theta} \right) = \frac{\theta}{1+\theta} \left(\frac{w_t + \Pi_t}{w_t} \right) \\ N_t &= 1 - L_t = 1 - \frac{\theta}{1+\theta} \left(\frac{w_t + \Pi_t}{w_t} \right) \end{aligned}$$

Question:
Consider a representative agent with the utility function

$$U = \ln C_t + \theta \ln(1 - N_t)$$

The budget constraint is

$$C_t + w_t(1 - N_t) = w_t + \Pi_t$$

where w_t is the wage and Π_t is non-wage income.

b) Suppose that the government implements a lump sum subsidy to all workers. The budget constraint is now

$$C_t + w_t(1 - N_t) = w_t + \Pi_t + T_t$$

How are the optimal quantities of C_t and N_t affected by the introduction of the subsidy? Specifically, do people consume more or less leisure? What is the economic intuition for this?

Optimality condition:

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$$L_t = \frac{\theta}{w_t} C_t$$

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Plugging into budget constraint:

$$C_t + w_t L_t = w_t + \Pi_t + T_t$$

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Plugging into budget constraint:

$$C_t + w_t L_t = w_t + \Pi_t + T_t$$
$$C_t + w_t \frac{\theta}{w_t} C_t = w_t + \Pi_t + T_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned} C_t + w_t L_t &= w_t + \Pi_t + T_t \\ C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t + T_t \\ C_t &= \frac{w_t + \Pi_t + T_t}{1+\theta} \\ L_t &= \frac{\theta}{w_t} \frac{w_t + \Pi_t + T_t}{1+\theta} \end{aligned}$$

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned}
C_t + w_t L_t &= w_t + \Pi_t + T_t \\
C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t + T_t \\
C_t &= \frac{w_t + \Pi_t + {\color{red}T_t}}{1+\theta} \qquad > \frac{w_t + \Pi_t}{1+\theta} \qquad | T_t > 0 \\
L_t &= \frac{\theta}{1+\theta} \left(\frac{w_t + \Pi_t + {\color{red}T_t}}{w_t} \right)
\end{aligned}$$

Economic Intuition?

Optimality condition:

$$L_t = \frac{\theta}{w_t} C_t$$

Plugging into budget constraint:

$$\begin{aligned}
C_t + w_t L_t &= w_t + \Pi_t + T_t \\
C_t + w_t \frac{\theta}{w_t} C_t &= w_t + \Pi_t + T_t \\
C_t &= \frac{w_t + \Pi_t + {\color{red}T_t}}{1+\theta} && > \frac{w_t + \Pi_t}{1+\theta} && |T_t > 0 \\
L_t &= \frac{\theta}{1+\theta}\left(\frac{w_t + \Pi_t + {\color{red}T_t}}{w_t}\right) && > \frac{\theta}{1+\theta}\left(\frac{w_t + \Pi_t}{w_t}\right) && |T_t > 0
\end{aligned}$$

Economic Intuition?

- Pure income effect!

Question:
Consider a representative agent with the utility function

$$U = \ln C_t + \theta \ln(1 - N_t)$$

The budget constraint is

$$C_t + w_t(1 - N_t) = w_t + \Pi_t$$

where w_t is the wage and Π_t is non-wage income.

c) Instead of a lump sum subsidy, suppose the government subsidies work. With the subsidy, the workers receive an effective wage rate of $w_t(1 + \tau_t)$. The budget constraint is

$$C_t + w_t(1 + \tau_t)(1 - N_t) = w_t(1 + \tau_t) + \Pi_t$$

How are the optimal quantities of C and N affected by the introduction of the subsidy? Specifically, do people consume more or less leisure? What is the economic intuition for this?

Household problem:

$$\max_{C_t, L_t} U = \ln C_t + \theta \ln L_t$$
$$\text{s.t.} \quad C_t + w_t(1+\tau_t)L_t = w_t(1+\tau_t) + \Pi_t$$

Household problem:

$$\max_{C_t, L_t} U = \ln C_t + \theta \ln L_t$$

$$\text{s.t.} \quad C_t + w_t(1+\tau_t)L_t = w_t(1+\tau_t) + \Pi_t$$

Lagrange:

$$\mathcal{L} = \ln C_t + \theta \ln L_t + \lambda(C_t + w_t(1+\tau_t)L_t - w_t(1+\tau_t) - \Pi_t)$$

Household problem:

$$\max_{C_t, L_t} U = \ln C_t + \theta \ln L_t$$
$$\text{s.t.} \quad C_t + w_t(1+\tau_t)L_t = w_t(1+\tau_t) + \Pi_t$$

Lagrange:

$$\mathcal{L} = \ln C_t + \theta \ln L_t + \lambda(C_t + w_t(1+\tau_t)L_t - w_t(1+\tau_t) - \Pi_t)$$

