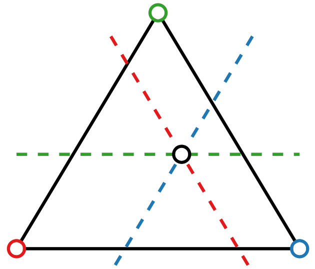
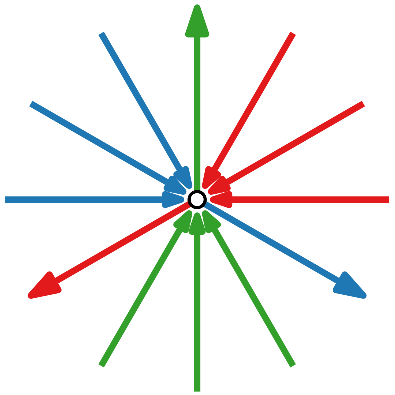
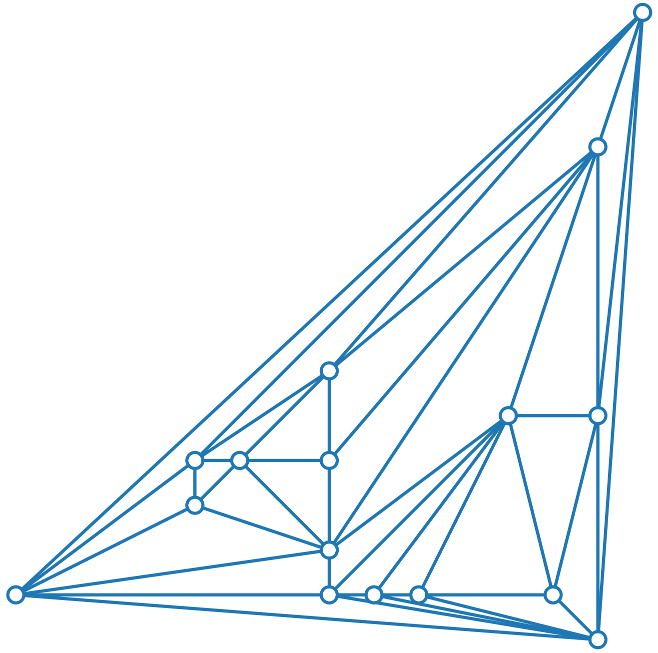
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 4:
Straight-Line Drawings of Planar Graphs II: Schnyder Woods







Part I:
Barycentric Representation

Alexander Wolff

Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

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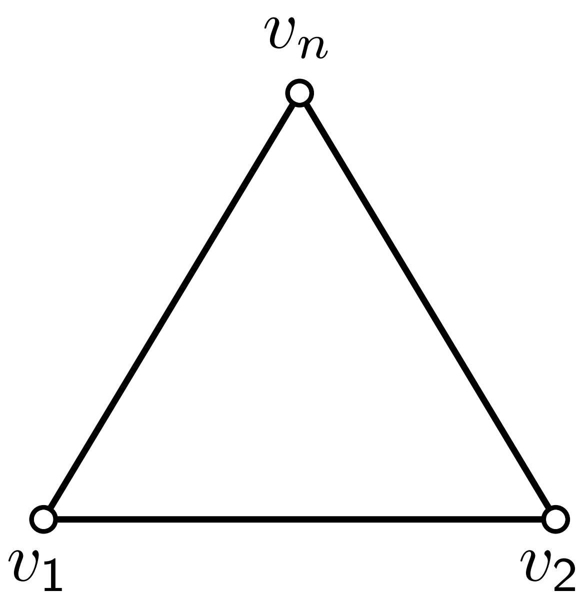

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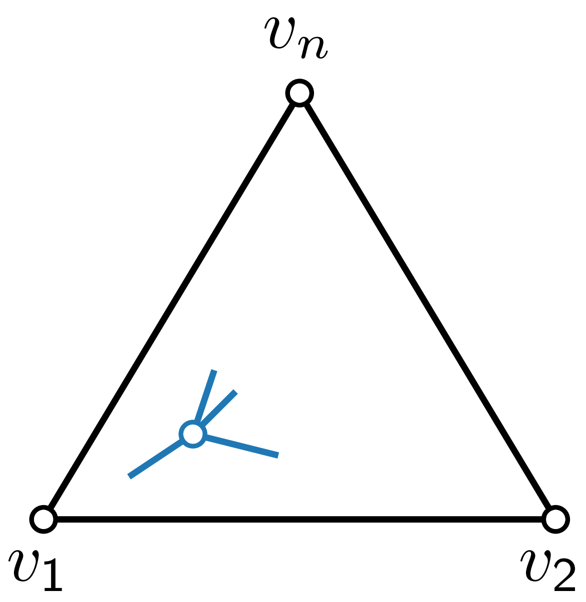

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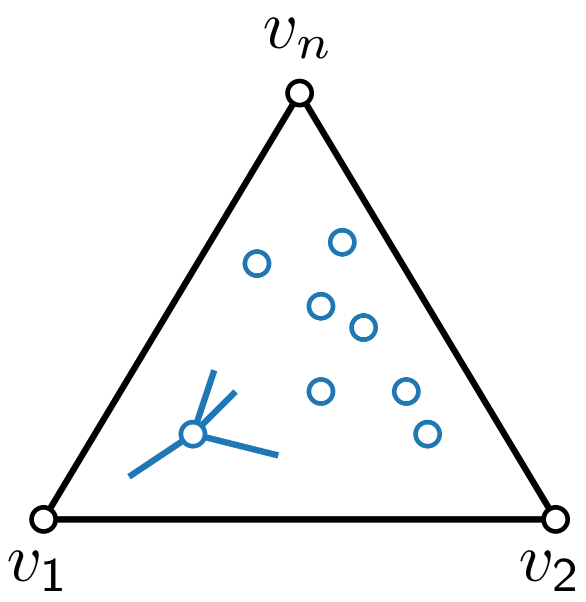

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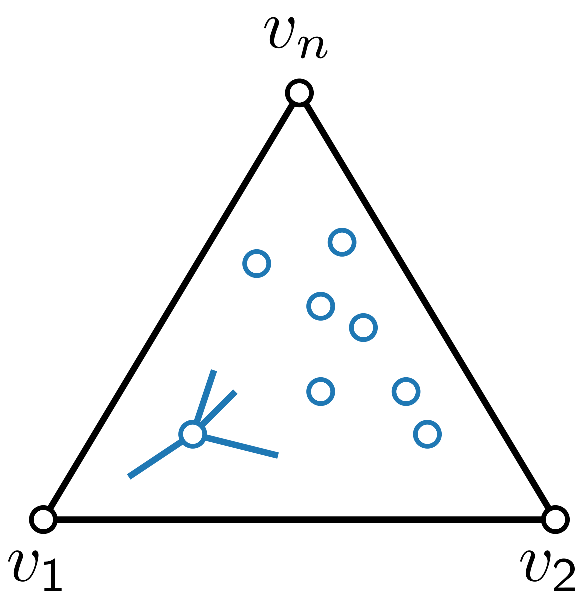

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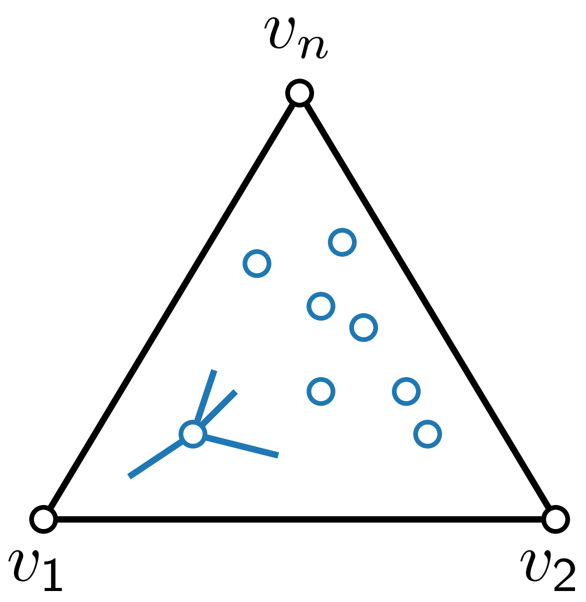

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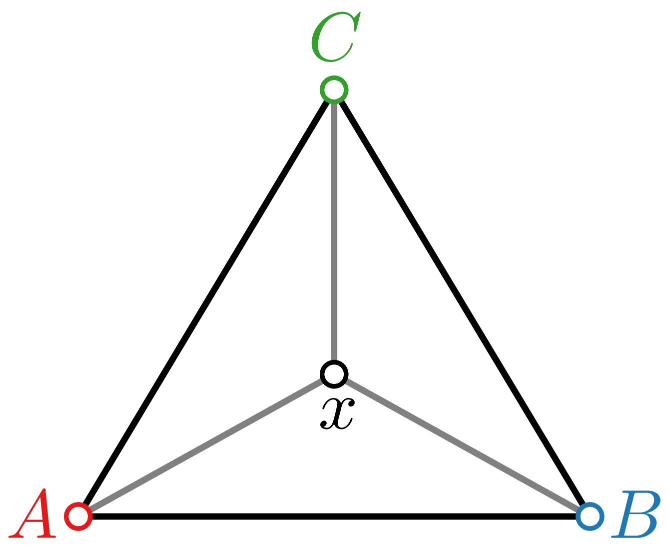
Recall: barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$



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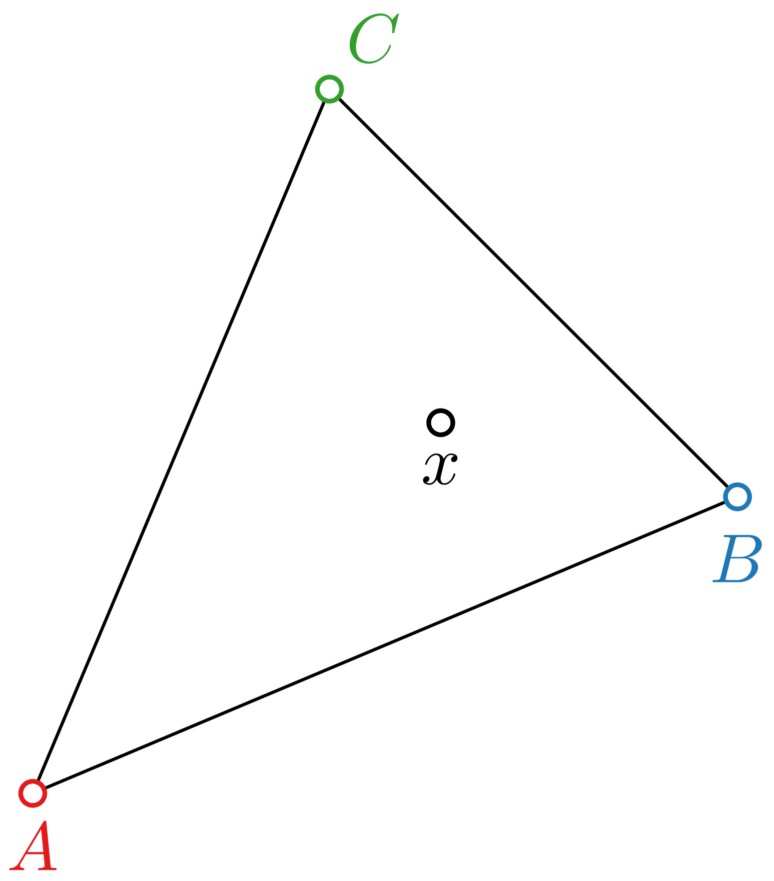

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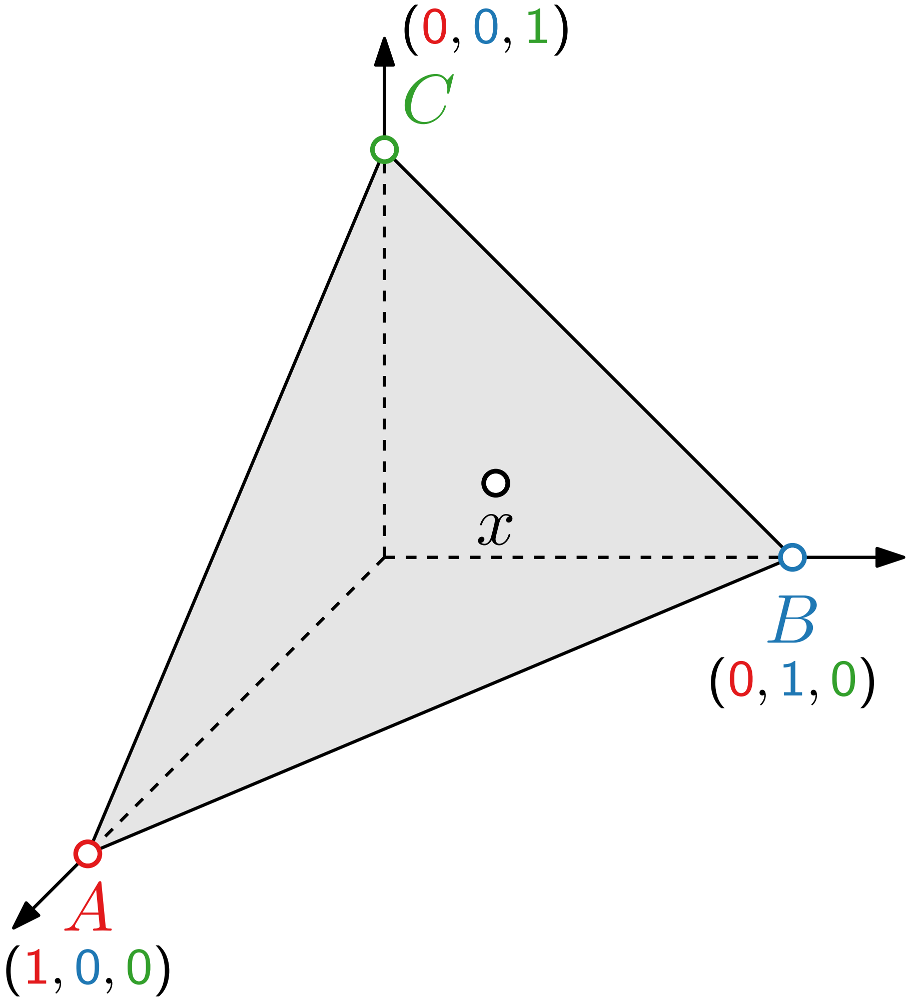

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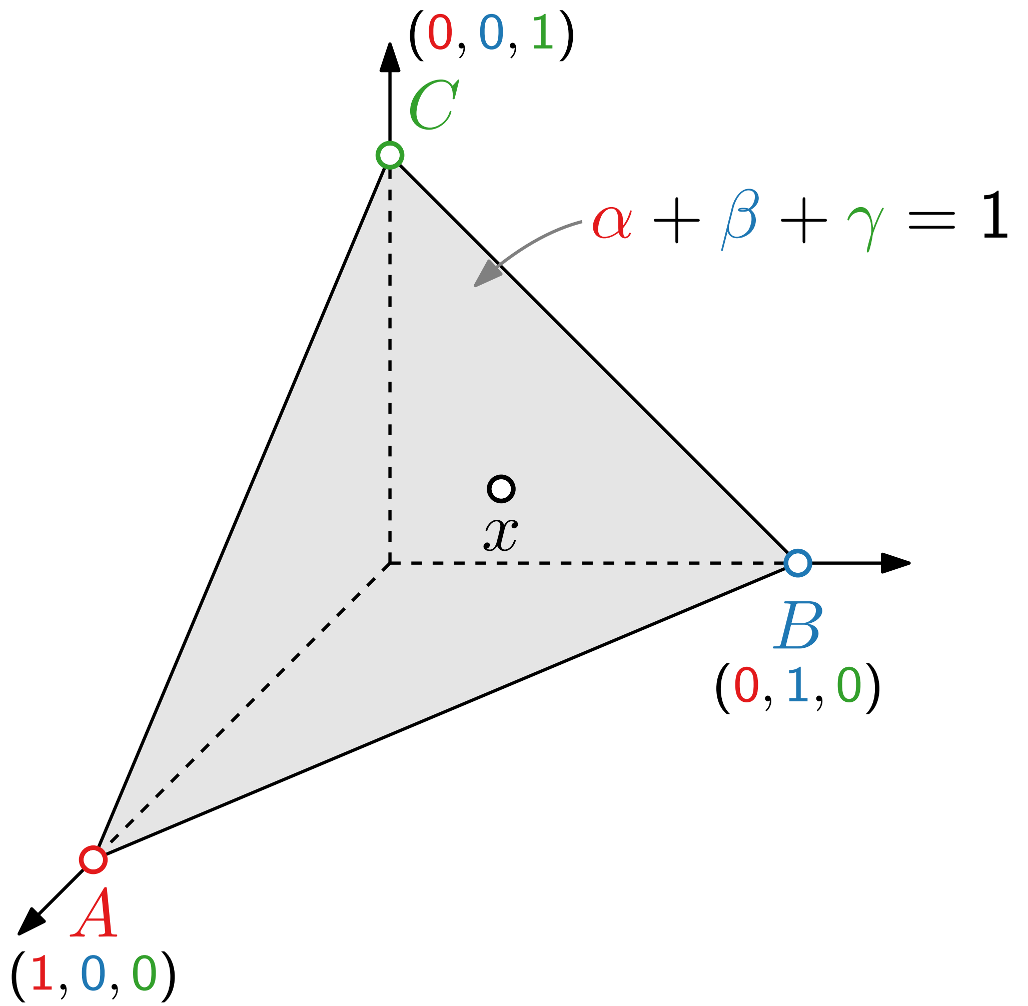

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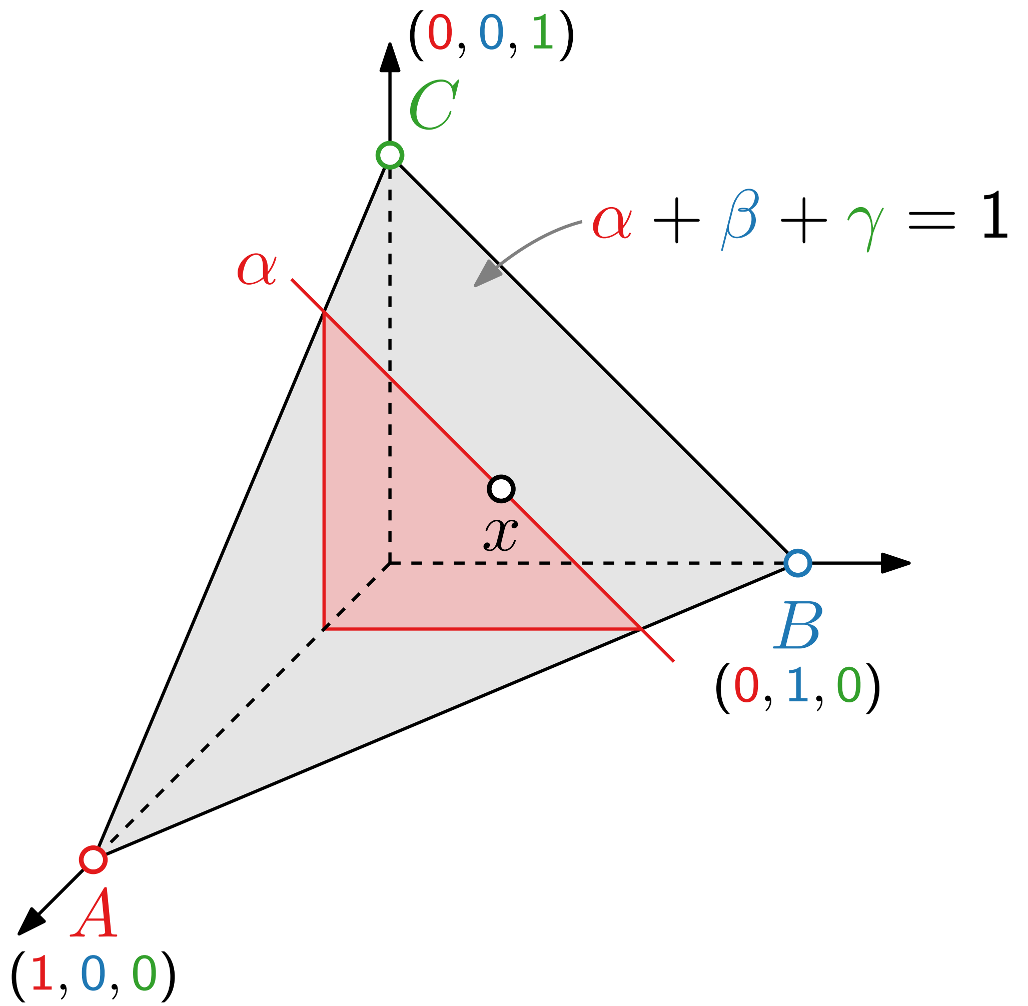

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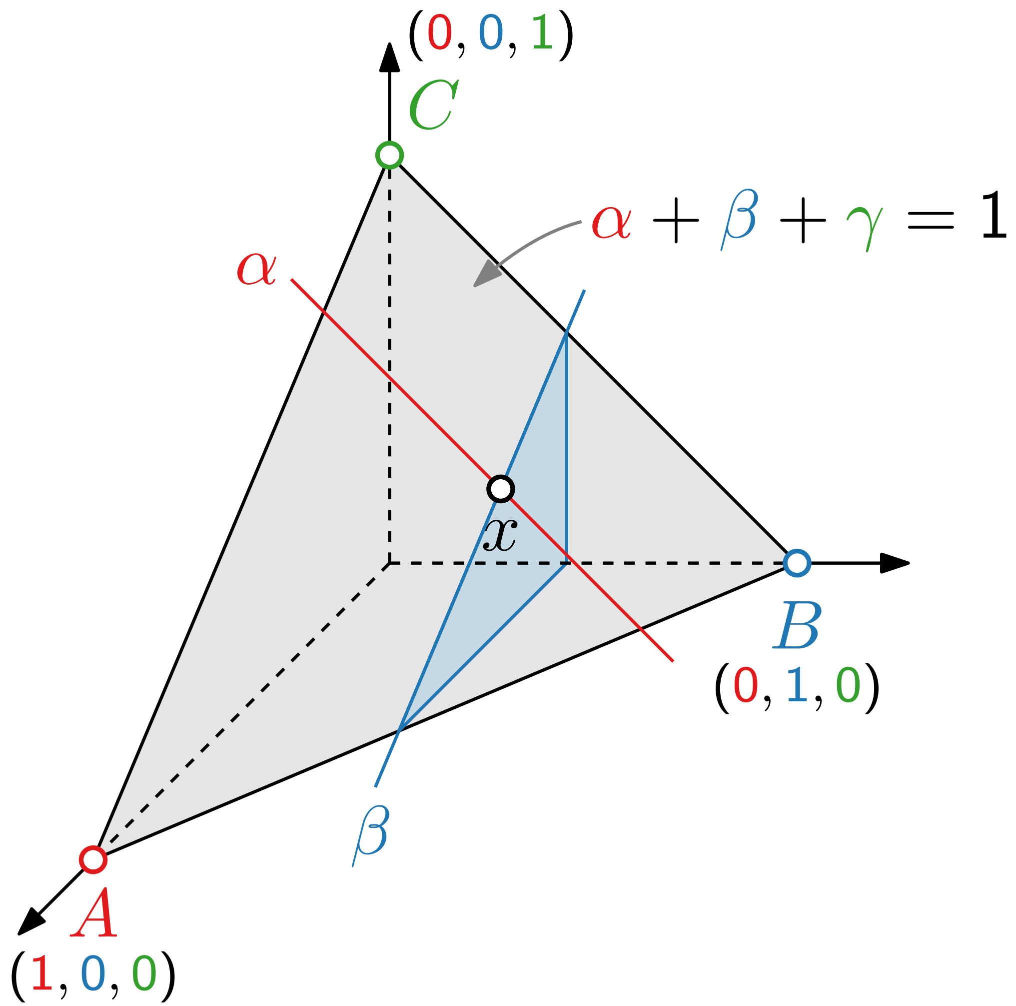

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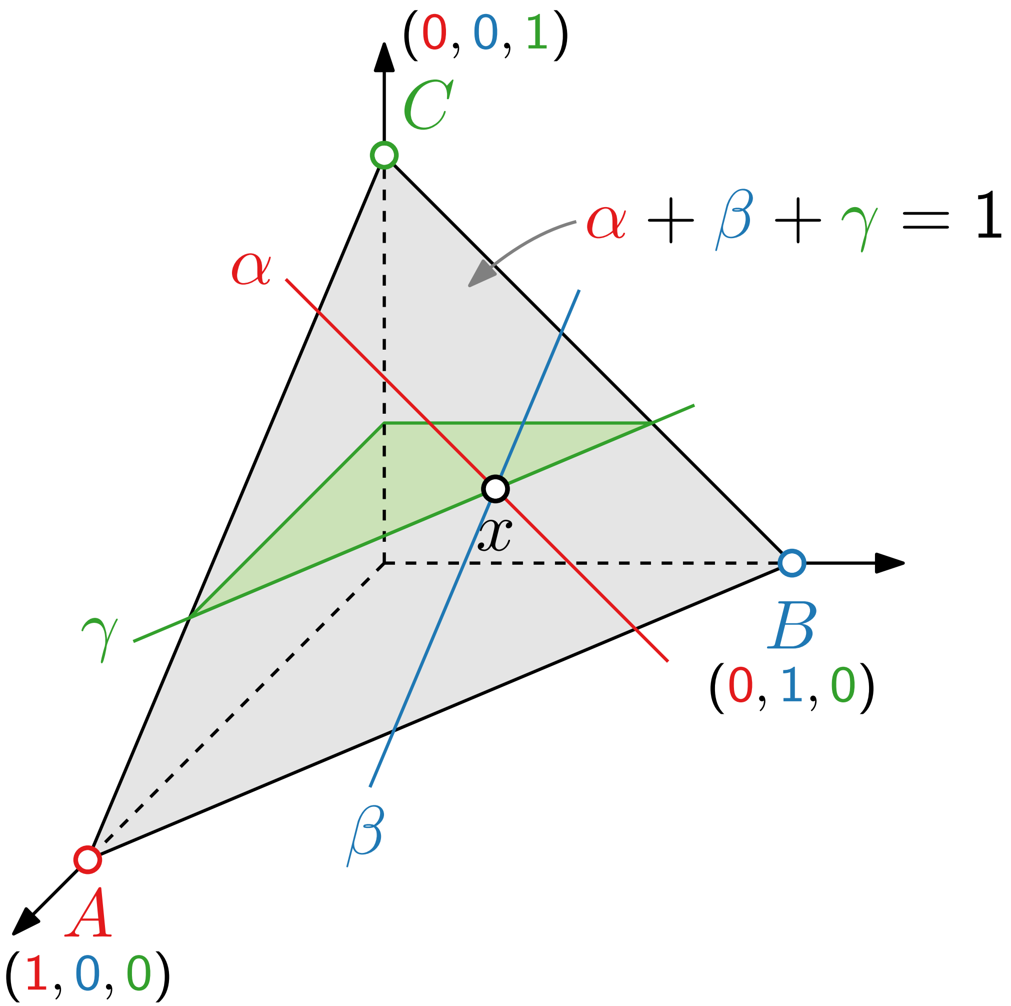

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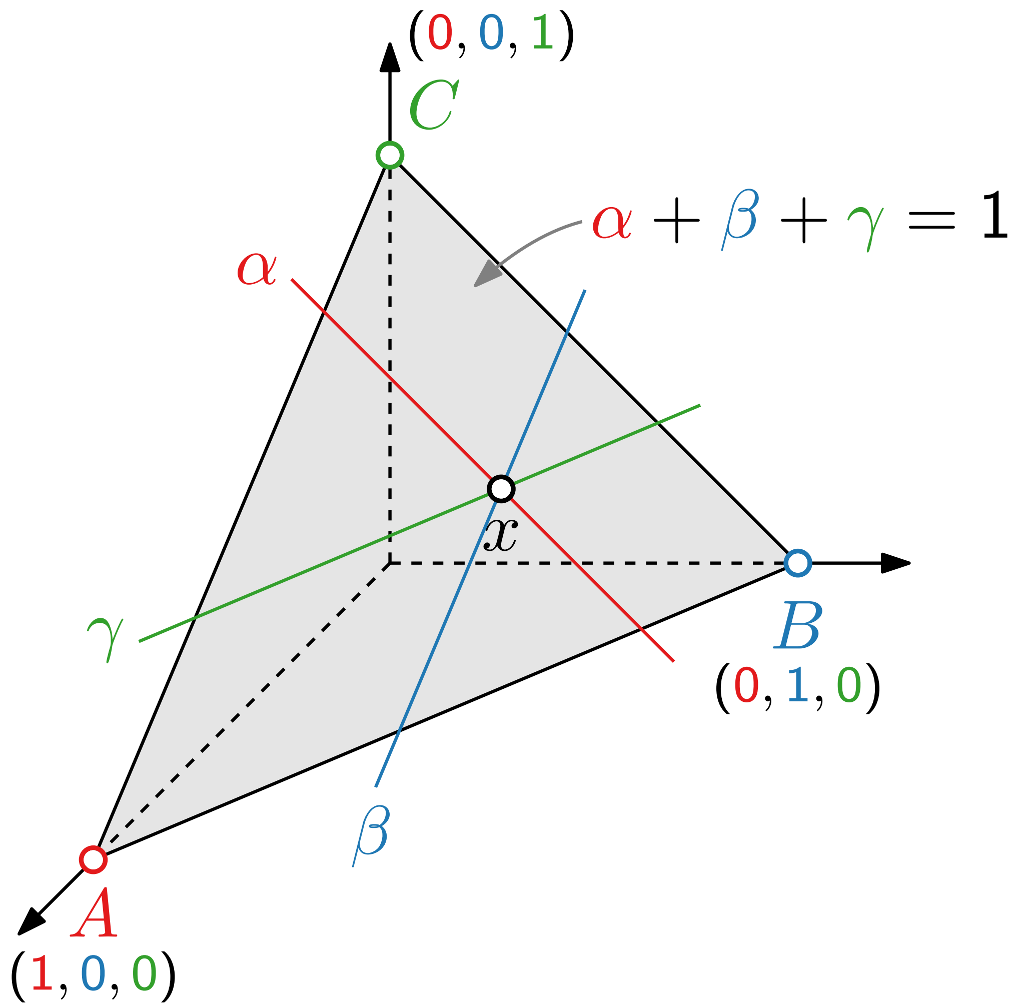

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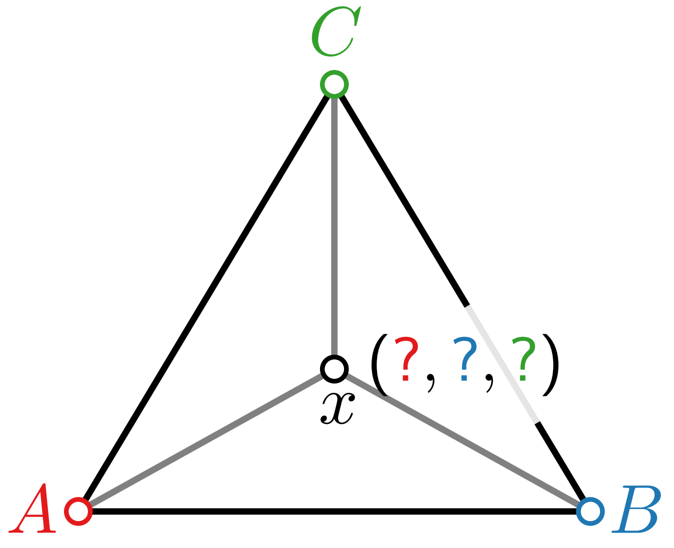

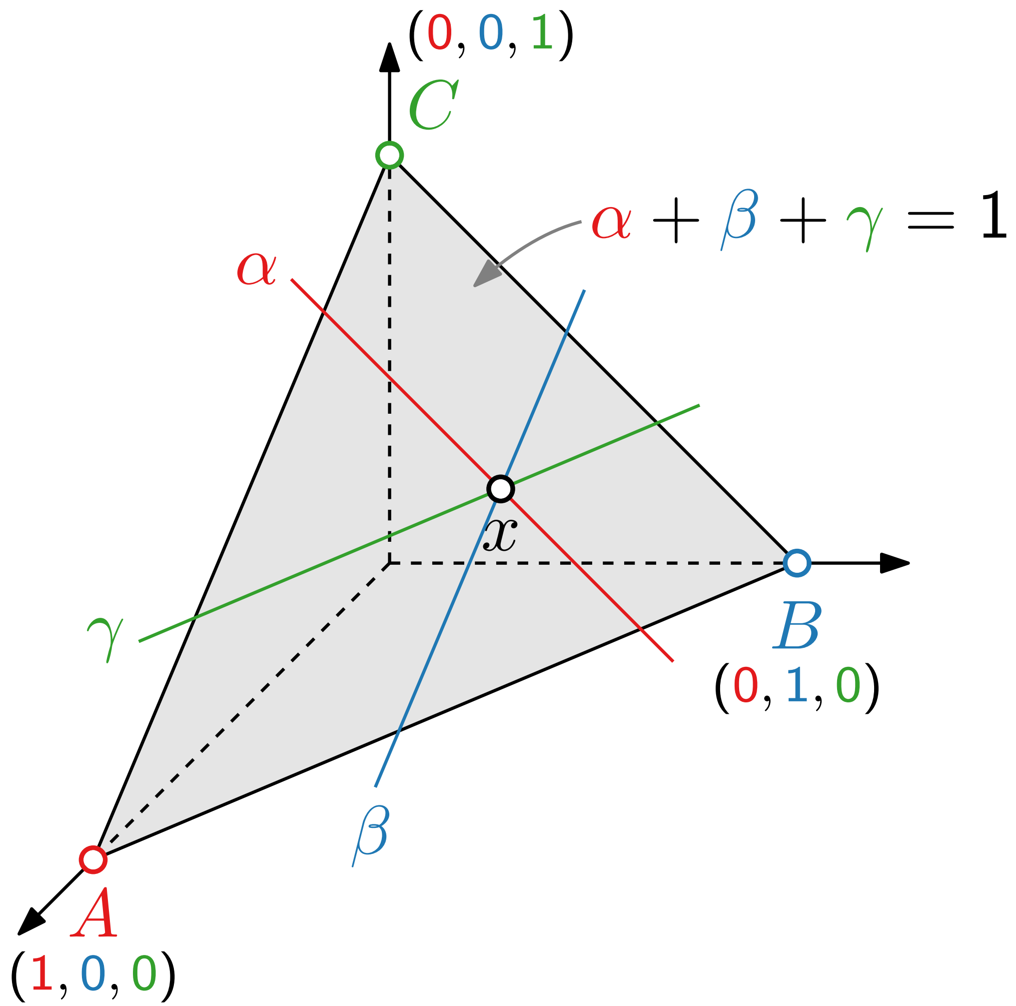

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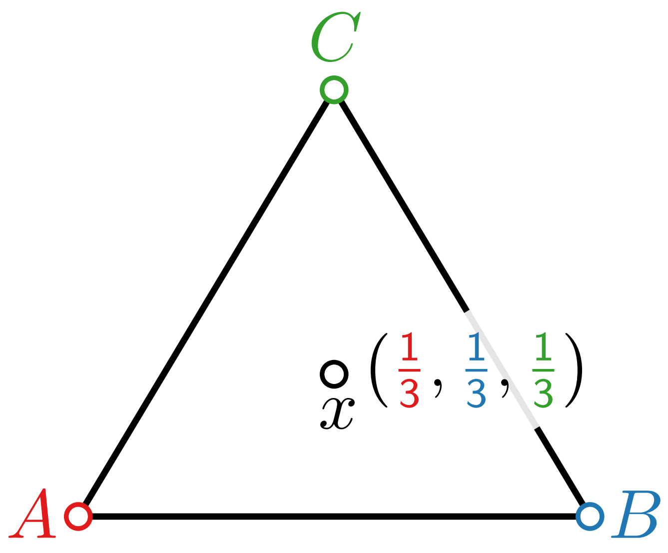

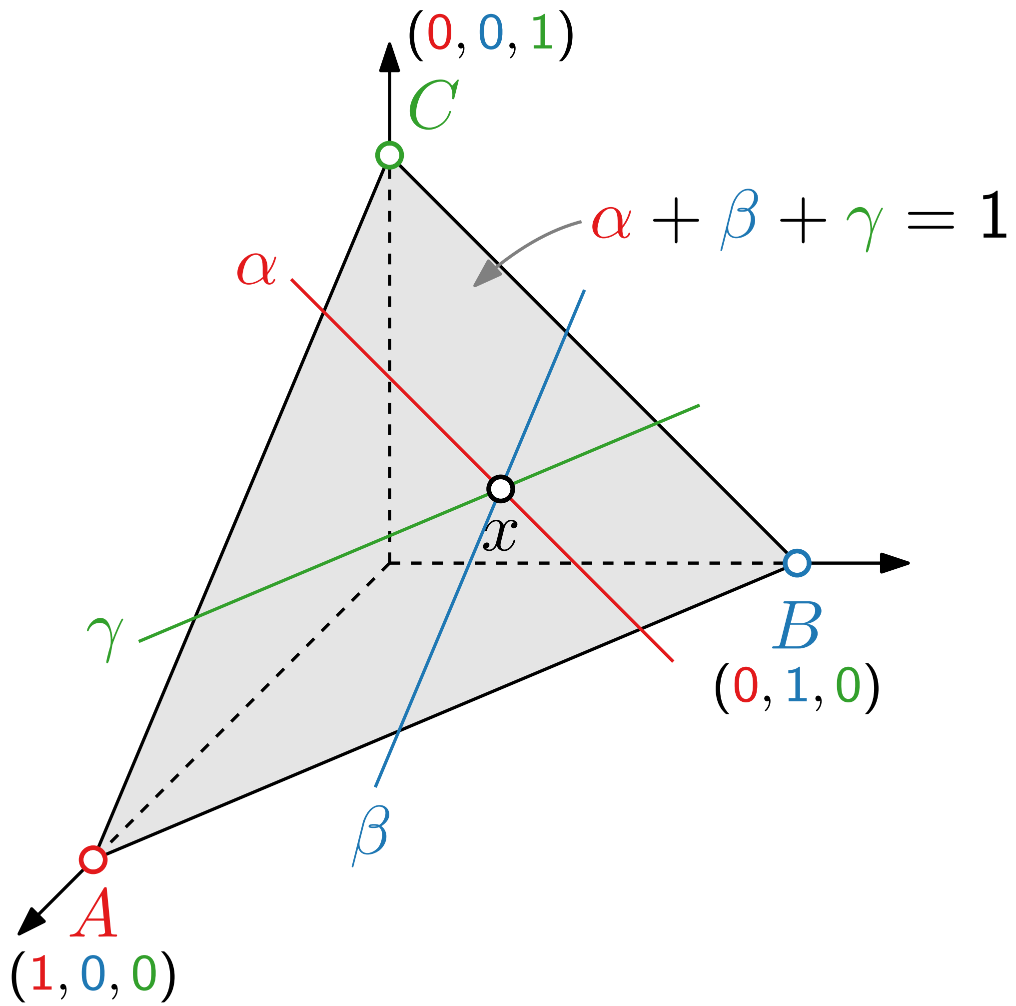

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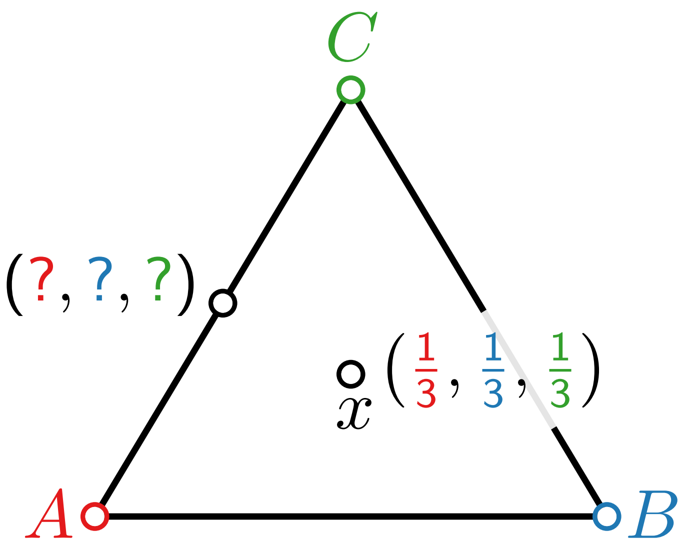

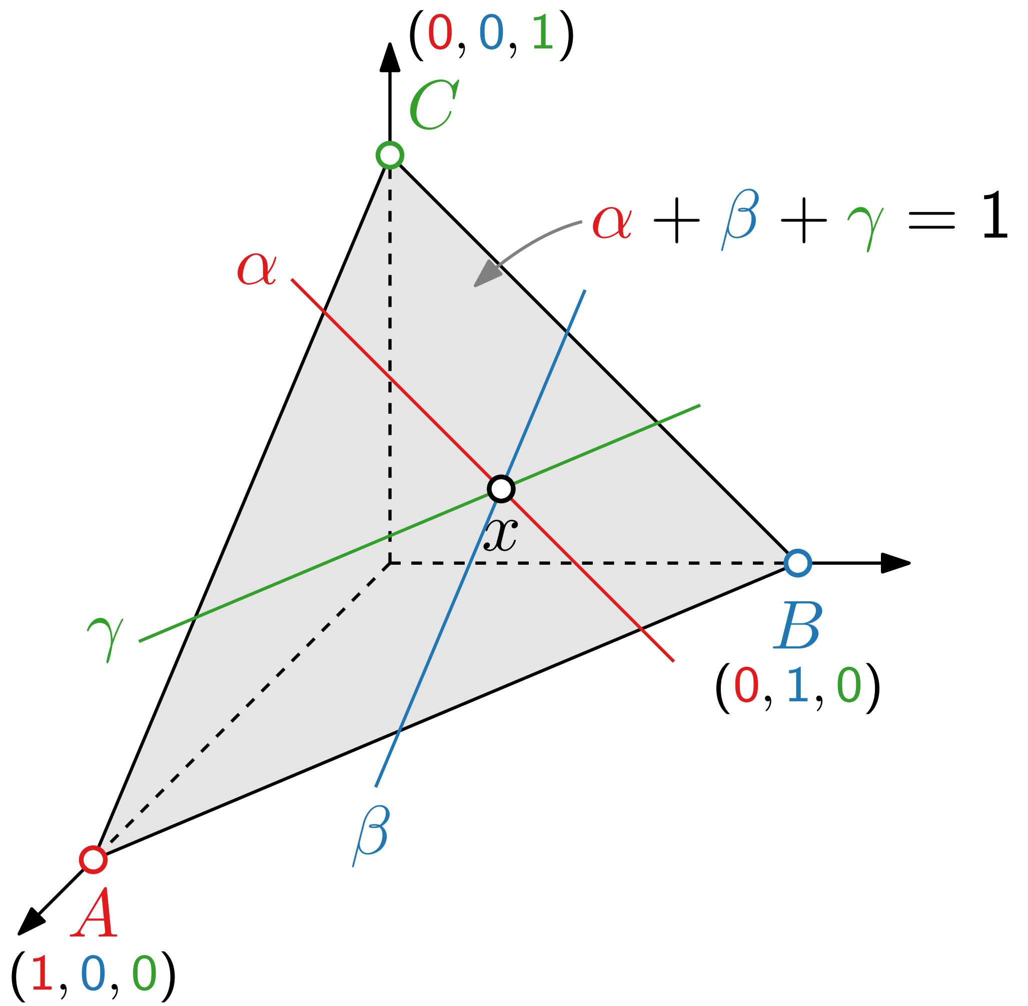

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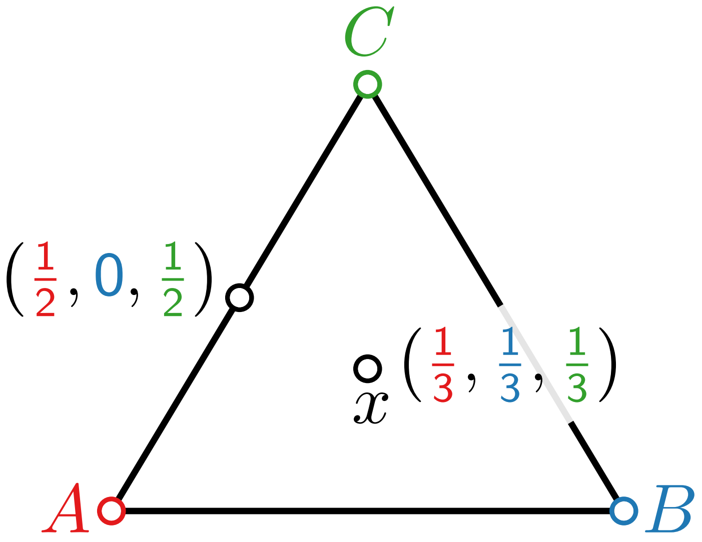

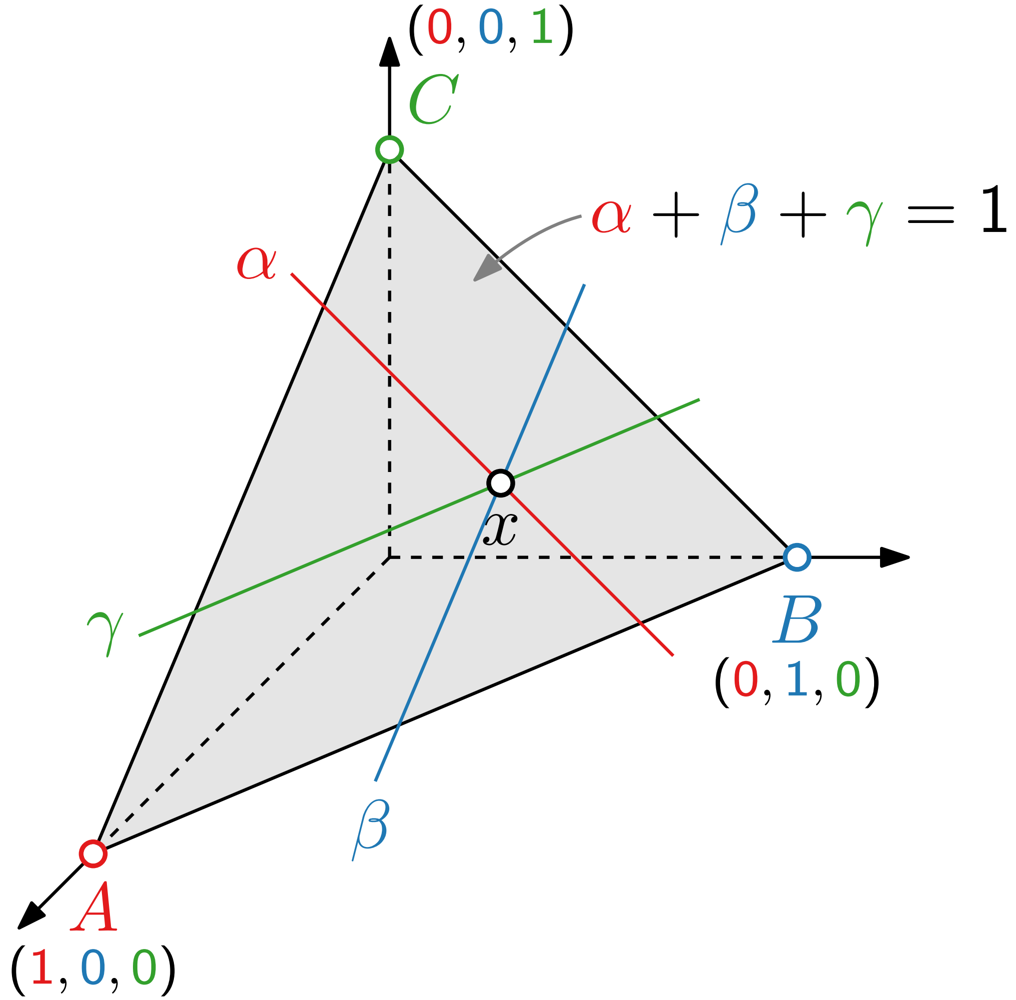

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A **barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of G:

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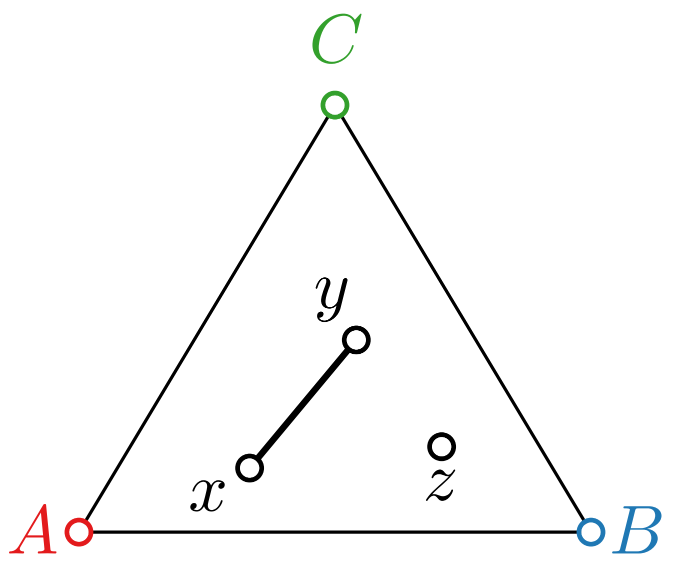

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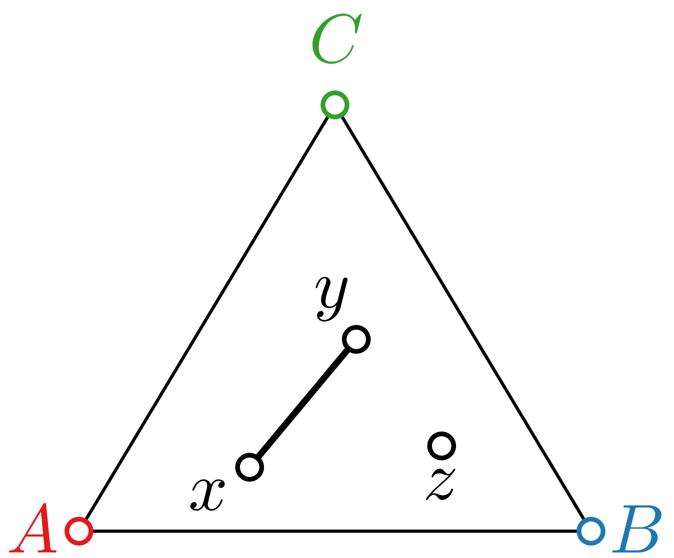

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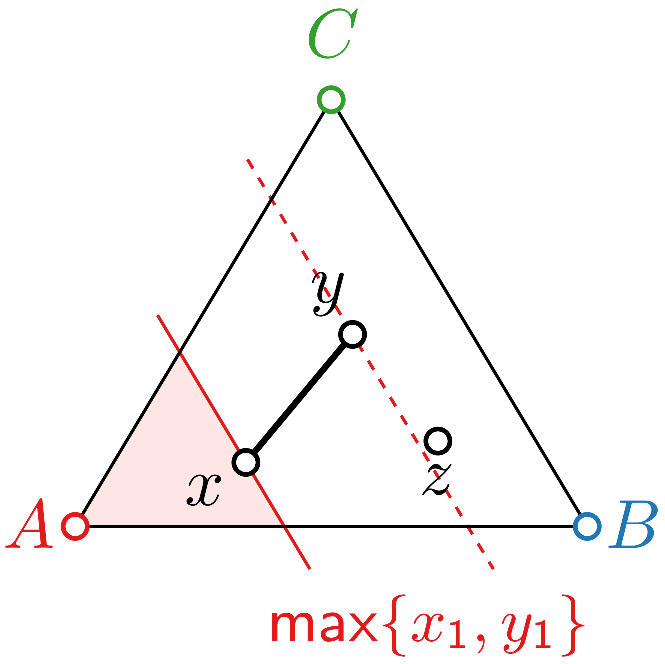

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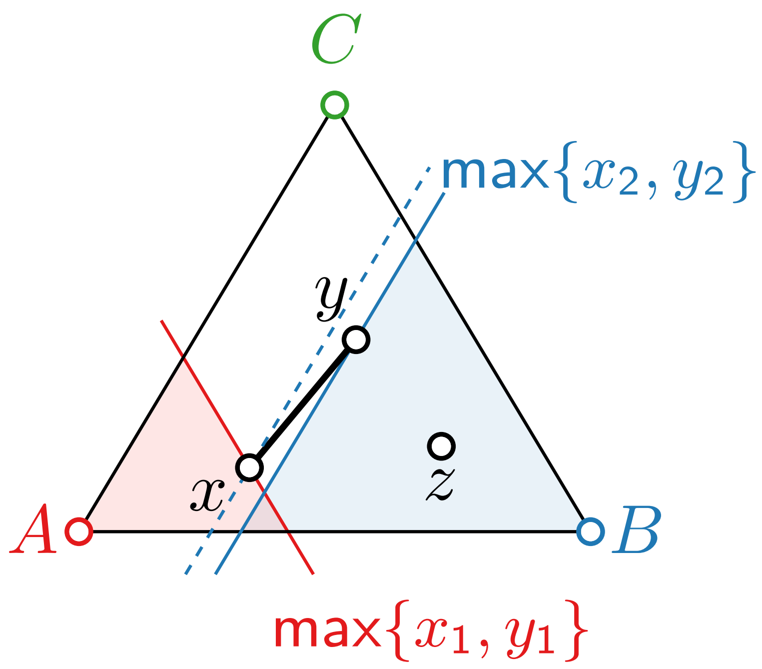

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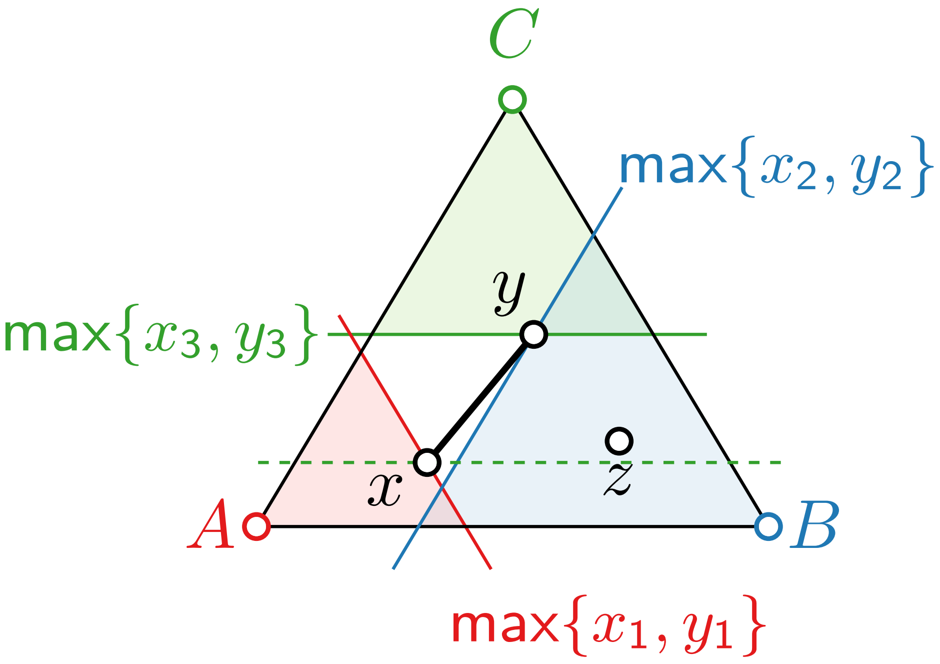

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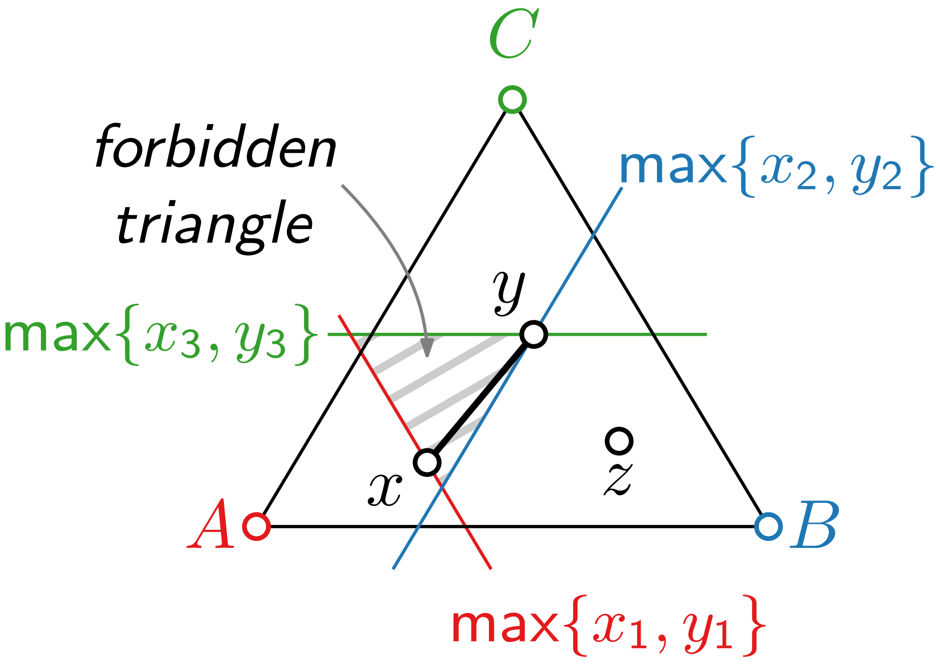
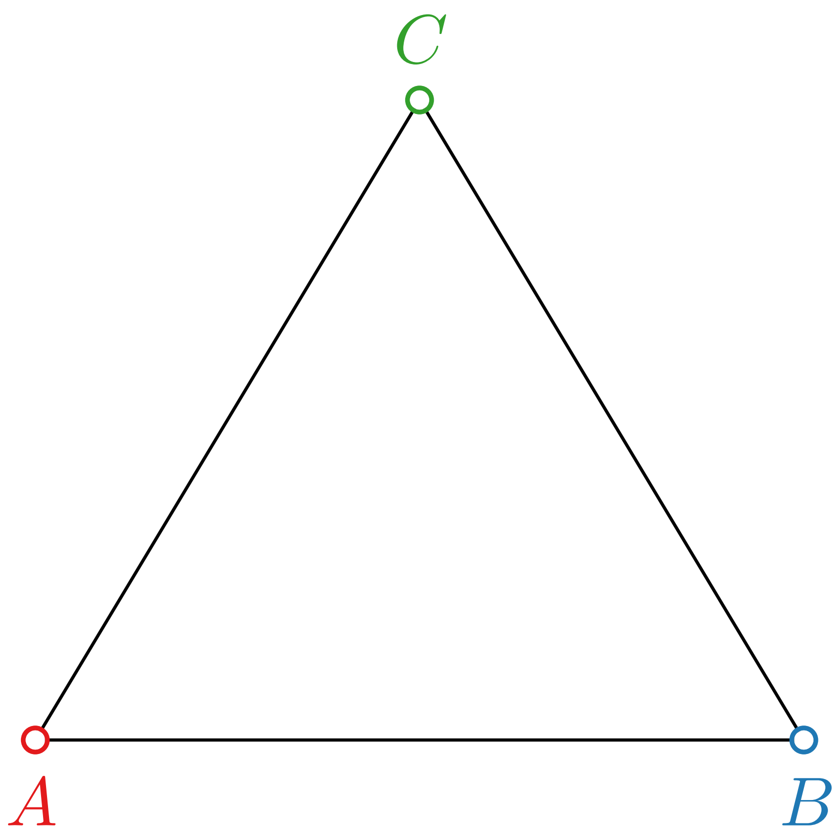


Barycentric Representations of Planar Graphs

Lemma.
Let $f : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G, and let $A, B, C \in \mathbb{R}^2$ be in general position.



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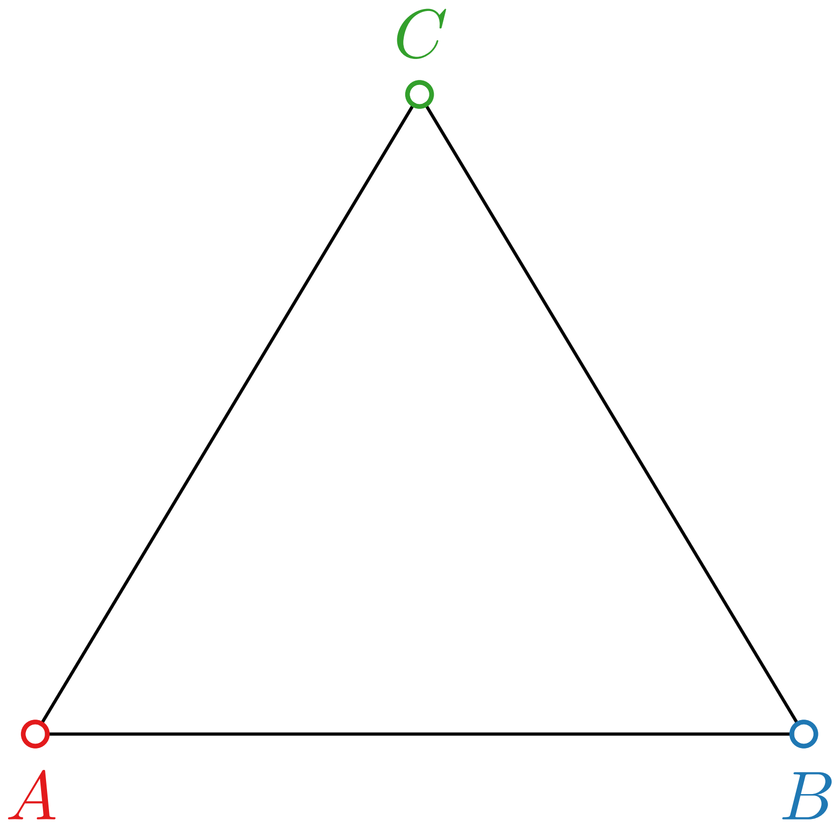
$$\phi\colon v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.



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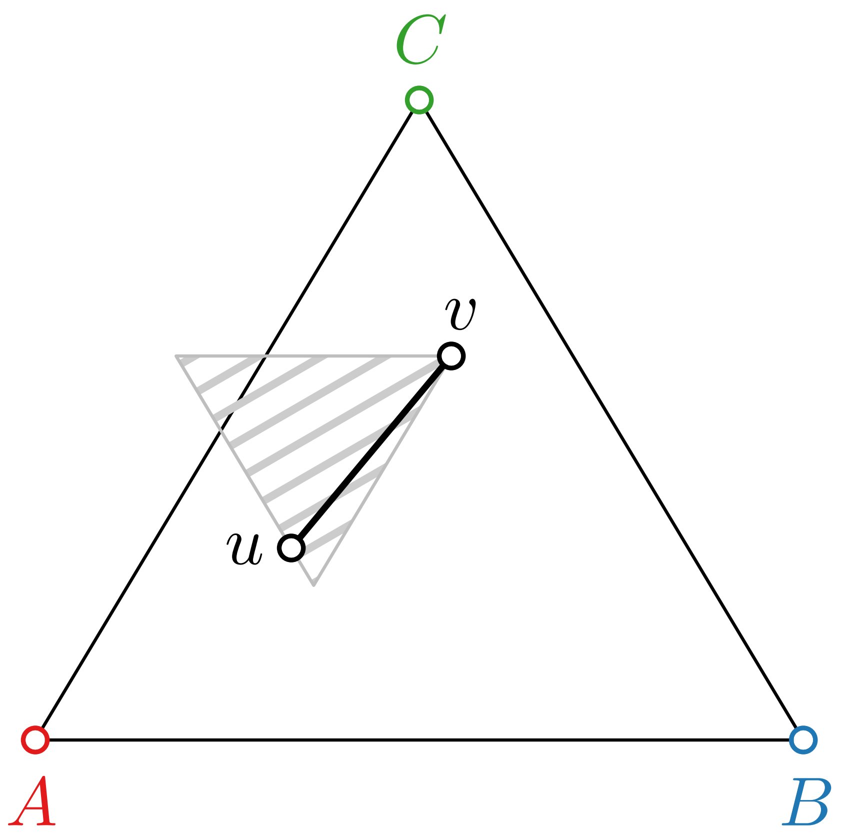

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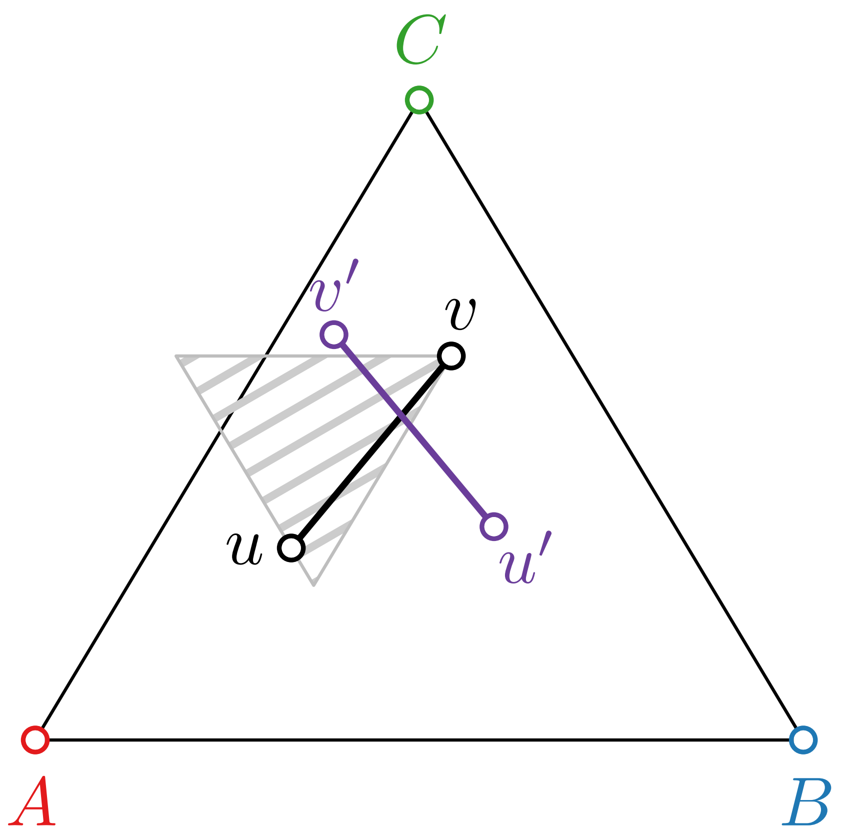

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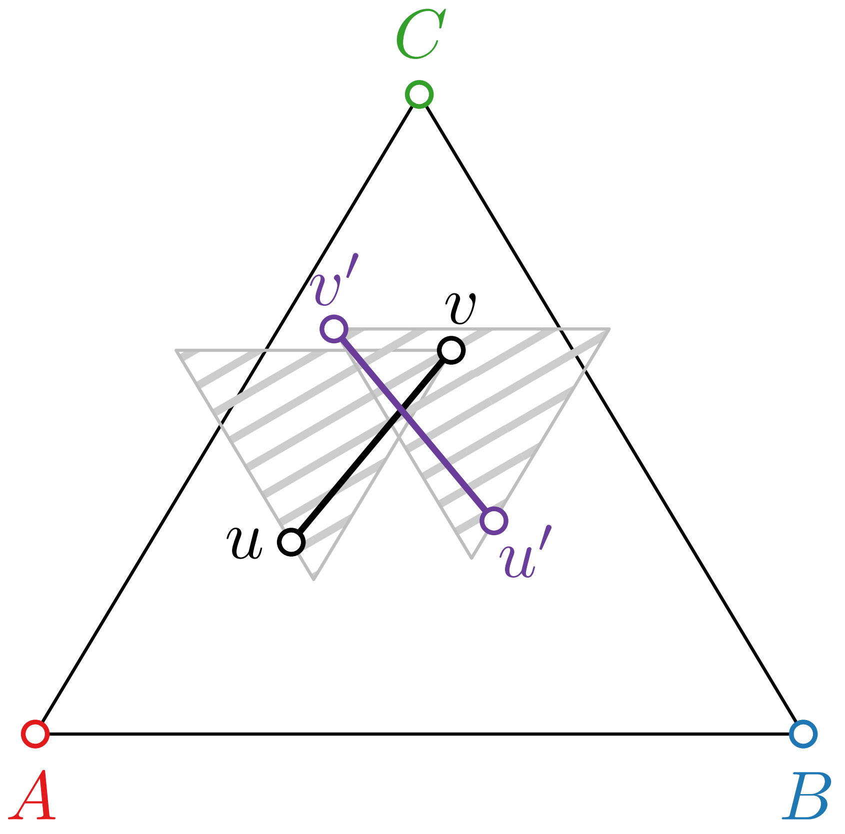

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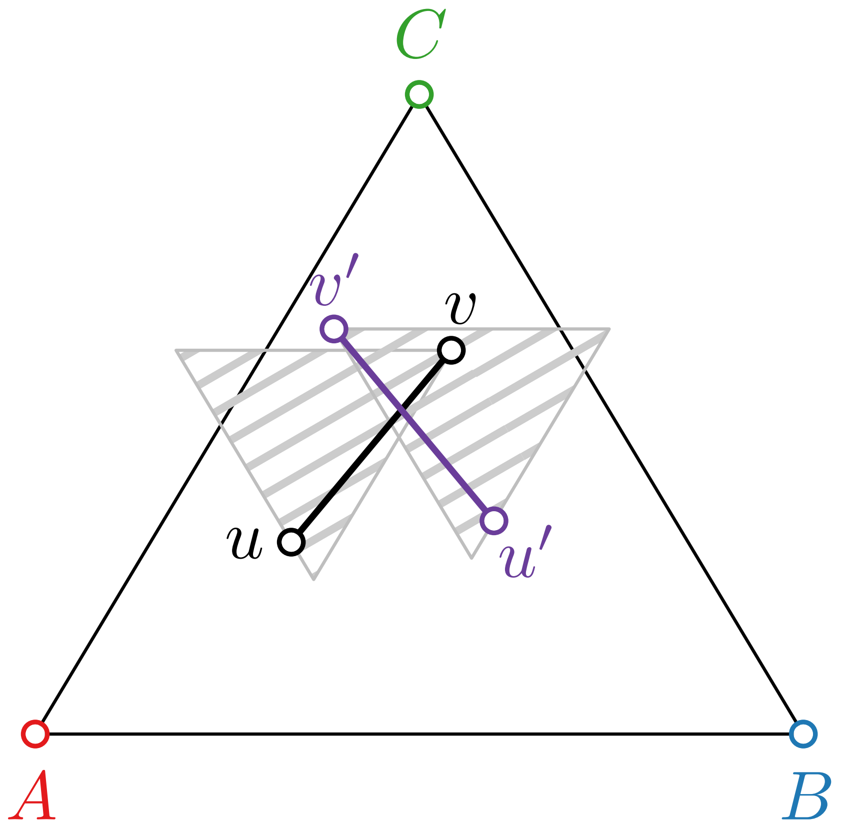
$u'_i > u_i, v_i \quad v'_j > u_j, v_j \quad u_k > u'_k, v'_k \quad v_l > u'_l, v'_l$



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Let $f: v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G, and let $A, B, C \in \mathbb{R}^2$ be in general position. Then the mapping

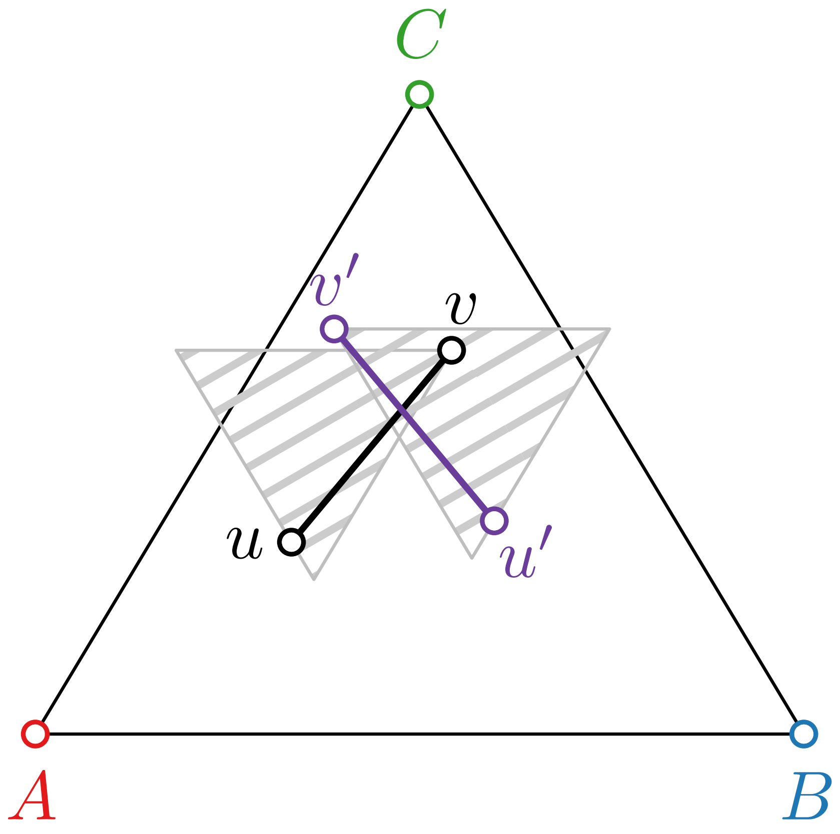
$$\phi\colon v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.

- No vertex x can lie on an edge $\{u, v\}$.
- No pair of edges $\{u, v\}$ and $\{u', v'\}$ crosses:

$u'_i > u_i, v_i \quad v'_j > u_j, v_j \quad u_k > u'_k, v'_k \quad v_l > u'_l, v'_l$

$\Rightarrow \{i, j\} \cap \{k, l\} = \emptyset$



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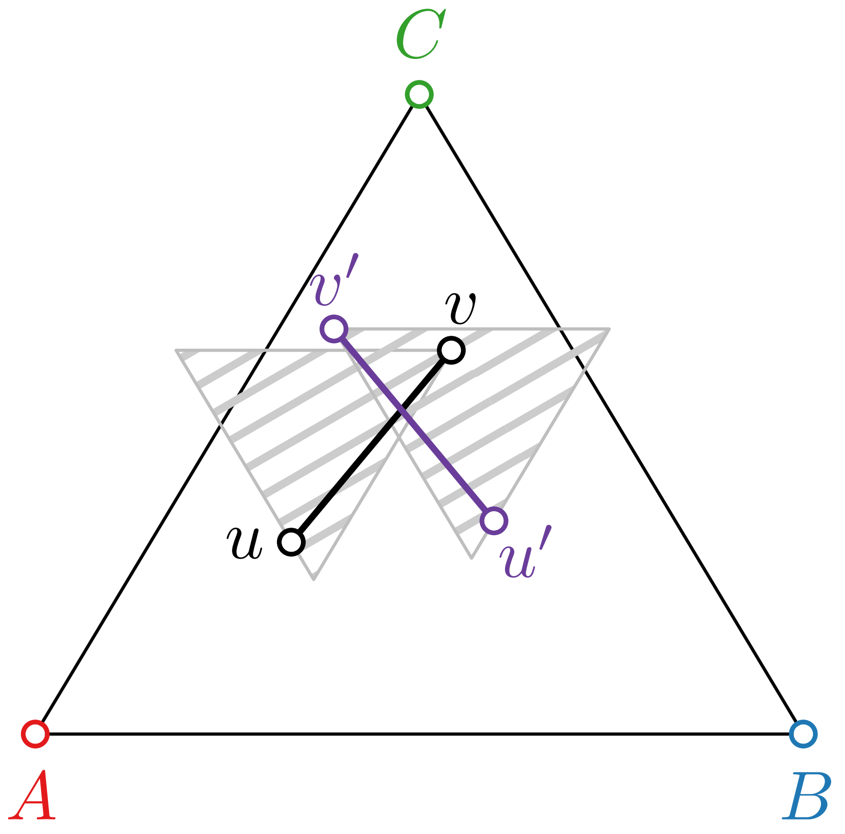

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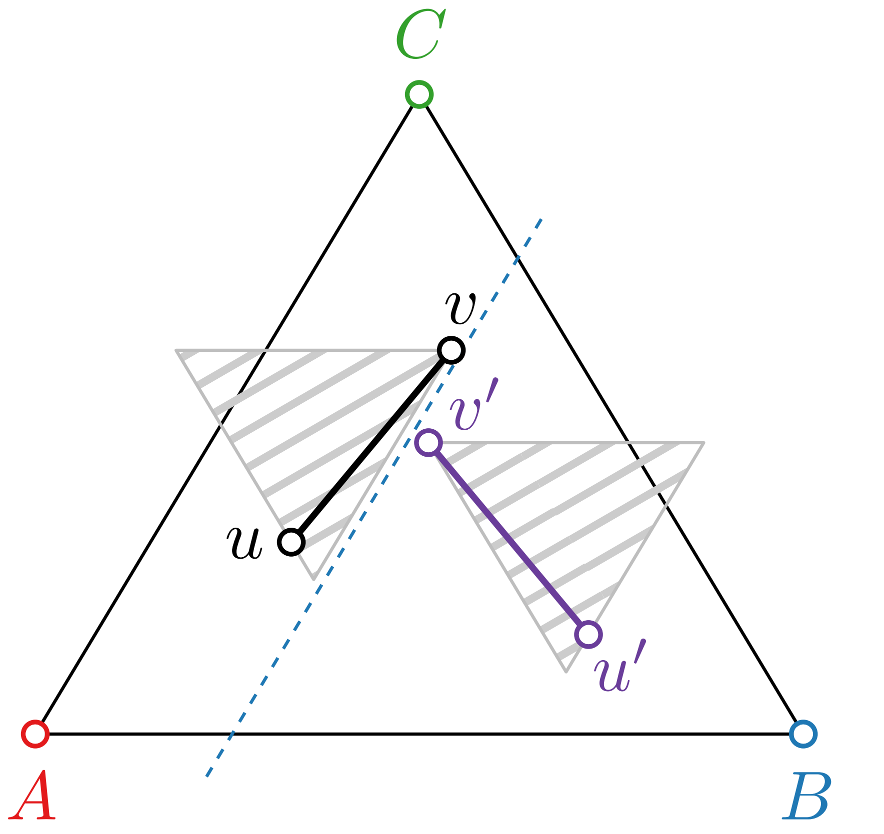

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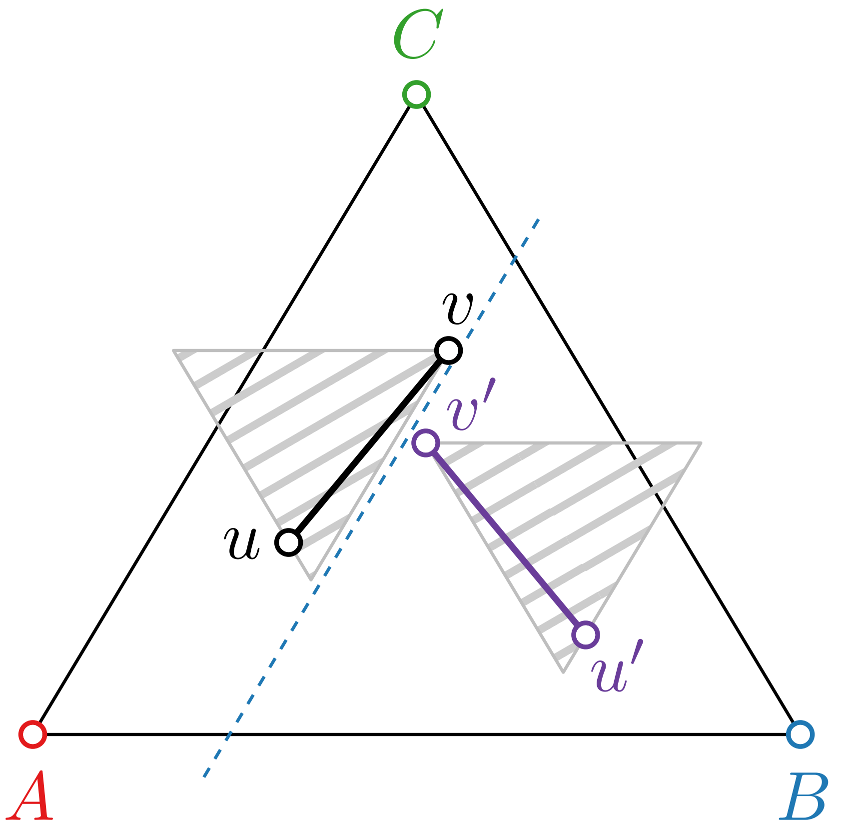
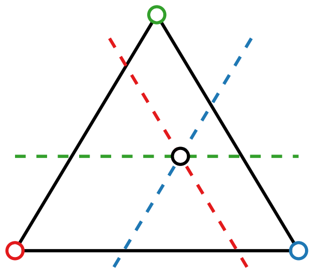


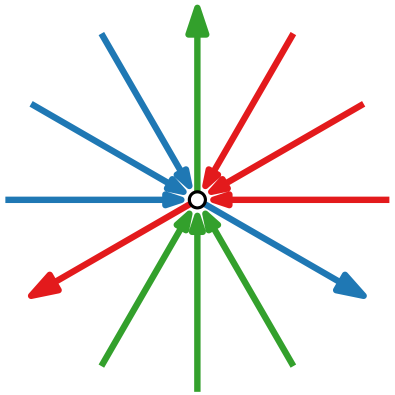
How to find a barycentric representation?

Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

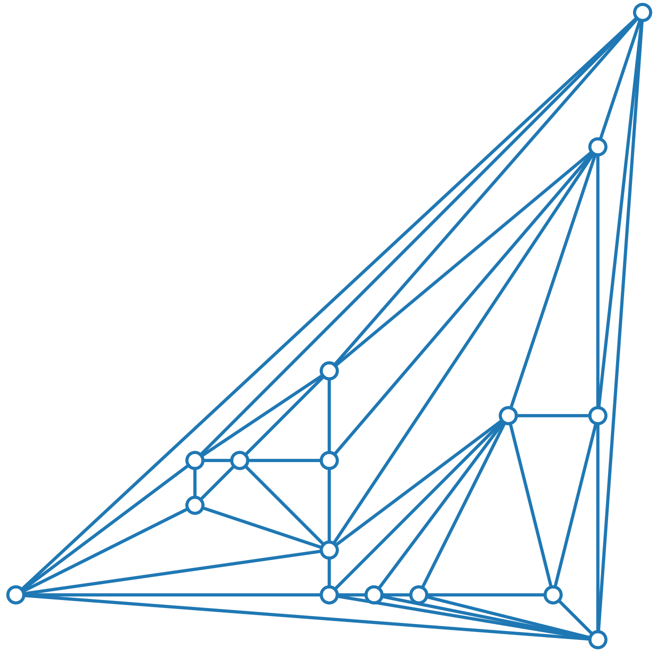
Lecture 4:
Straight-Line Drawings of Planar Graphs II:
Schnyder Woods





Part II:
Schnyder Woods

Alexander Wolff



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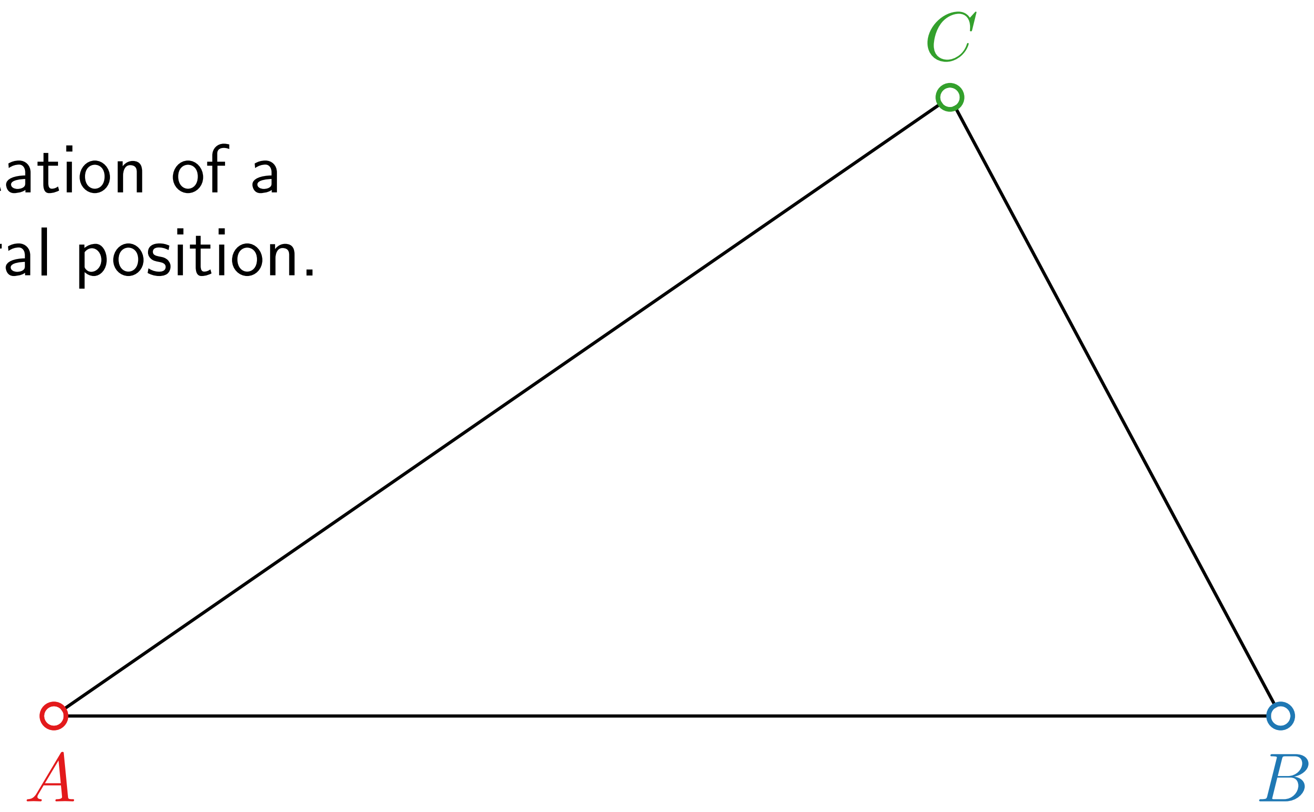

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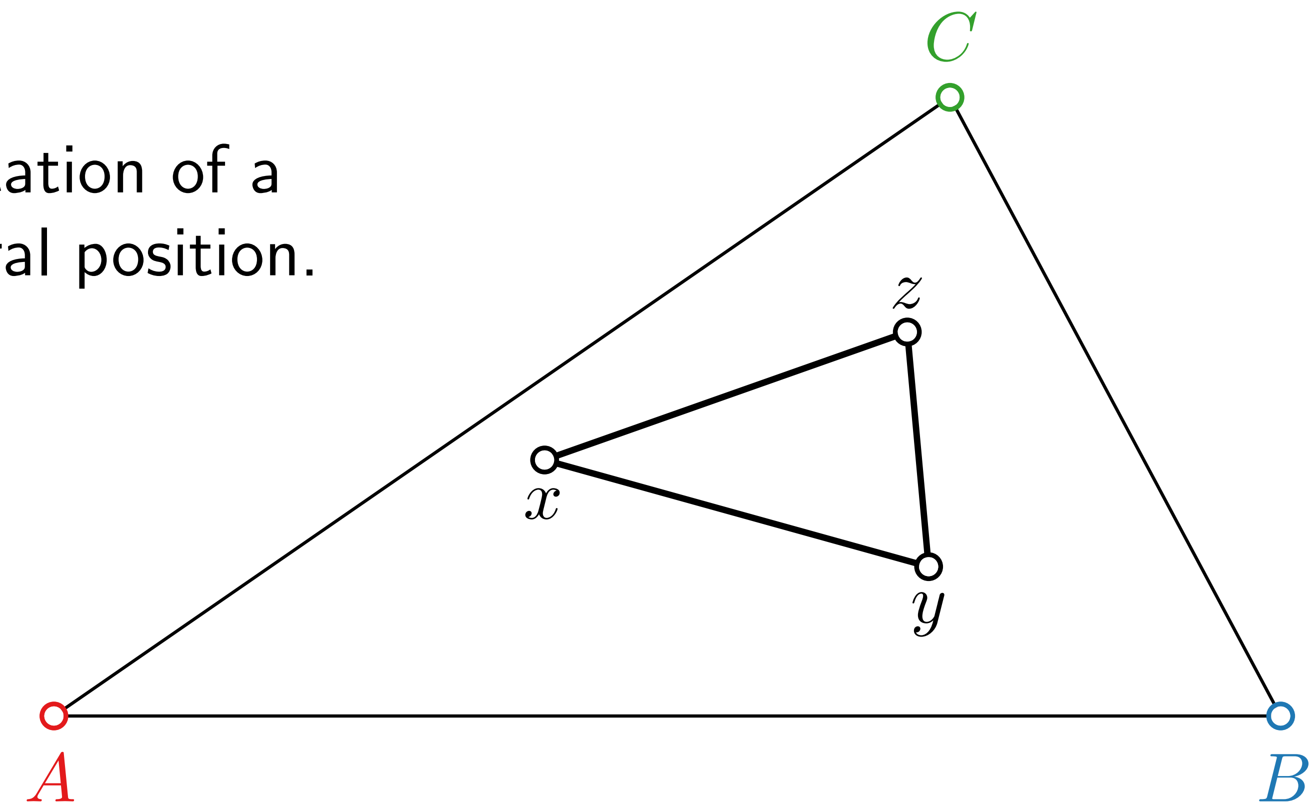

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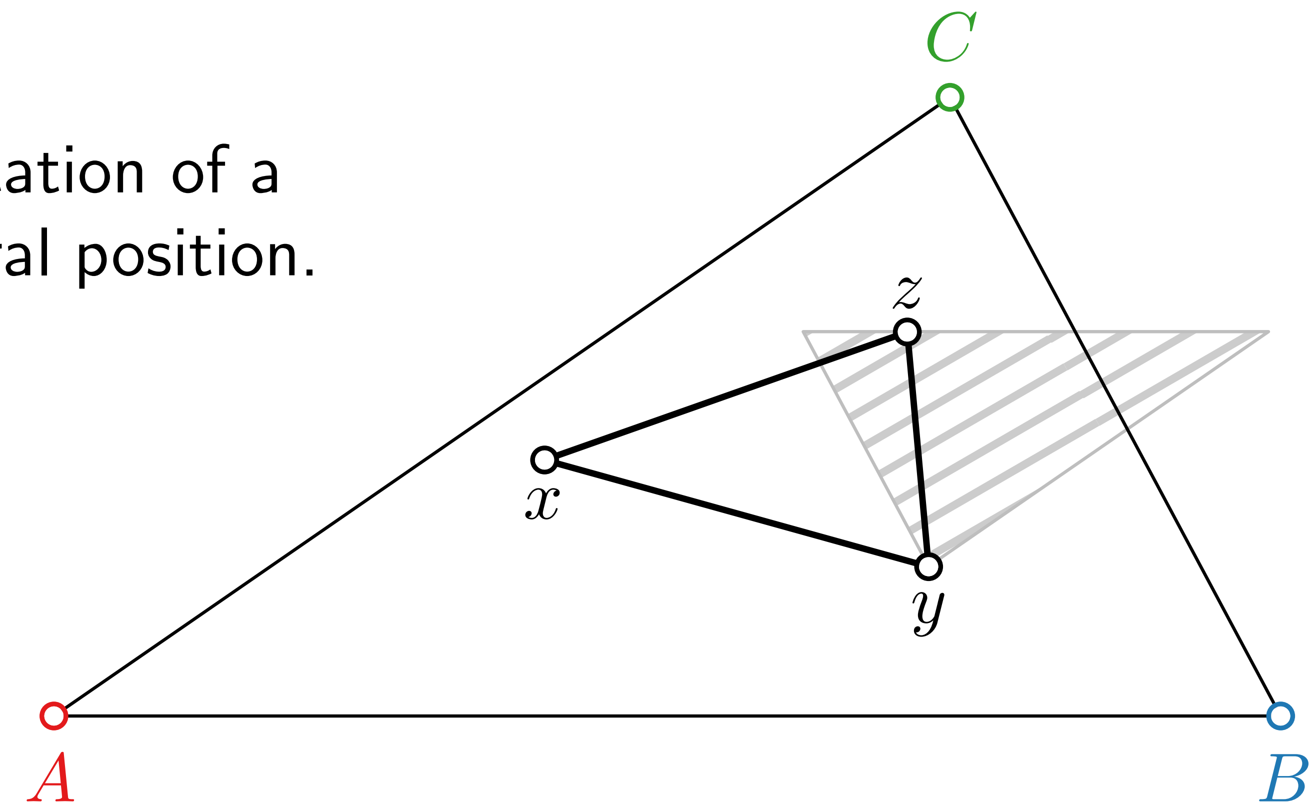

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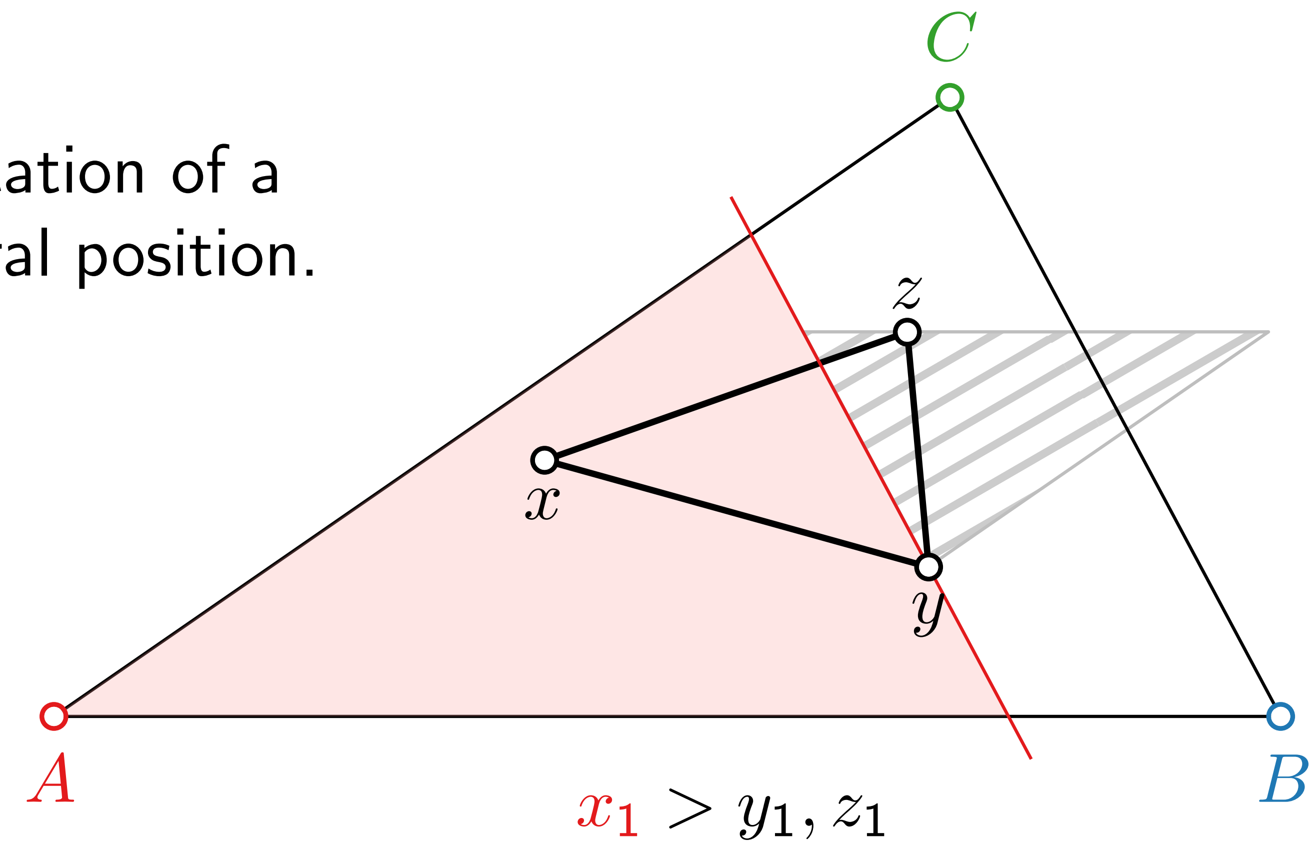

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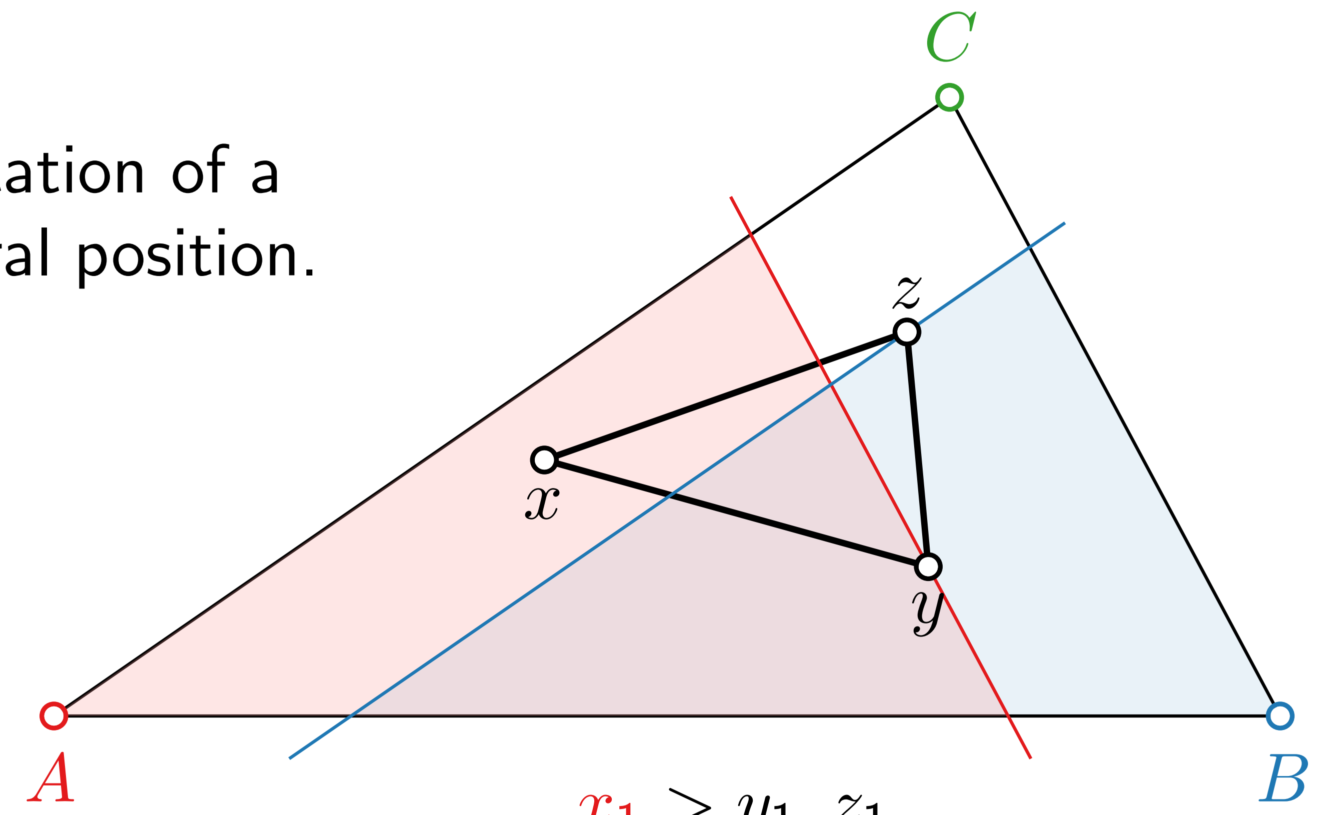

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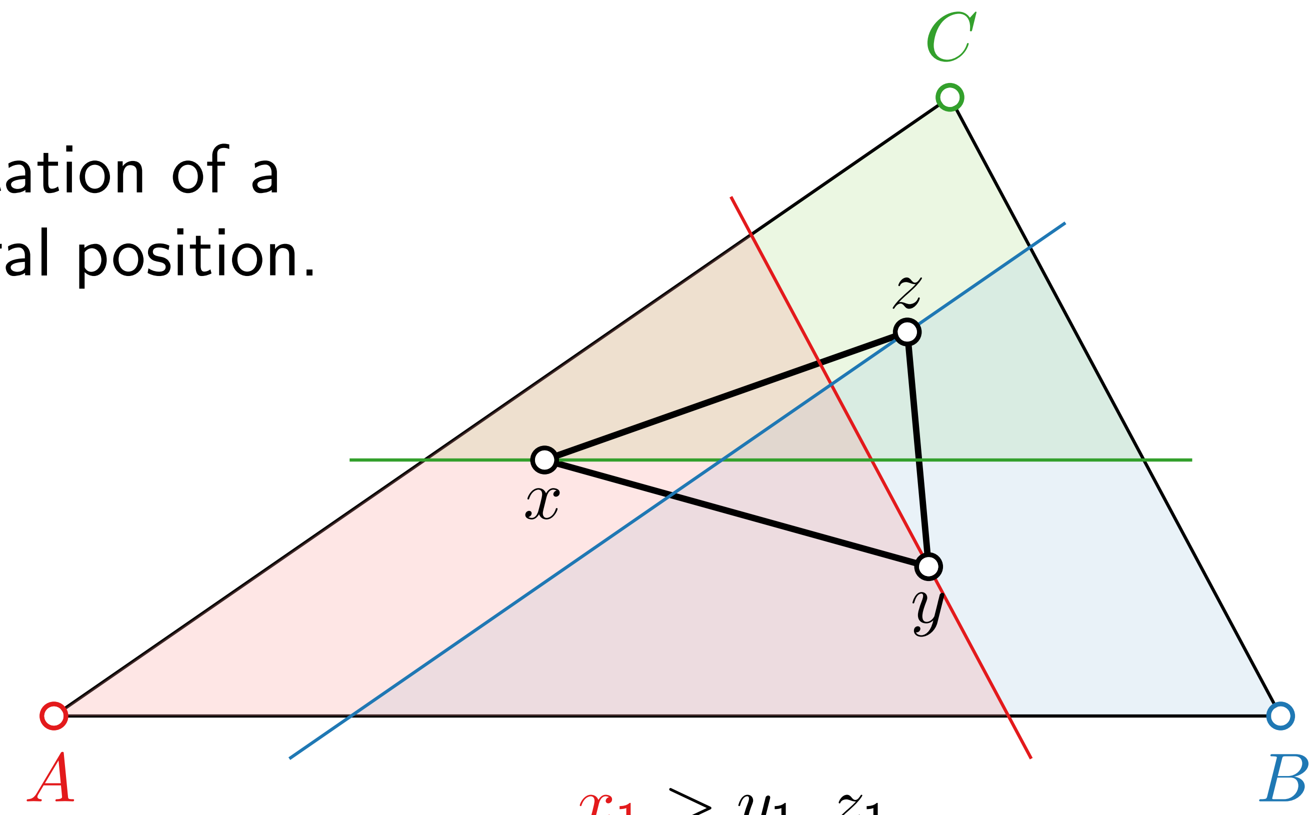

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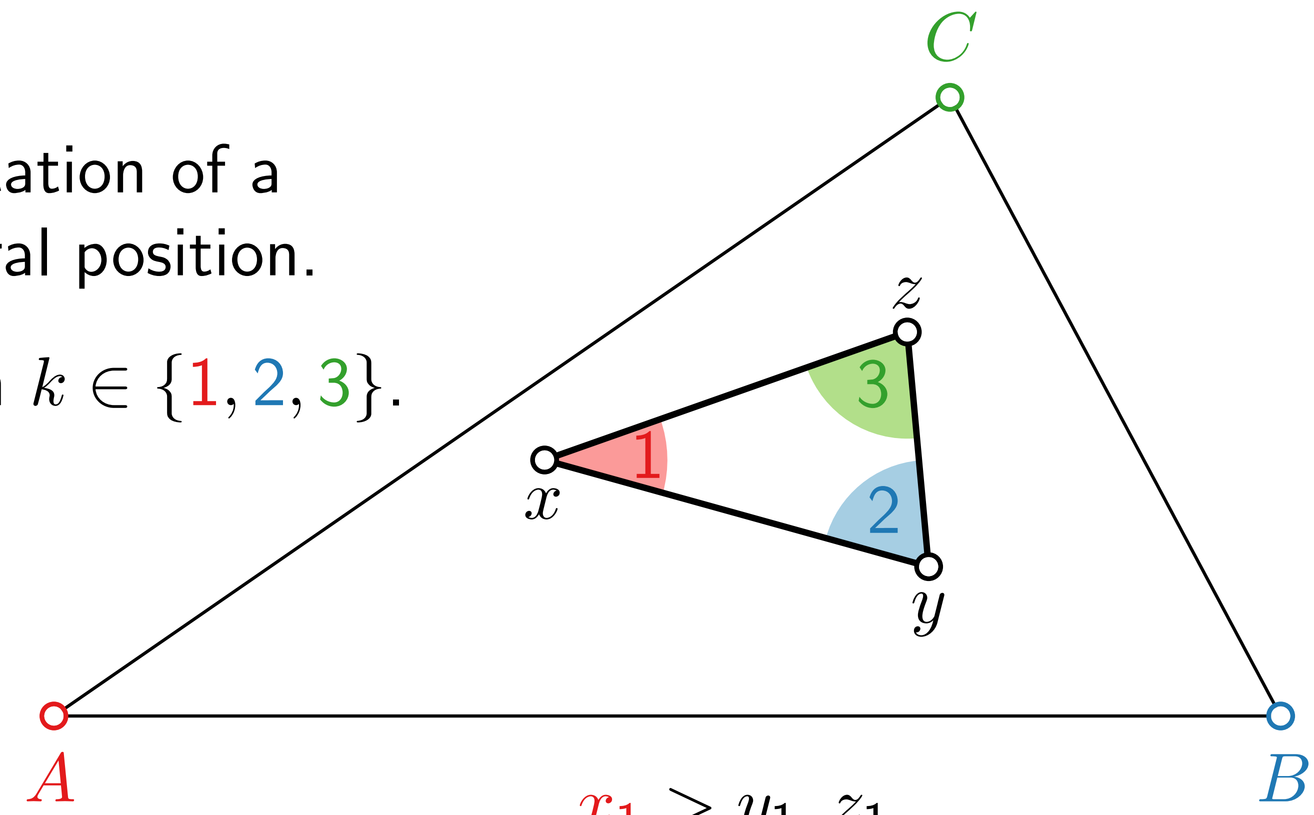

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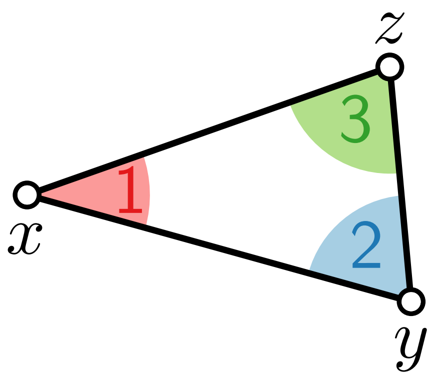

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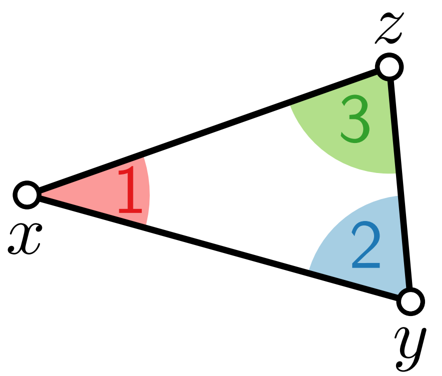

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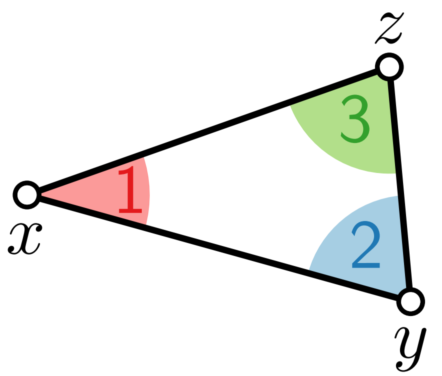

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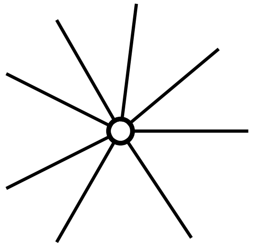

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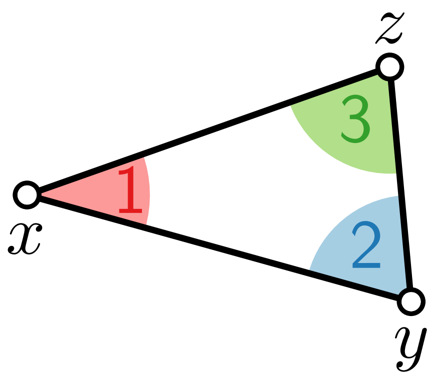

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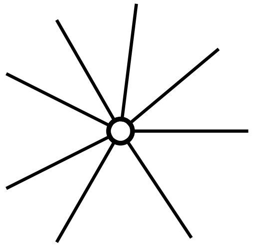

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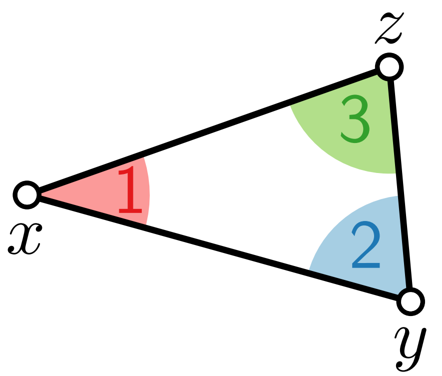

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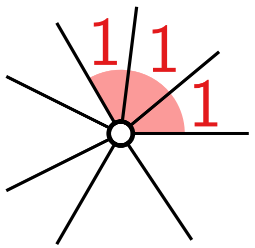

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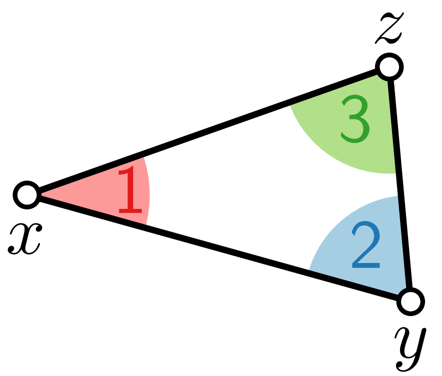

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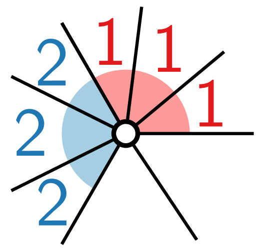

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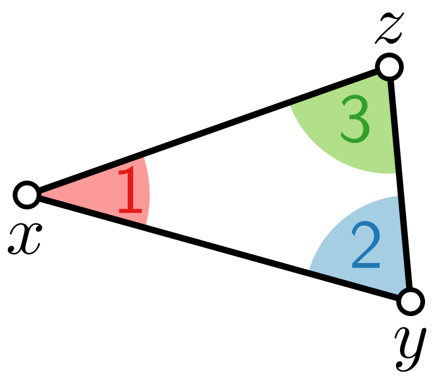

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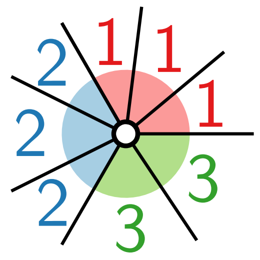
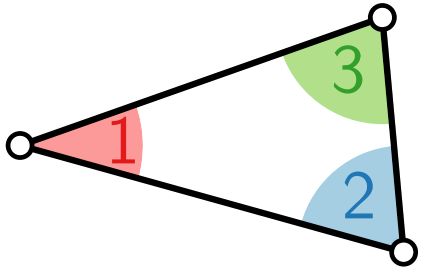
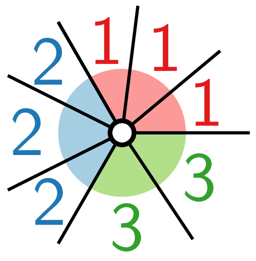


Schnyder Wood

A Schnyder labeling induces an edge labeling.