Optimality condition:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} + \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial L_t} = \frac{\theta}{L_t} + \lambda w_t(1+\tau_t) = 0$$
$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Plugging into budget constraint: C_t

$$C_t + w_t(1+\tau_t)L_t = w_t(1+\tau_t) + \Pi_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Plugging into budget constraint: C_t

$$C_t + w_t(1+\tau_t)L_t = w_t(1+\tau_t) + \Pi_t$$
$$C_t + w_t(1+\tau_t)\frac{\theta}{w_t(1+\tau_t)} C_t = w_t(1+\tau_t) + \Pi_t$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Plugging into budget constraint: C_t

$$\begin{aligned} C_t + w_t(1+\tau_t)L_t &= w_t(1+\tau_t) + \Pi_t \\ C_t + w_t(1+\tau_t)\frac{\theta}{w_t(1+\tau_t)}C_t &= w_t(1+\tau_t) + \Pi_t \\ C_t &= \frac{w_t(1+\tau_t) + \Pi_t}{1+\theta} \end{aligned}$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Plugging into budget constraint: C_t

$$\begin{aligned} C_t + w_t(1+\tau_t)L_t &= w_t(1+\tau_t) + \Pi_t \\ C_t + w_t(1+\tau_t)\frac{\theta}{w_t(1+\tau_t)}C_t &= w_t(1+\tau_t) + \Pi_t \\ C_t &= \frac{w_t(1+\tau_t)+\Pi_t}{1+\theta} > \frac{w_t+\Pi_t}{1+\theta} \quad | \tau_t > 0 \end{aligned}$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Optimal L_t:

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Optimal L_t:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} \left(\frac{w_t(1+\tau_t) + \Pi_t}{1+\theta} \right)$$

Optimality condition:

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Optimal L_t:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} \left(\frac{w_t(1+\tau_t) + \Pi_t}{1+\theta} \right)$$
$$L_t = \frac{\theta}{1+\theta} \left(\frac{w_t(1+\tau_t) + \Pi_t}{w_t(1+\tau_t)} \right)$$

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Optimal L_t:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} \left(\frac{w_t(1+\tau_t) + \Pi_t}{1+\theta} \right)$$

$$L_t = \frac{\theta}{1+\theta} \left(\frac{w_t(1+\tau_t) + \Pi_t}{w_t(1+\tau_t)} \right) < \frac{\theta}{1+\theta} \left(\frac{w_t + \Pi_t}{w_t} \right) \qquad | \tau_t > 0$$

Economic Intuition?

Optimality condition:

$$L_t = \frac{\theta}{w_t(1+\tau_t)} C_t$$

Optimal L_t:

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Economic Intuition?

- Substitution effect

Question:
Ricardian Equivalence

a) Explain the extent you agree with this statement: Ricardian Equivalence shows that government deficits do not matter.

b) List the assumptions of the Ricardian Equivalence theorem.

IBC of the government:

$$G_t + \frac{G_{t+1}}{1+r_t} = T_t + \frac{T_{t+1}}{1+r_t}$$

IBC of the households:

$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t - T_t + D_t + \frac{w_{t+1}N_{t+1} - T_{t+1} + D_{t+1}}{1+r_t}$$

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$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t + D_t + \frac{w_{t+1}N_{t+1} + D_{t+1}}{1+r_t} - \left[T_t + \frac{T_{t+1}}{1+r_t}\right]$$

IBC of the government:

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$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t + D_t + \frac{w_{t+1}N_{t+1} + D_{t+1}}{1+r_t} - \left[T_t + \frac{T_{t+1}}{1+r_t}\right]$$

$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t + D_t + \frac{w_{t+1}N_{t+1} + D_{t+1}}{1+r_t} - \left[G_t + \frac{G_{t+1}}{1+r_t}\right]$$

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$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t + D_t + \frac{w_{t+1}N_{t+1} + D_{t+1}}{1+r_t} - \left[T_t + \frac{T_{t+1}}{1+r_t}\right]$$

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$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t - G_t + D_t + \frac{w_{t+1}N_{t+1} - G_{t+1} + D_{t+1}}{1+r_t}$$

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$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t + D_t + \frac{w_{t+1}N_{t+1} + D_{t+1}}{1+r_t} - \left[T_t + \frac{T_{t+1}}{1+r_t}\right]$$

$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t + D_t + \frac{w_{t+1}N_{t+1} + D_{t+1}}{1+r_t} - \left[G_t + \frac{G_{t+1}}{1+r_t}\right]$$

$$C_t + \frac{C_{t+1}}{1+r_t} = w_t N_t - G_t + D_t + \frac{w_{t+1}N_{t+1} - G_{t+1} + D_{t+1}}{1+r_t}$$

- From household's perspective: the government balances its budget every period

Equilibrium conditions (demand side):

$$C_t = C^d(Y_t - G_t, Y_{t+1} - G_{t+1}, r_t)$$
$$Y_t = C_t + I_t + G_t$$

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- Taxes and government debt do not appear in equilibrium conditions $\Rightarrow$ only G_t and G_{t+1} are relevant for the determination of equilibrium prices and quantities
- **Ricardian Equivalence:** the method of how the government is financing G_t and G_{t+1} is irrelevant for understanding the effects of changes in government expenditures
- Corollary: the *level* of government debt is irrelevant for understanding the equilibrium behavior of the economy

Assumptions:

- Lump sum taxes
- No liquidity constraints
- Households are forward looking and they believe that the government's intertemporal budget constraint must hold
- Government and households have the same lifespan