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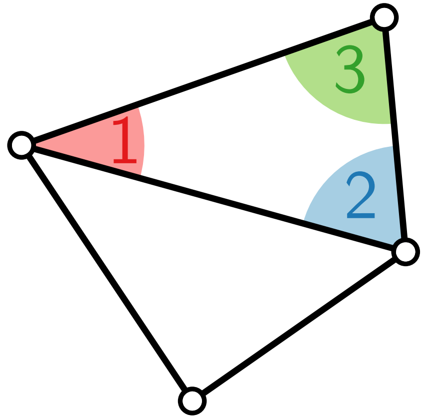

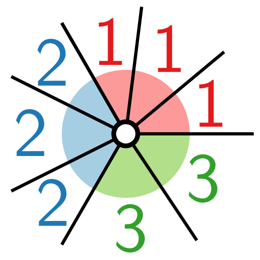

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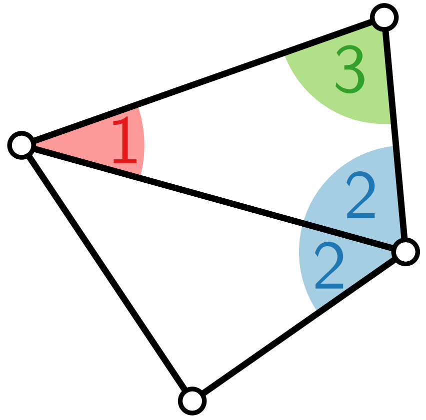

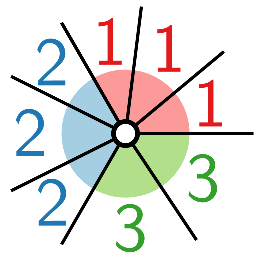

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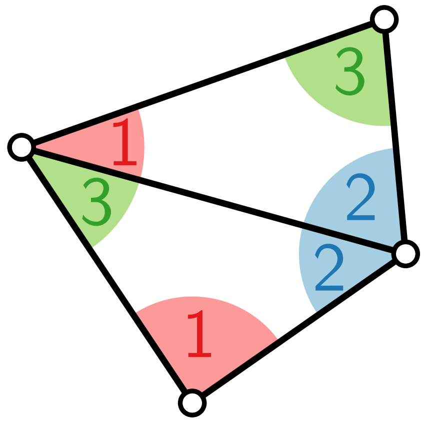

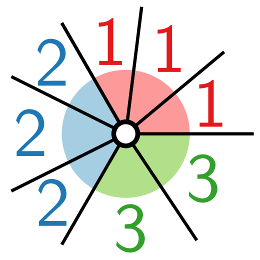

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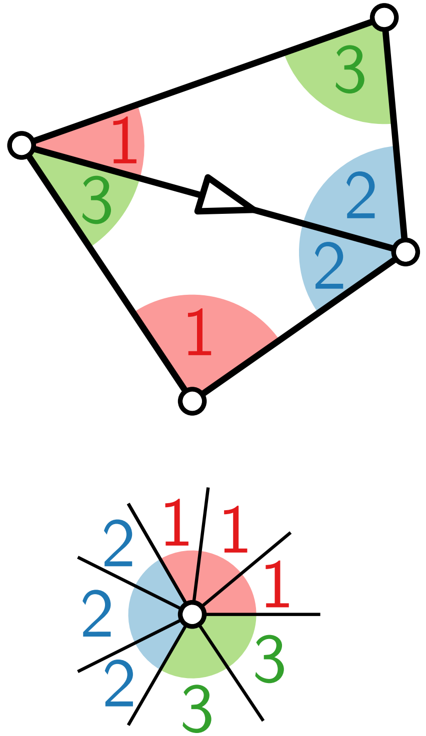

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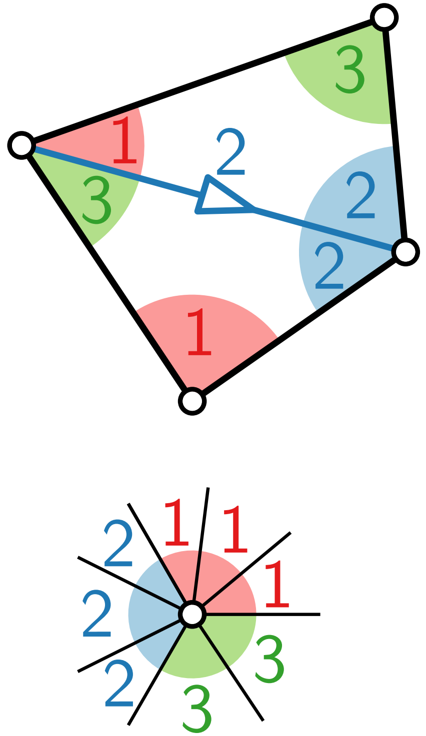

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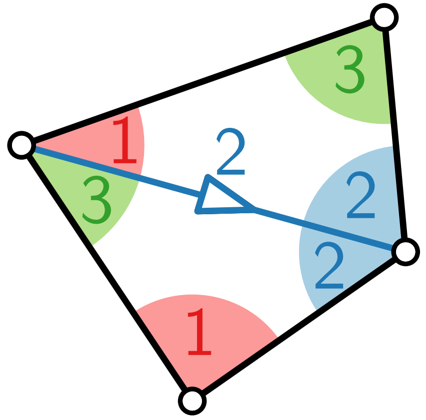

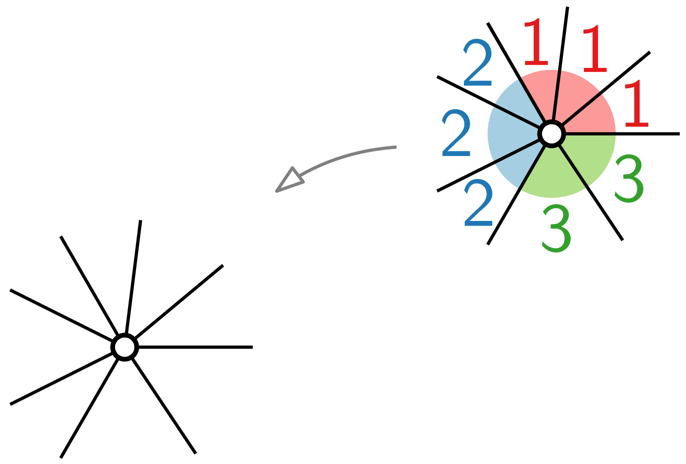

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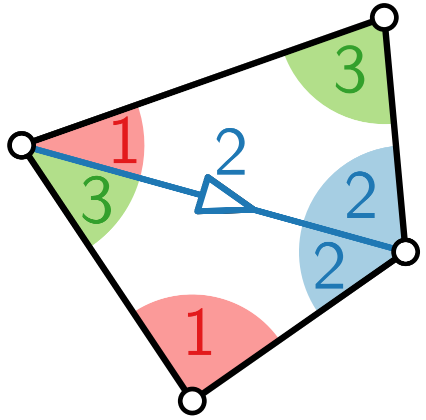

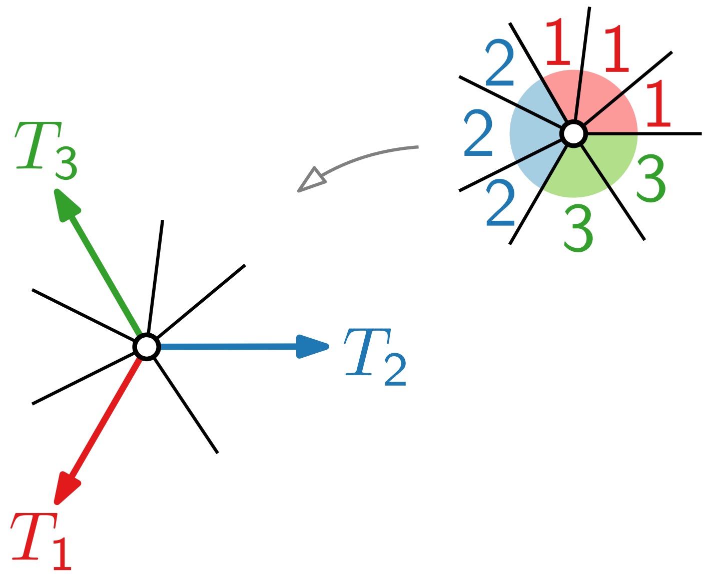

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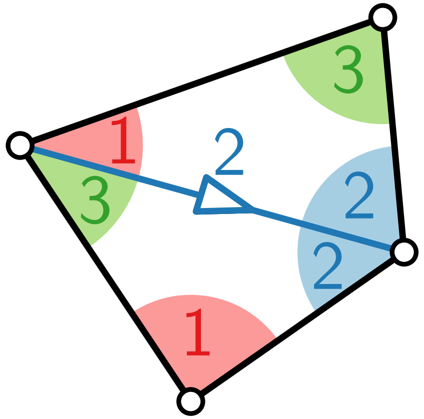

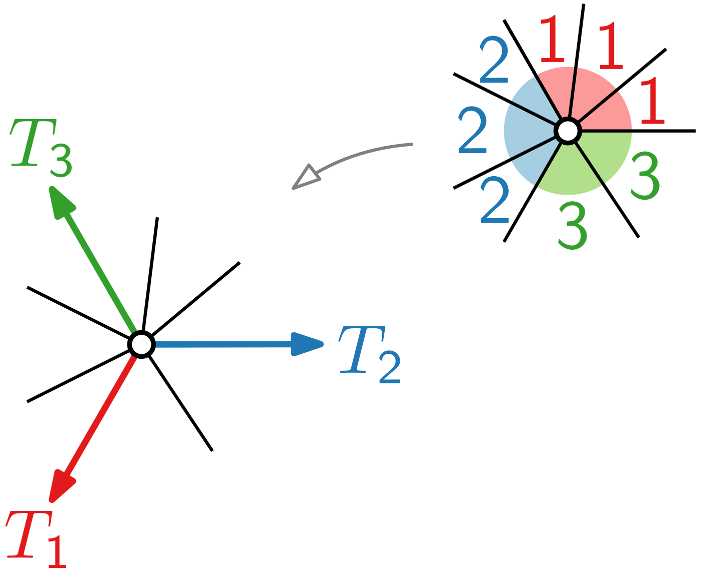

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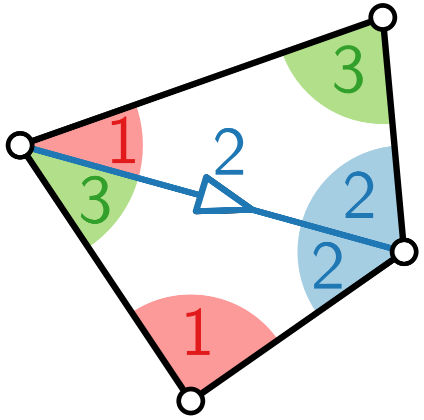

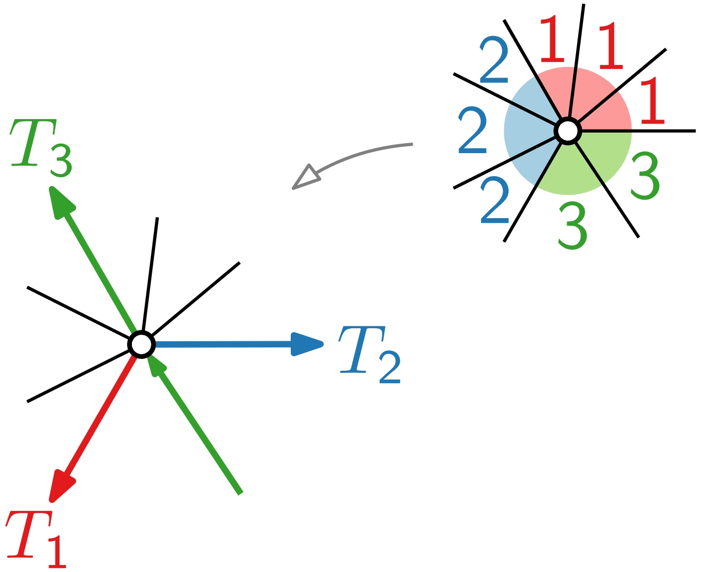

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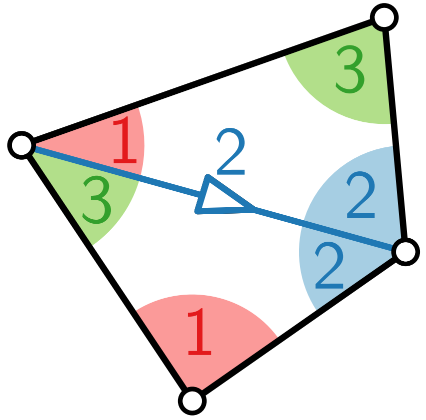
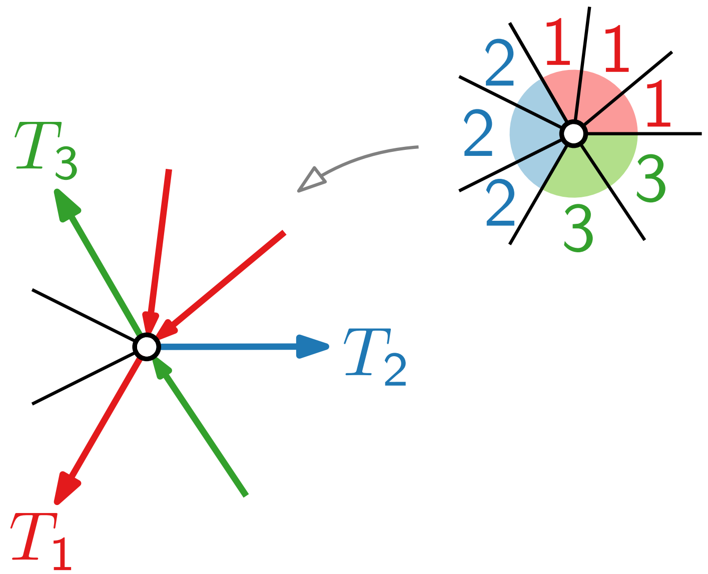
A **Schnyder wood** (or **realizer**) of a plane triangulation $G = (V, E)$ is a partition of the inner edges of E into three sets of oriented edges T_1, T_2, T_3 such that, for each inner vertex $v \in V$, it holds that

- v has one outgoing edge in each of T_1, T_2, and T_3.
- The ccw order of edges around v is:
  leaving in T_1, entering in T_3, leaving in T_2,
  entering in T_1,





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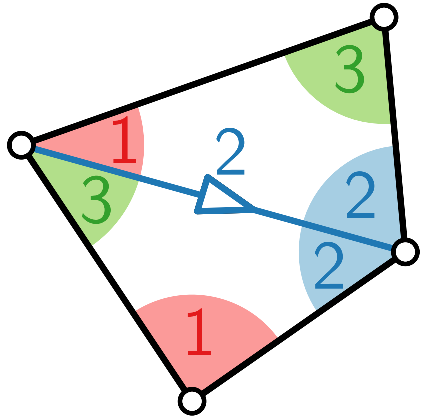
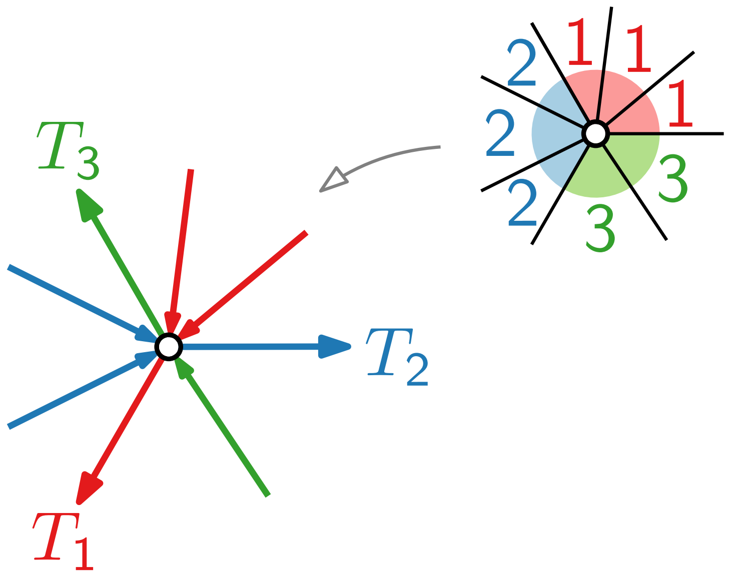
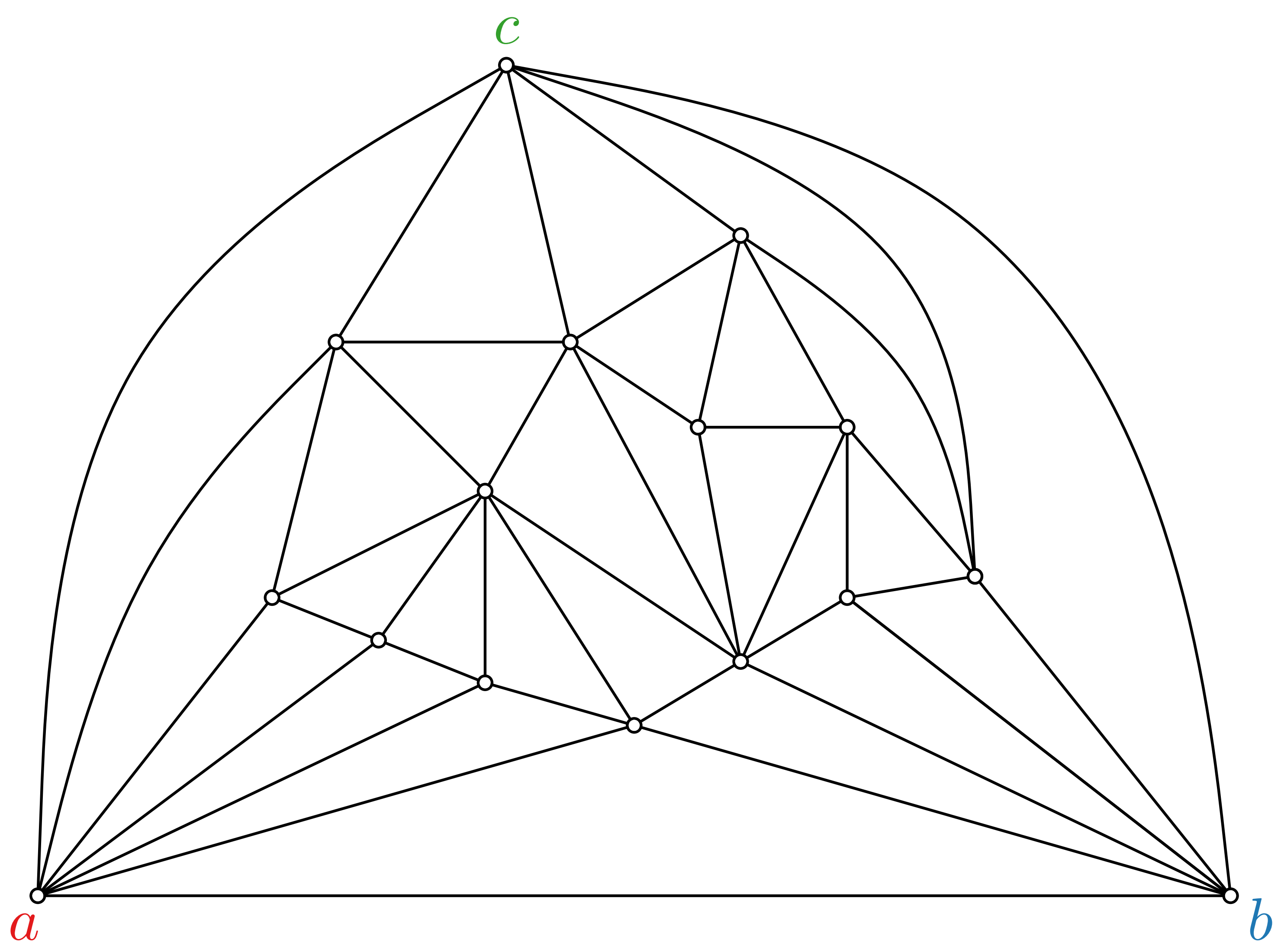
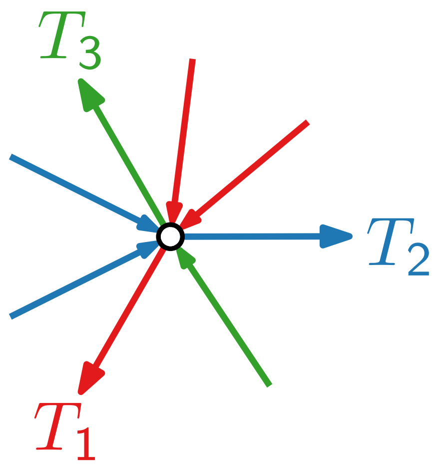




Schnyder Wood – Example and Properties





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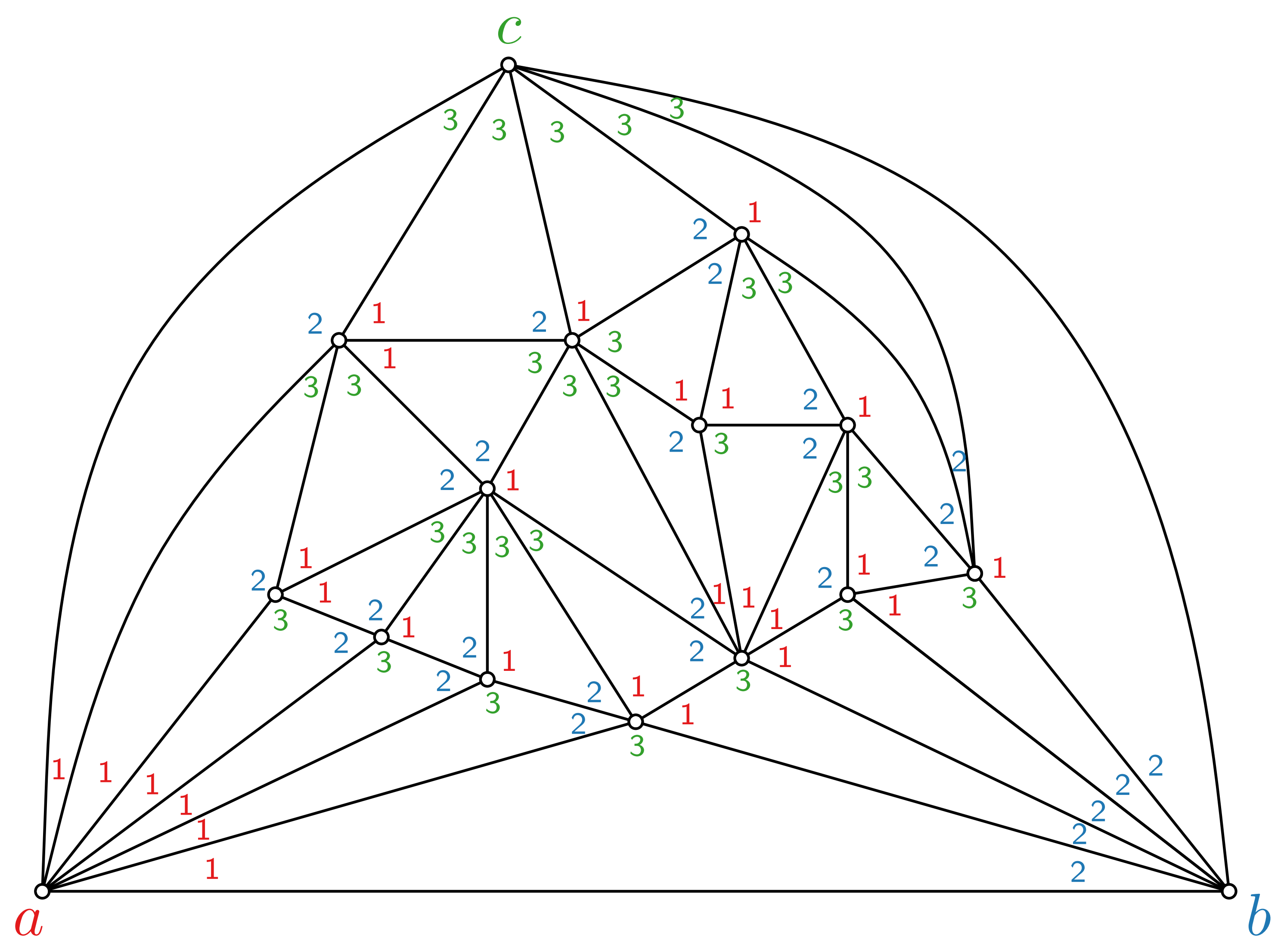

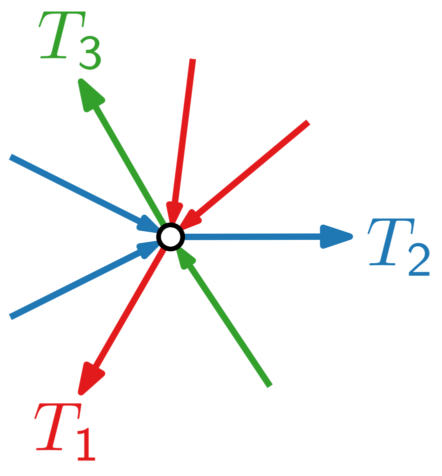

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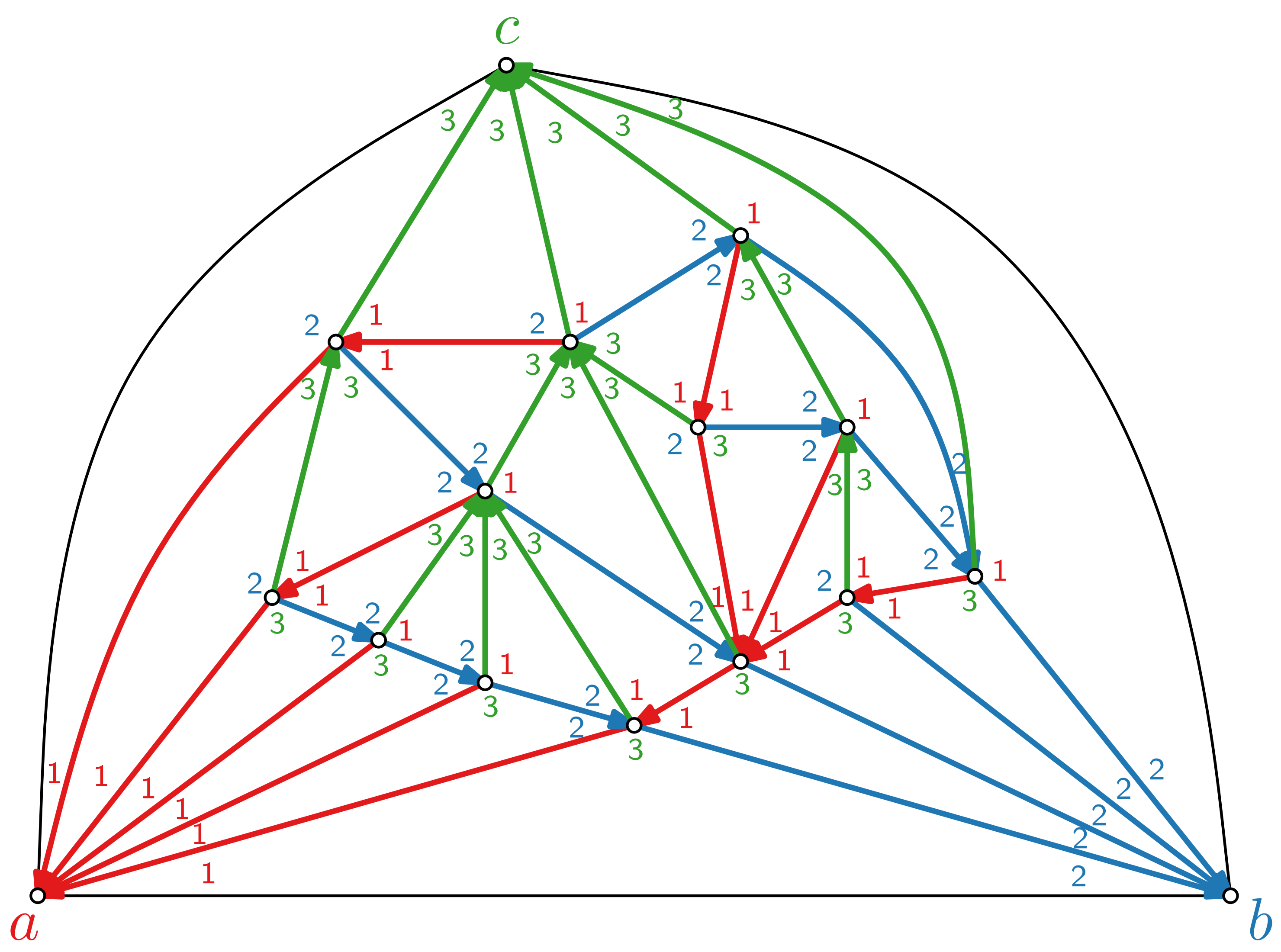

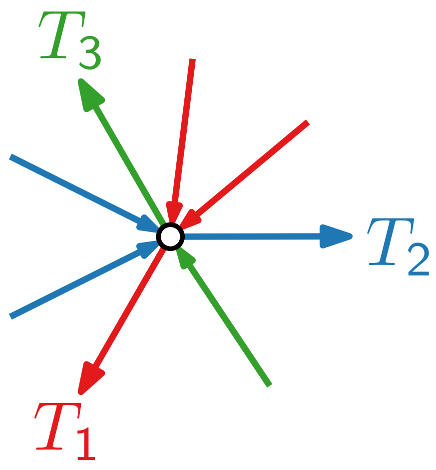

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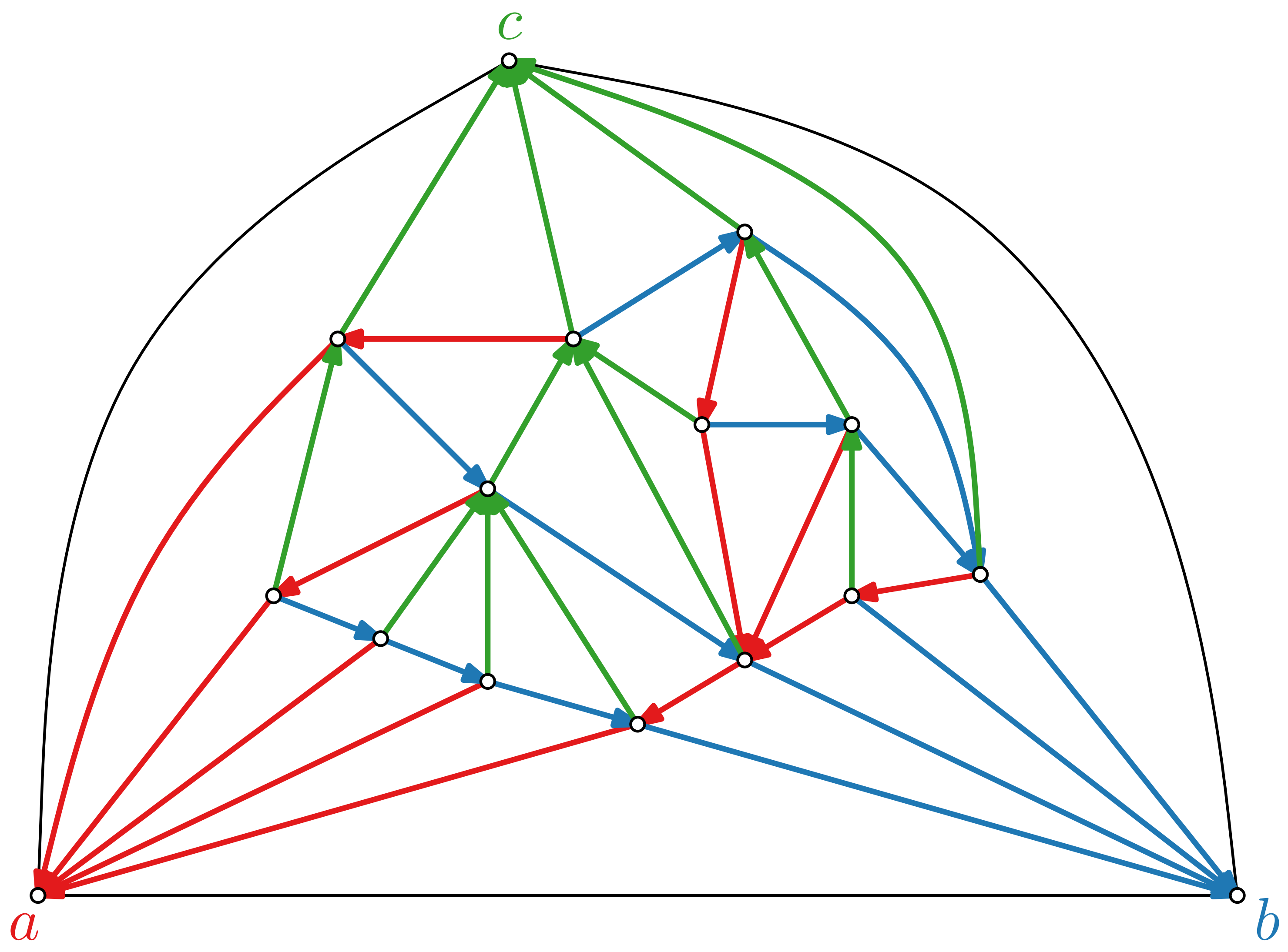

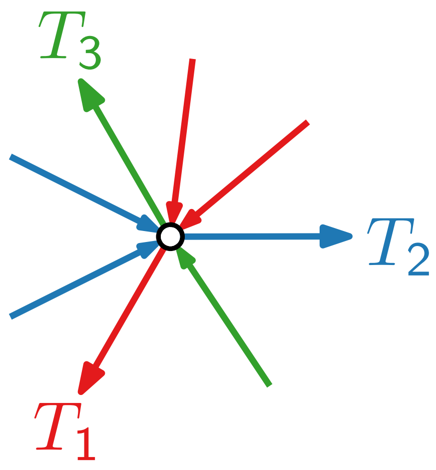

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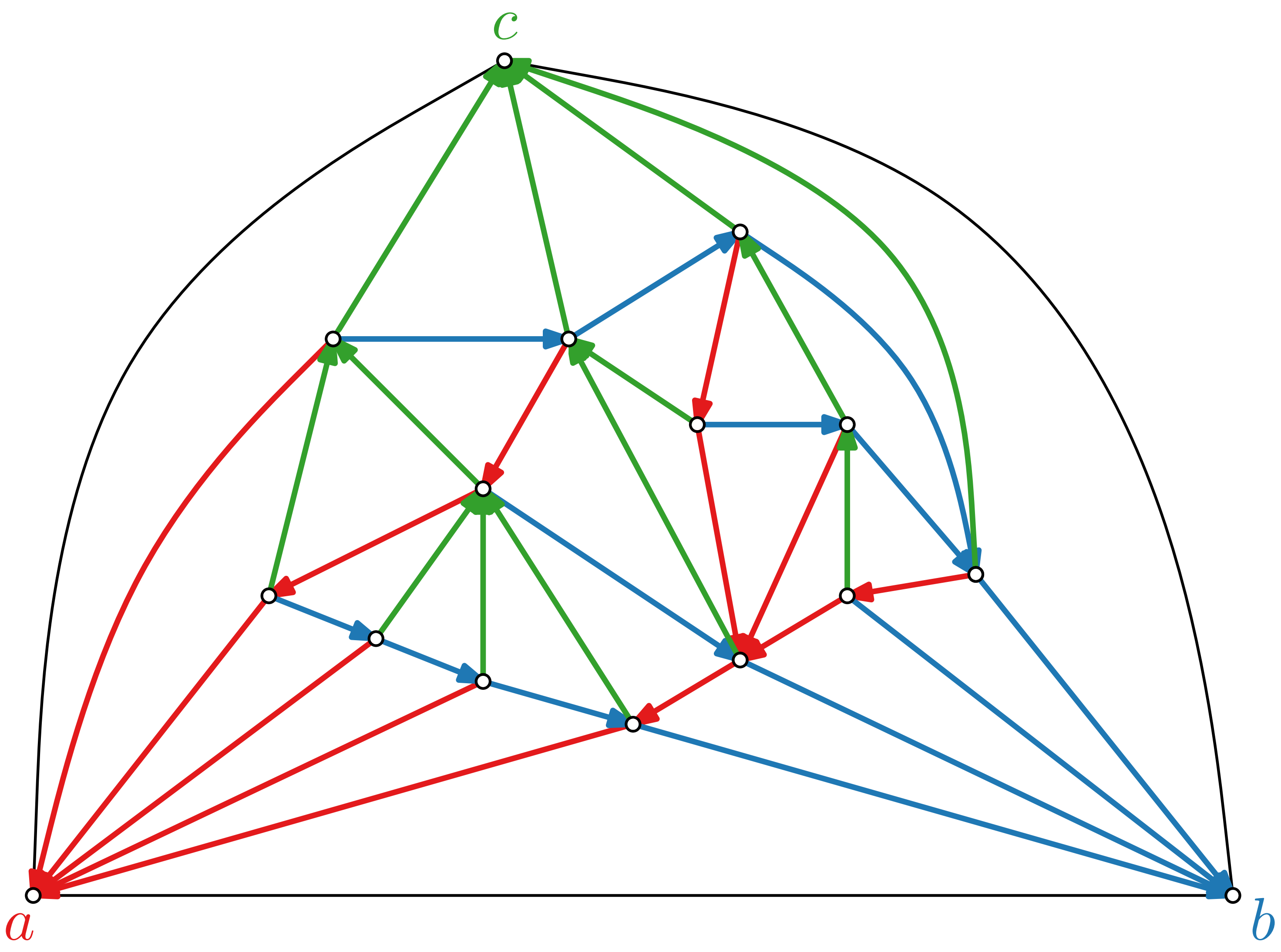

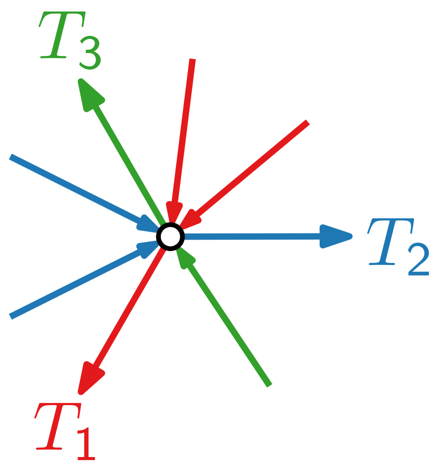

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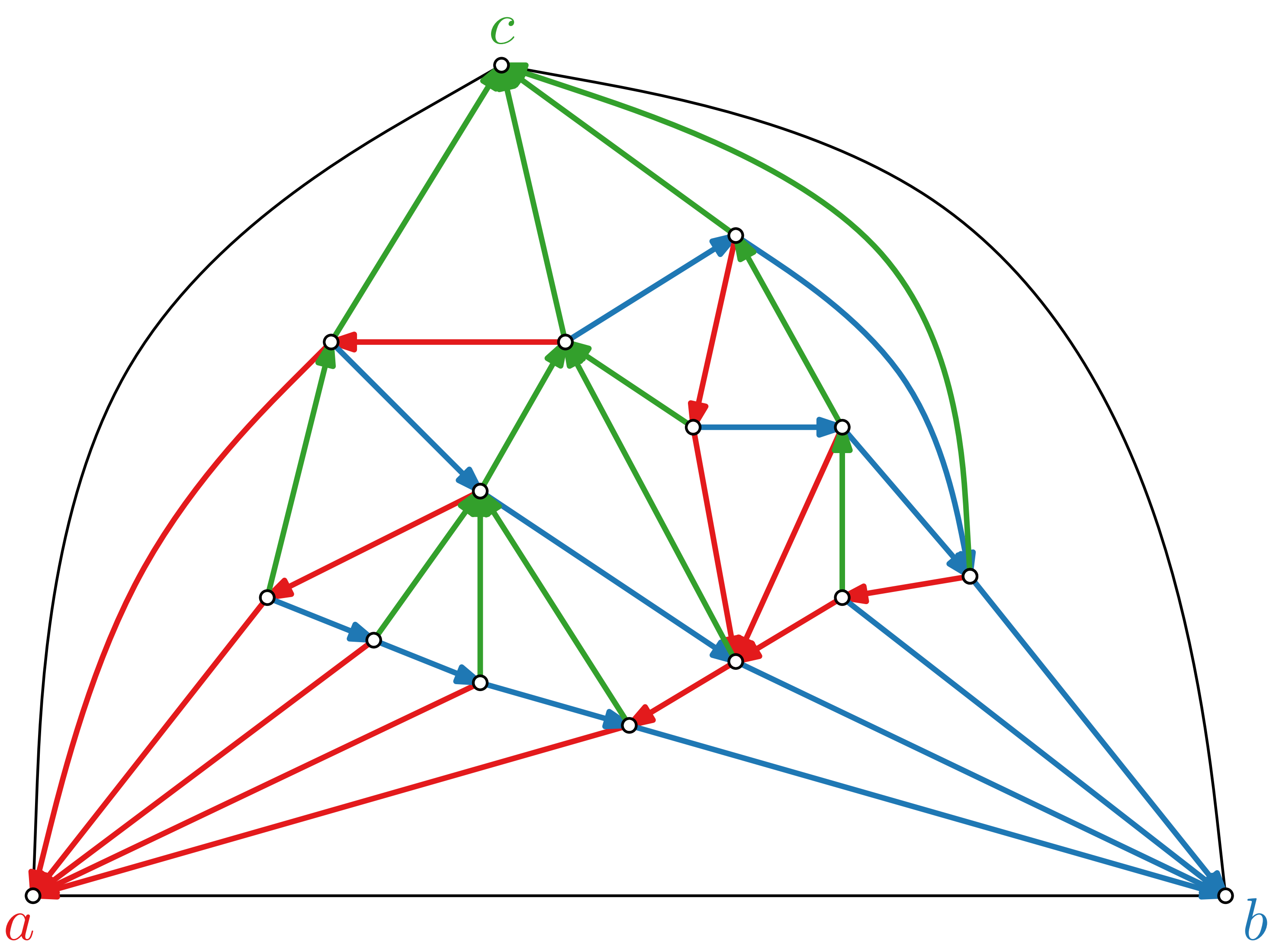

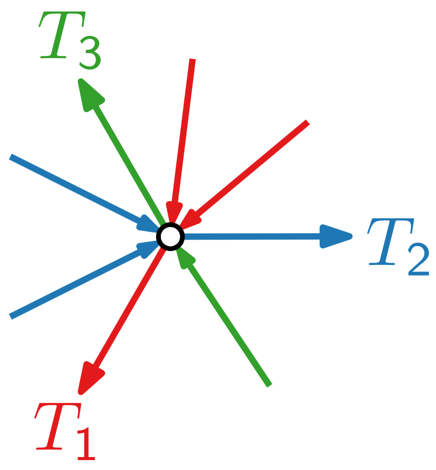

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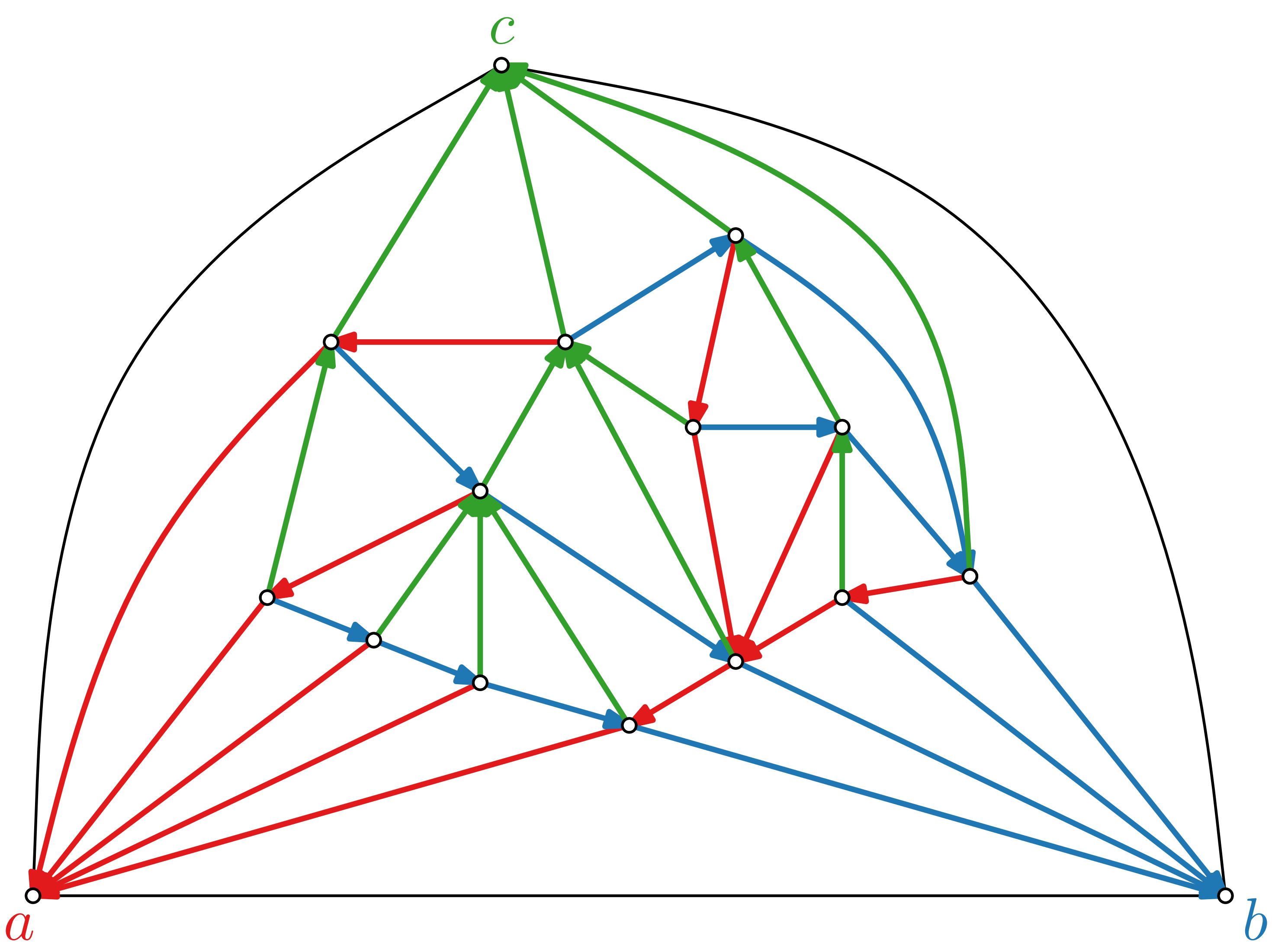

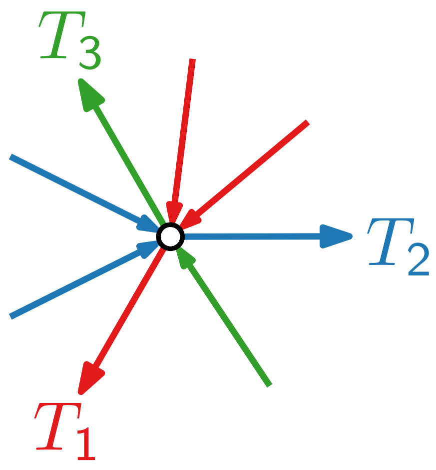

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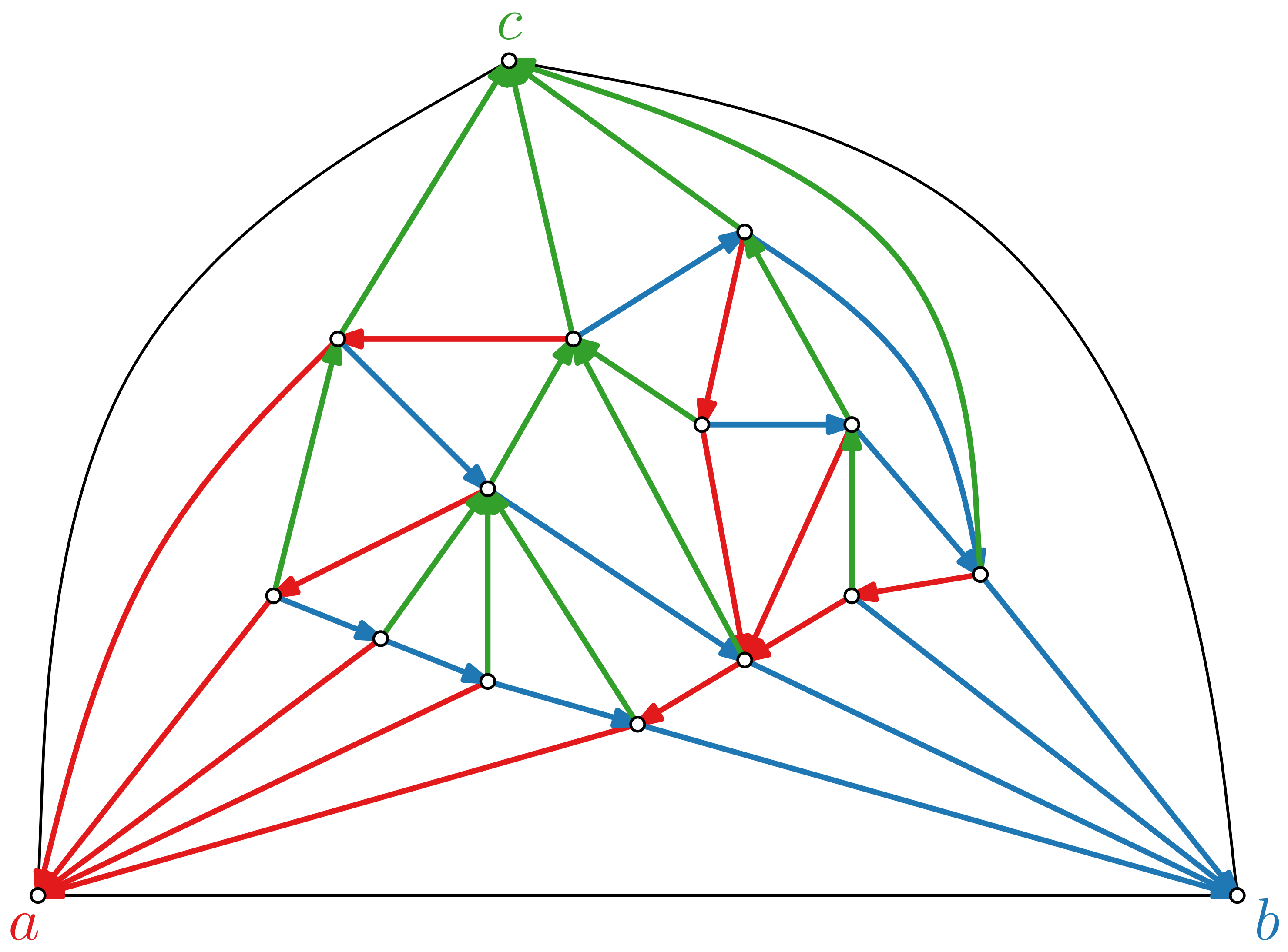
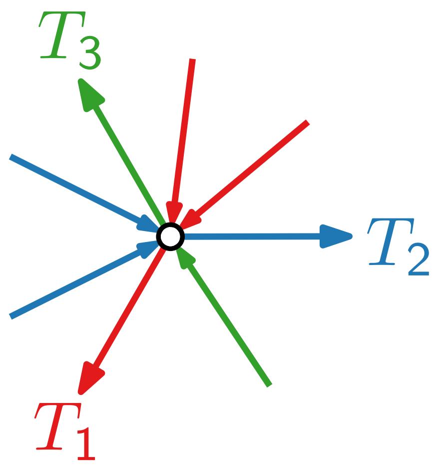
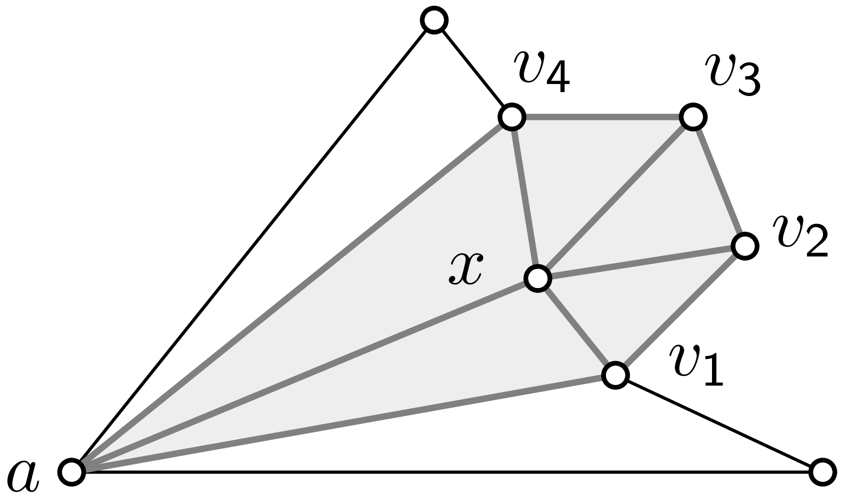




- All inner edges incident to a, b, and c are incoming in the same color.
- T_1, T_2, and T_3 are trees. Each spans all inner vertices and one outer vertex (its root).

Schnyder Wood – Existence



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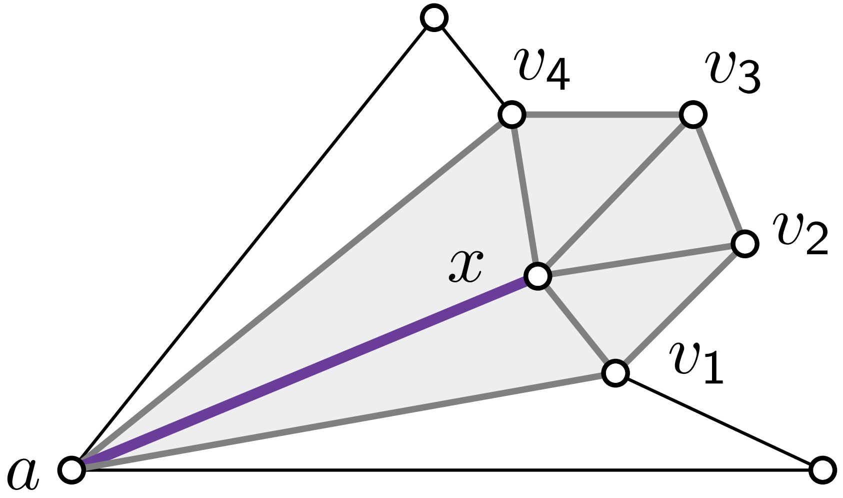

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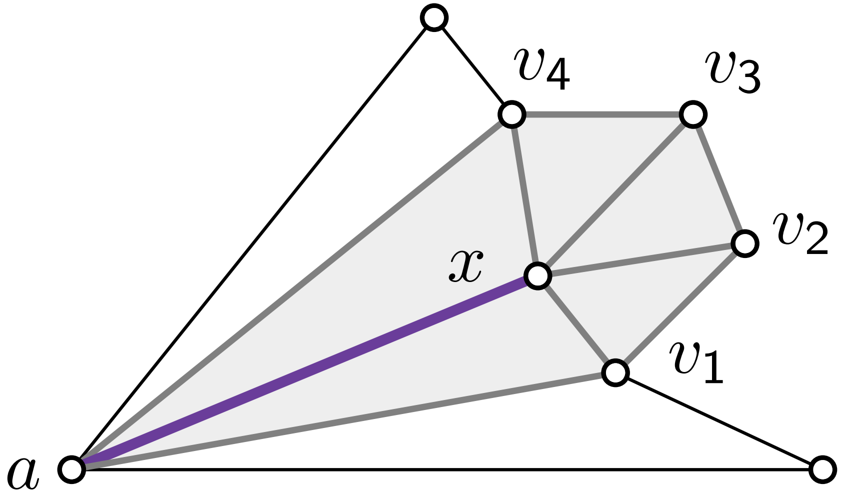

contracting

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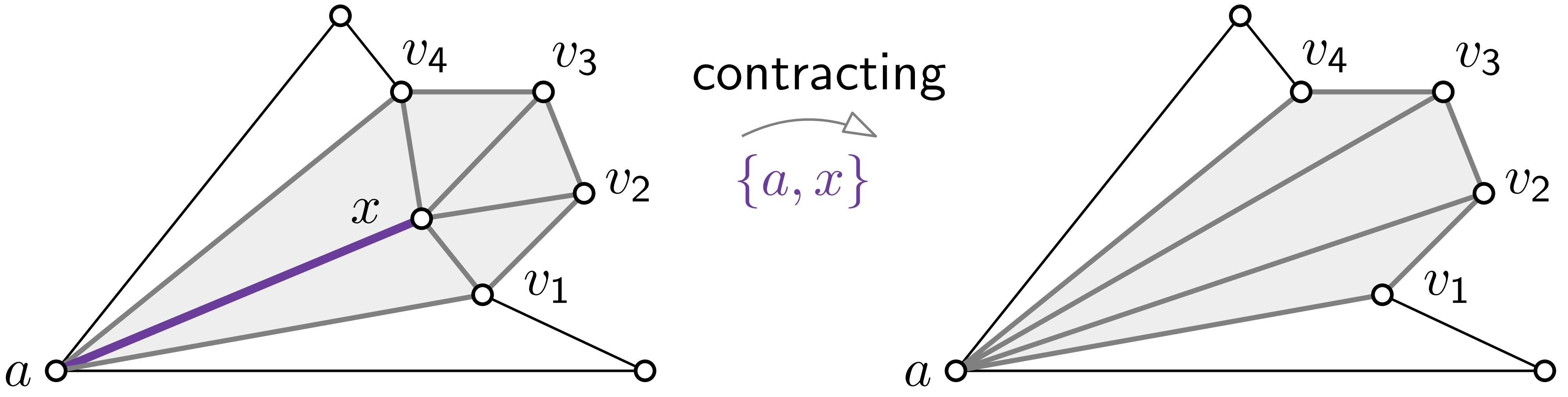

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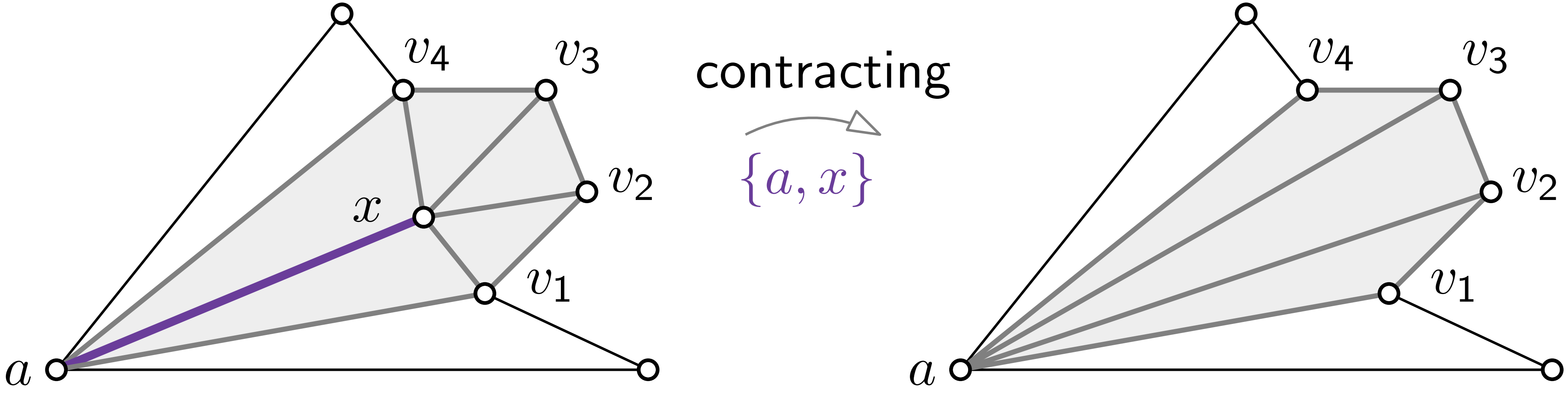

. . . requires that a and x have exactly two common neighbors.

Schnyder Wood – Existence

Lemma. [Kampen 1976]
Let G be a plane triangulation with vertices a, b, c on the outer face.
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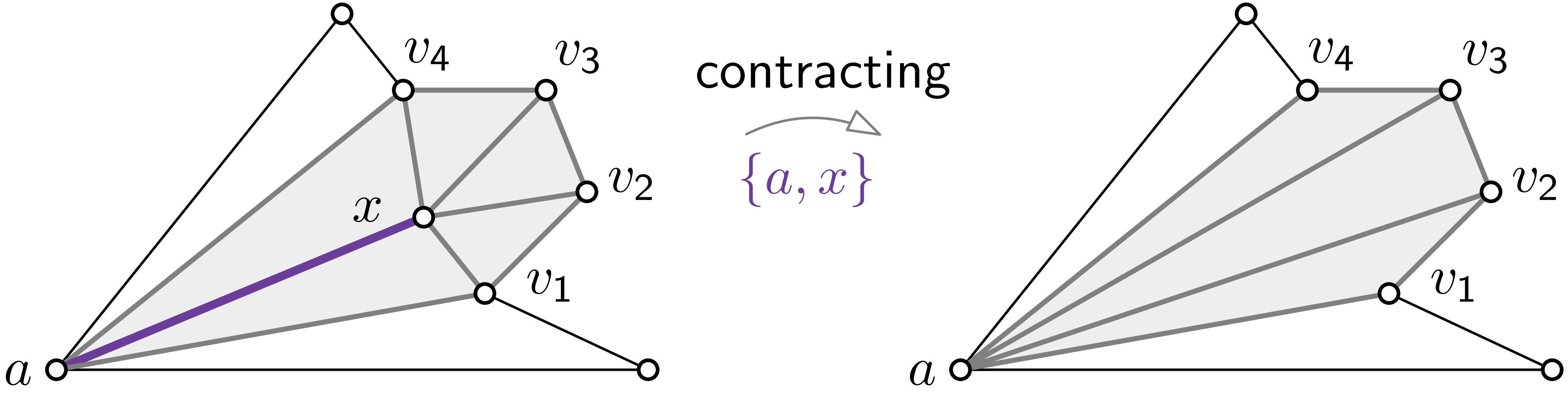

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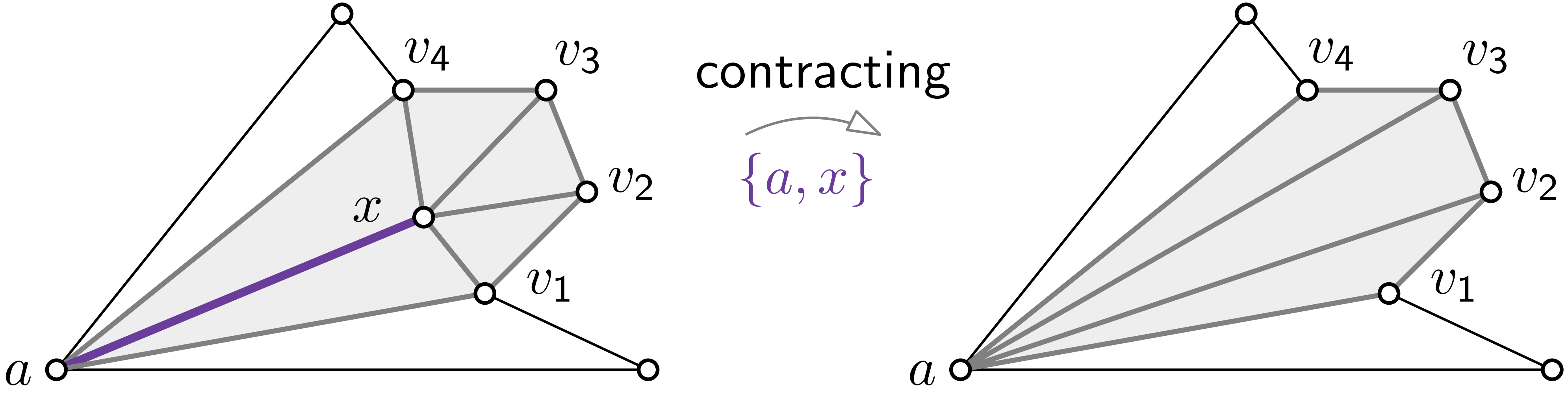

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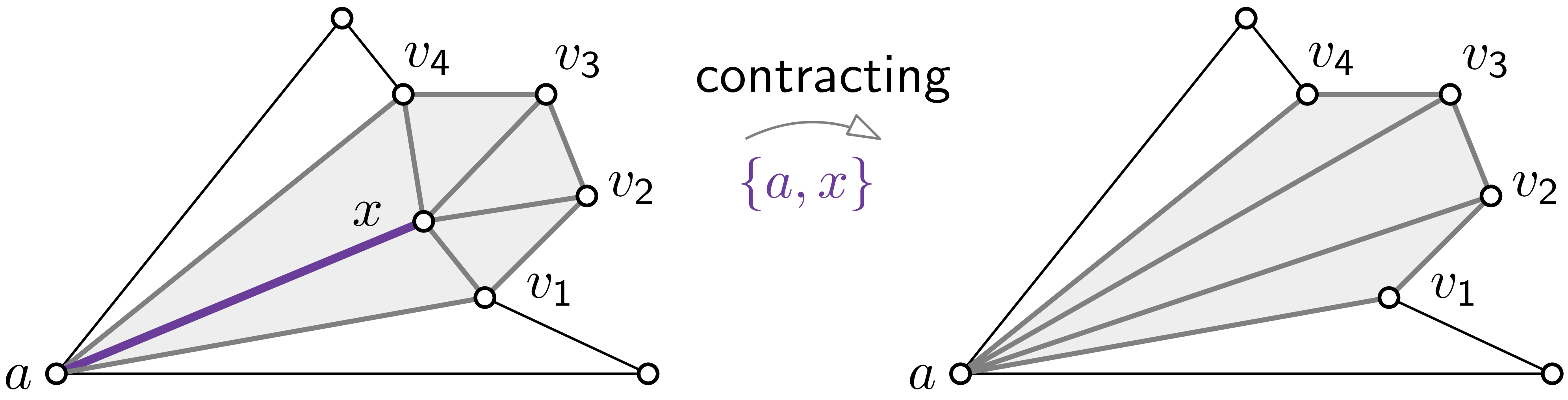

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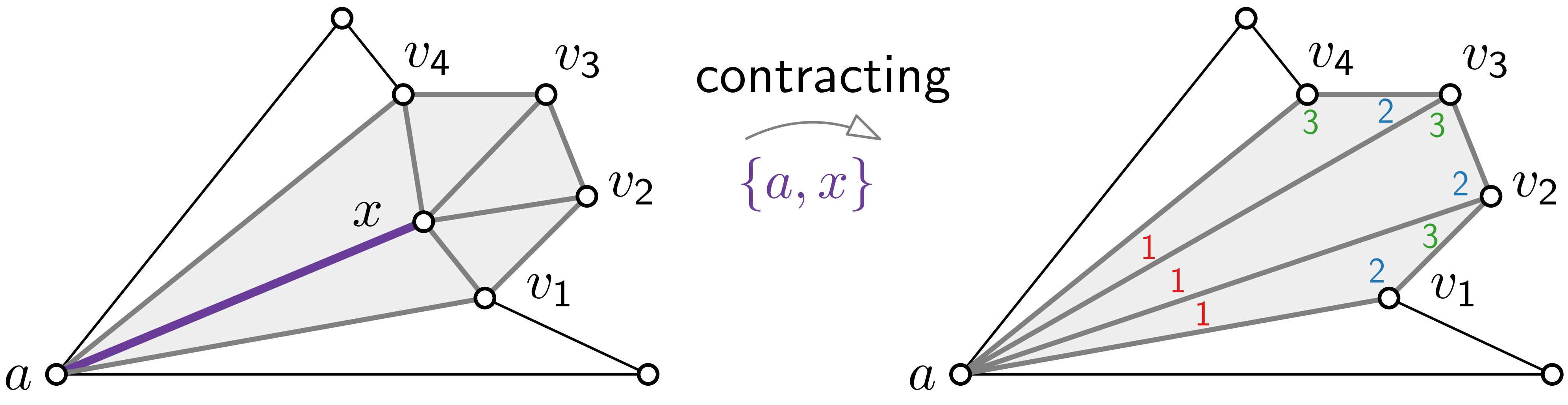

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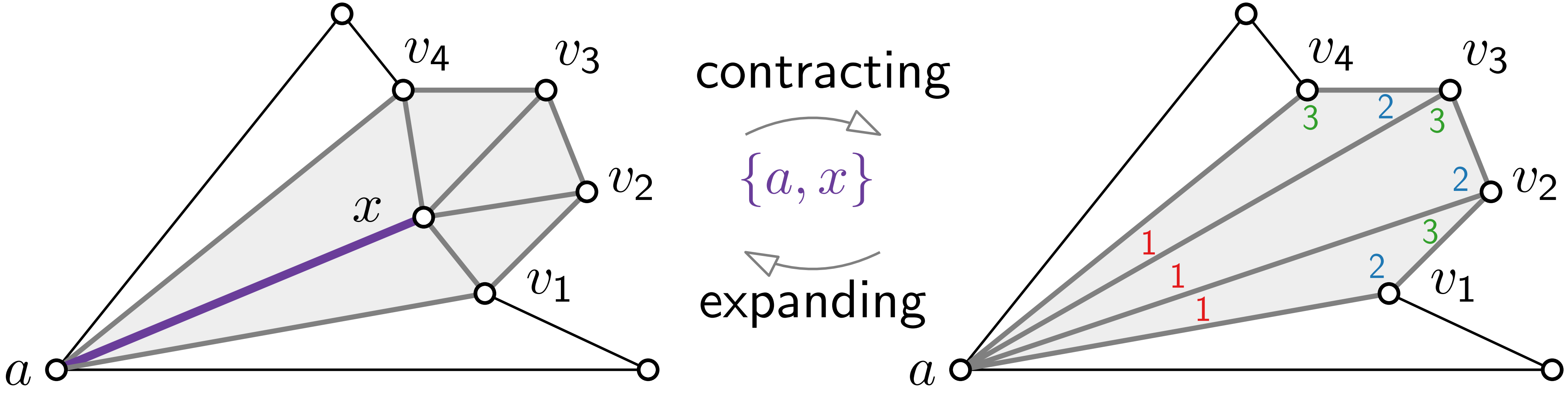

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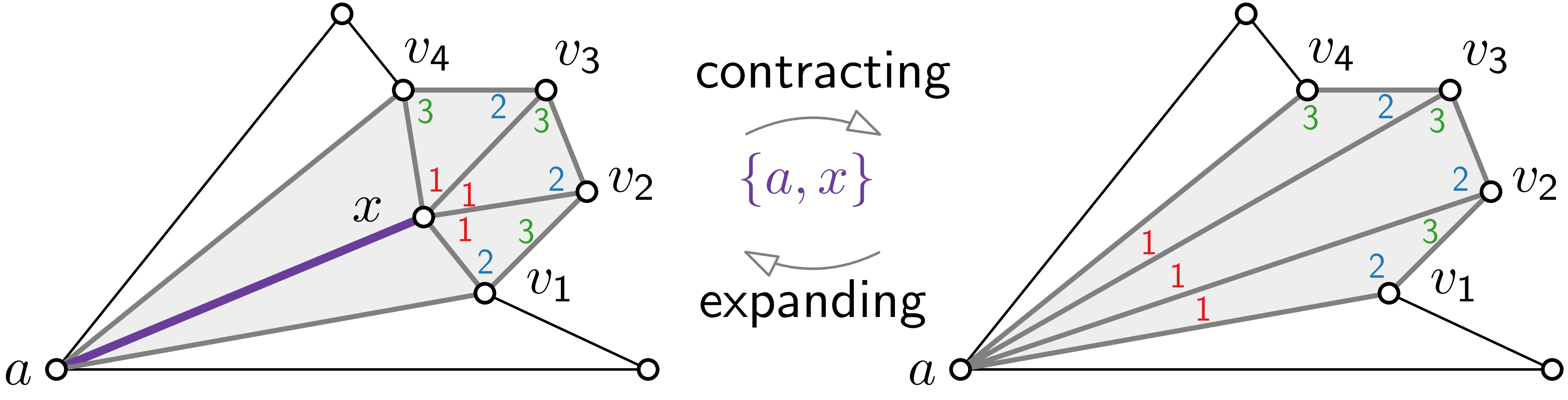

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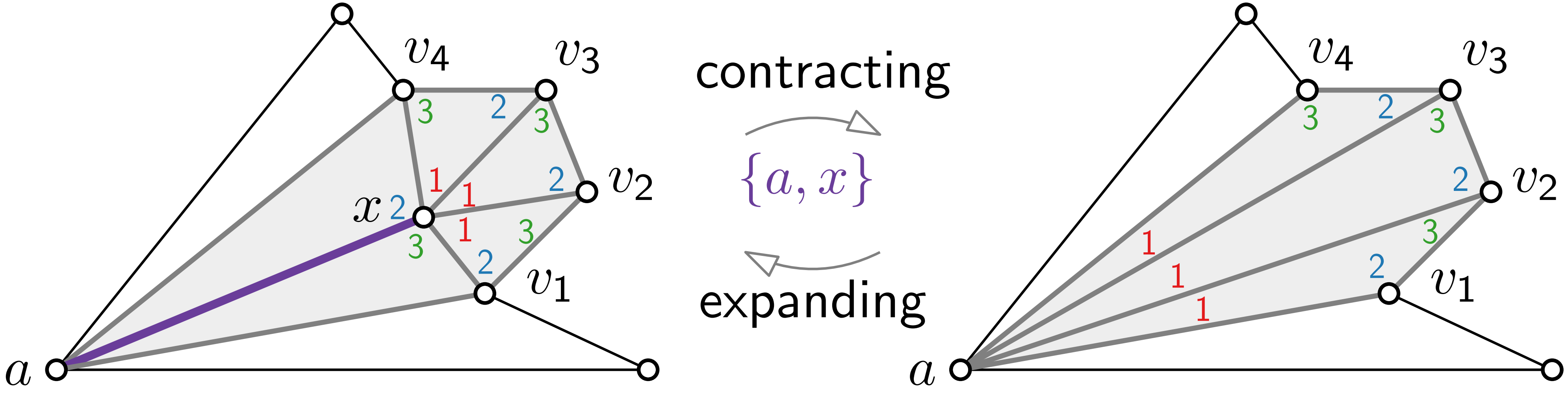

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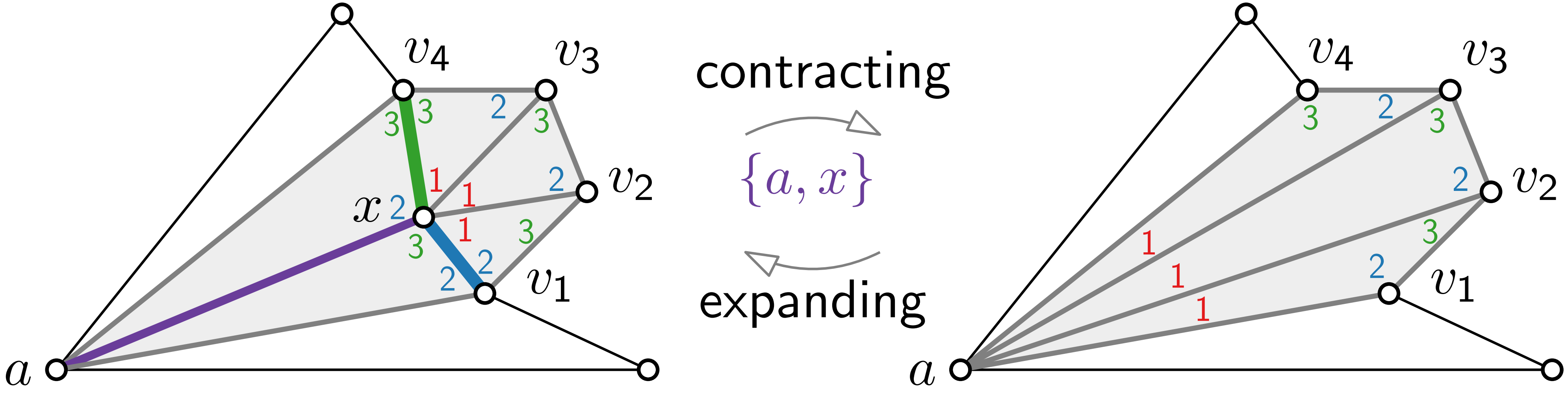

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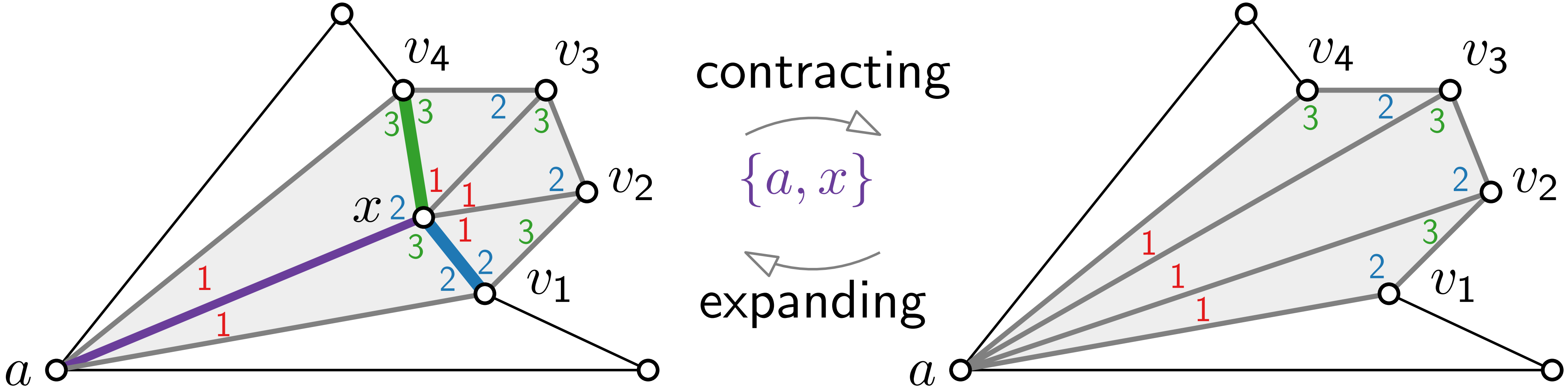

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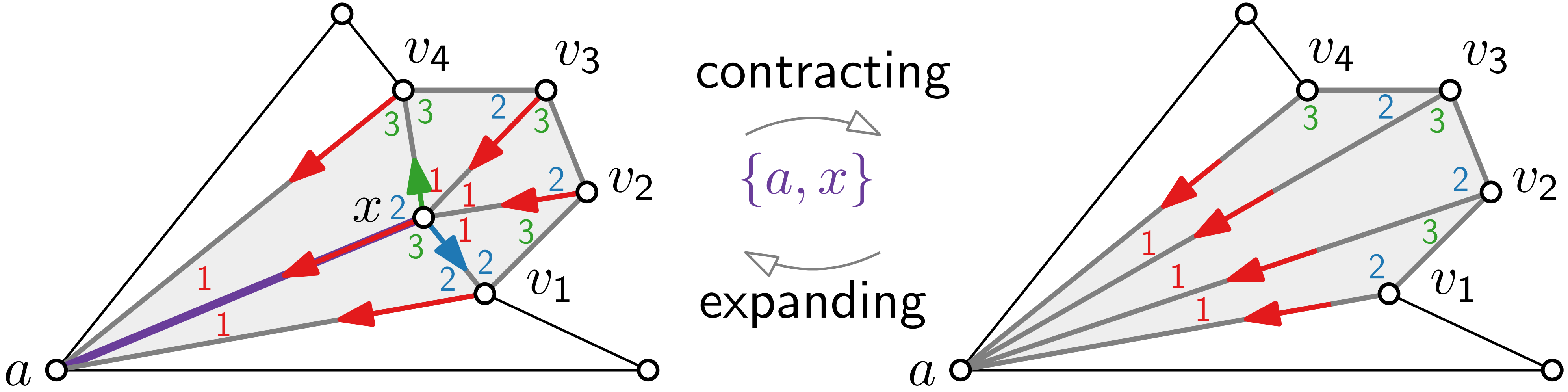

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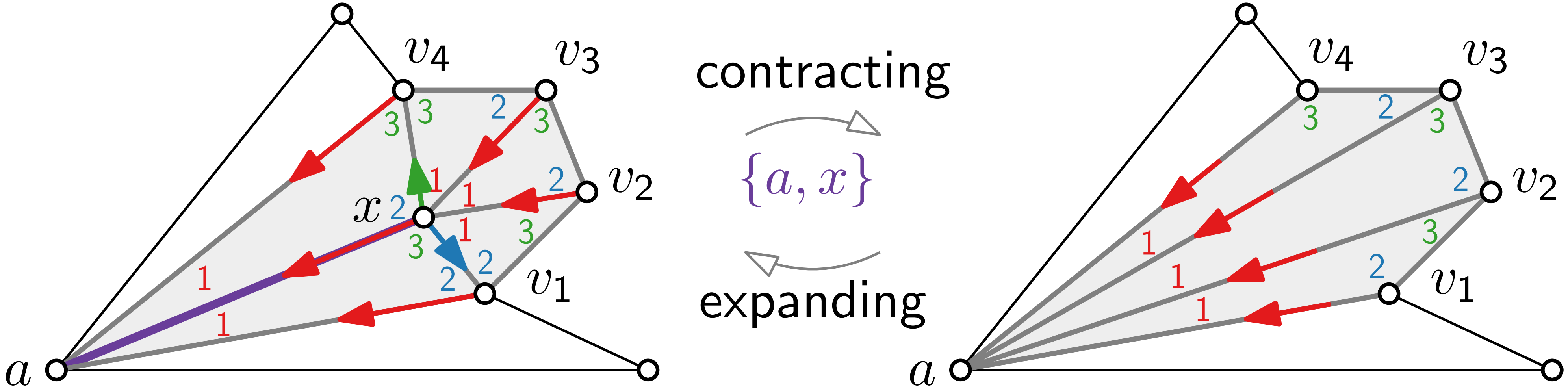

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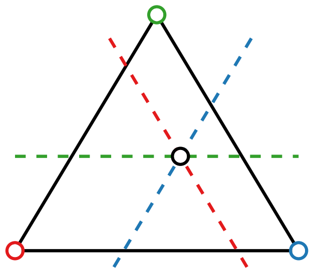
Constructive proof yields an algorithm for computing a Schnyder labeling. It can be implemented to run in $\mathcal{O}(n)$ time. . .

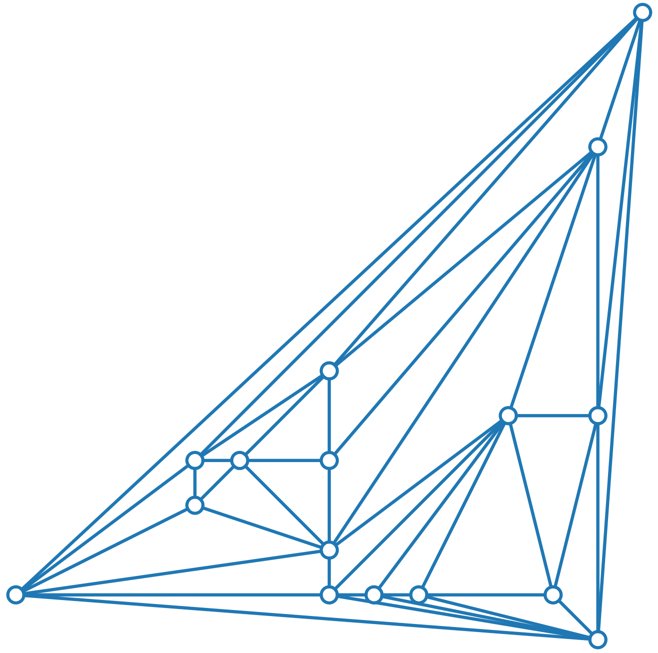
Exercise :-)

Visualization of Graphs

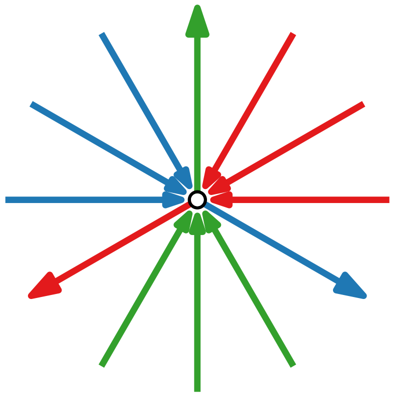
Lecture 4:
Straight-Line Drawings of Planar Graphs II: Schnyder Woods



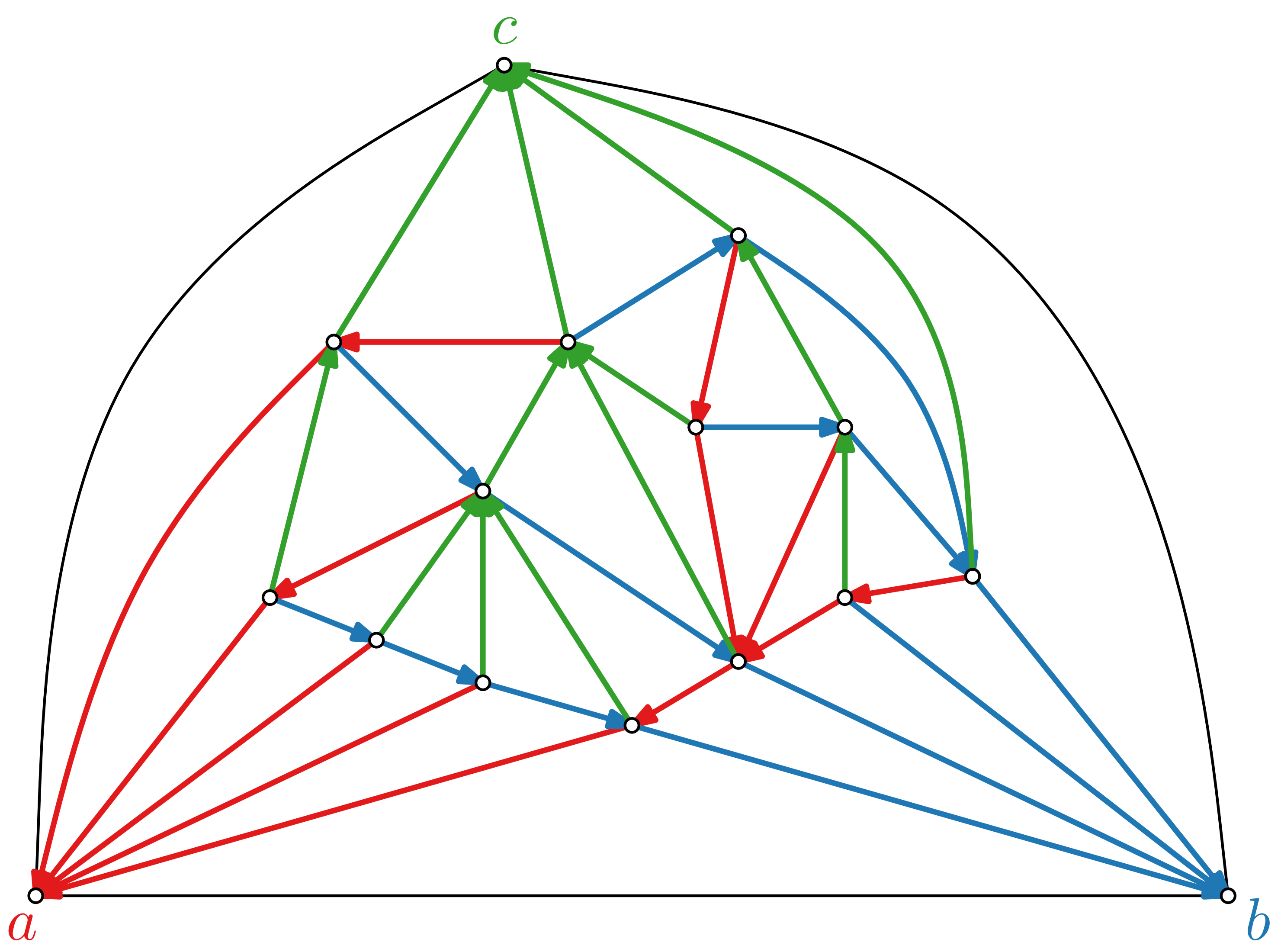
Part III:
Schnyder Drawings



Alexander Wolff



Schnyder Wood – More Properties



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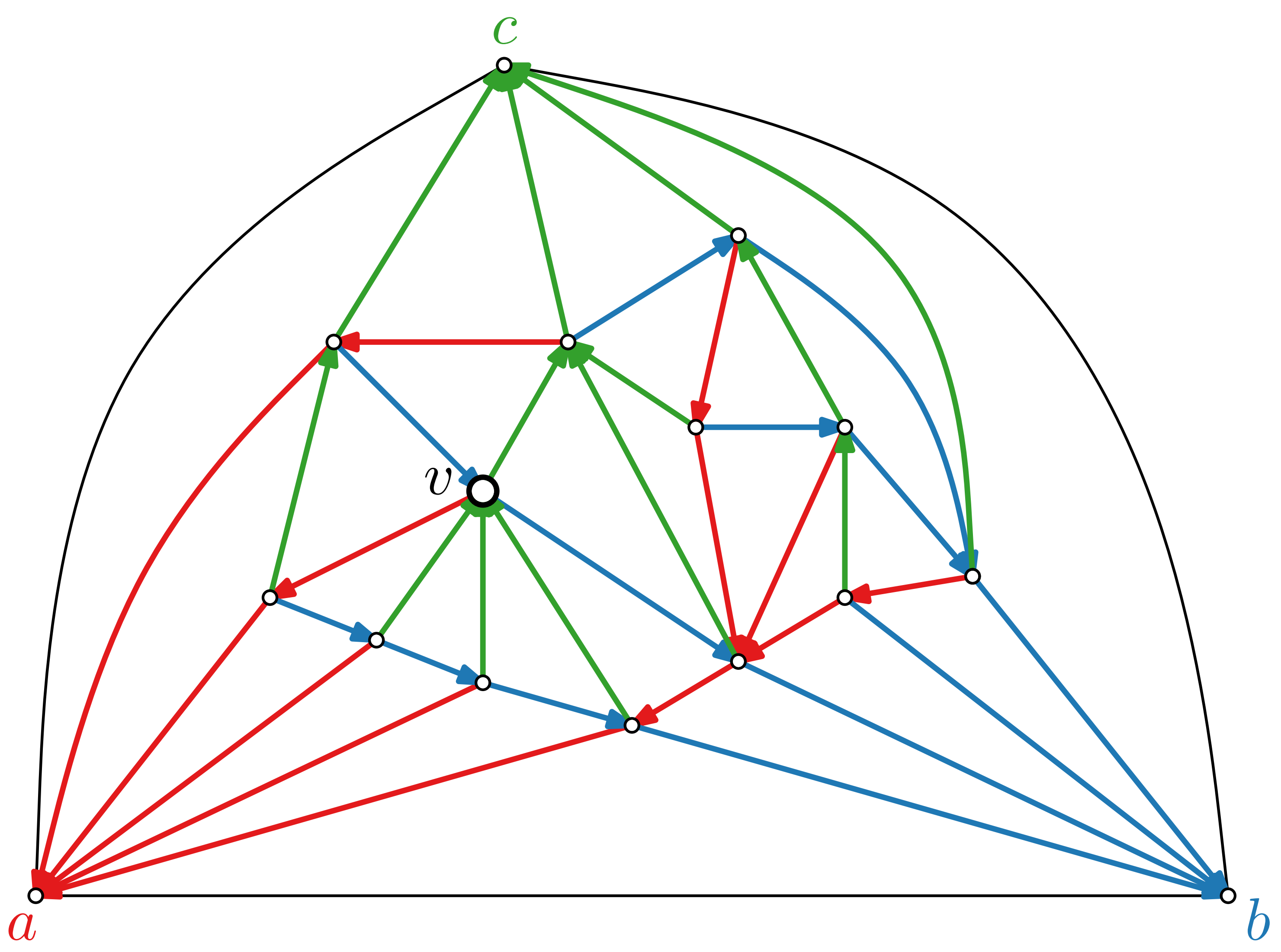

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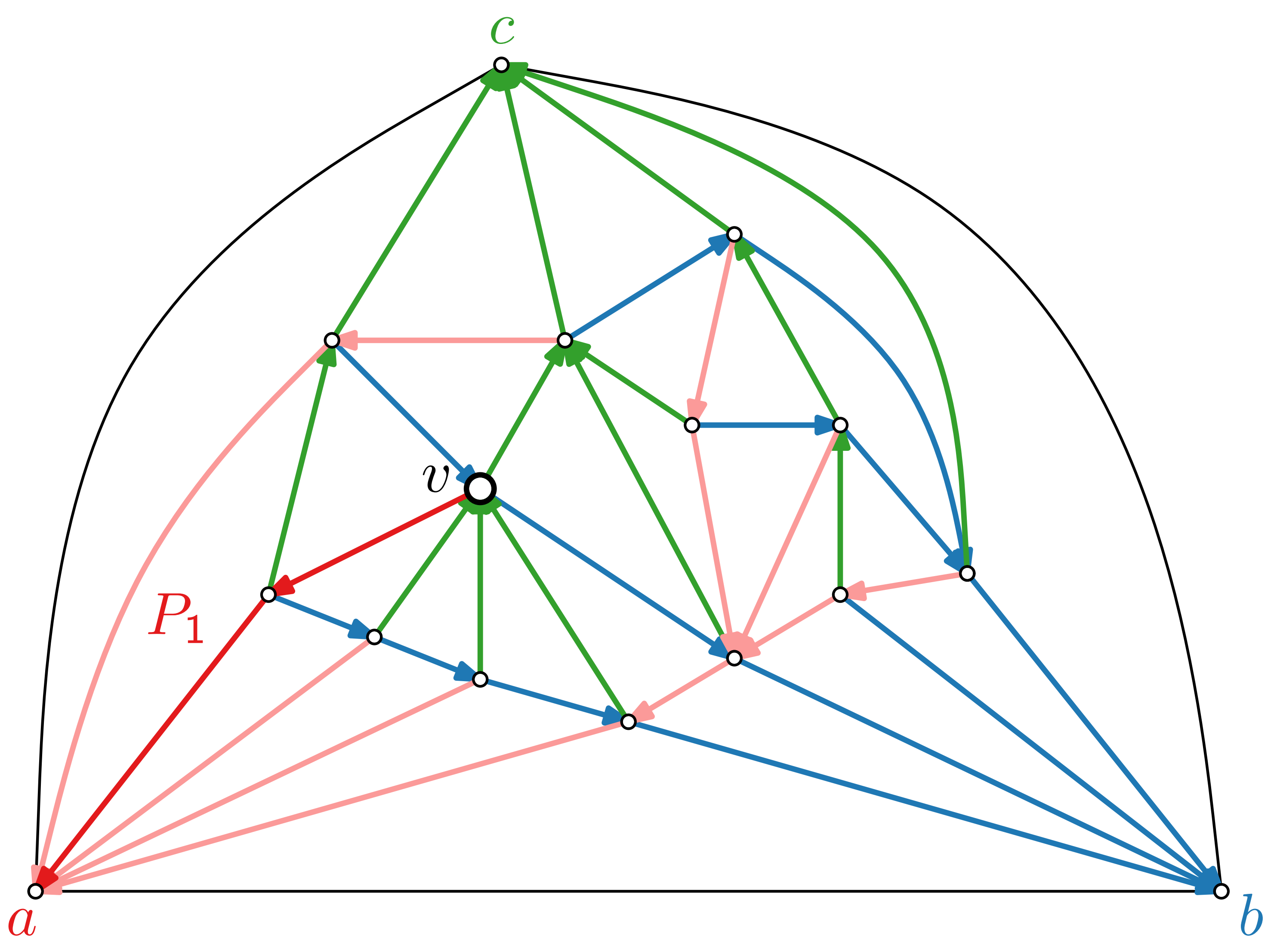

- From each vertex v there exists a directed red path $P_1(v)$ to a,

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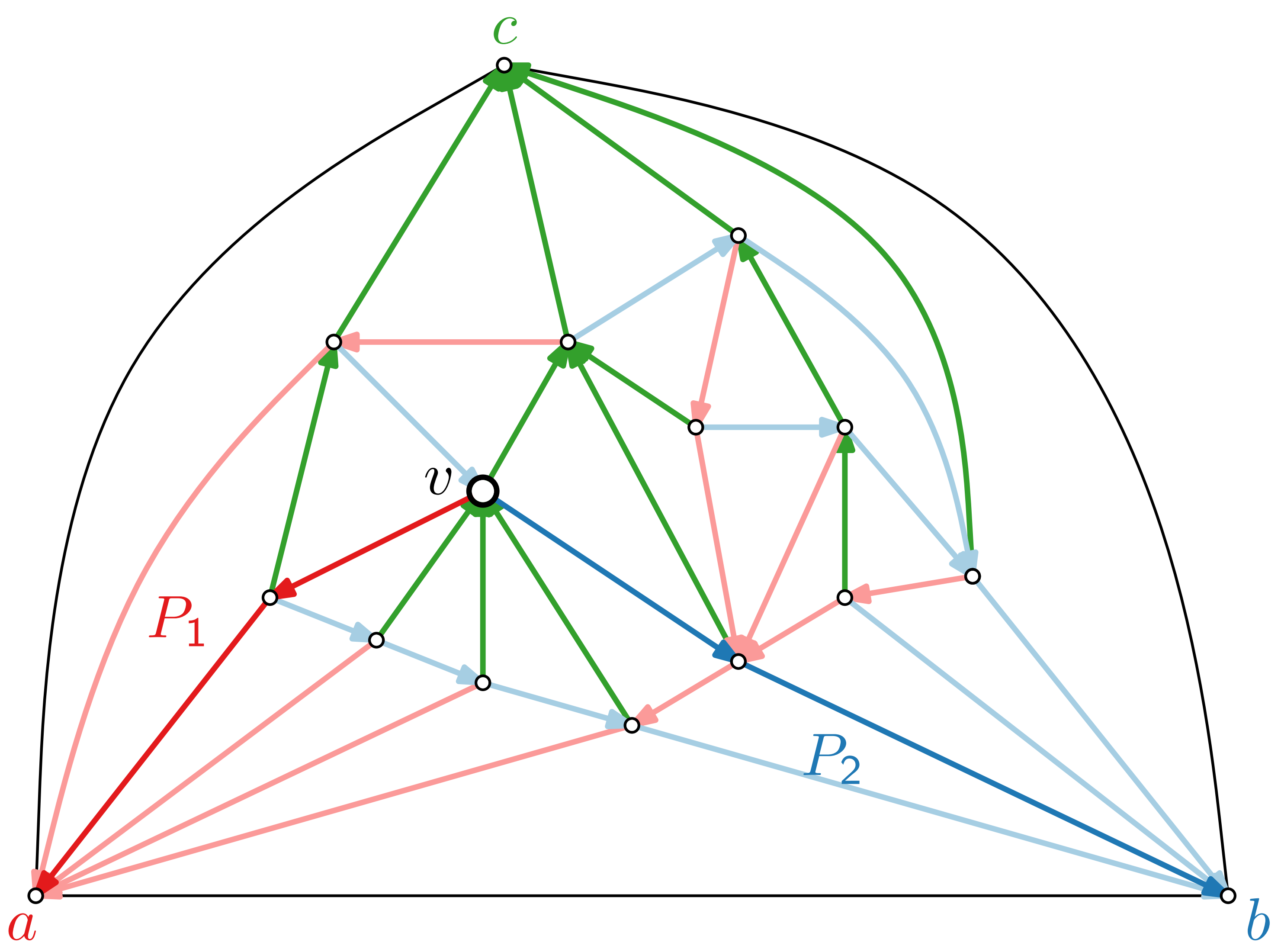

- From each vertex v there exists
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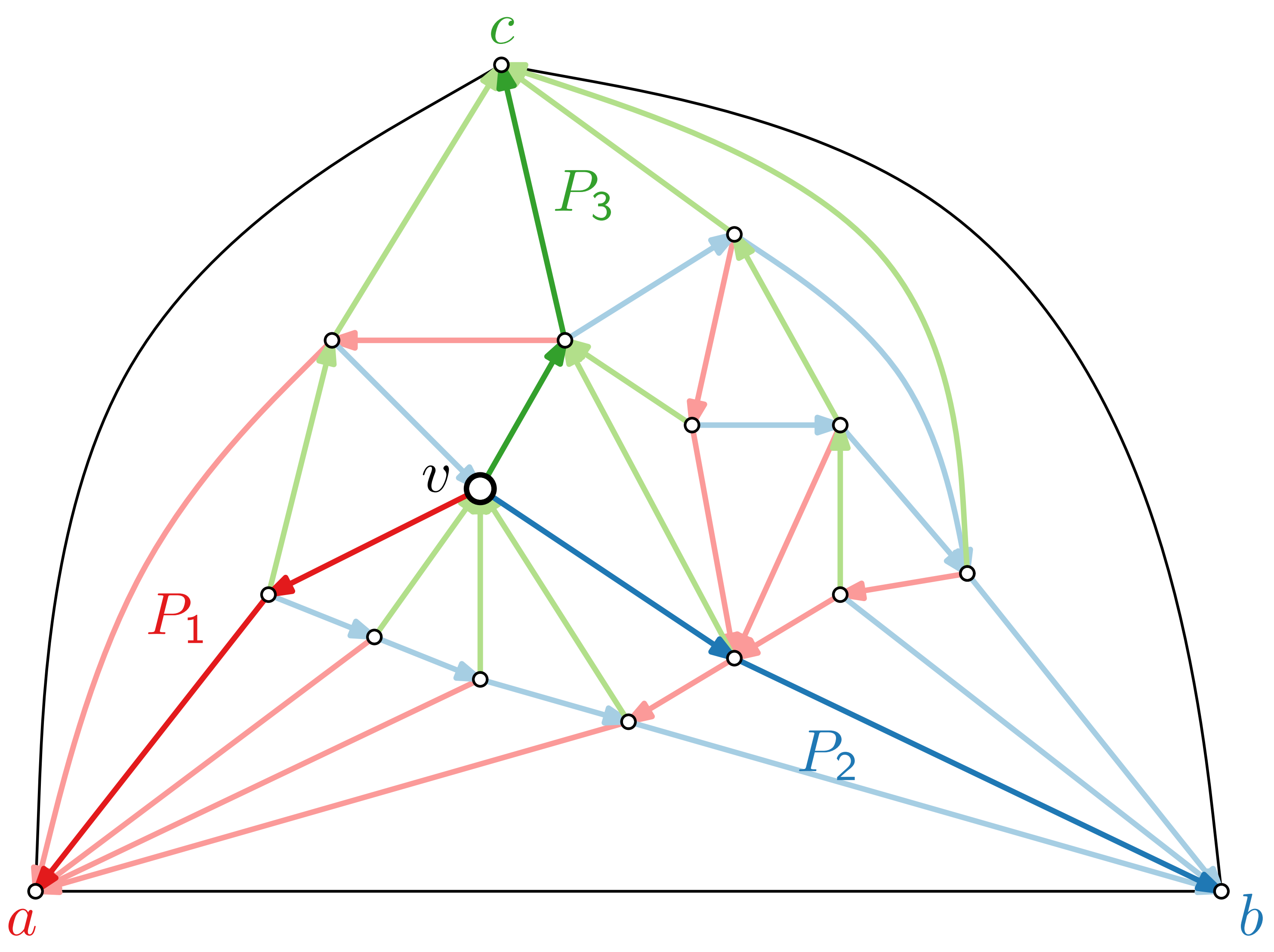

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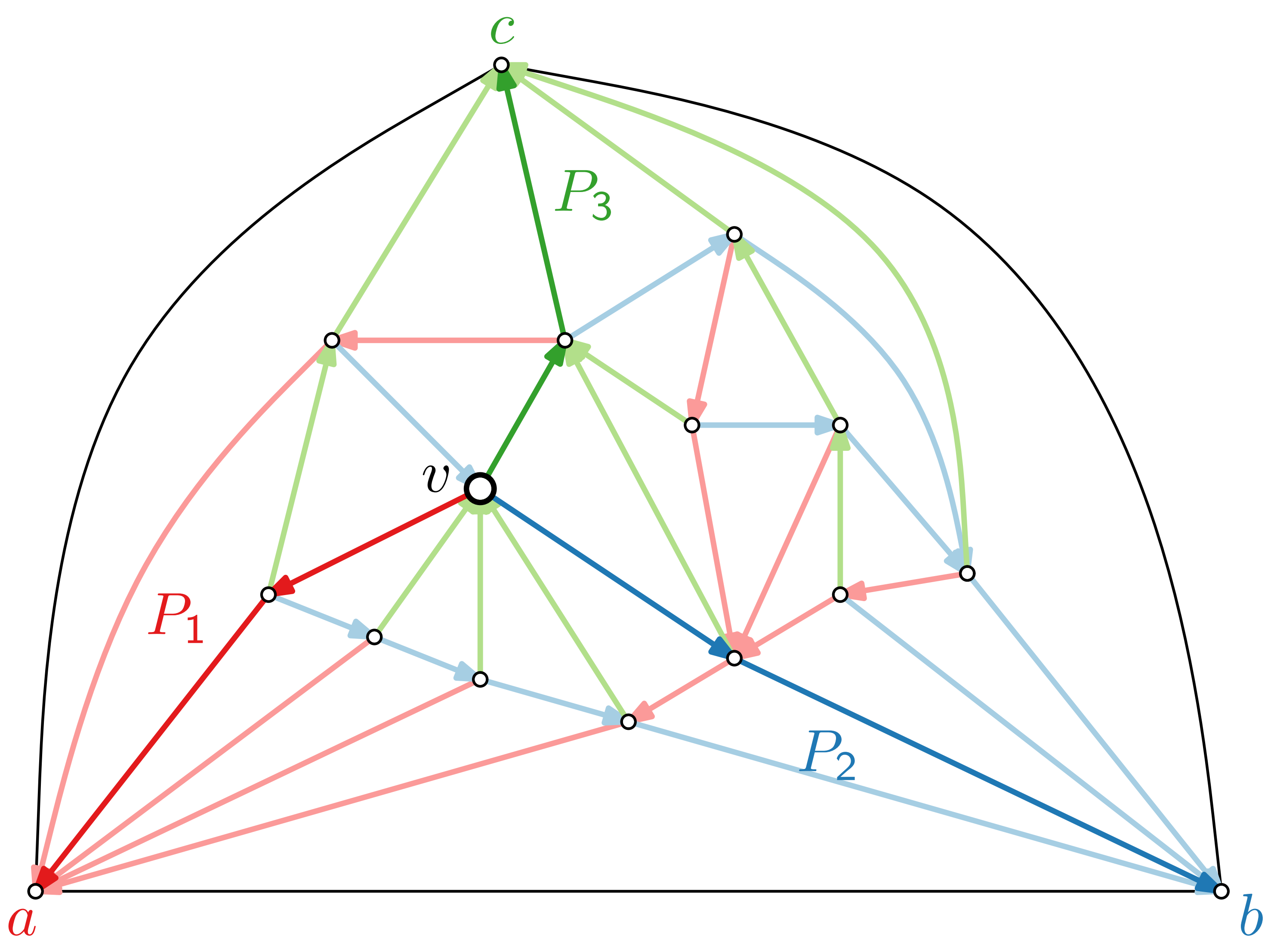

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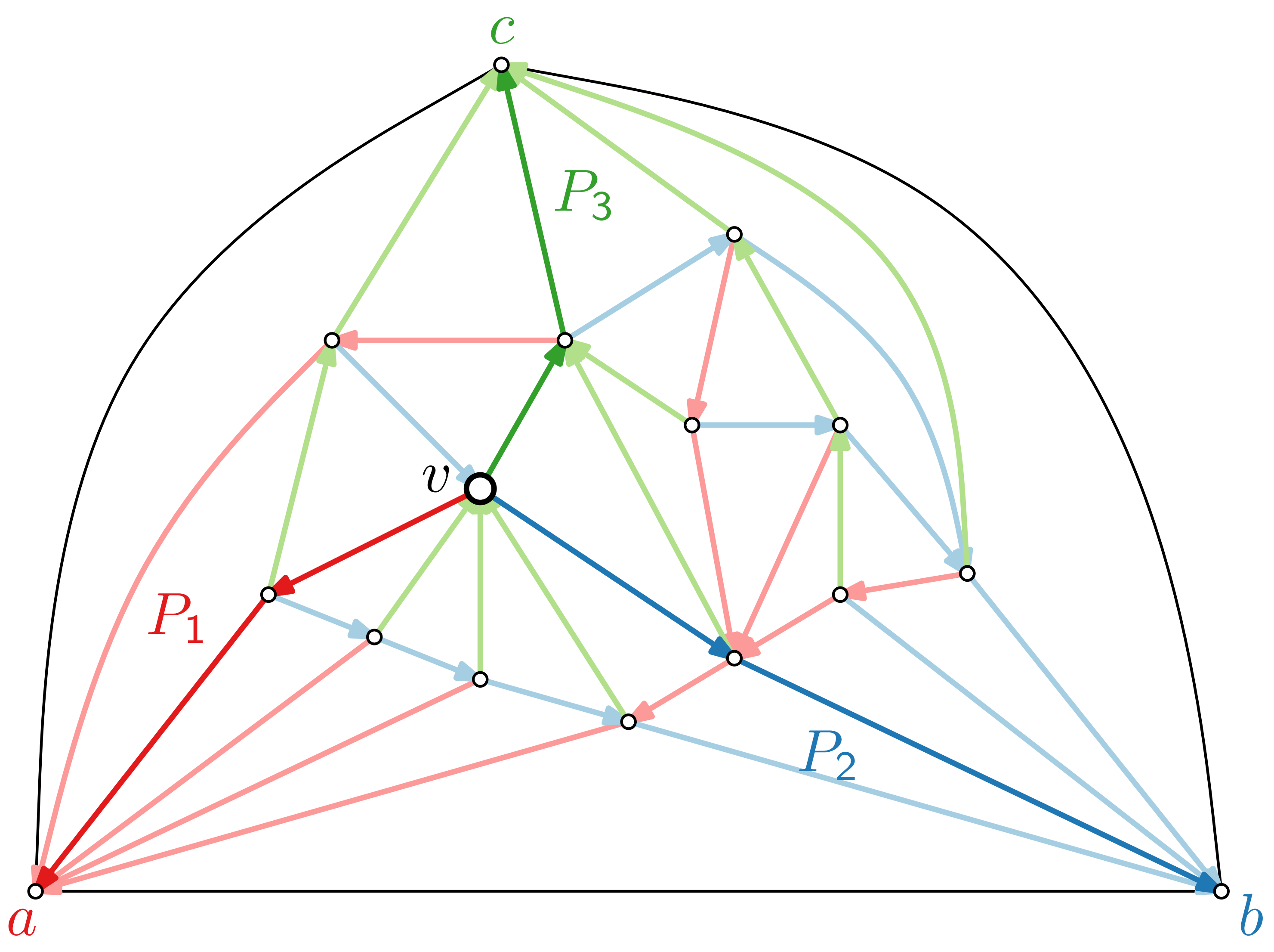

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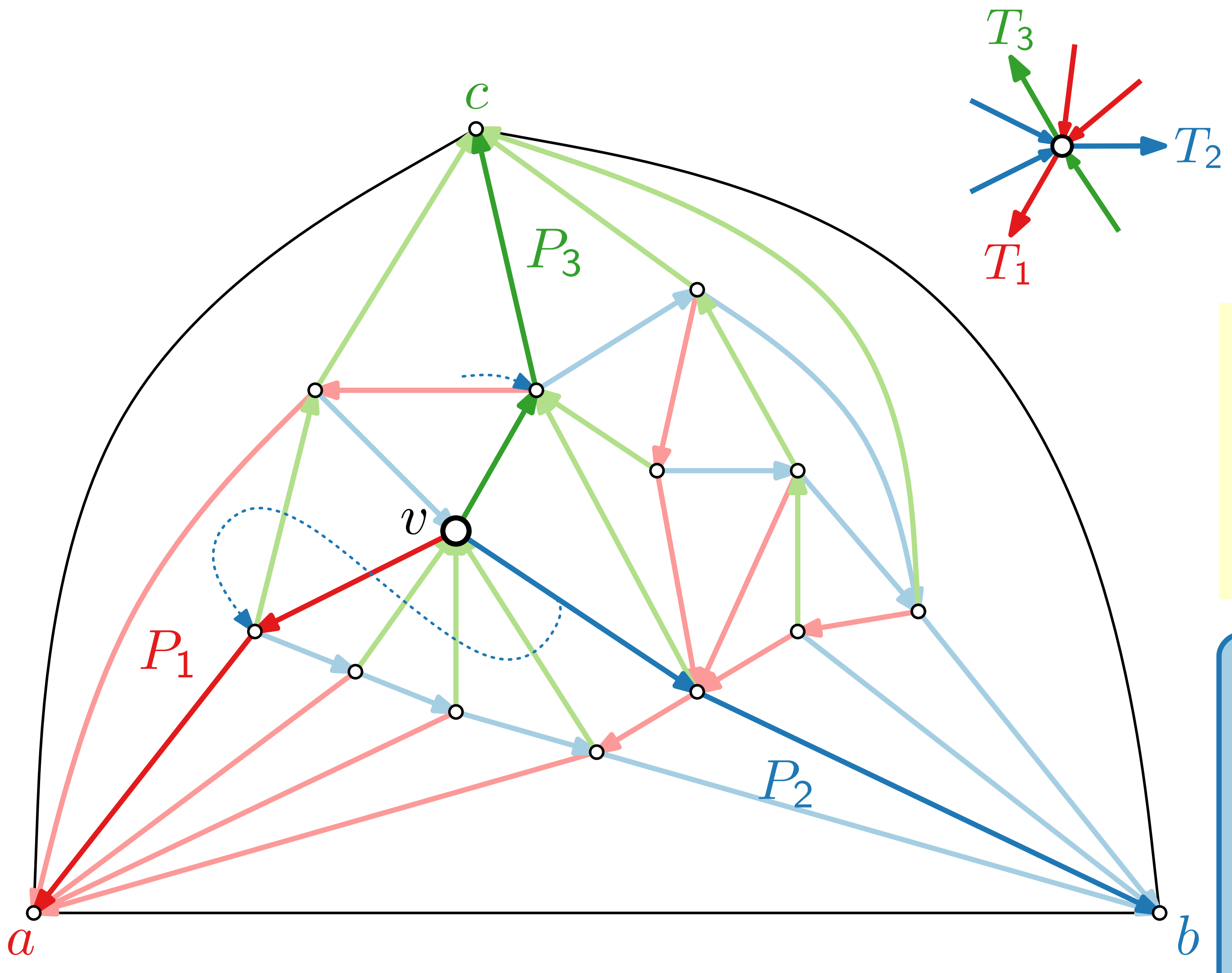

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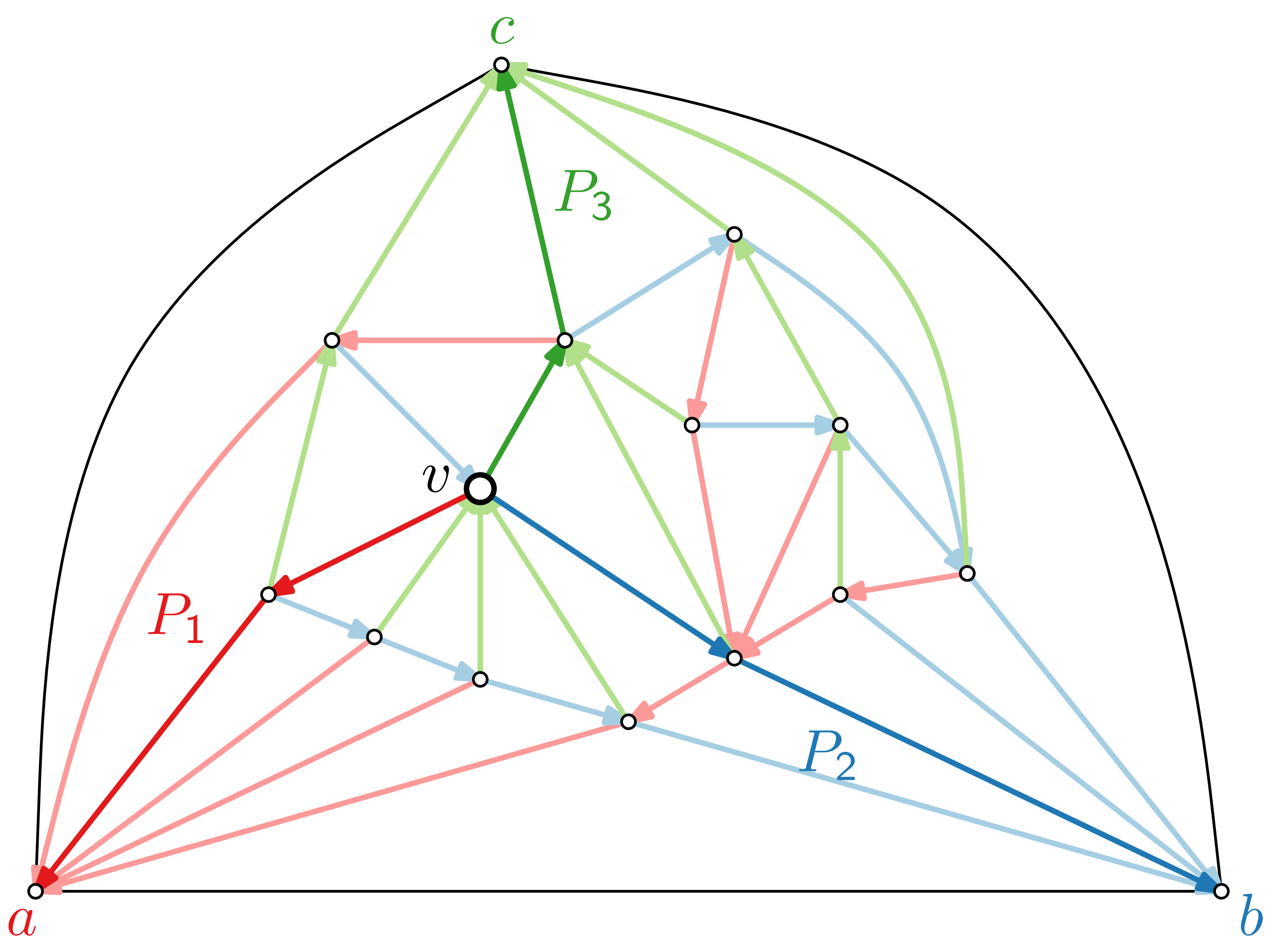

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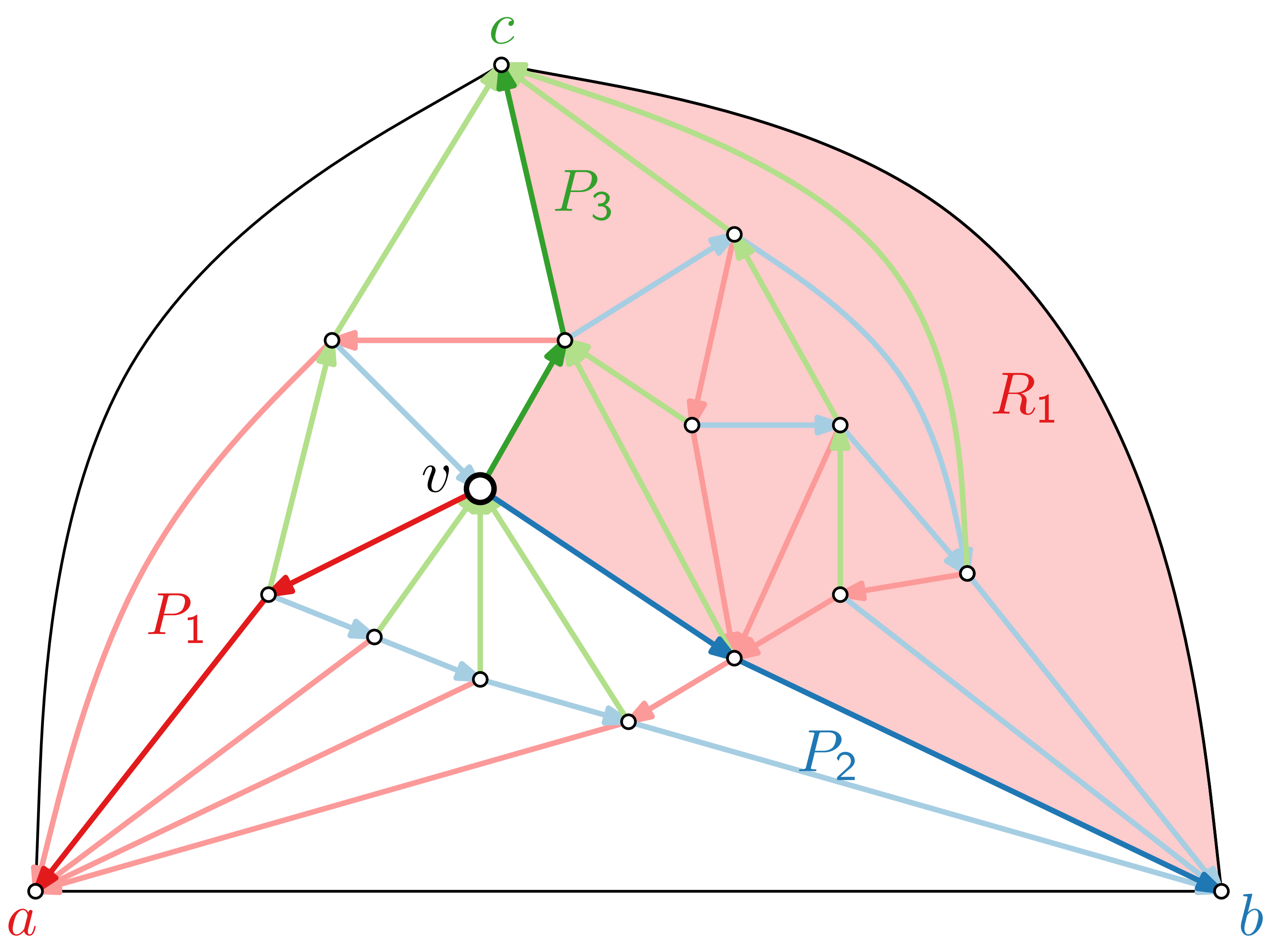

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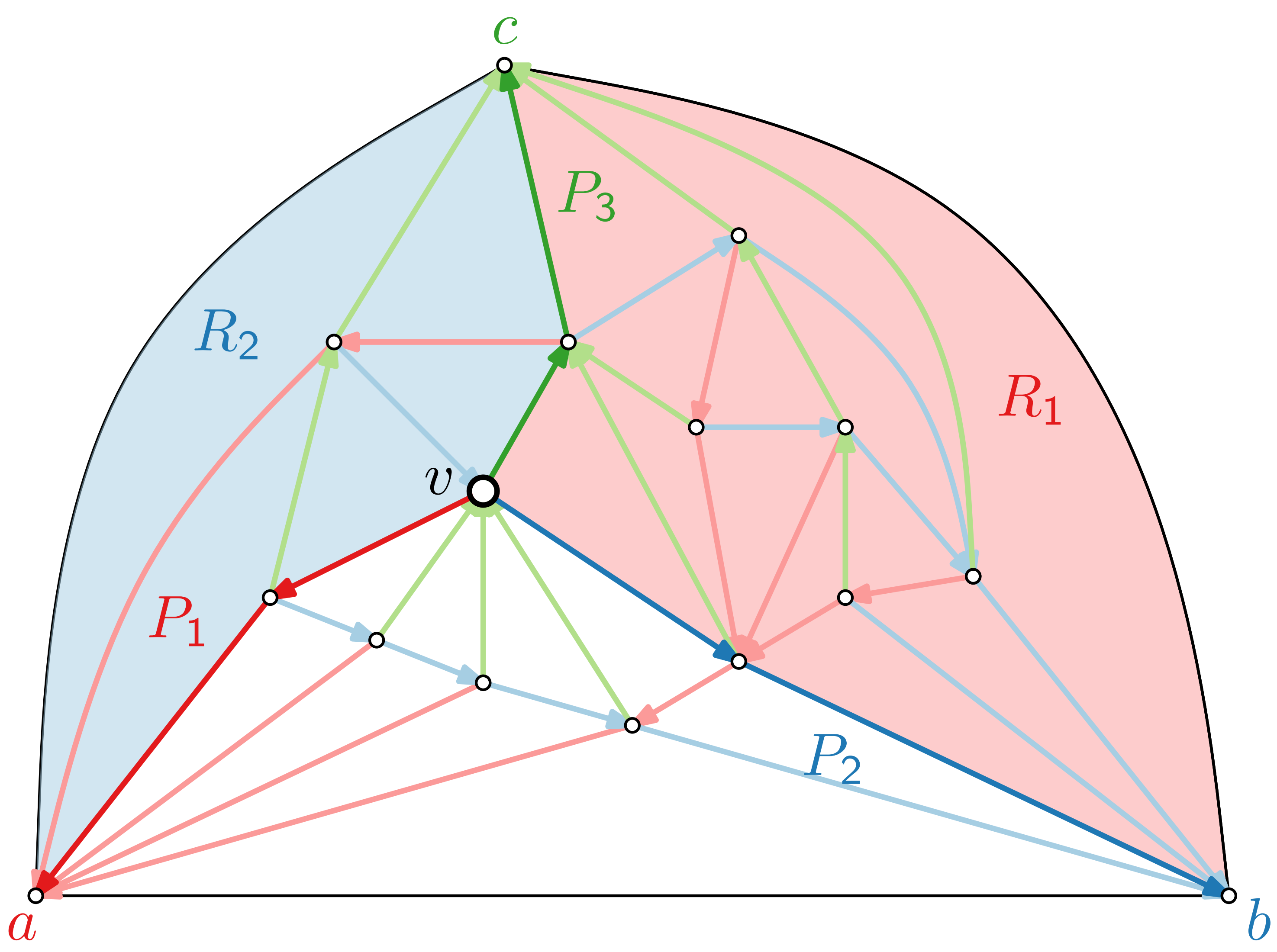

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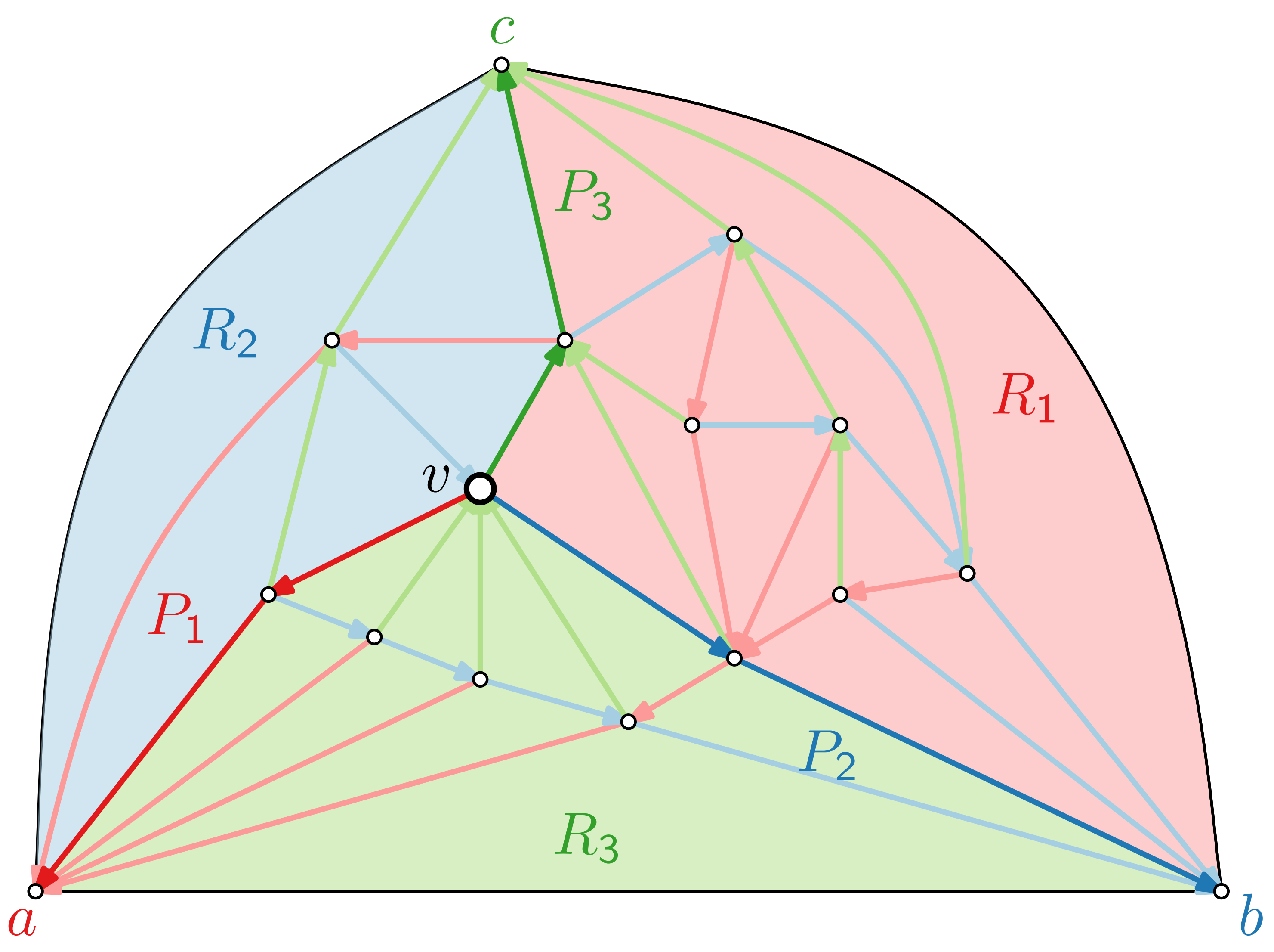

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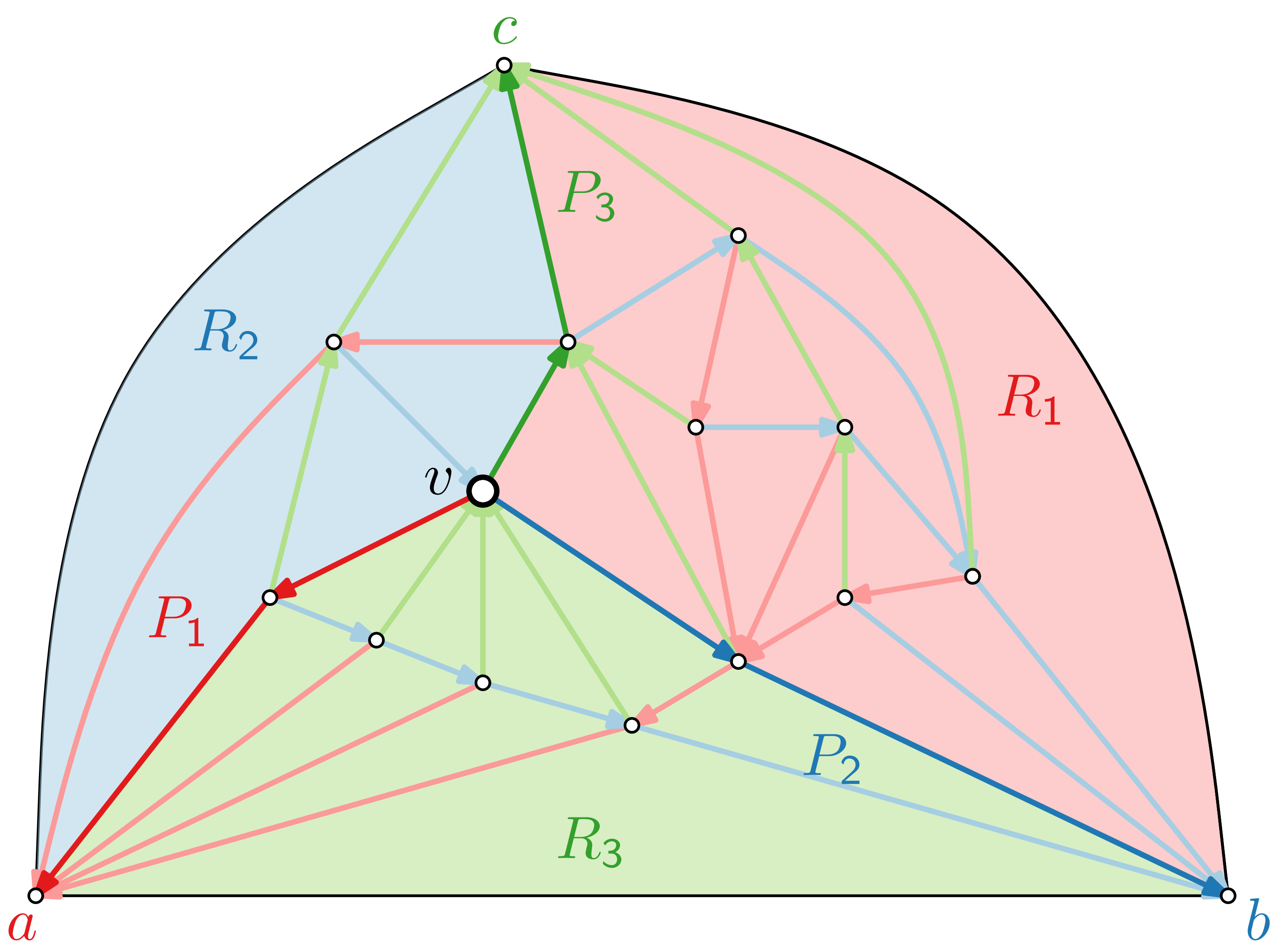

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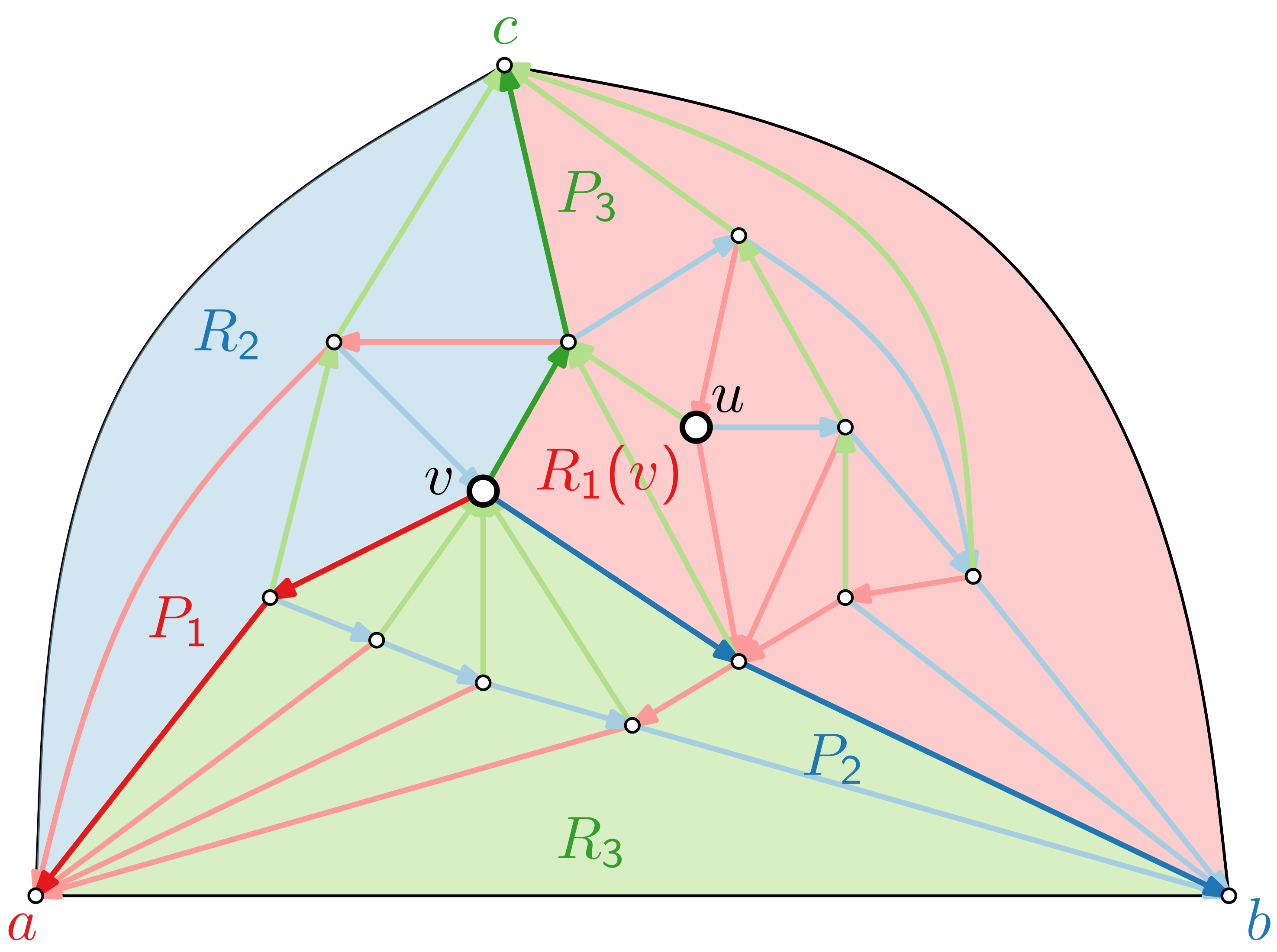

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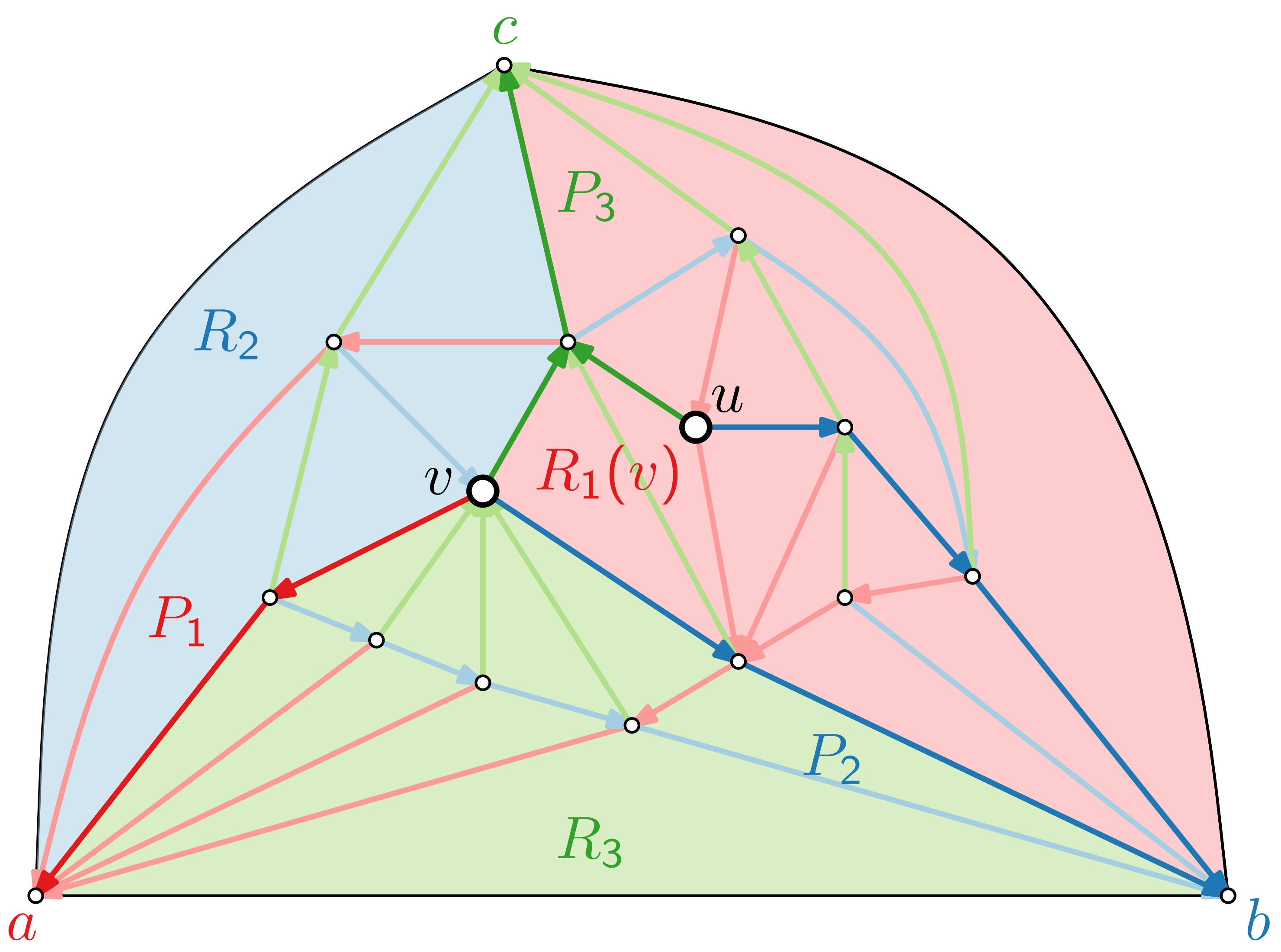

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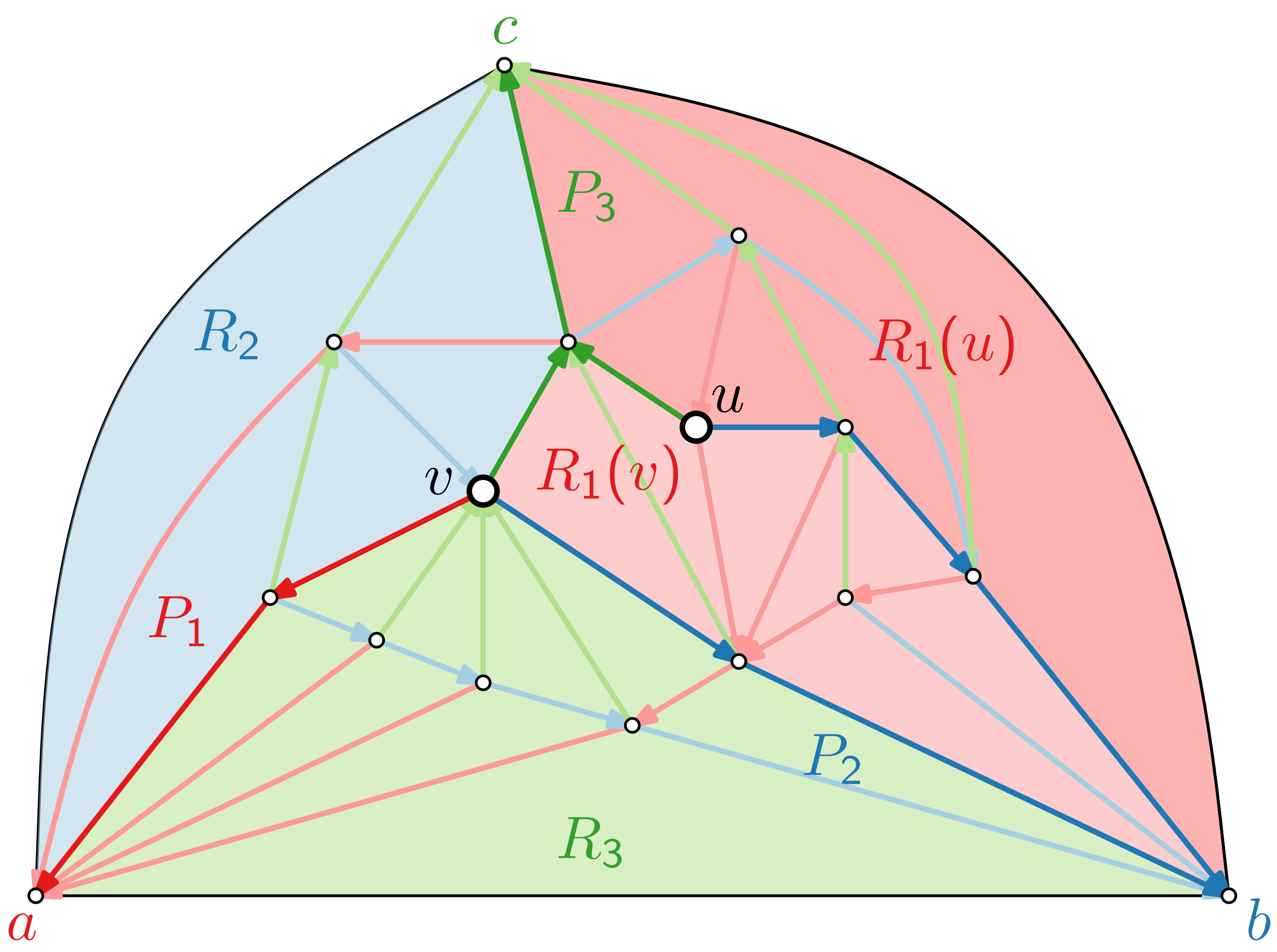

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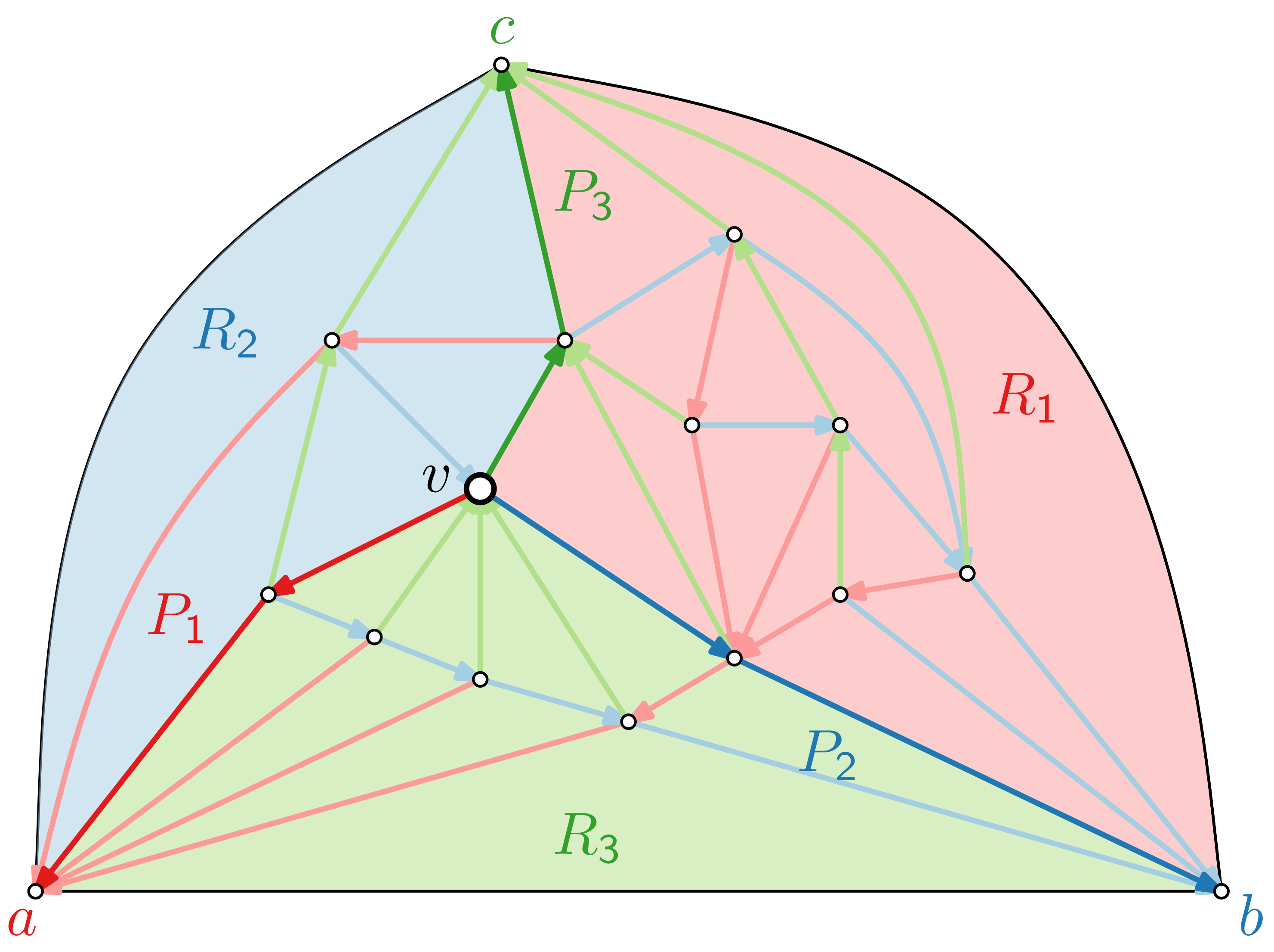

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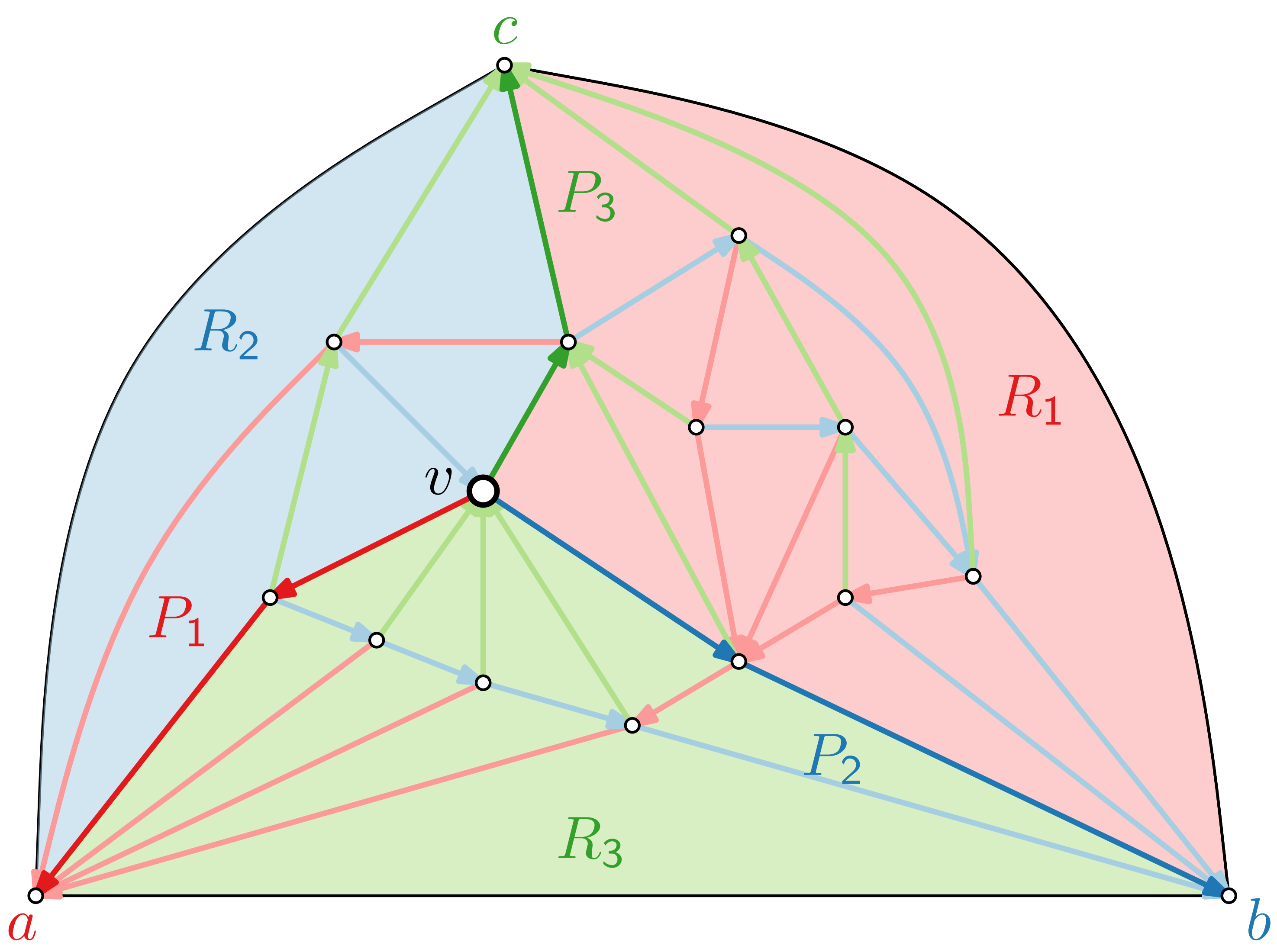

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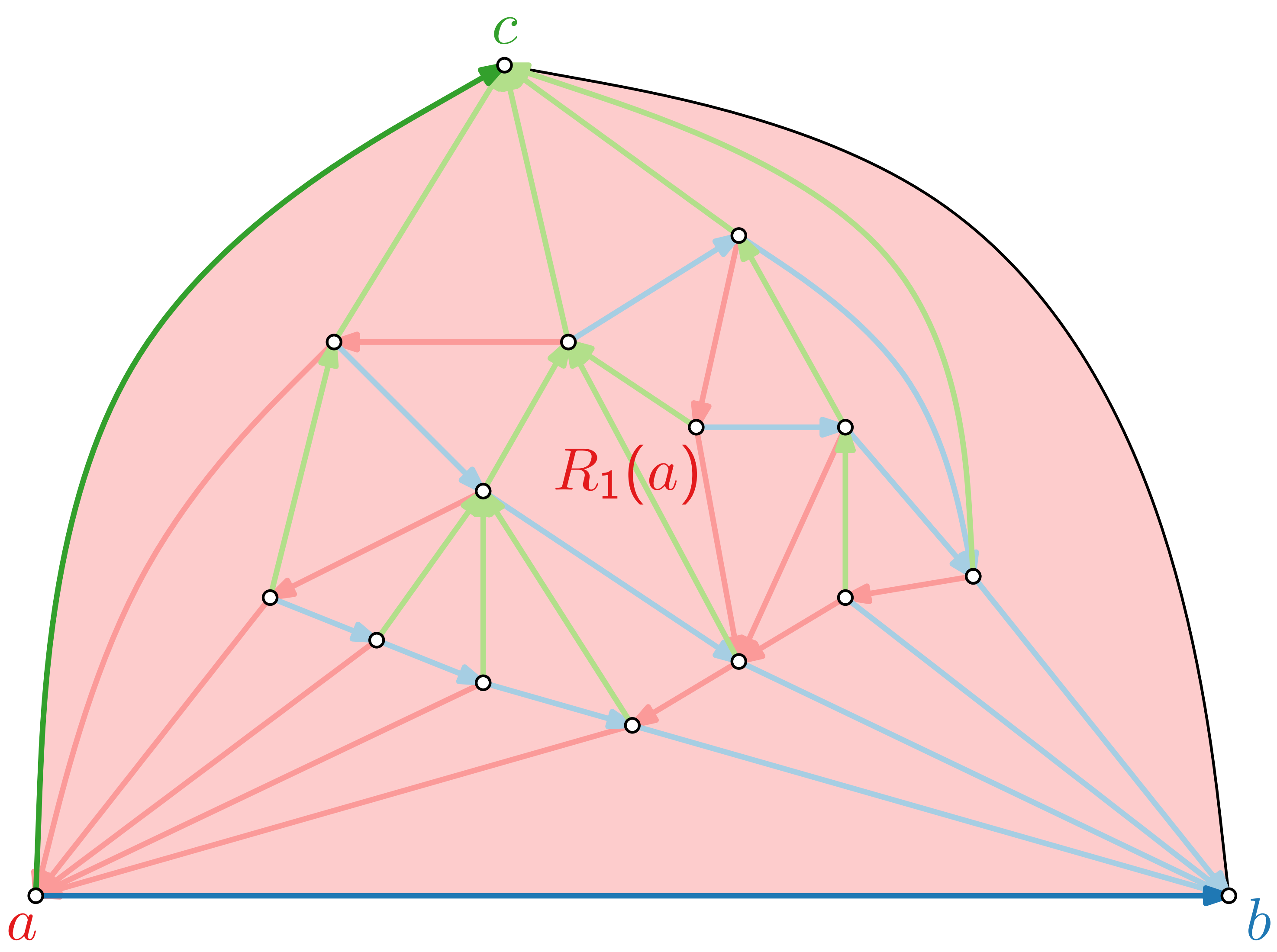

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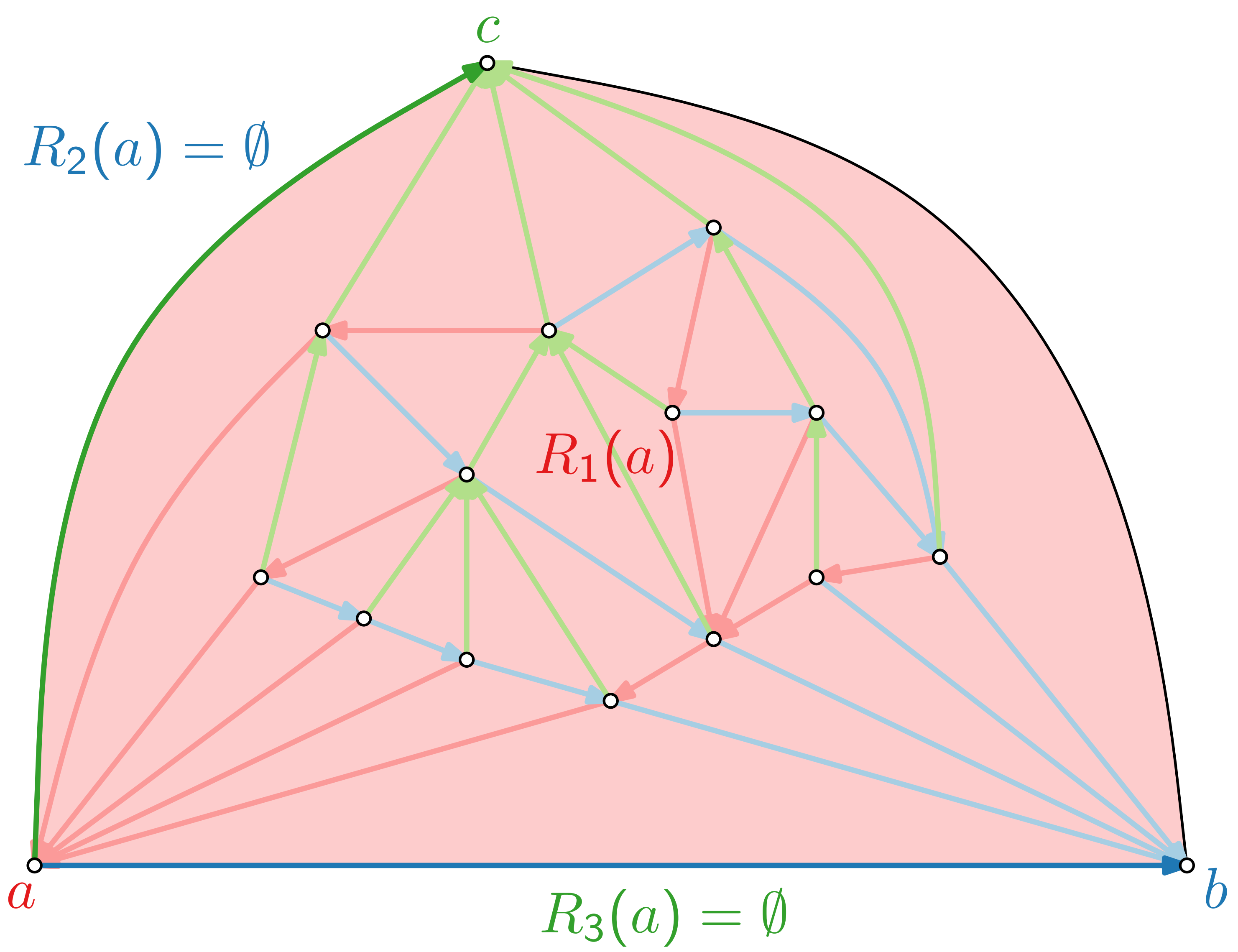

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Theorem. [Schnyder '90]

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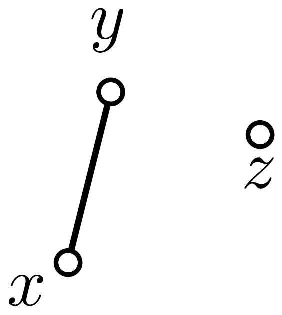

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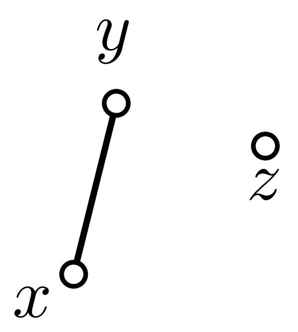

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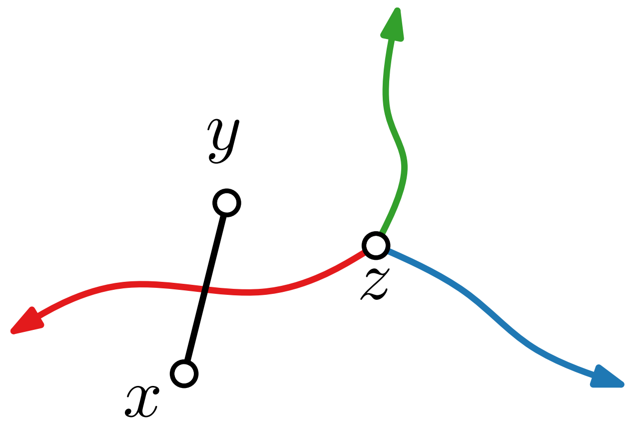

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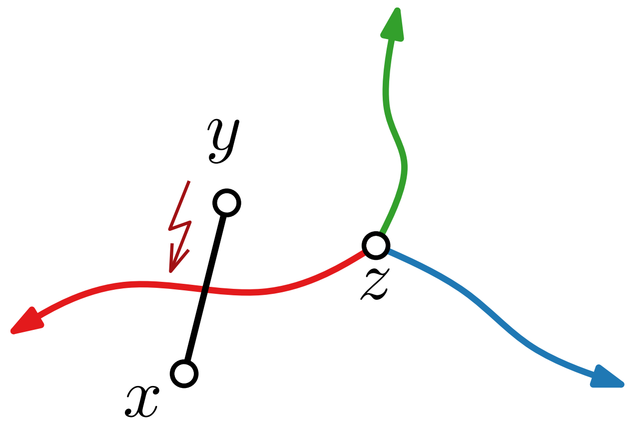

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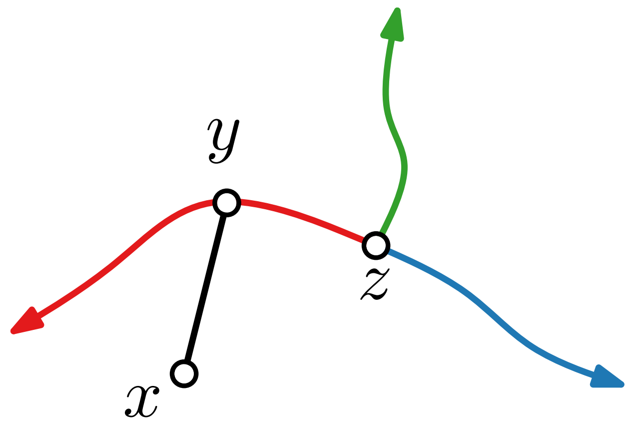

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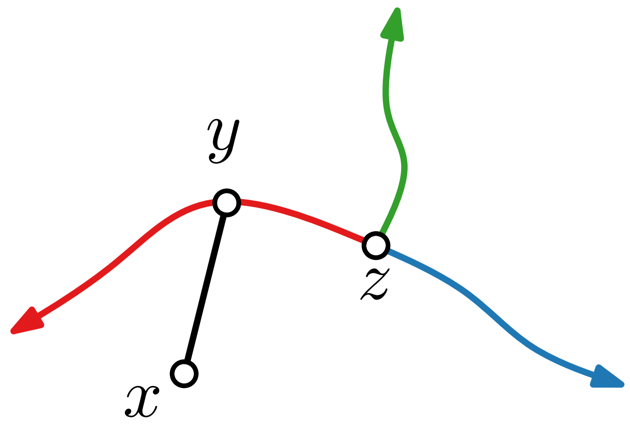

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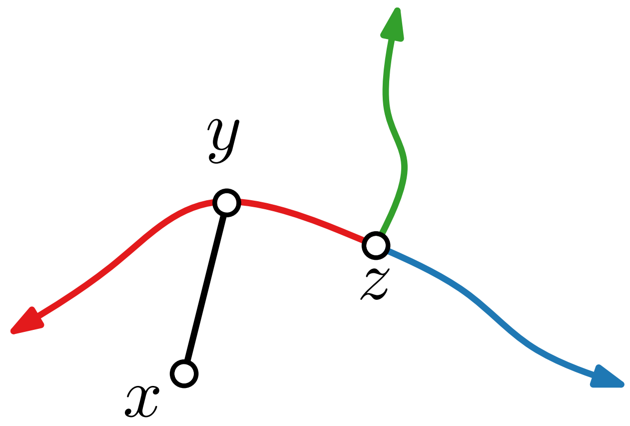

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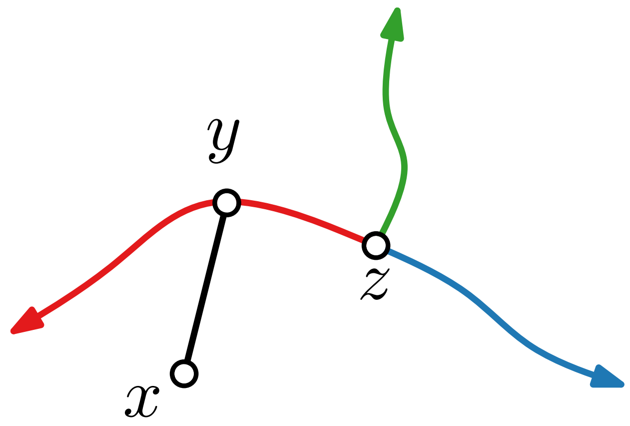

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Set $A = (0, 0)$, $B = (2n - 5, 0)$, and $C = (0, 2n - 5)$.

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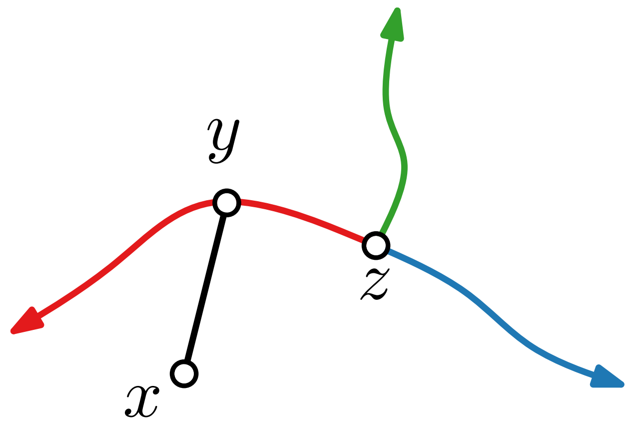

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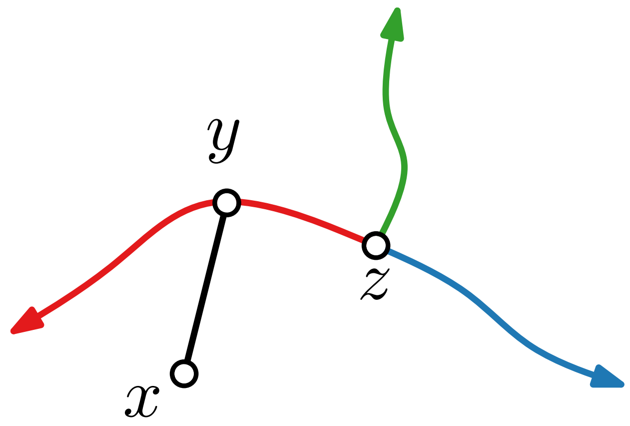

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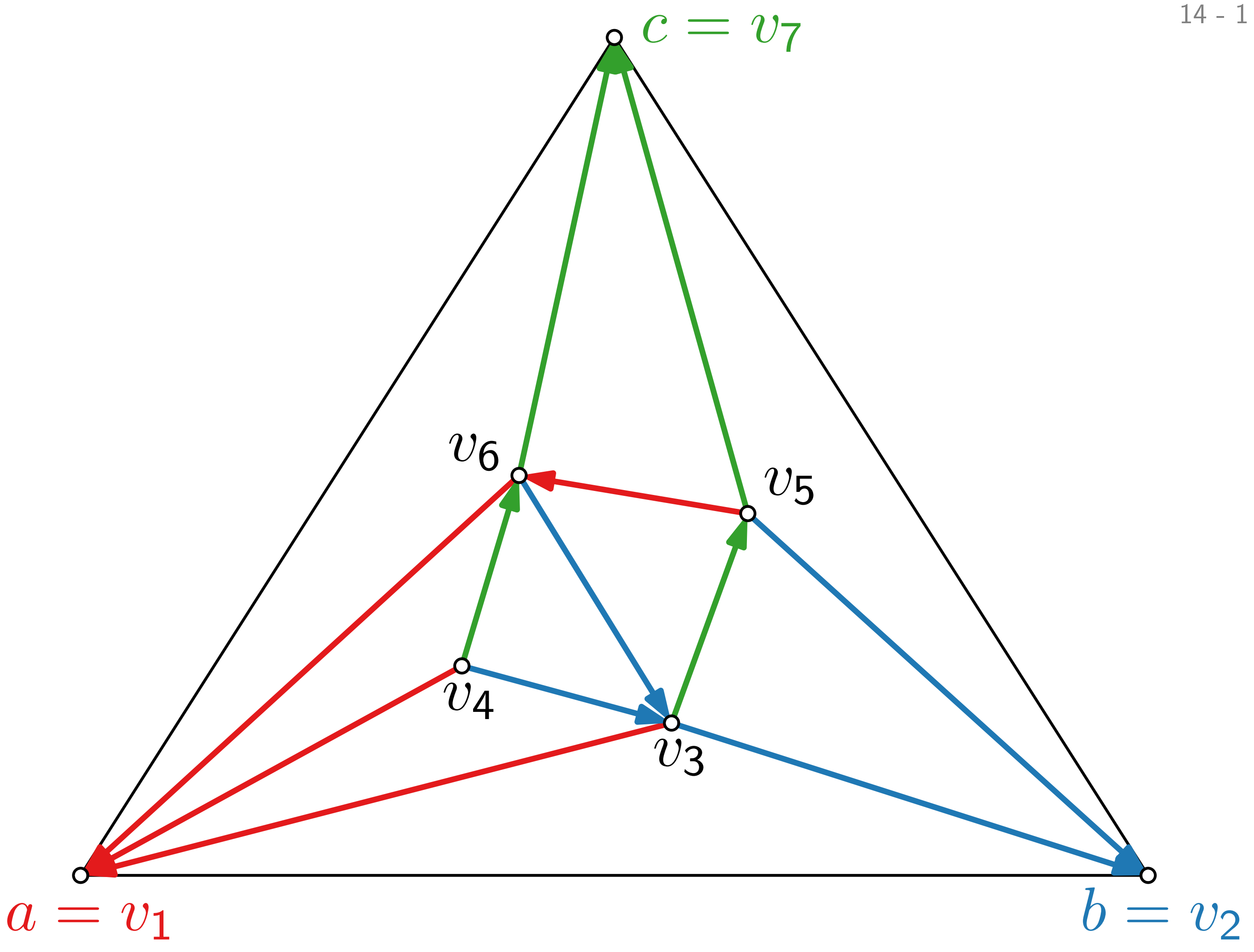

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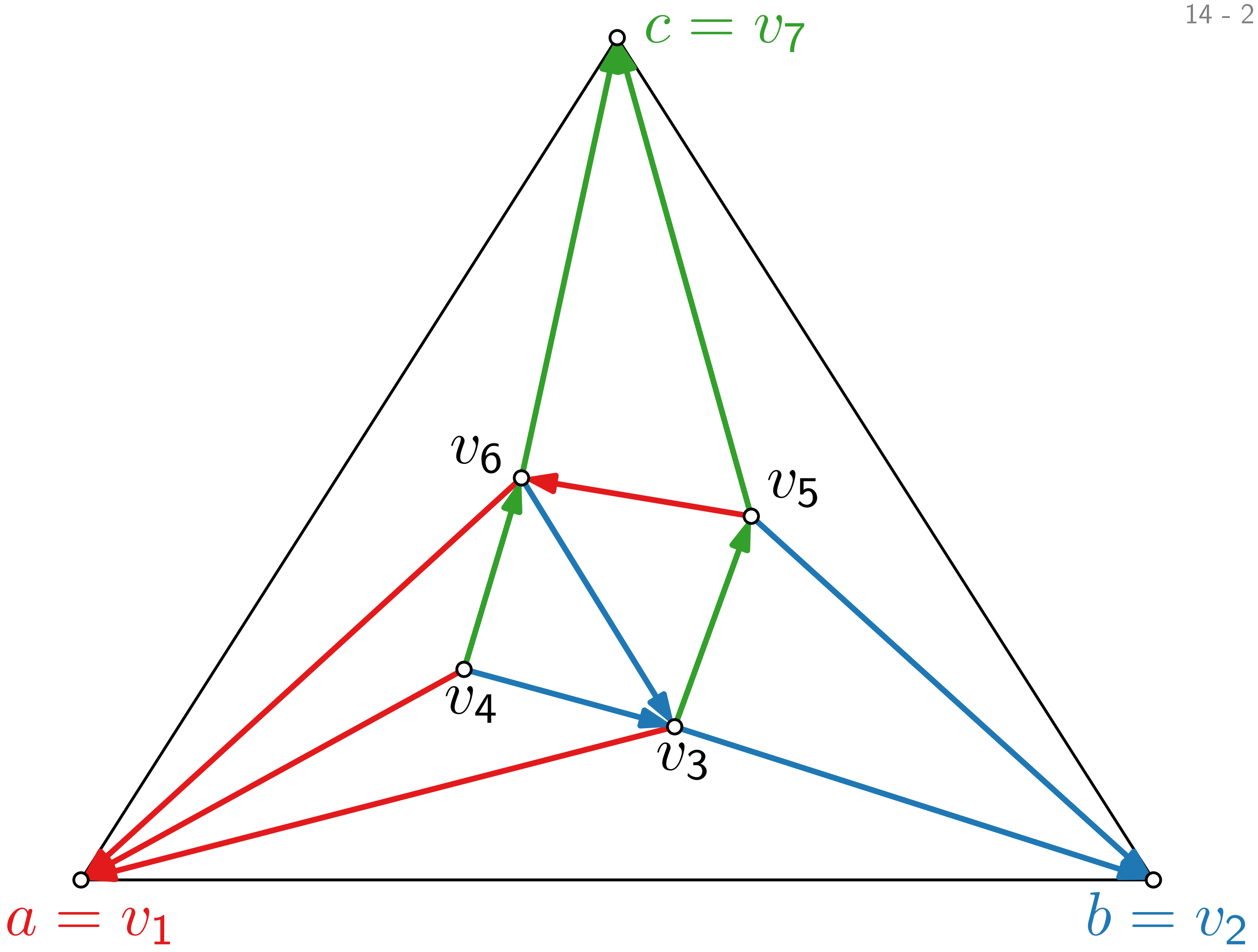

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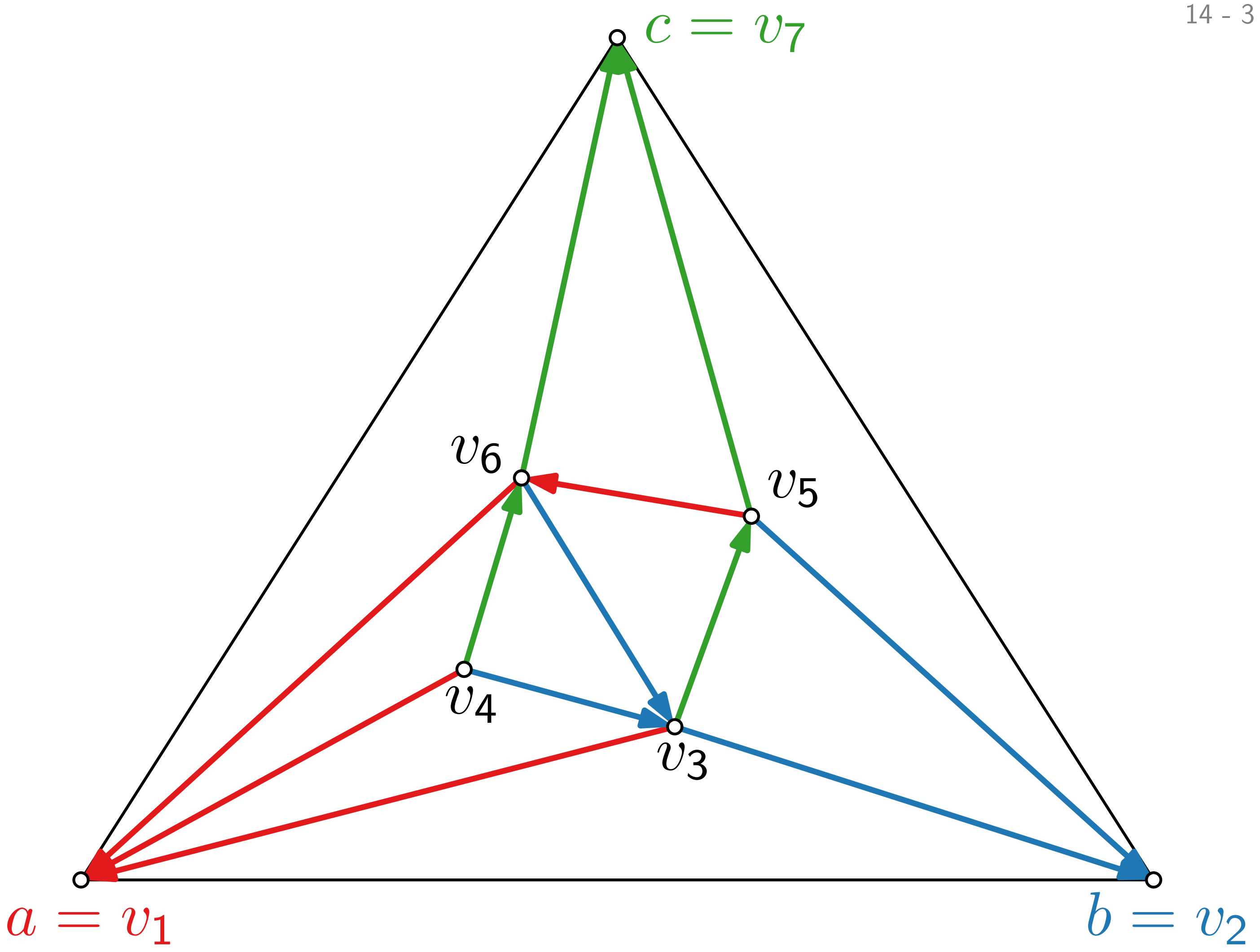

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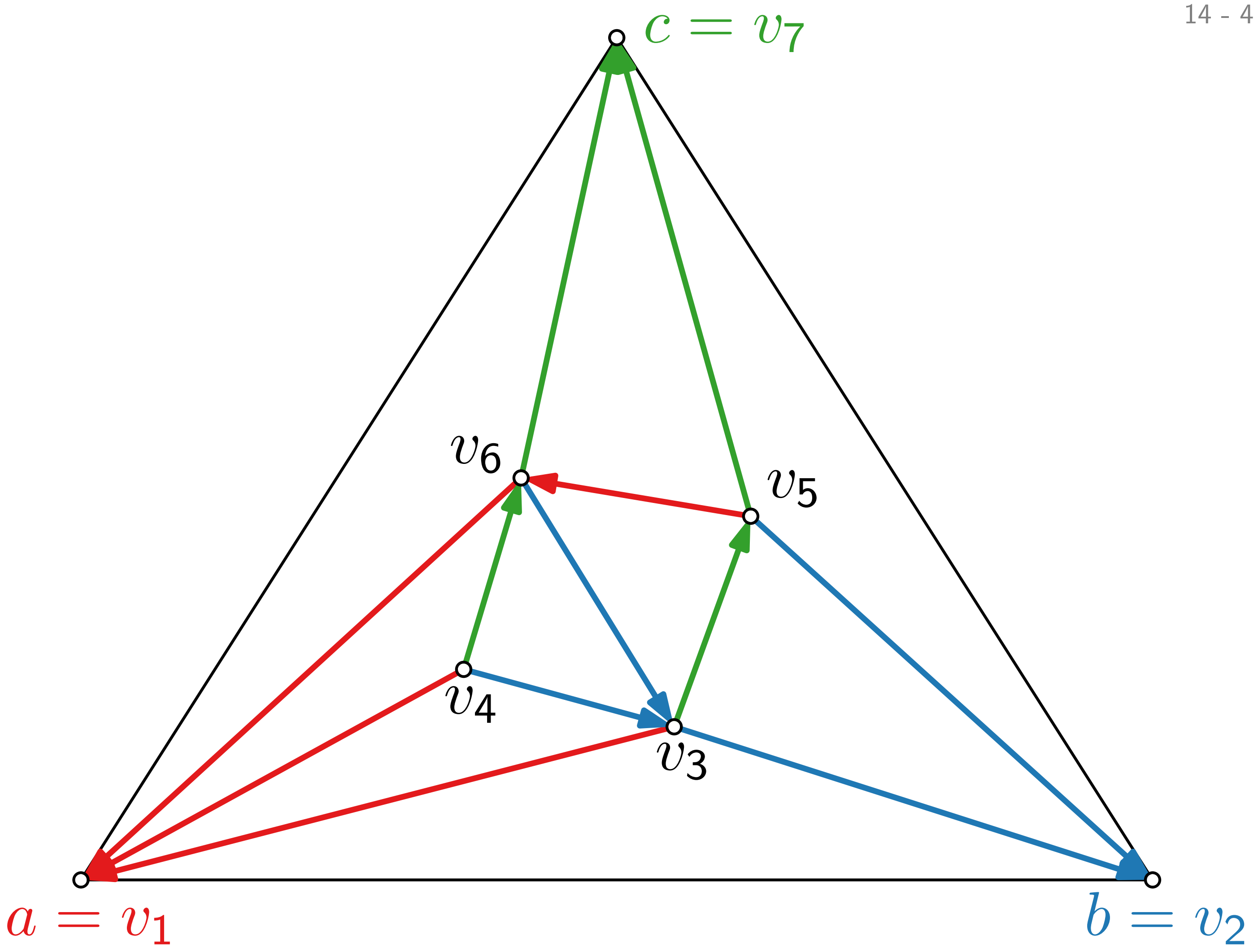

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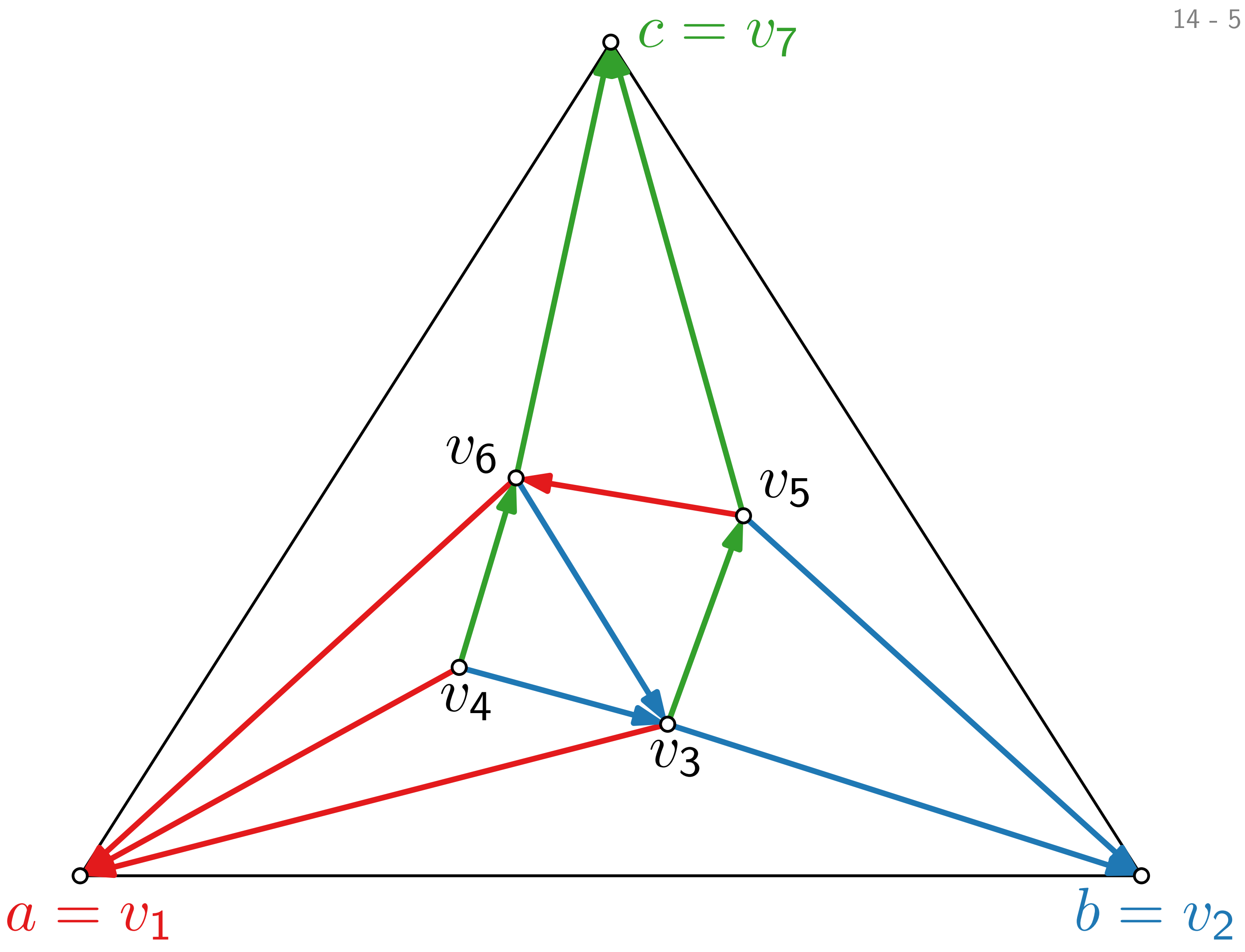

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$f(v_7) = (0, 0, 9)$

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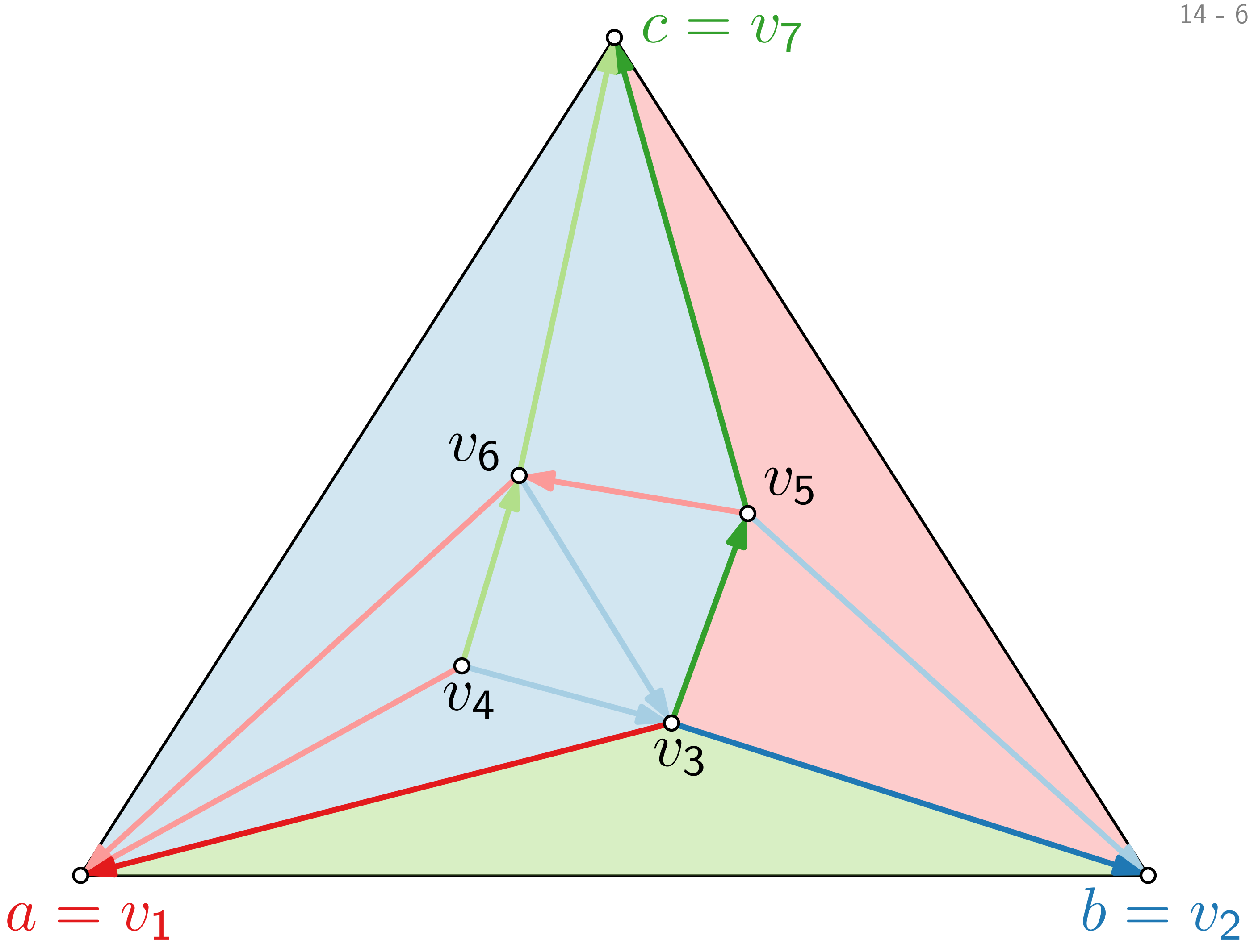

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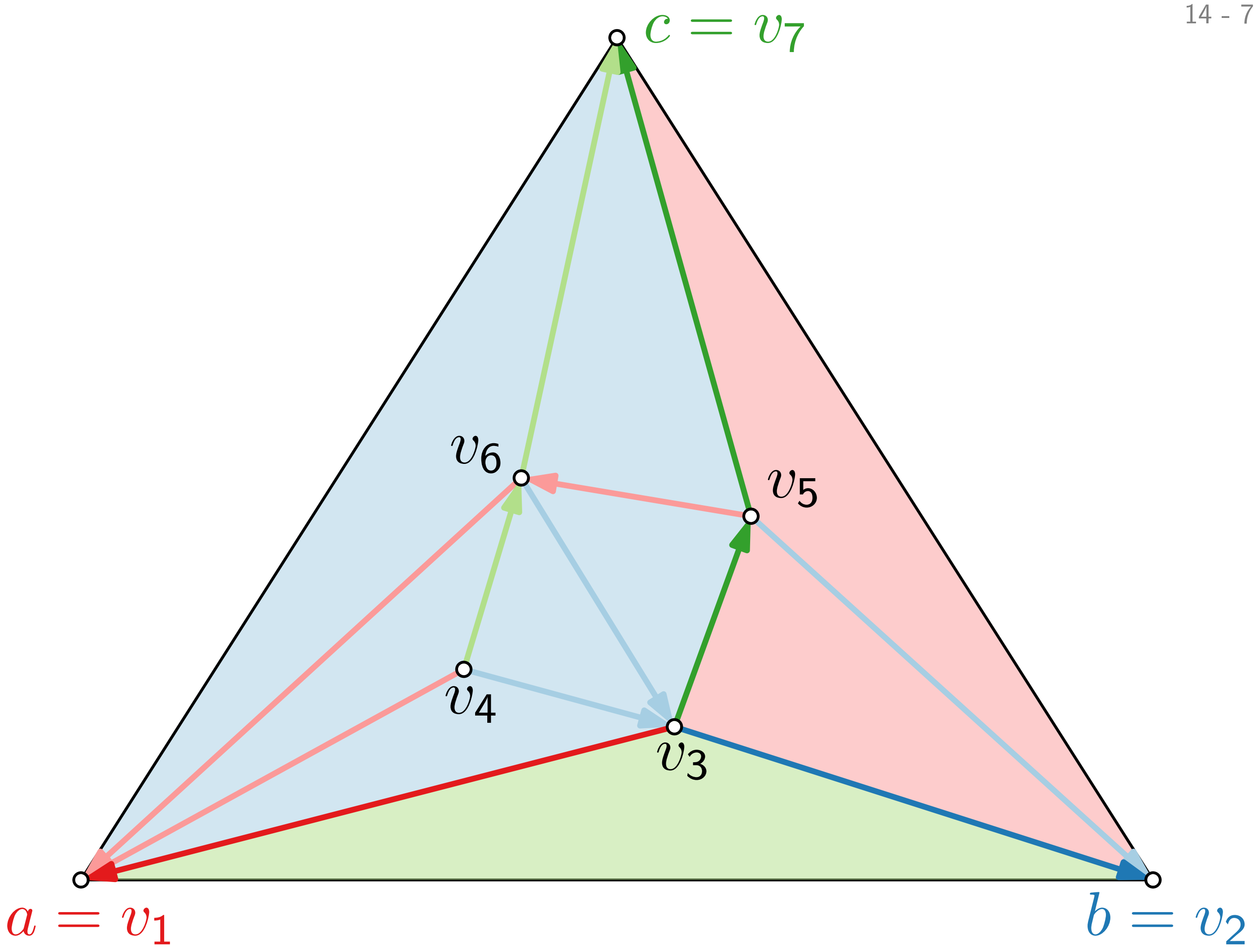

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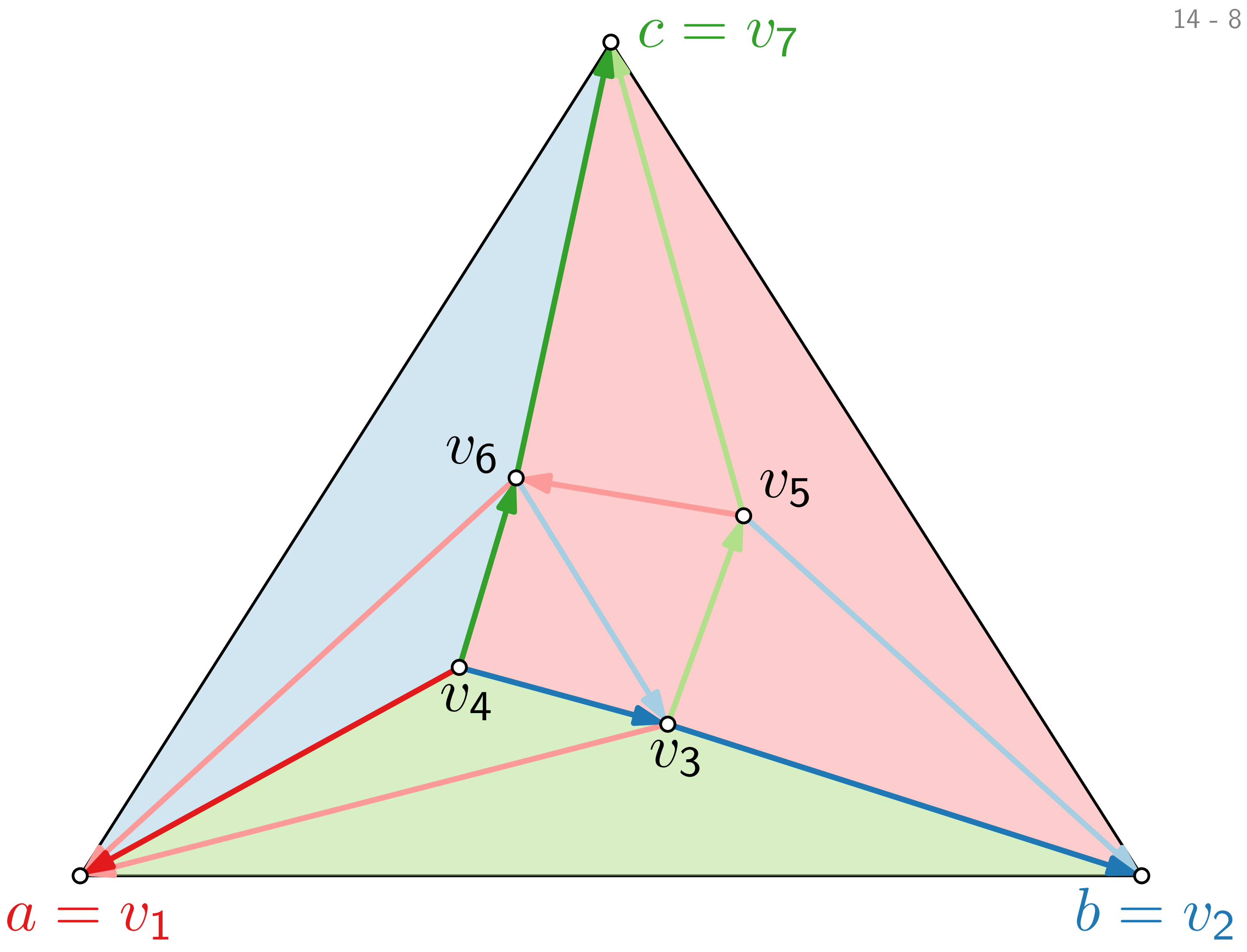

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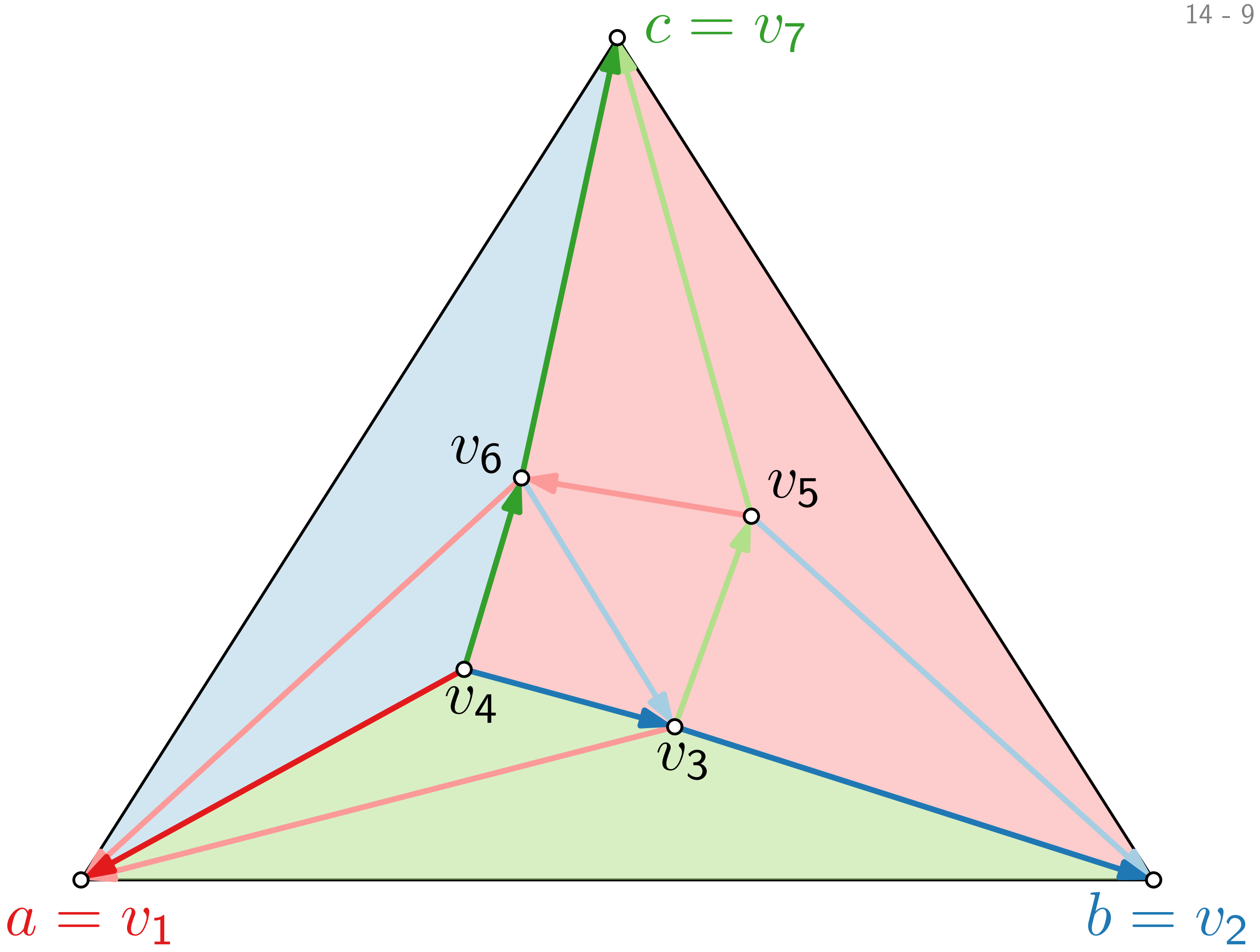

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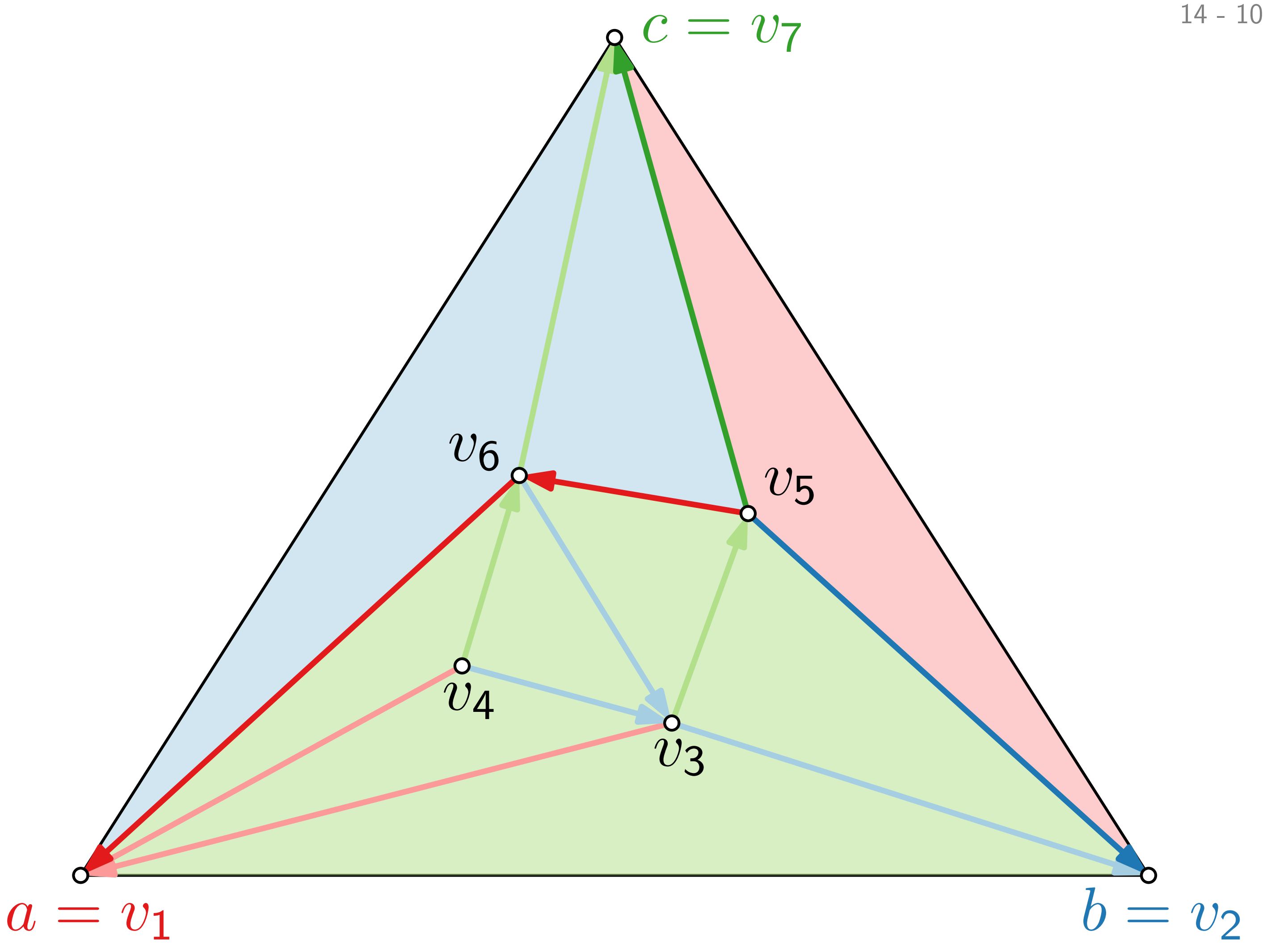

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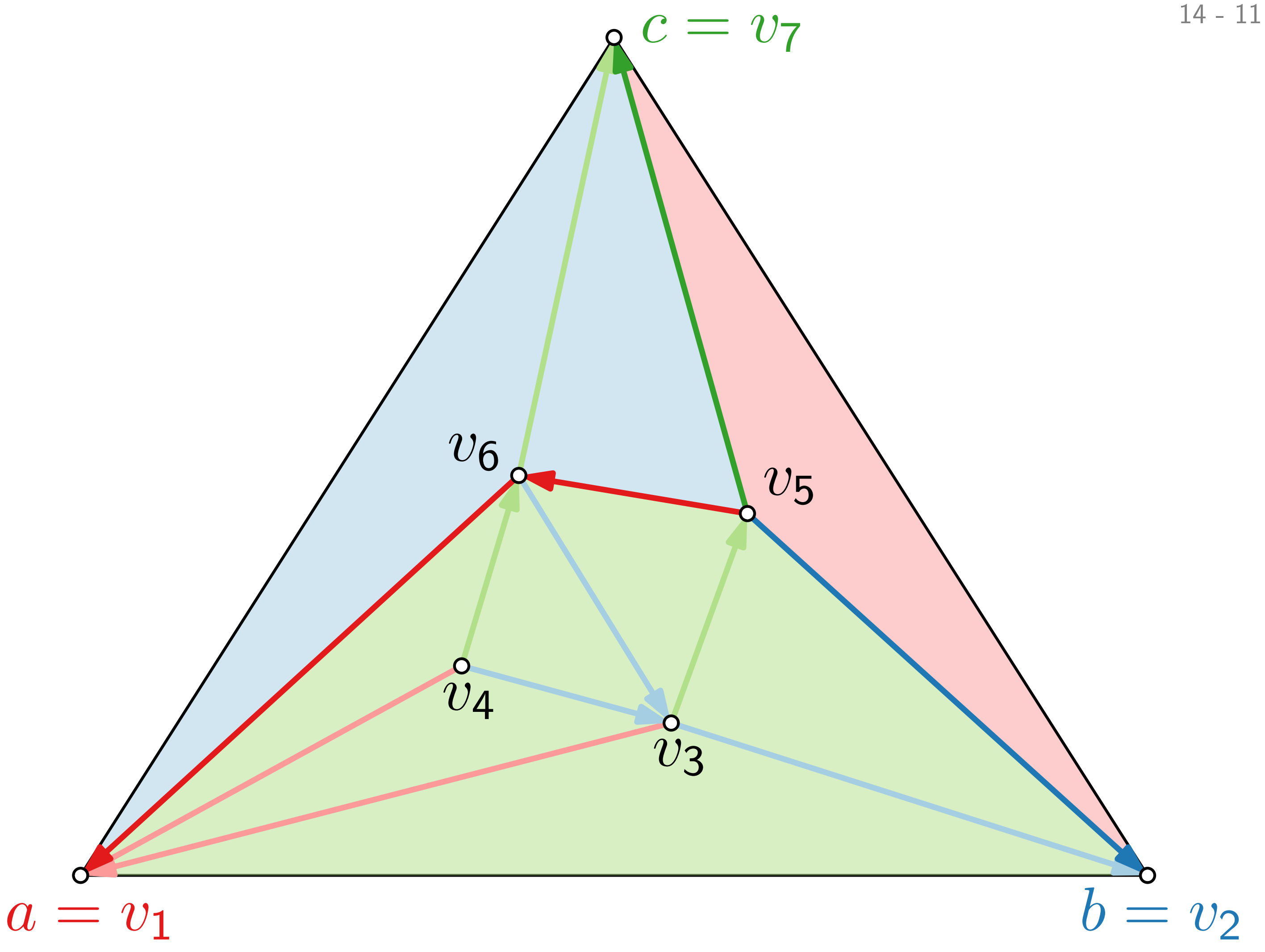

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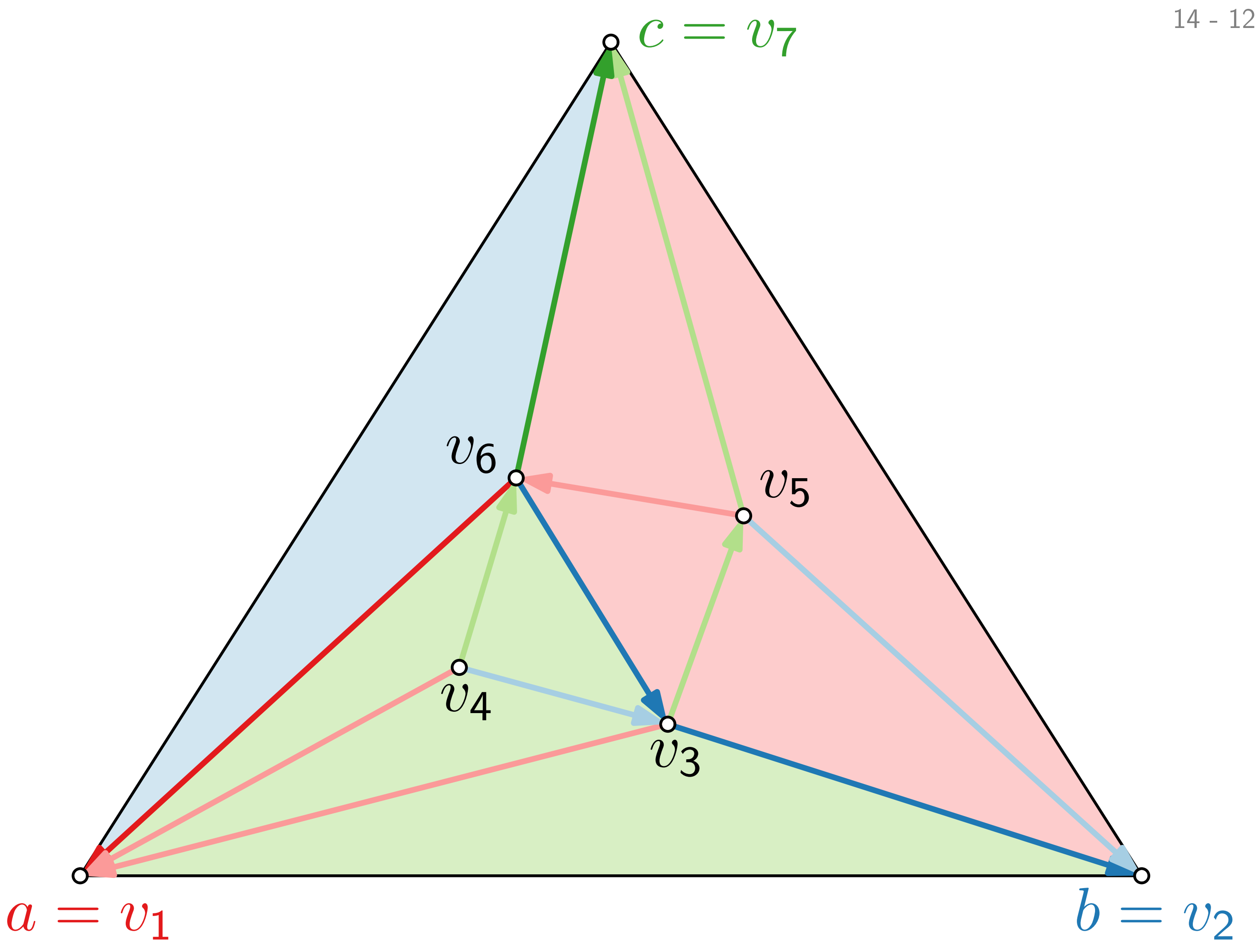

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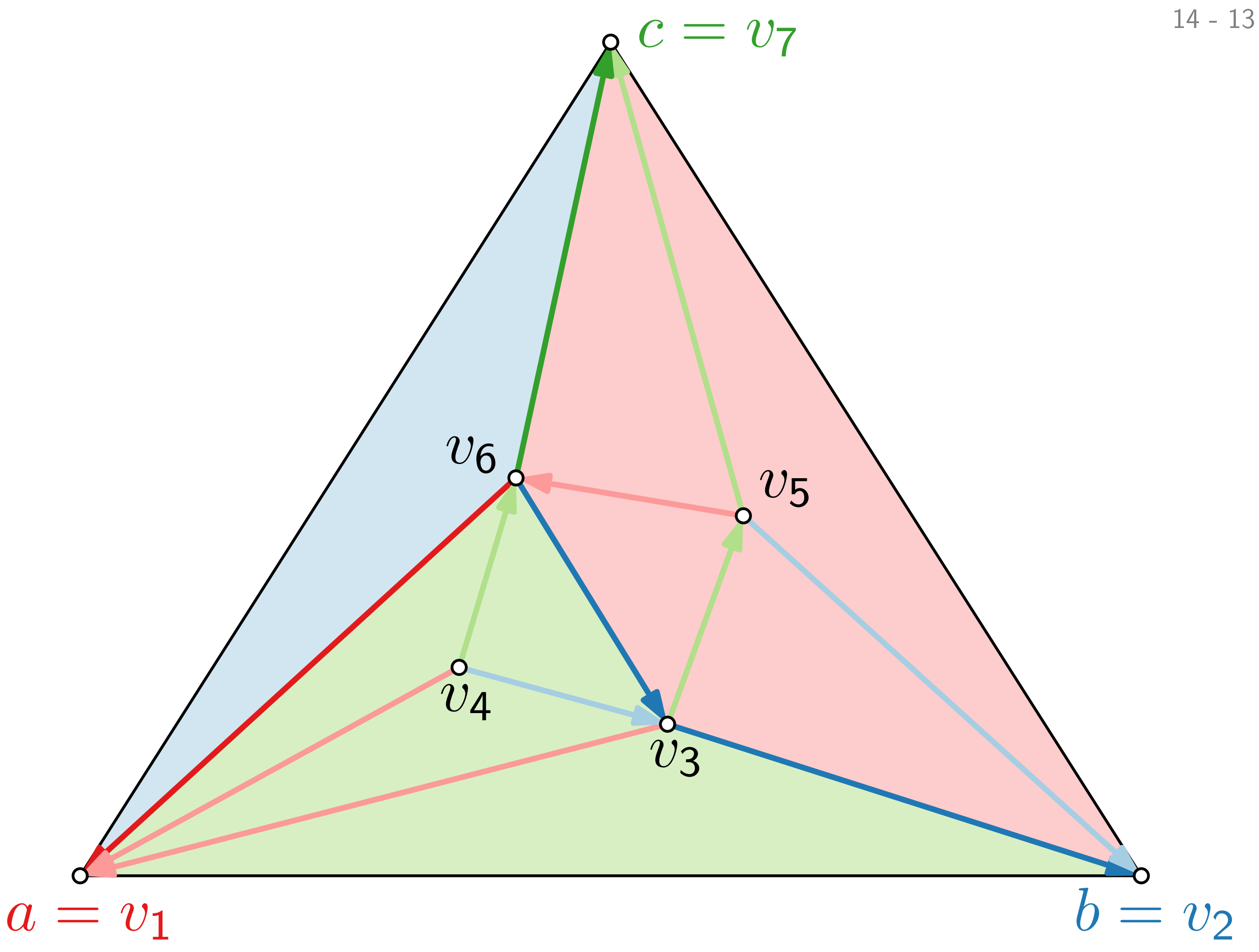

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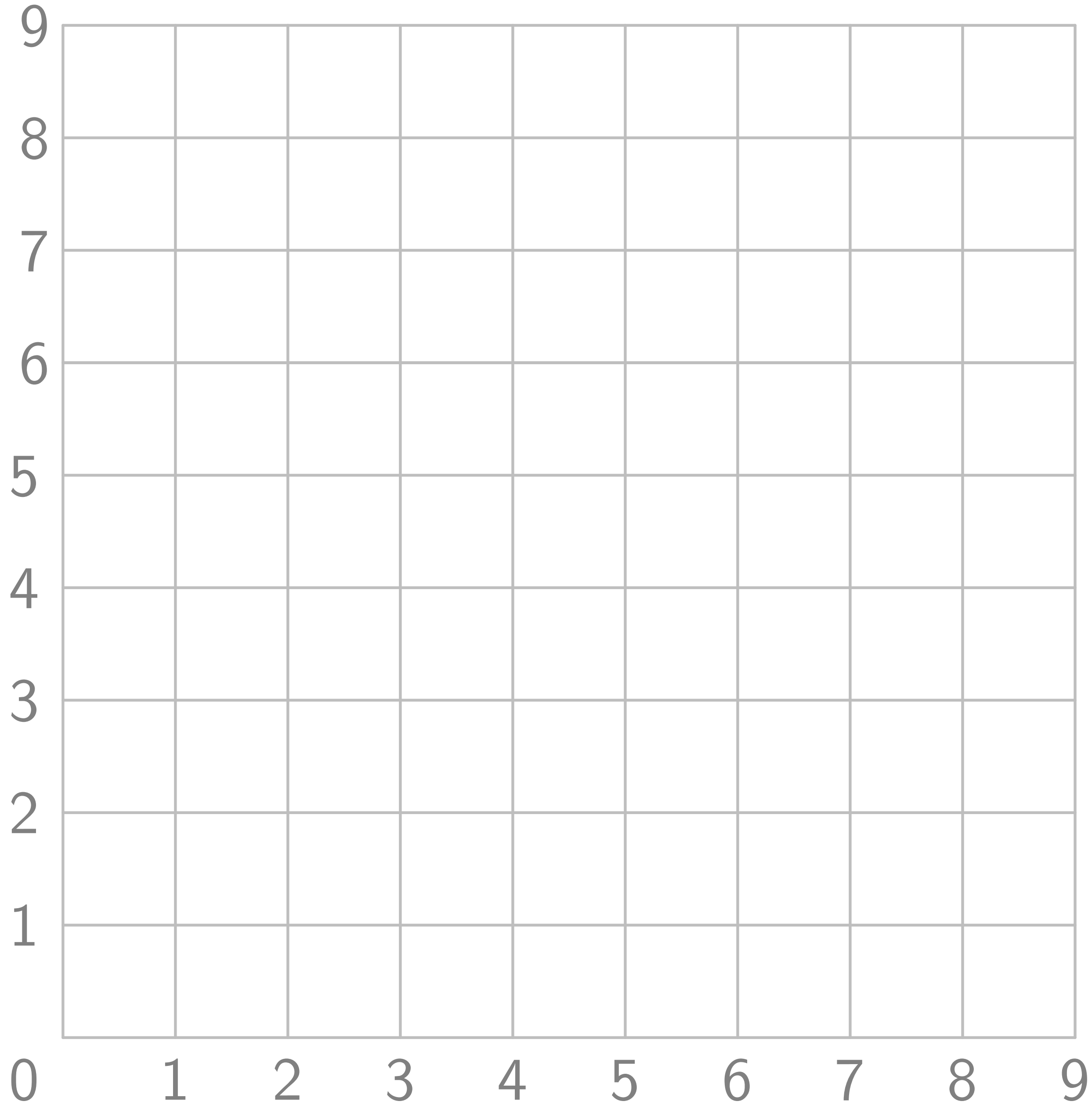

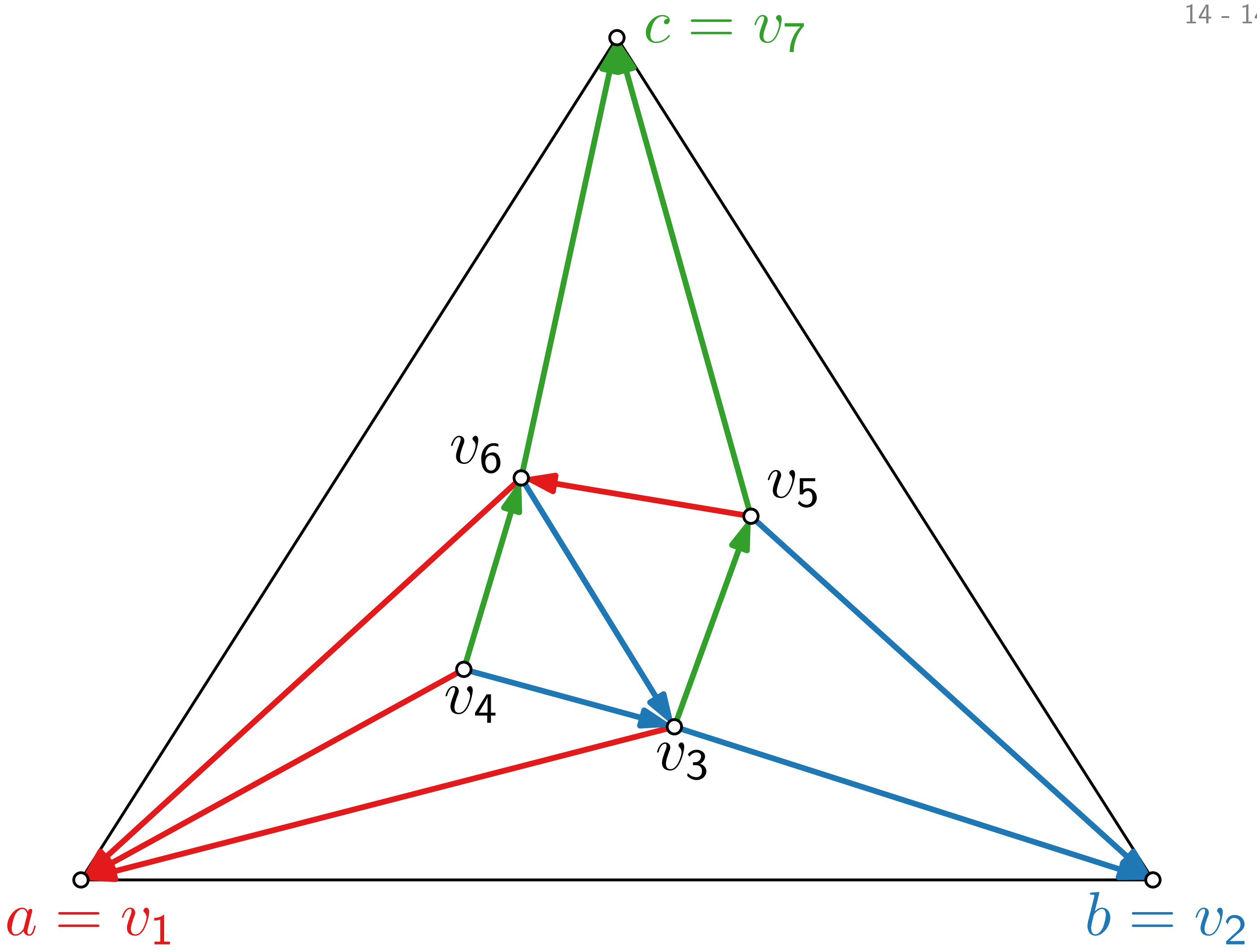

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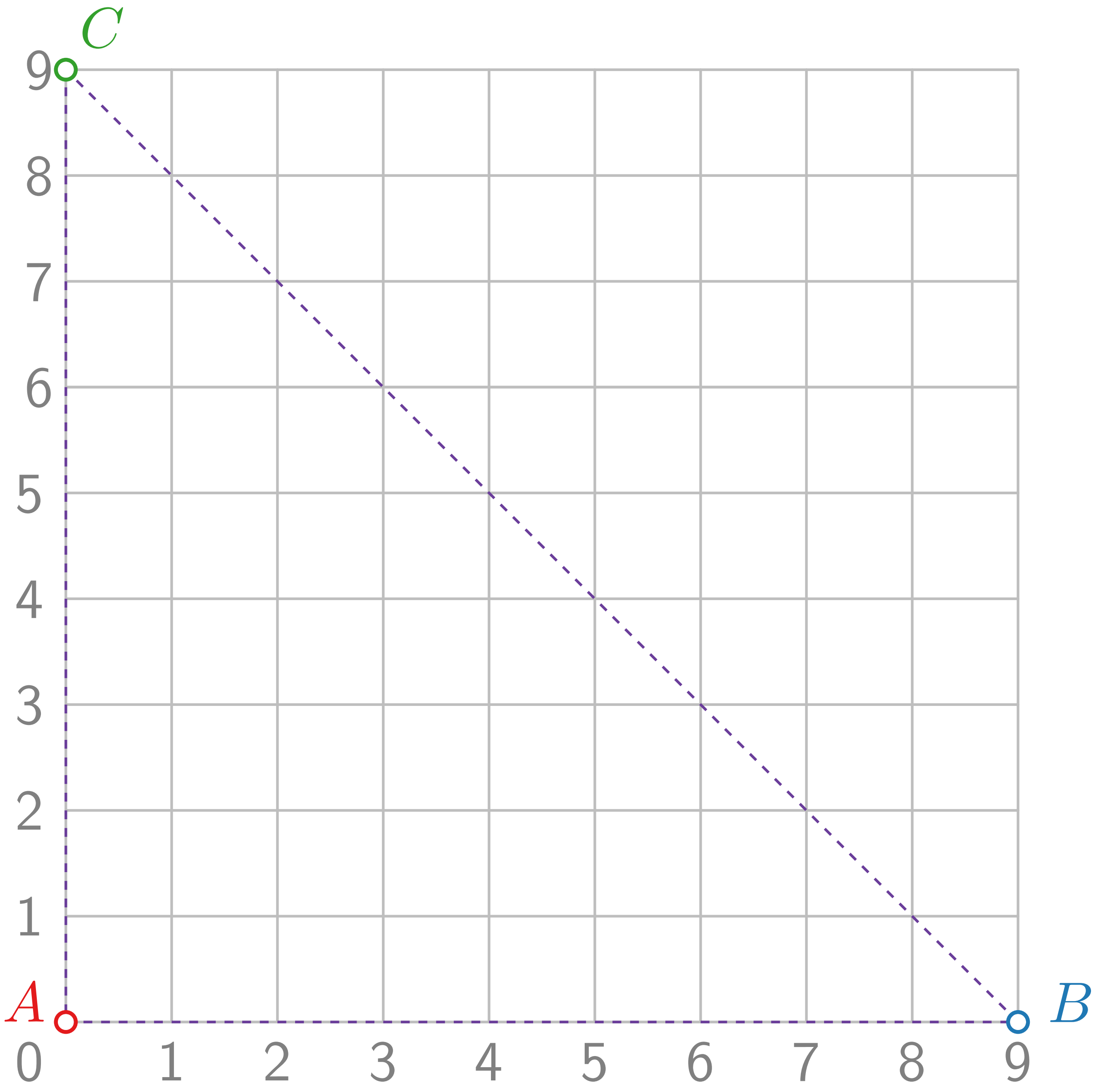

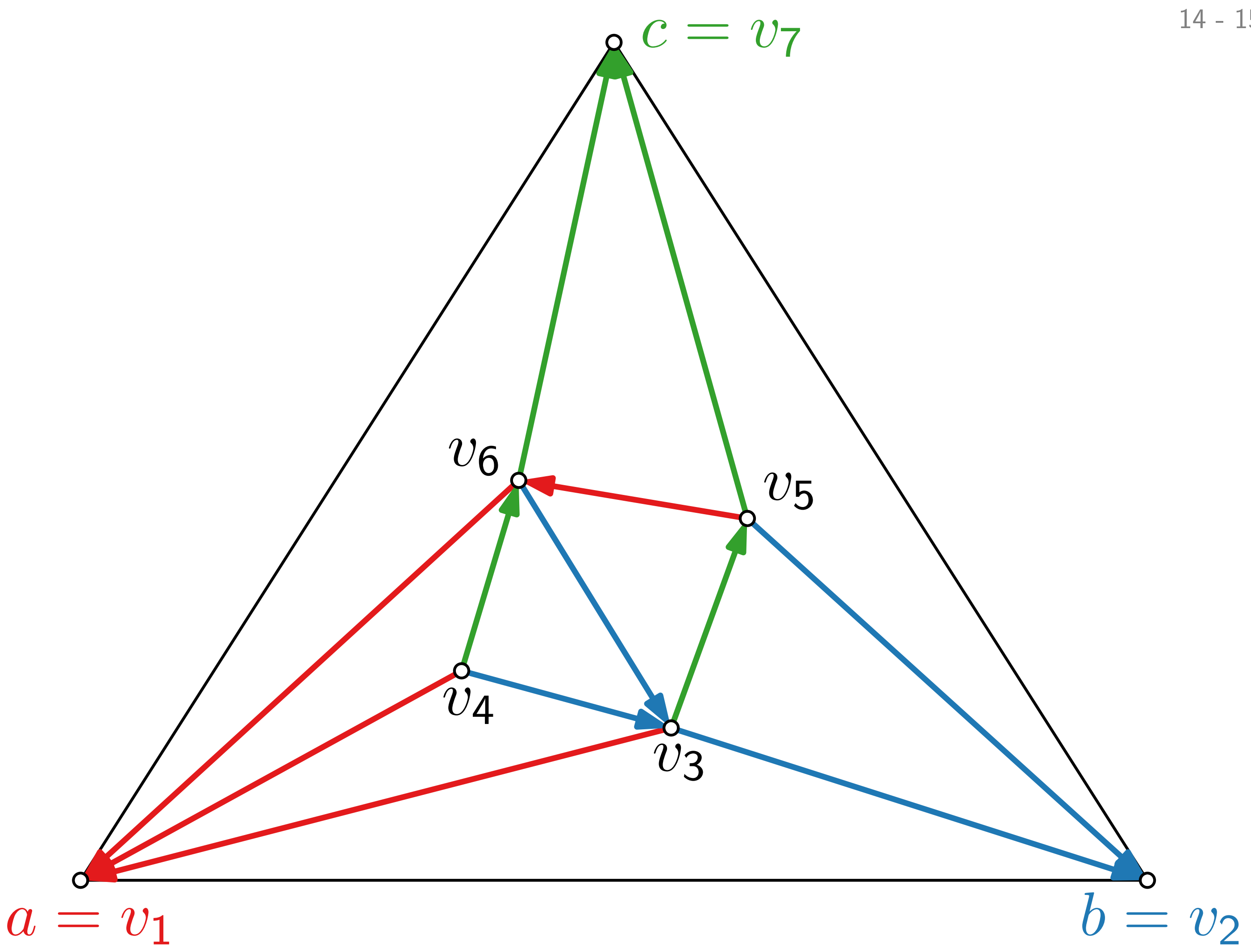

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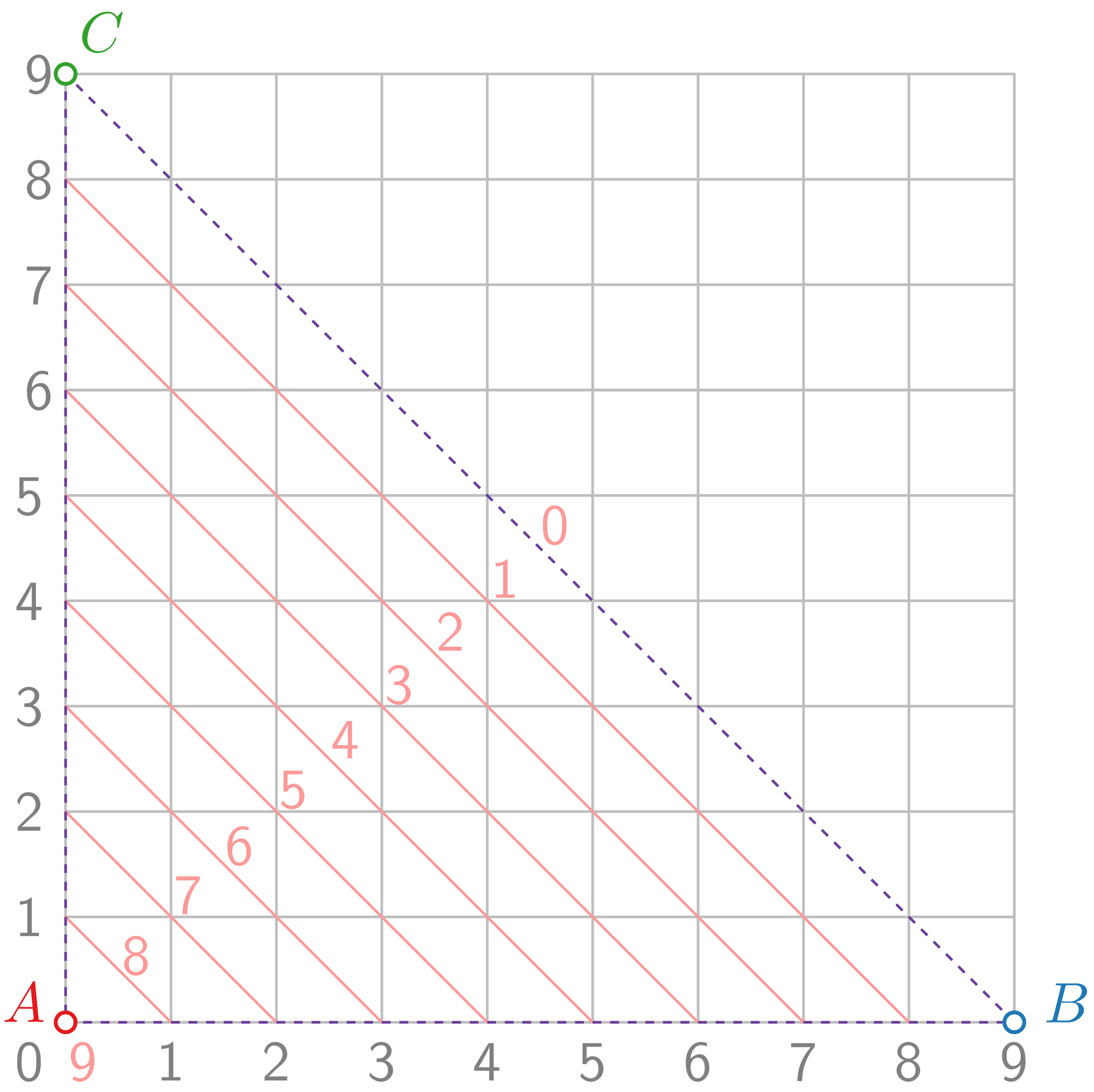

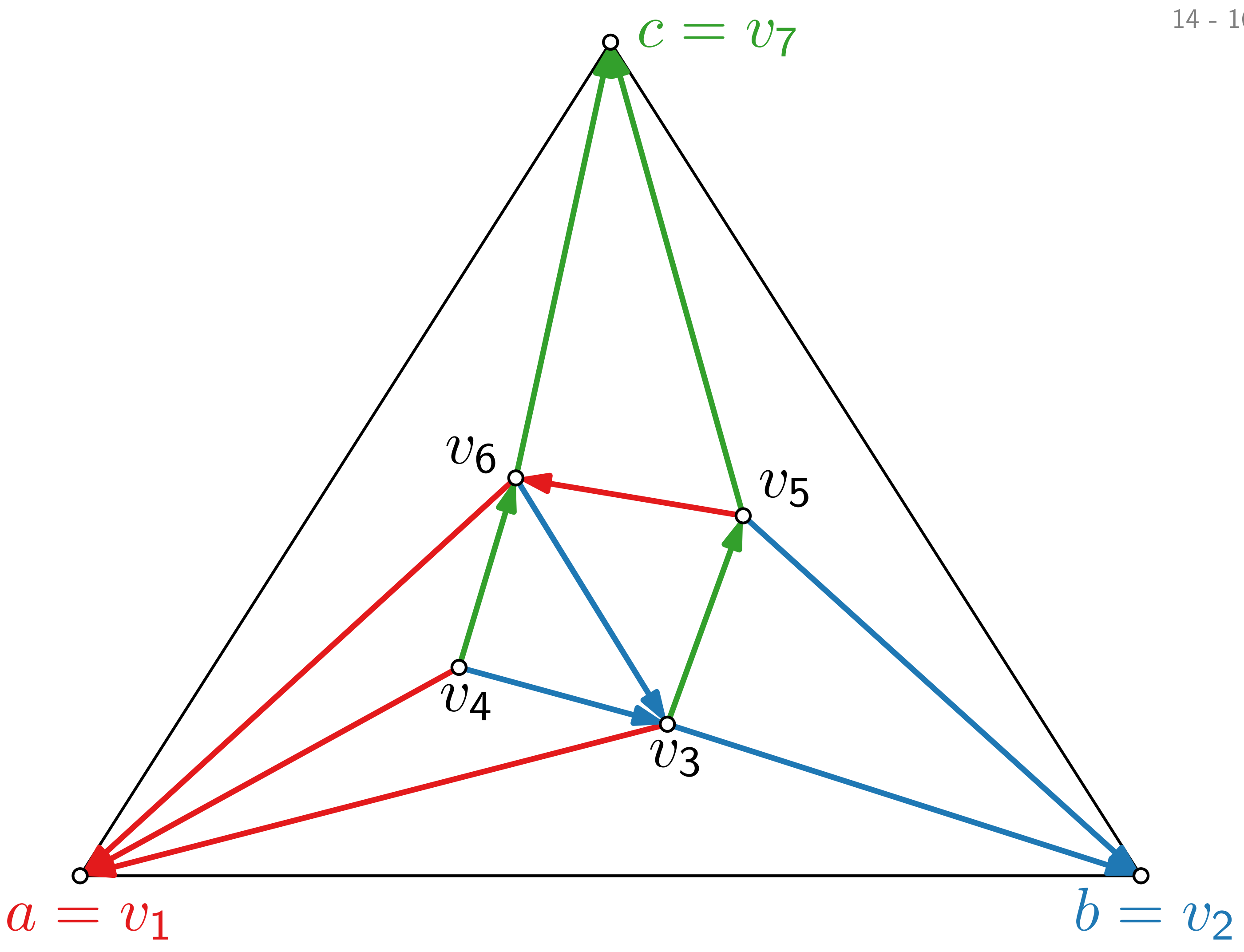

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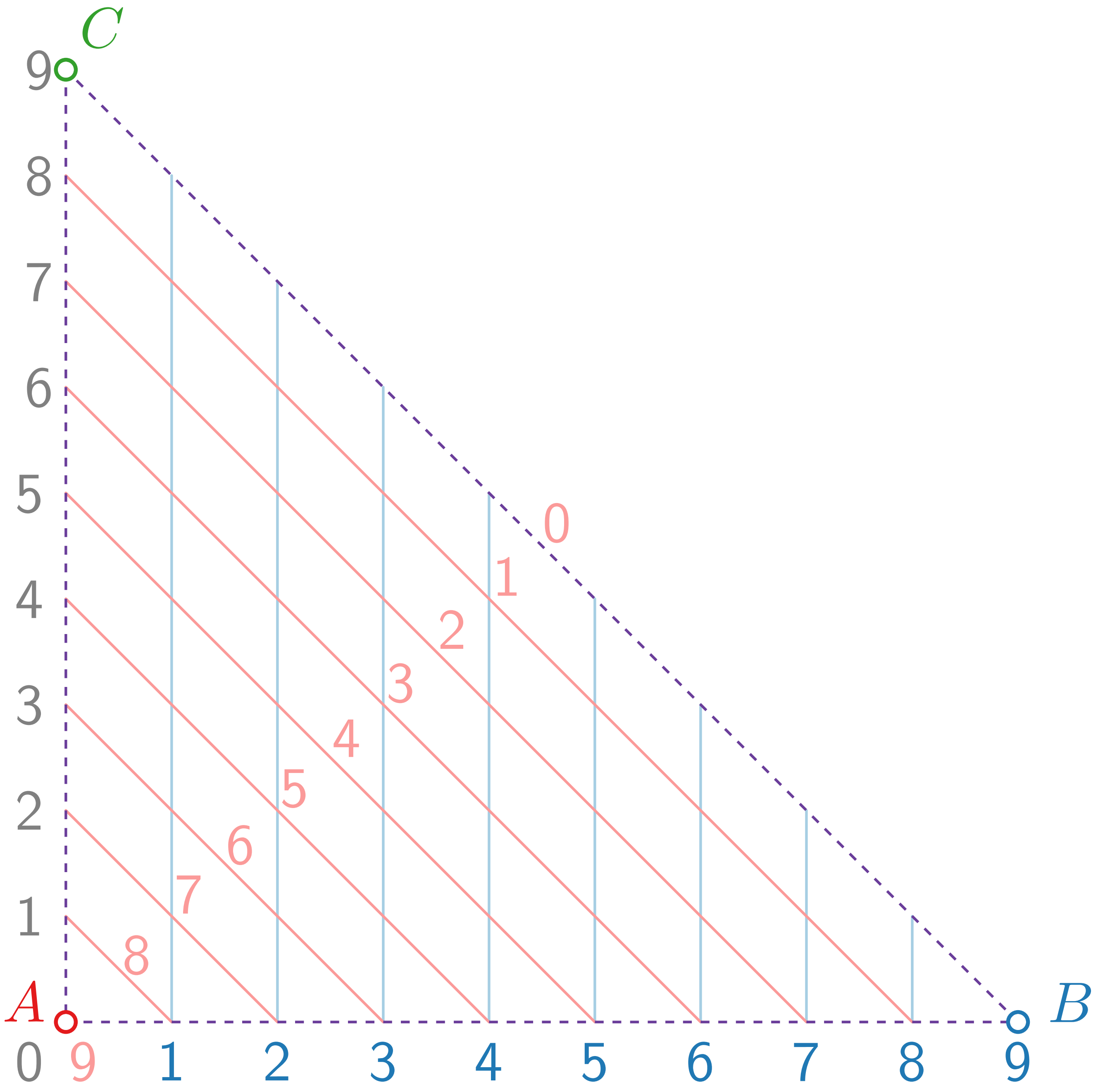

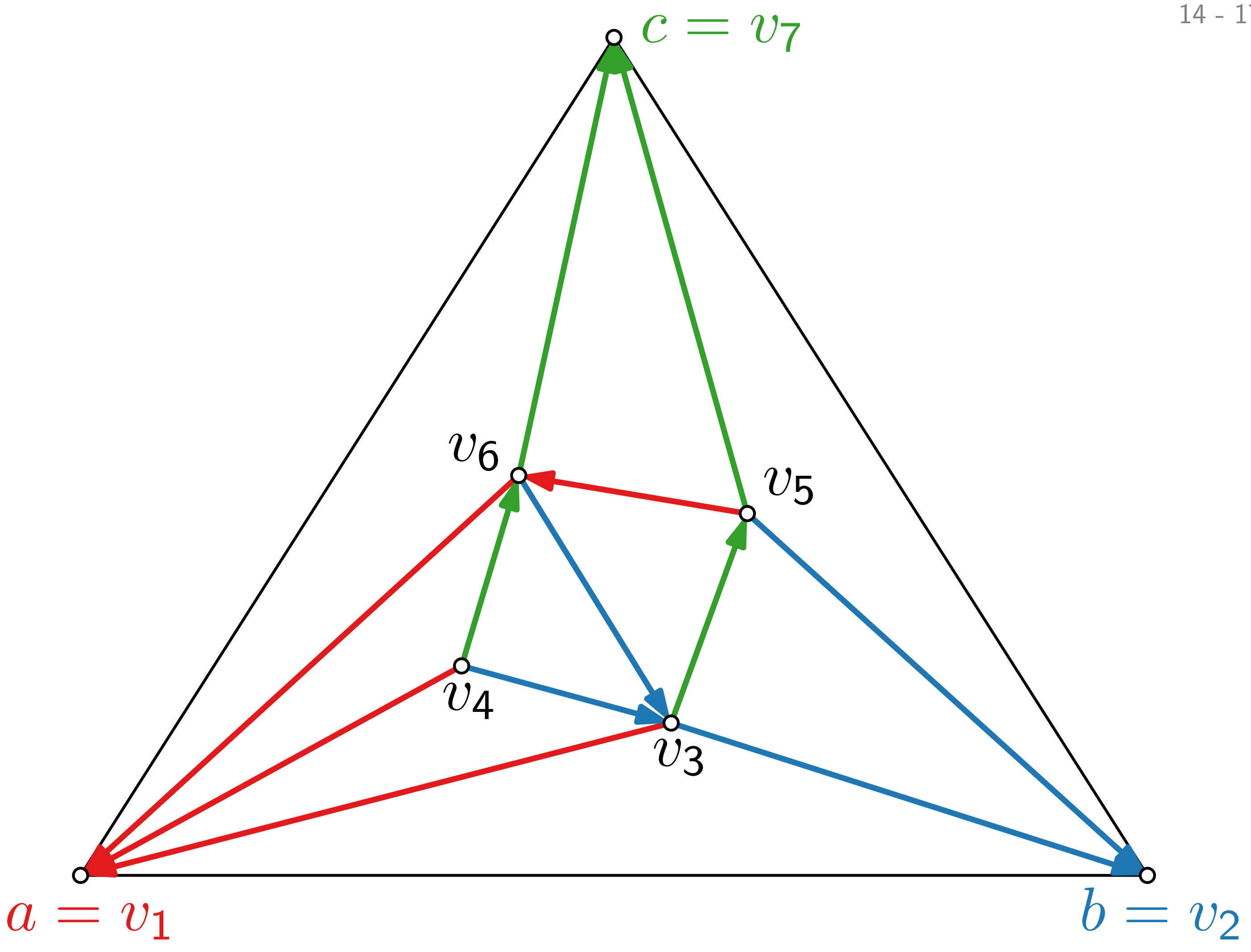

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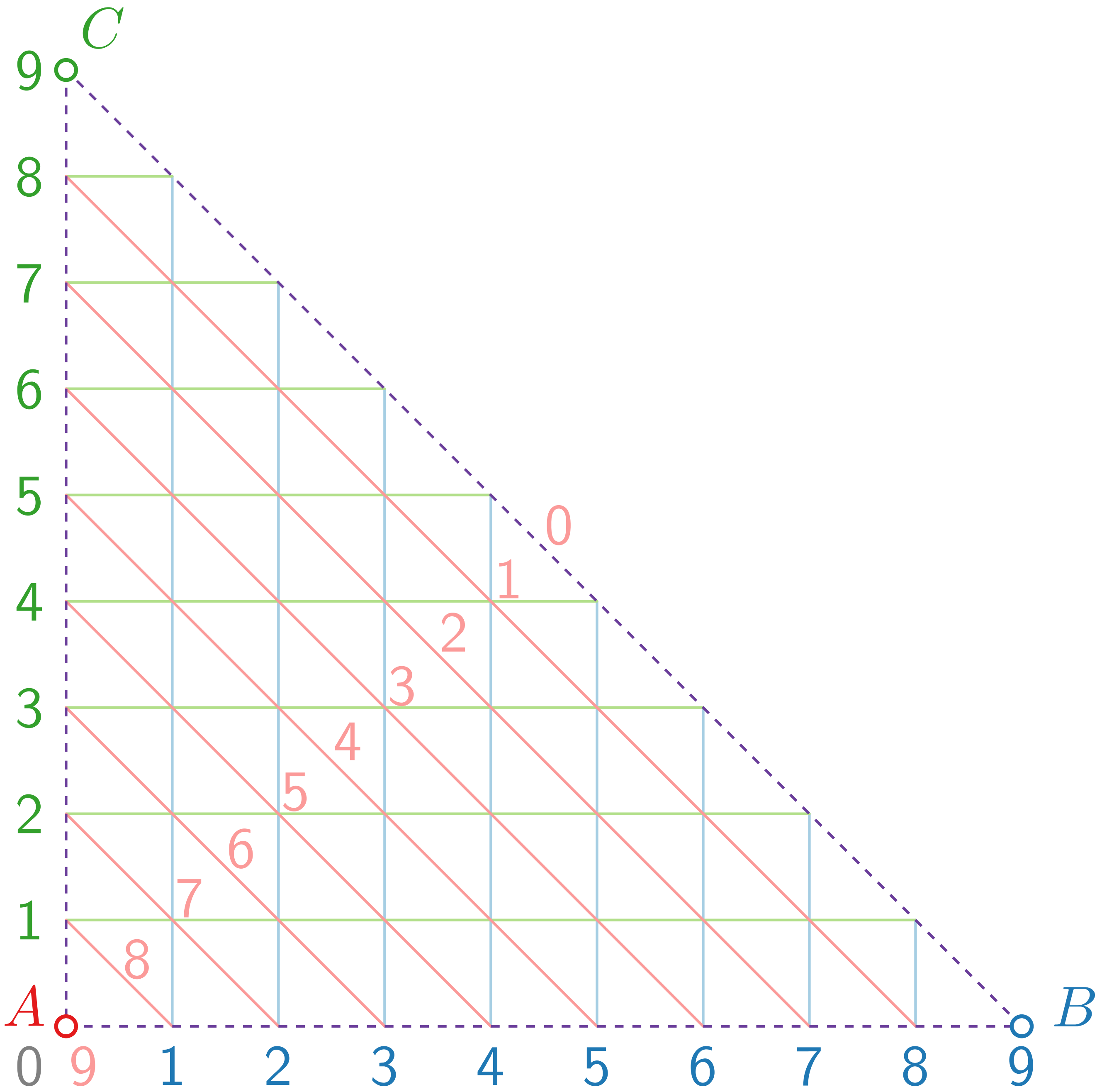

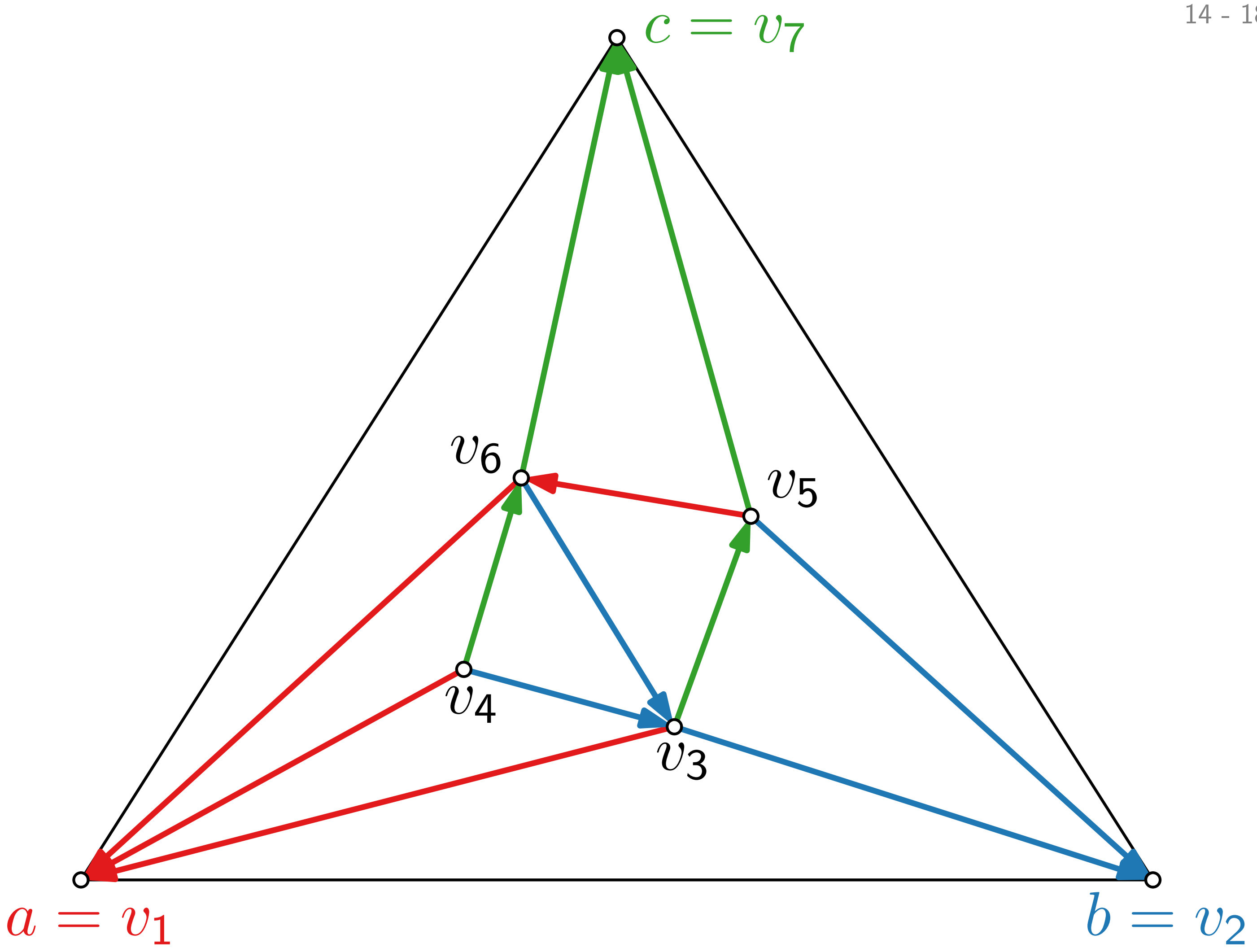

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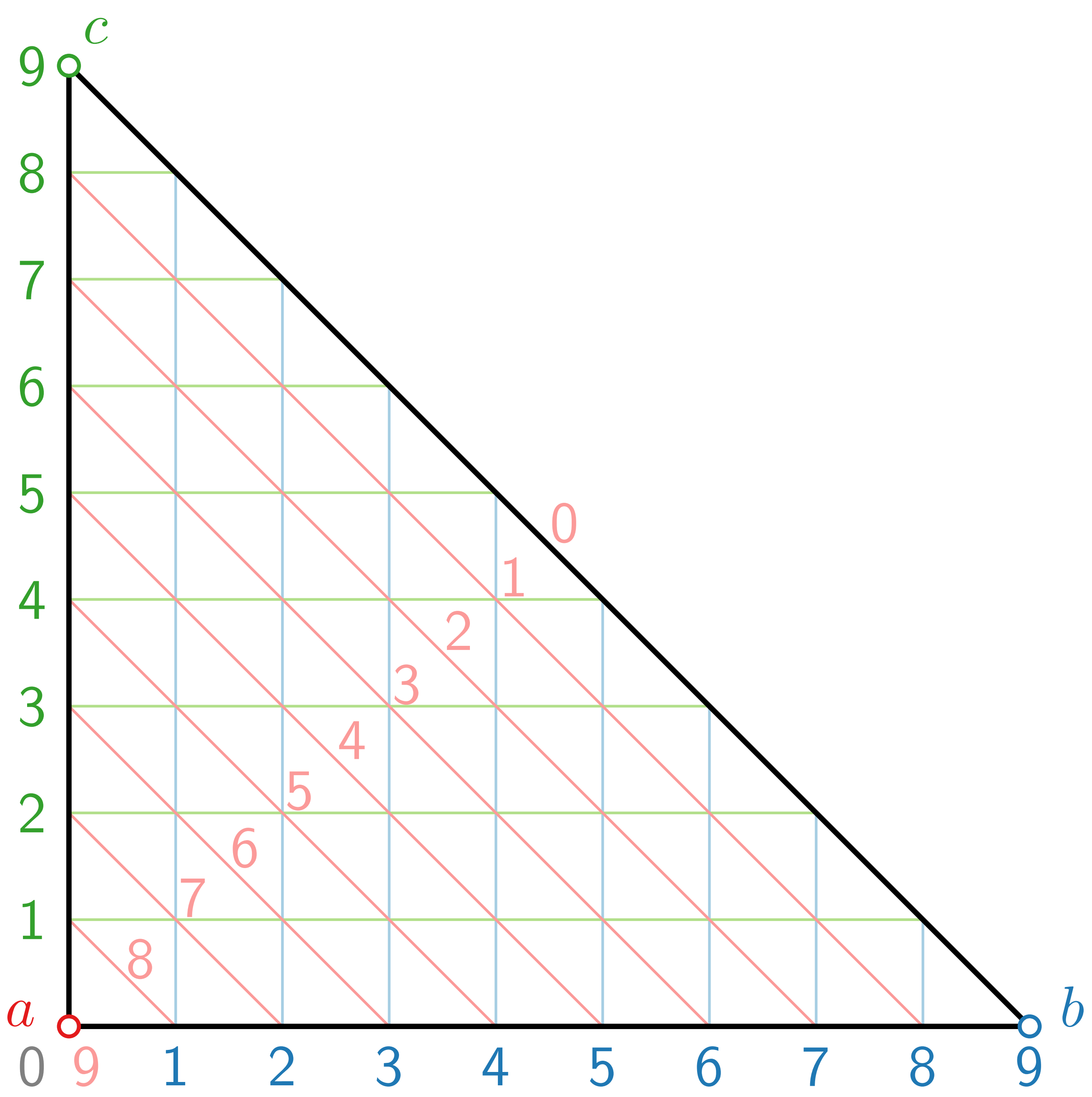

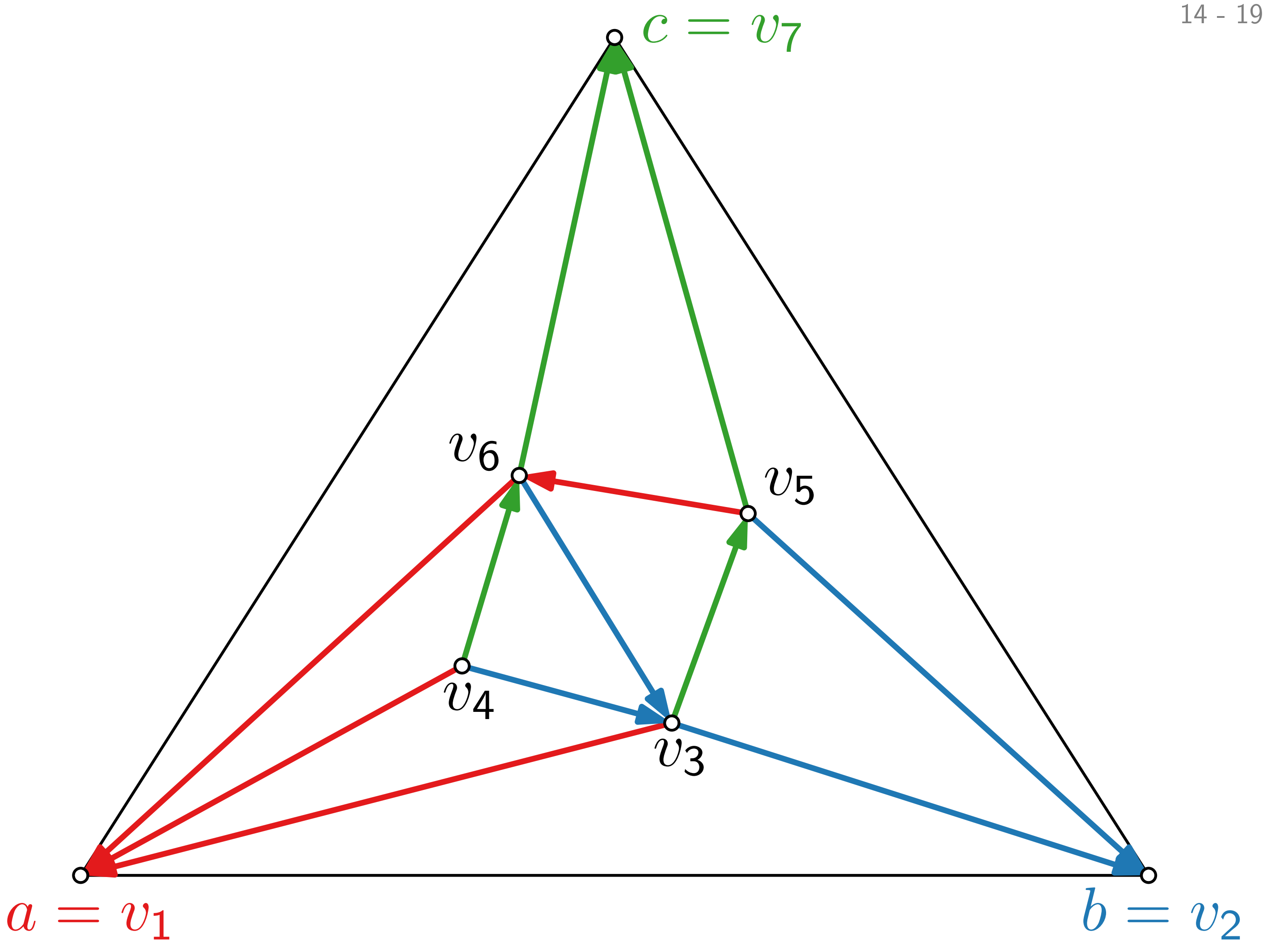

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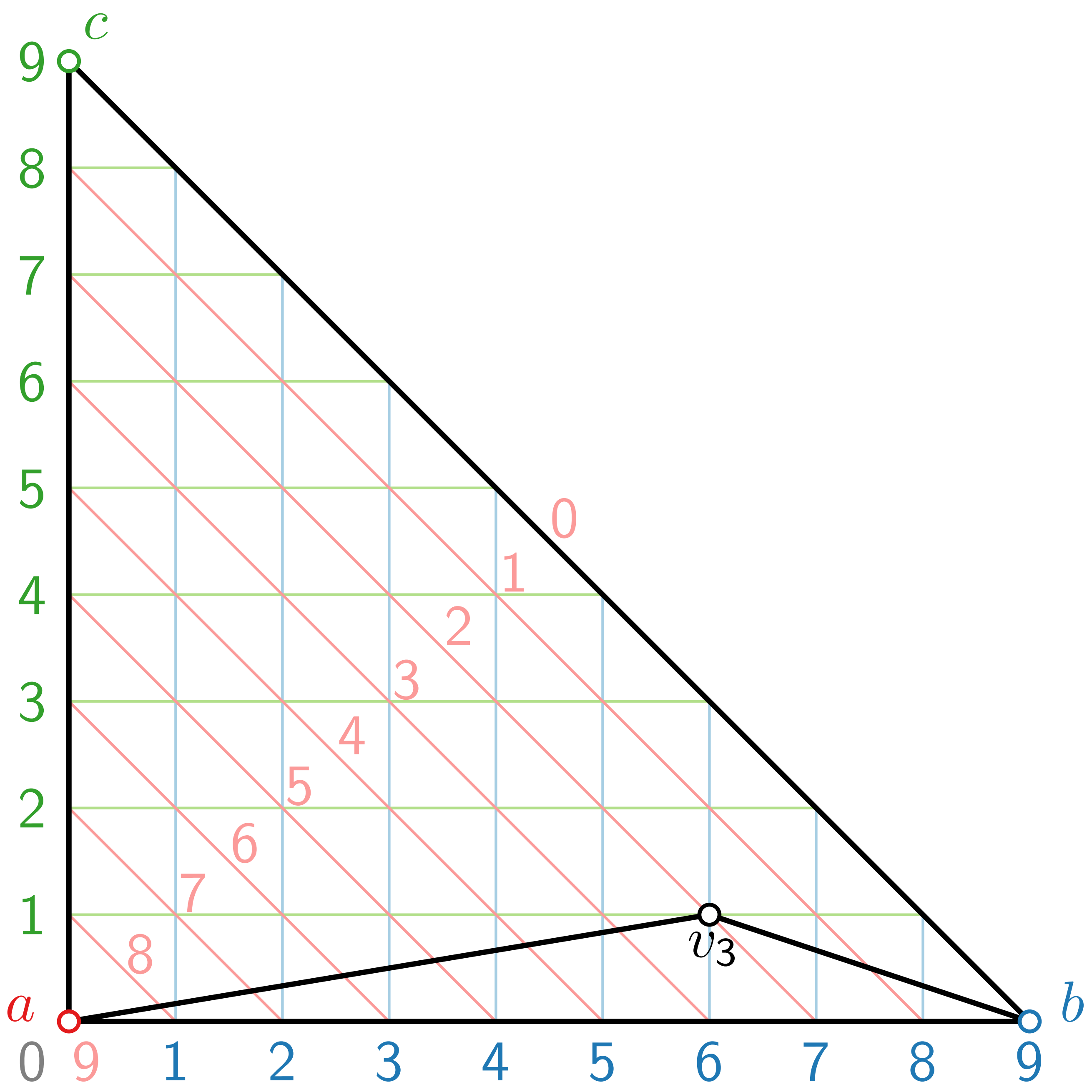

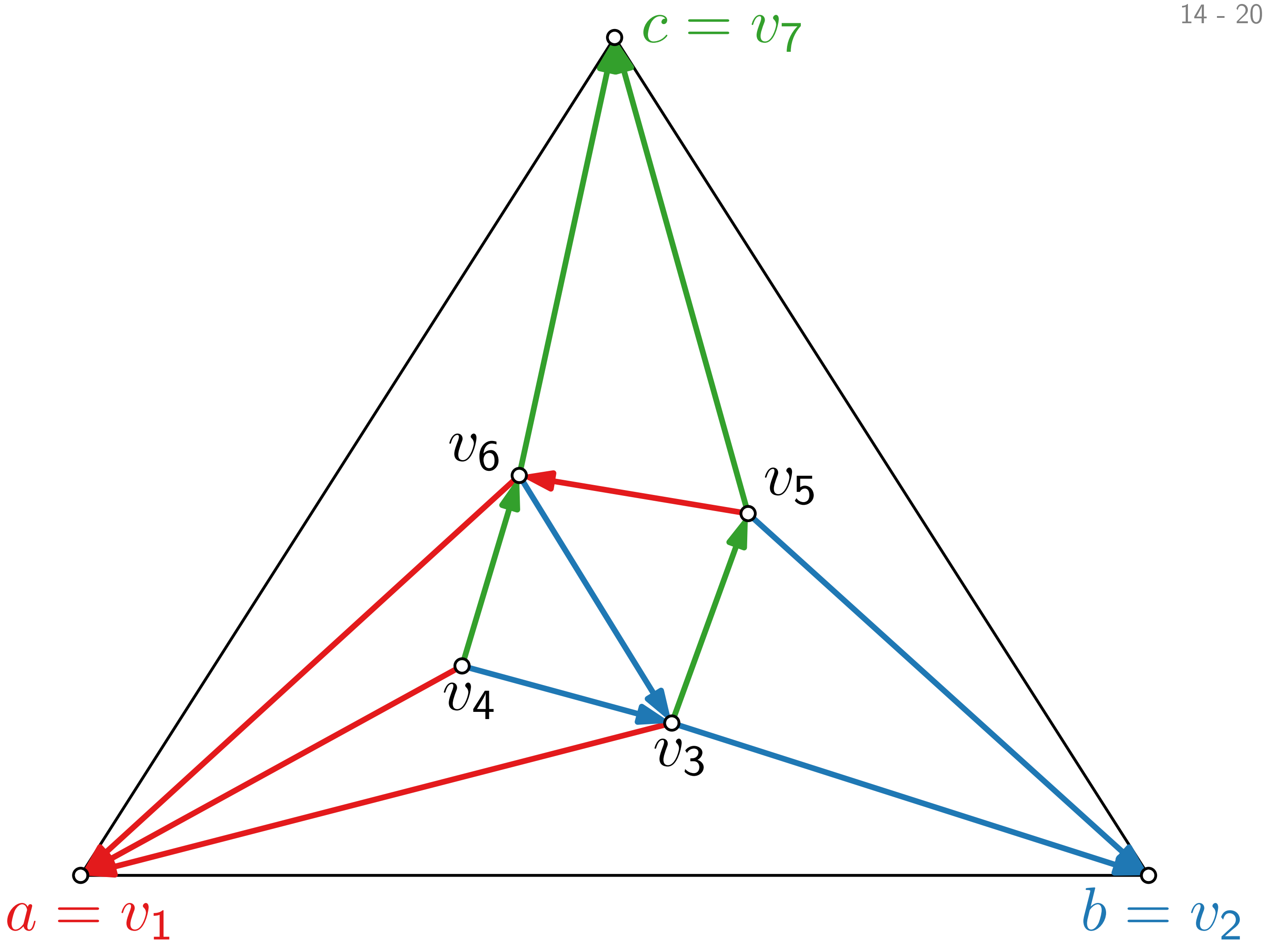

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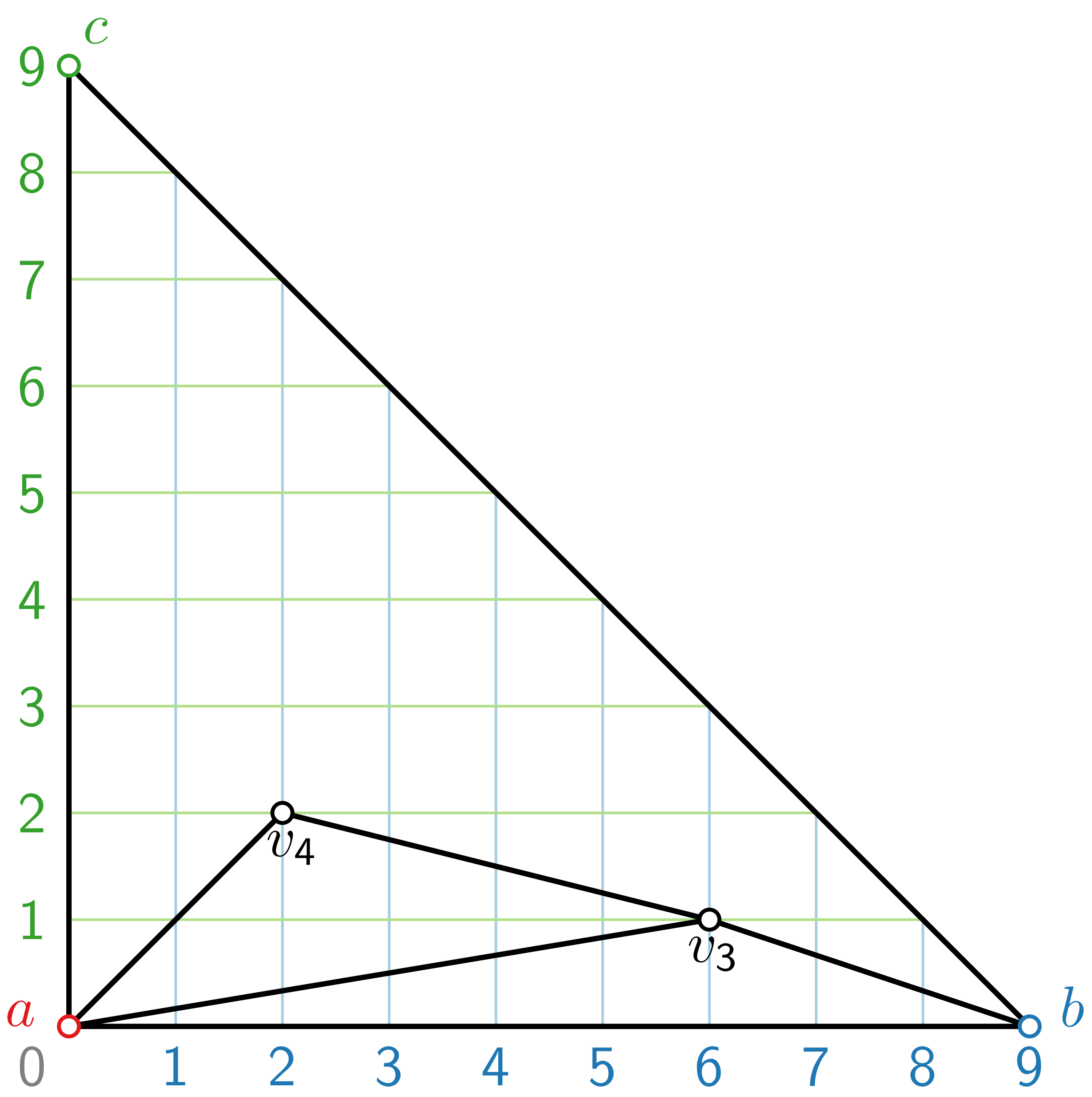

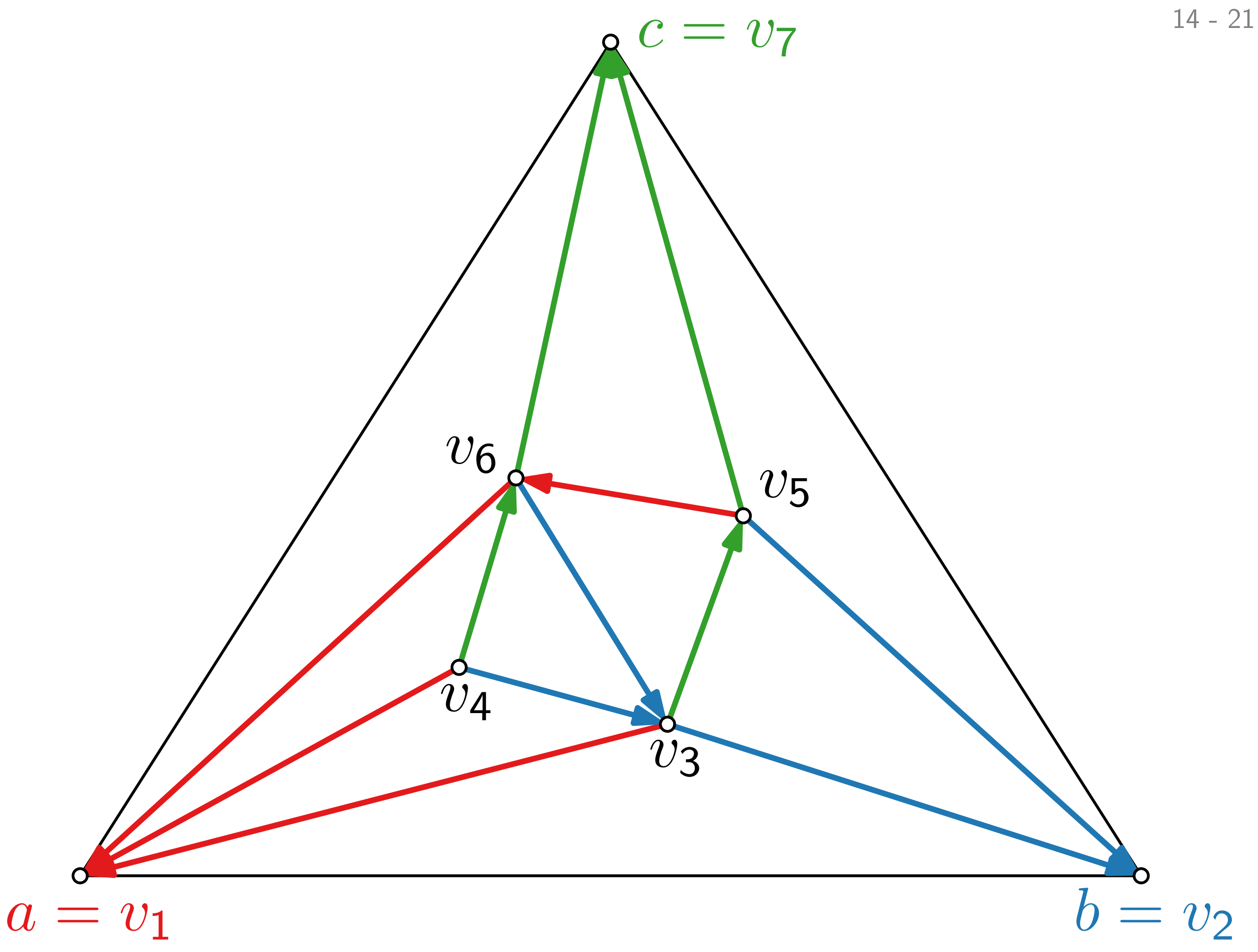

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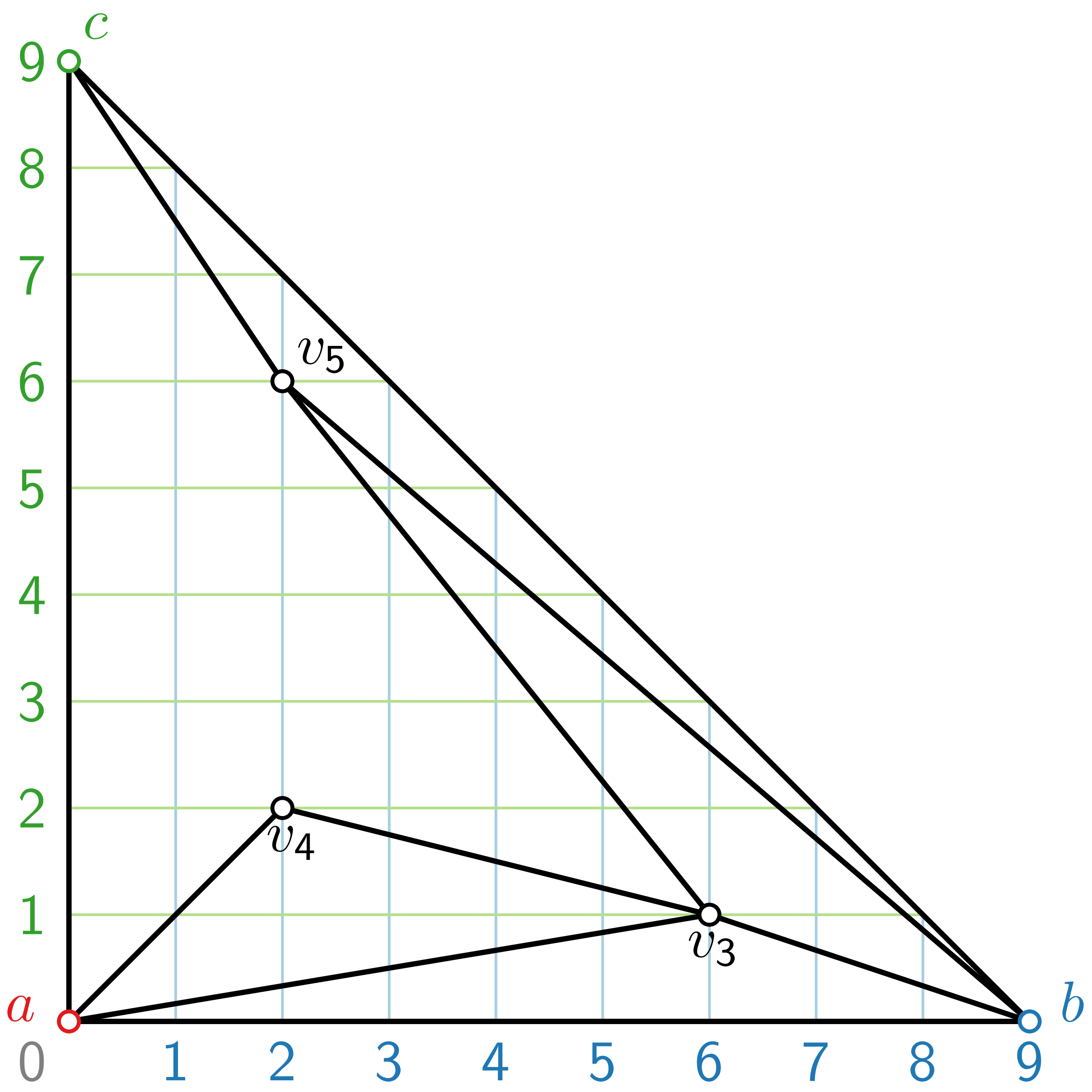

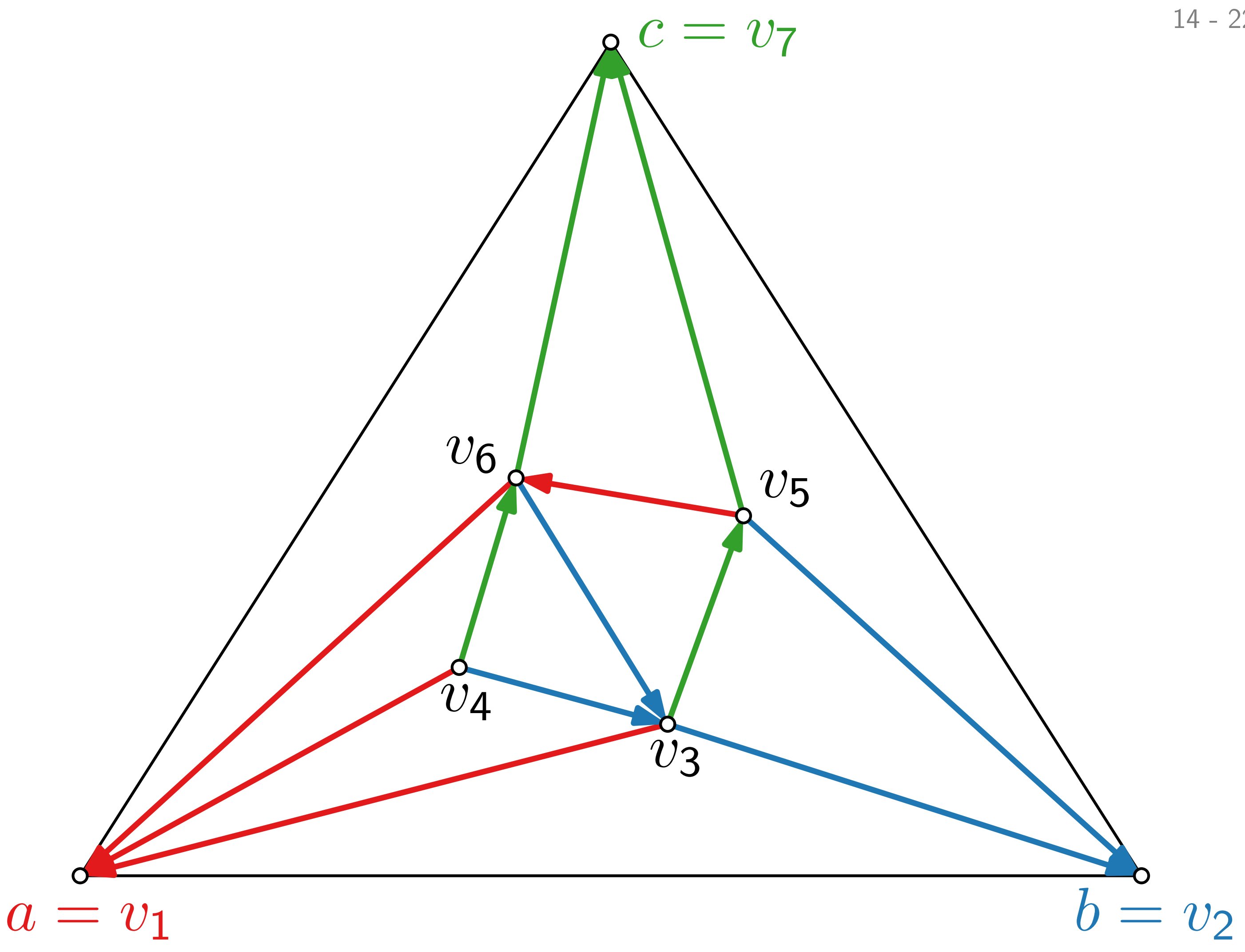

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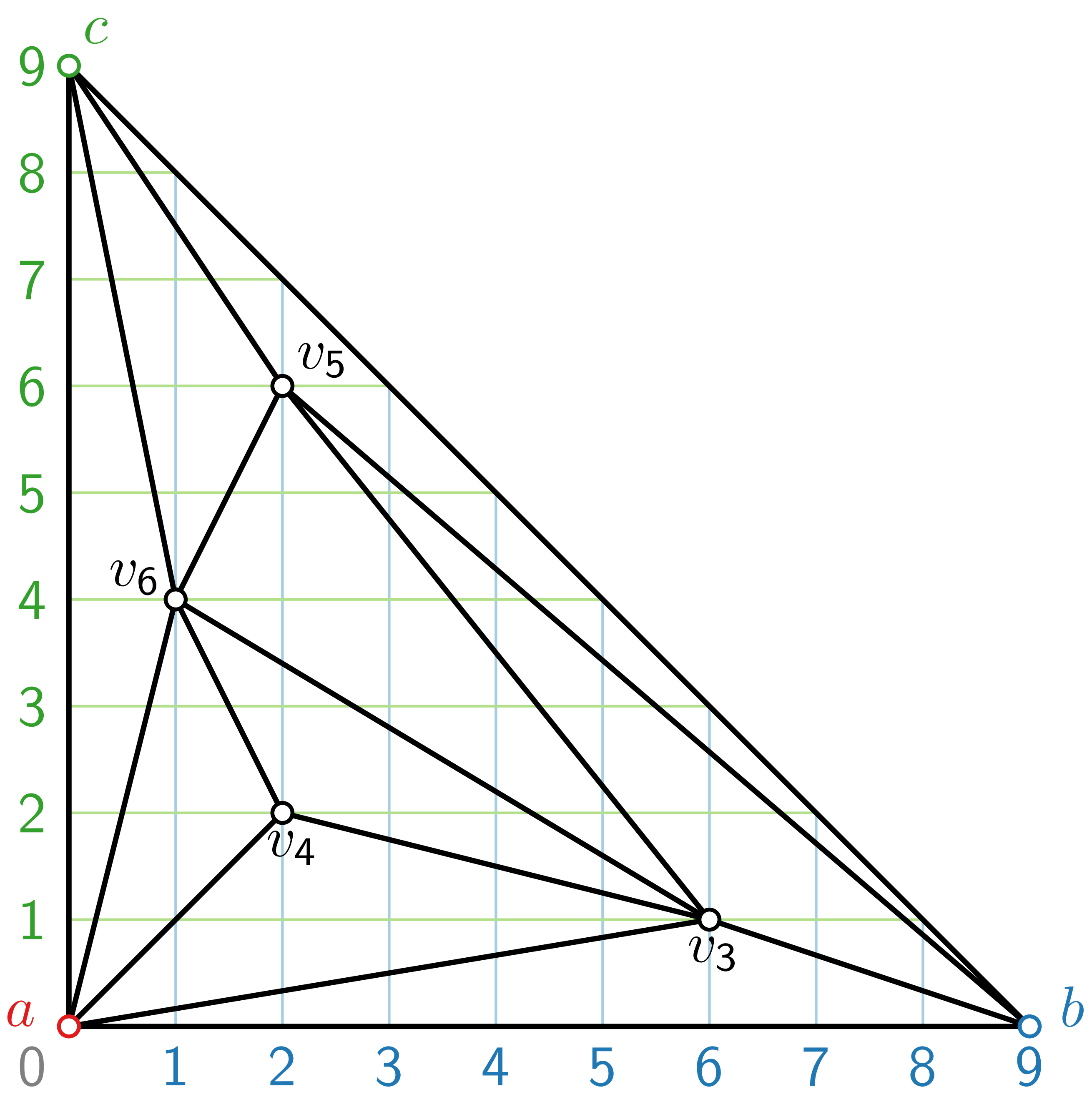

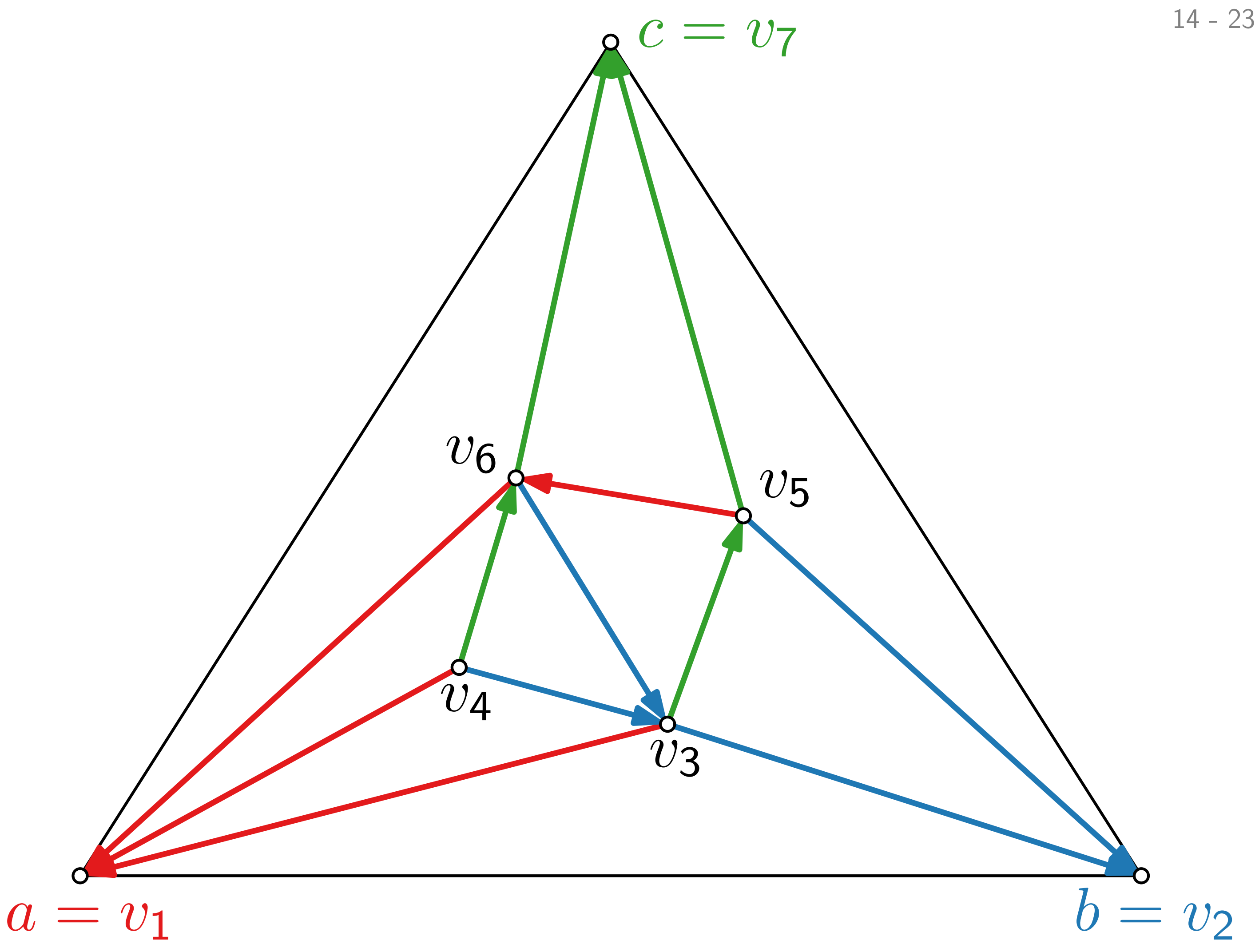

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A **weak barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to V:

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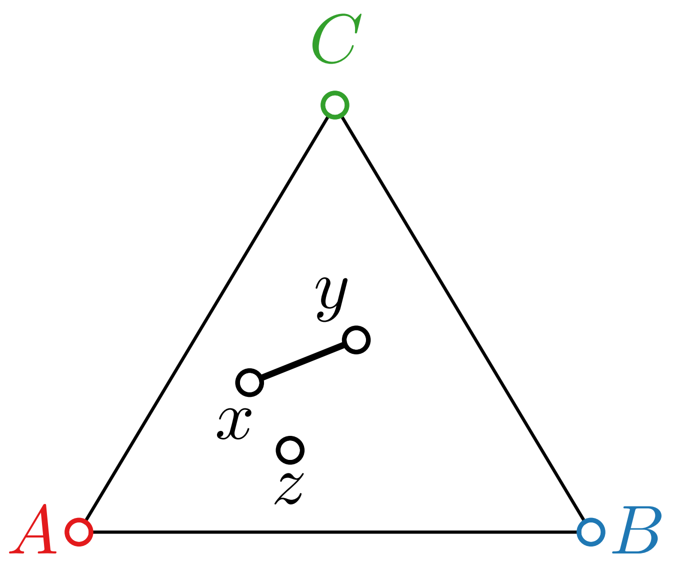

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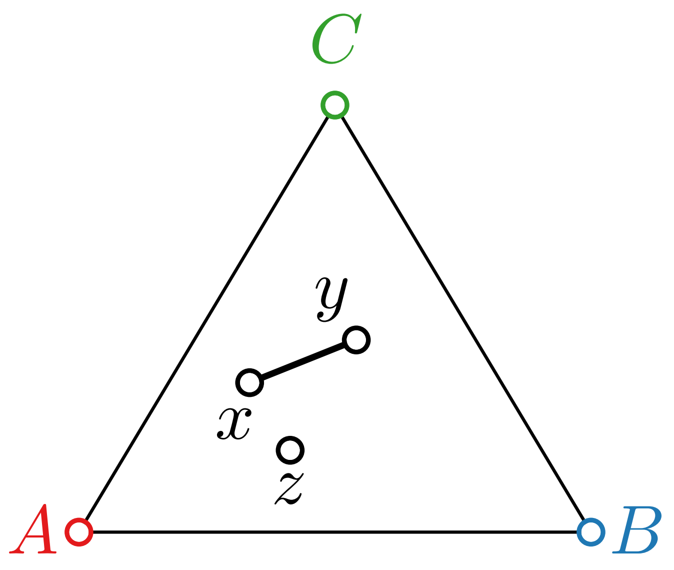

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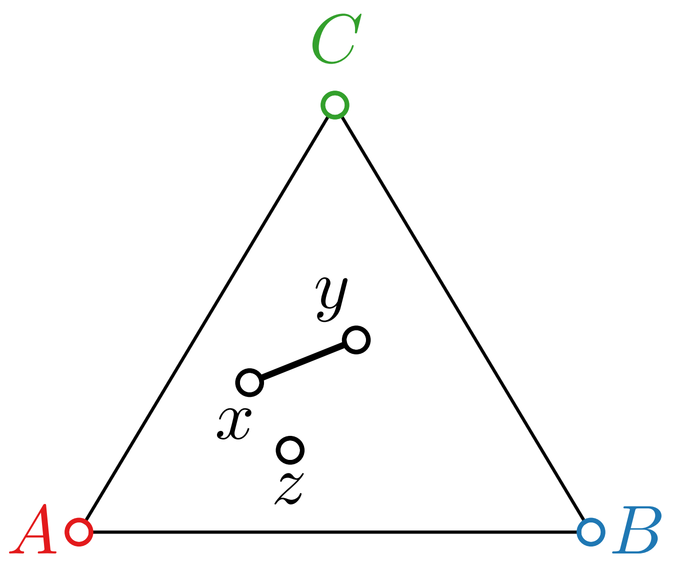

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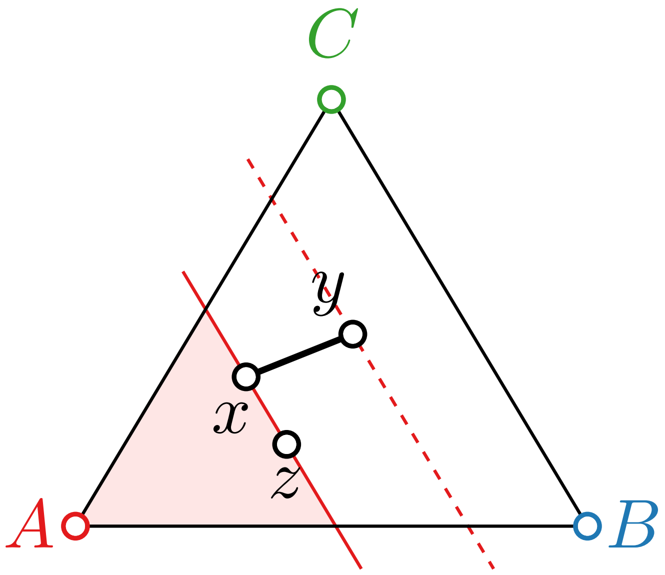

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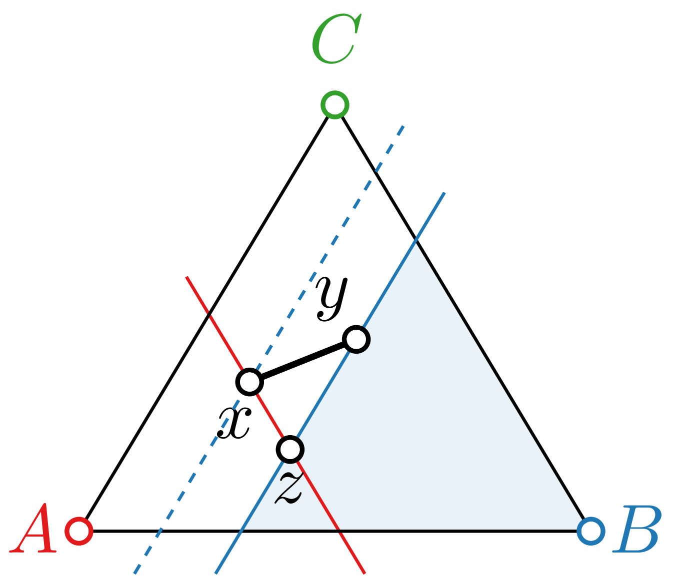

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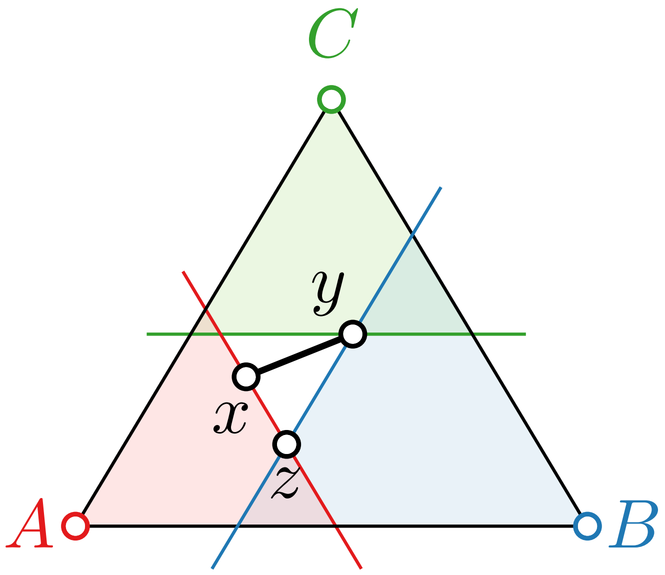

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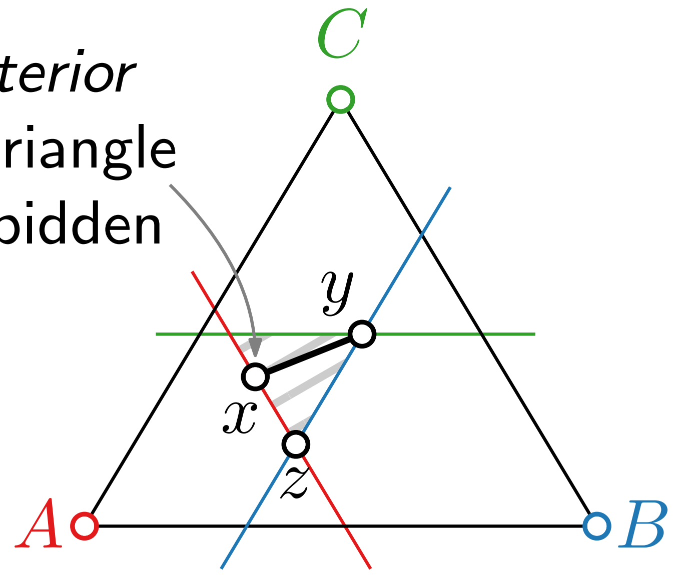

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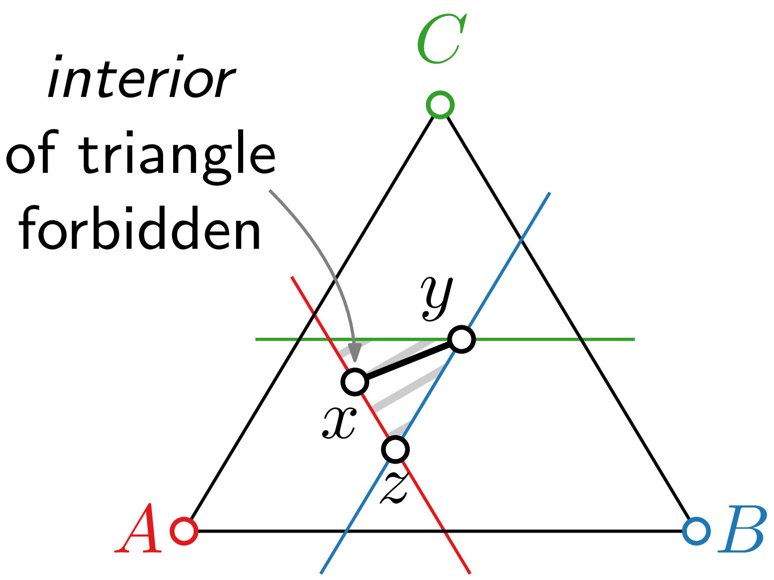

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Lemma.
For a weak barycentric representation $\phi\colon v \mapsto (v_1, v_2, v_3)$ and a triangle $\triangle ABC$, the mapping

$$f: v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.

Weak Barycentric Representation

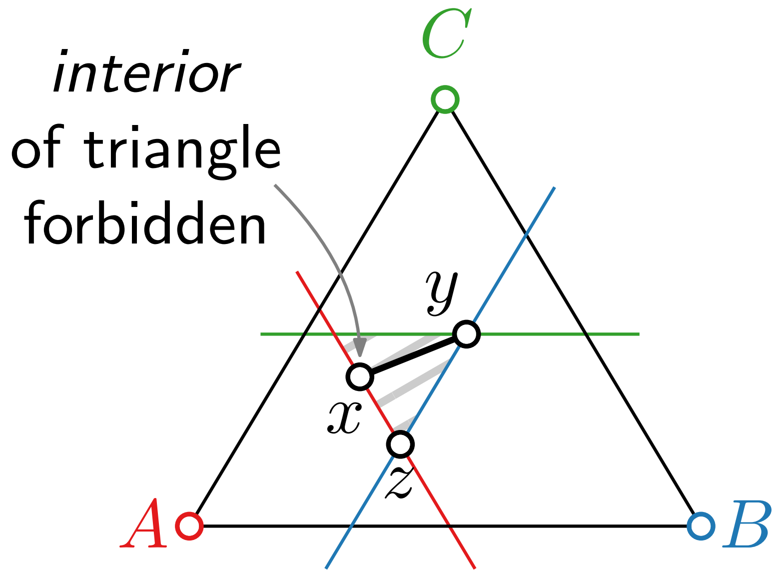
A **weak barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to V:

$$\phi\colon V \to \mathbb{R}^3_{\geq 0}, v \mapsto (v_1, v_2, v_3)$$

with the following properties:

(W1) $v_1 + v_2 + v_3 = 1$ for all $v \in V$,

(W2) for each $\{x, y\} \in E$ and each $z \in V \setminus \{x, y\}$ there exists a $k \in \{1, 2, 3\}$ with $(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$ and $(y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$.



i.e., either $y_k < z_k$ or $y_k = z_k$ and $y_{k+1} < z_{k+1}$

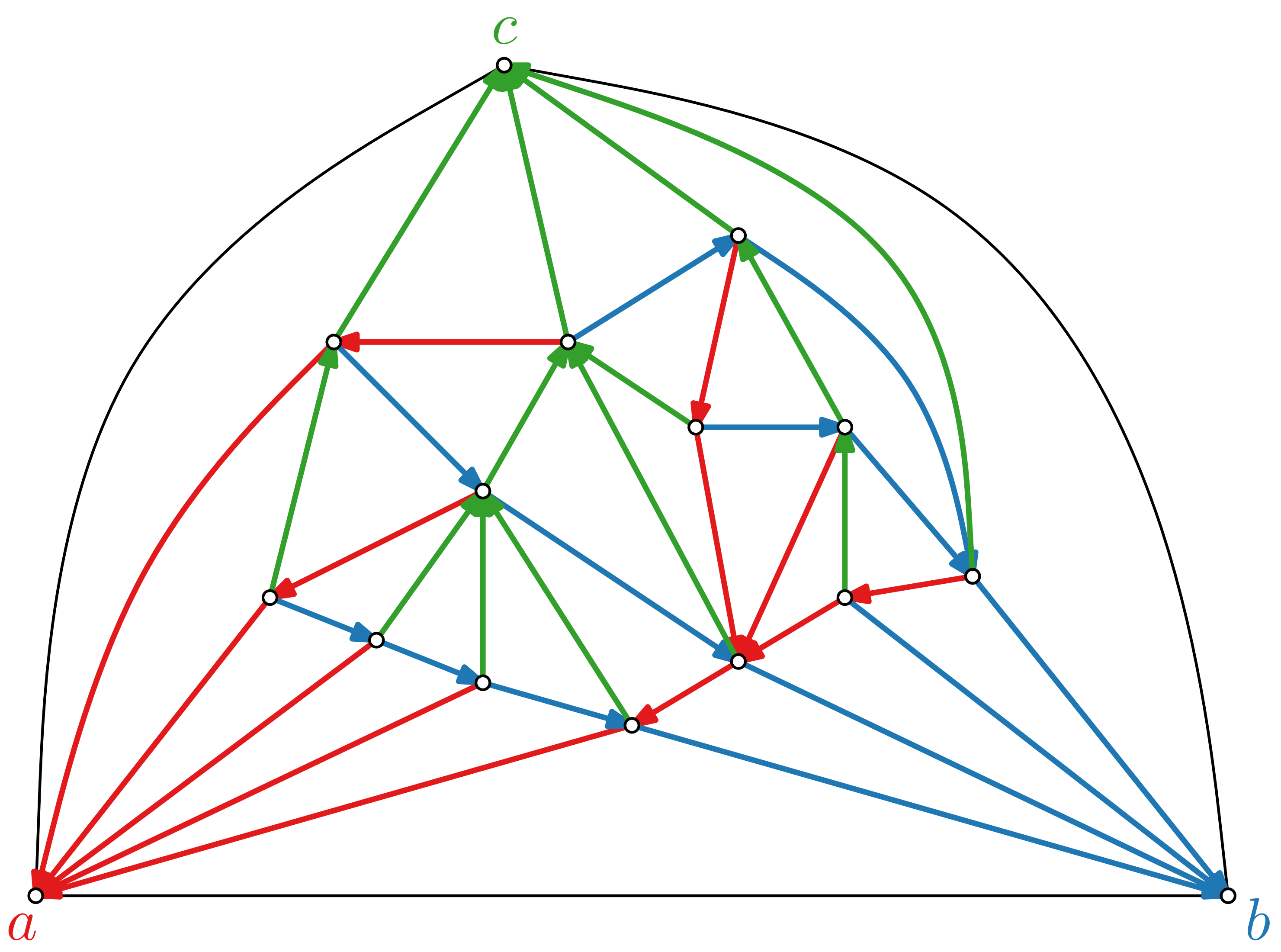
Lemma.
For a weak barycentric representation $\phi\colon v \mapsto (v_1, v_2, v_3)$ and a triangle $\triangle ABC$, the mapping

$$f : v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.

Proof as exercise.

Counting Vertices



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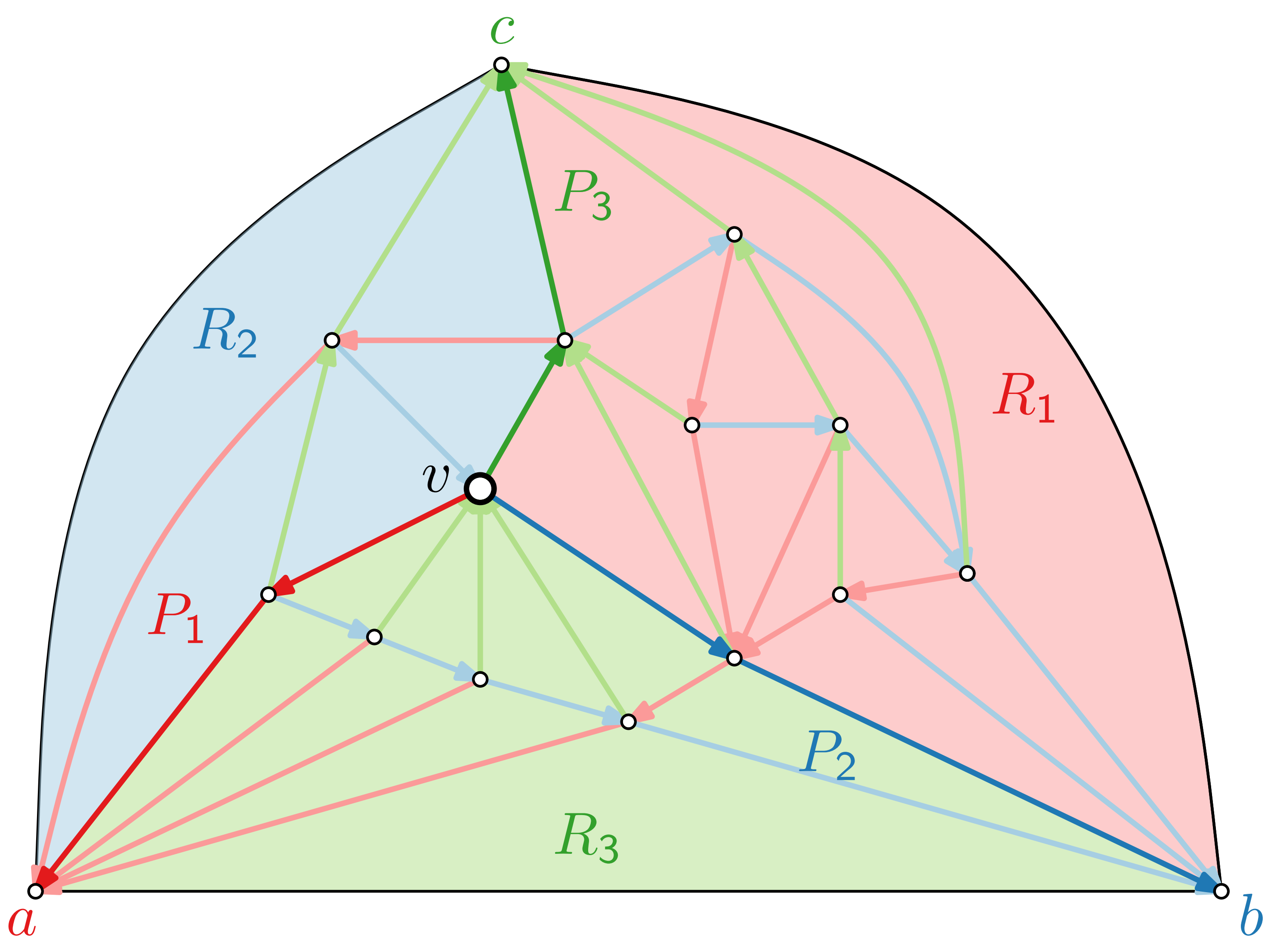

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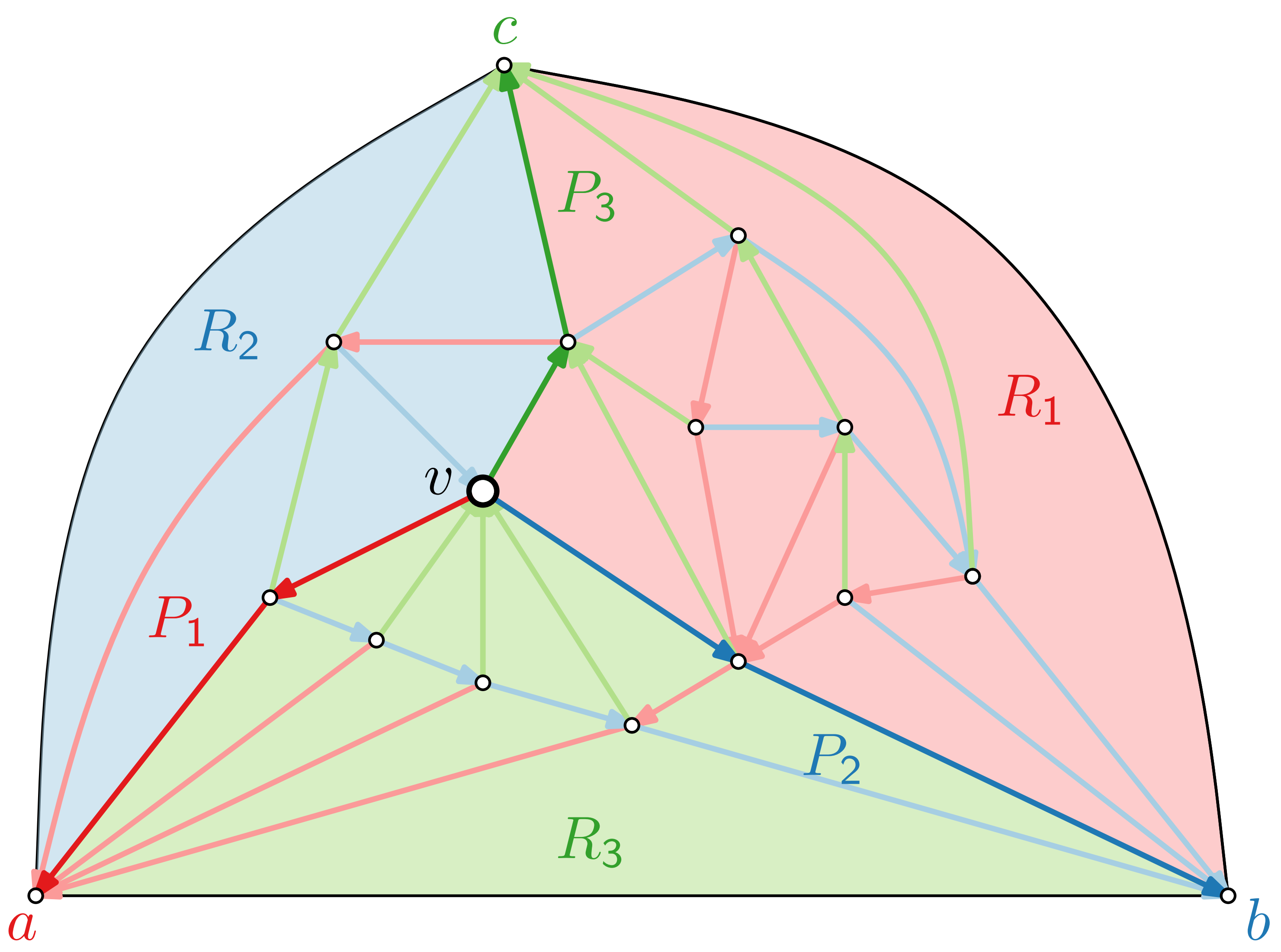

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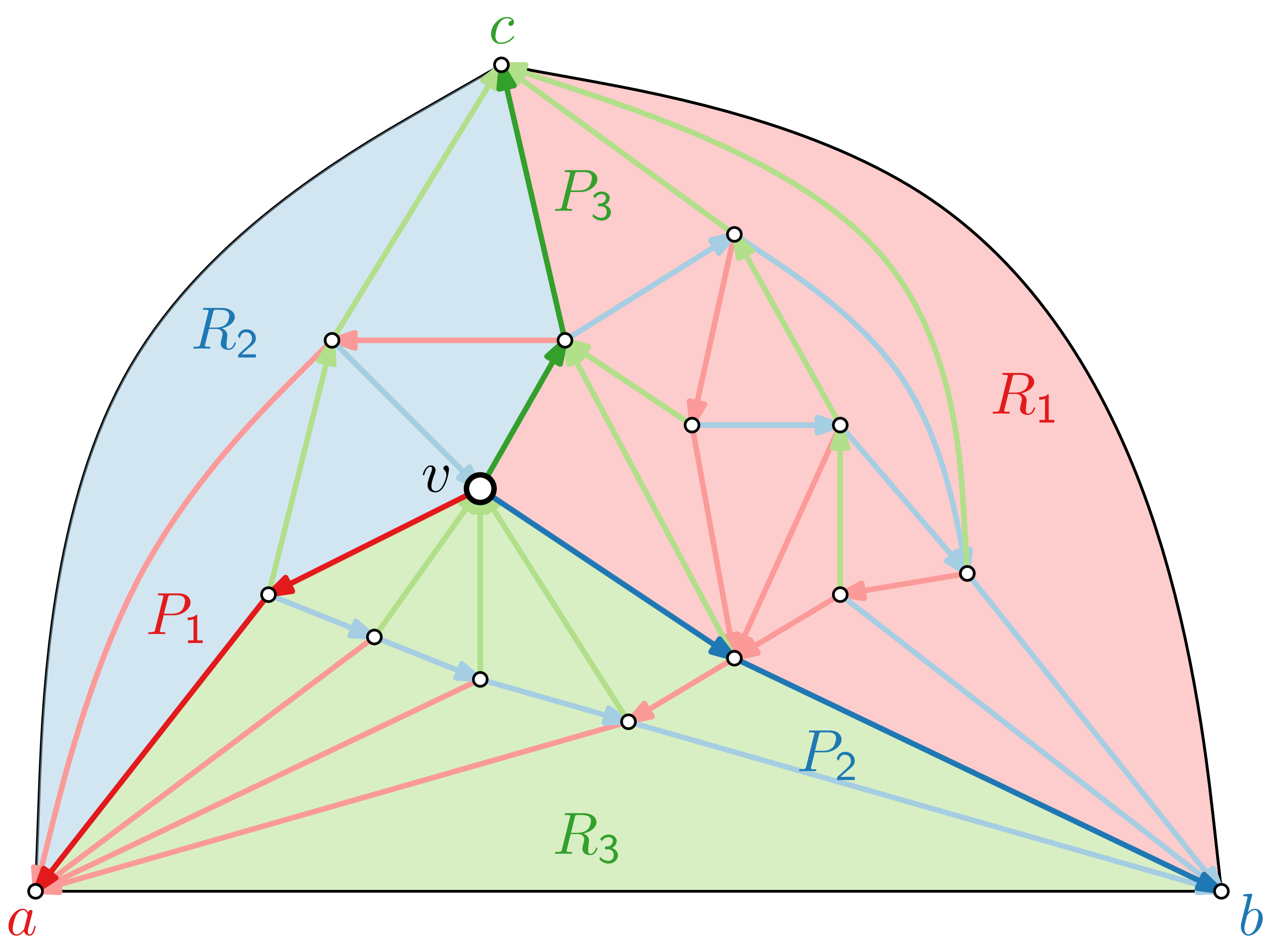

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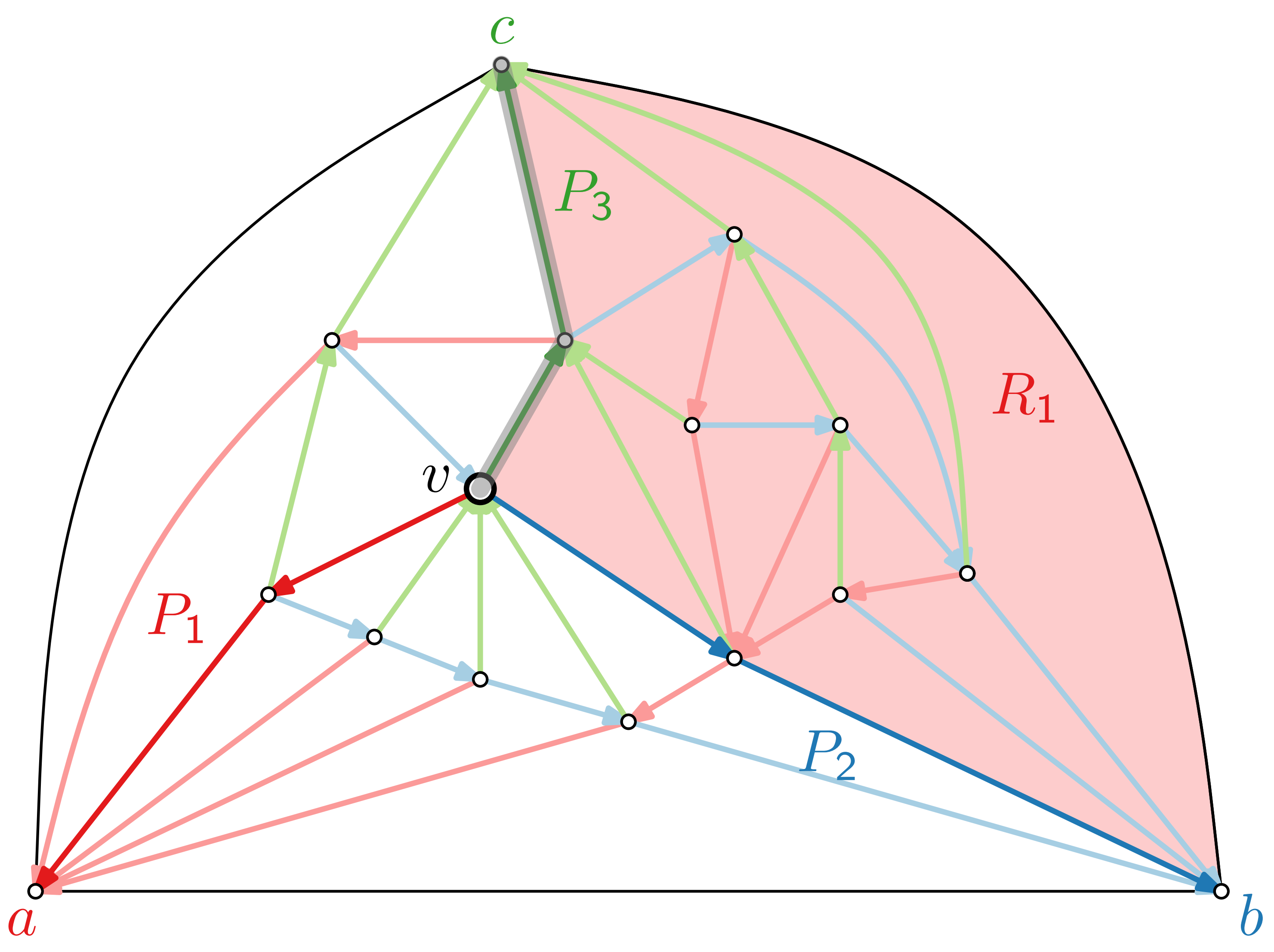

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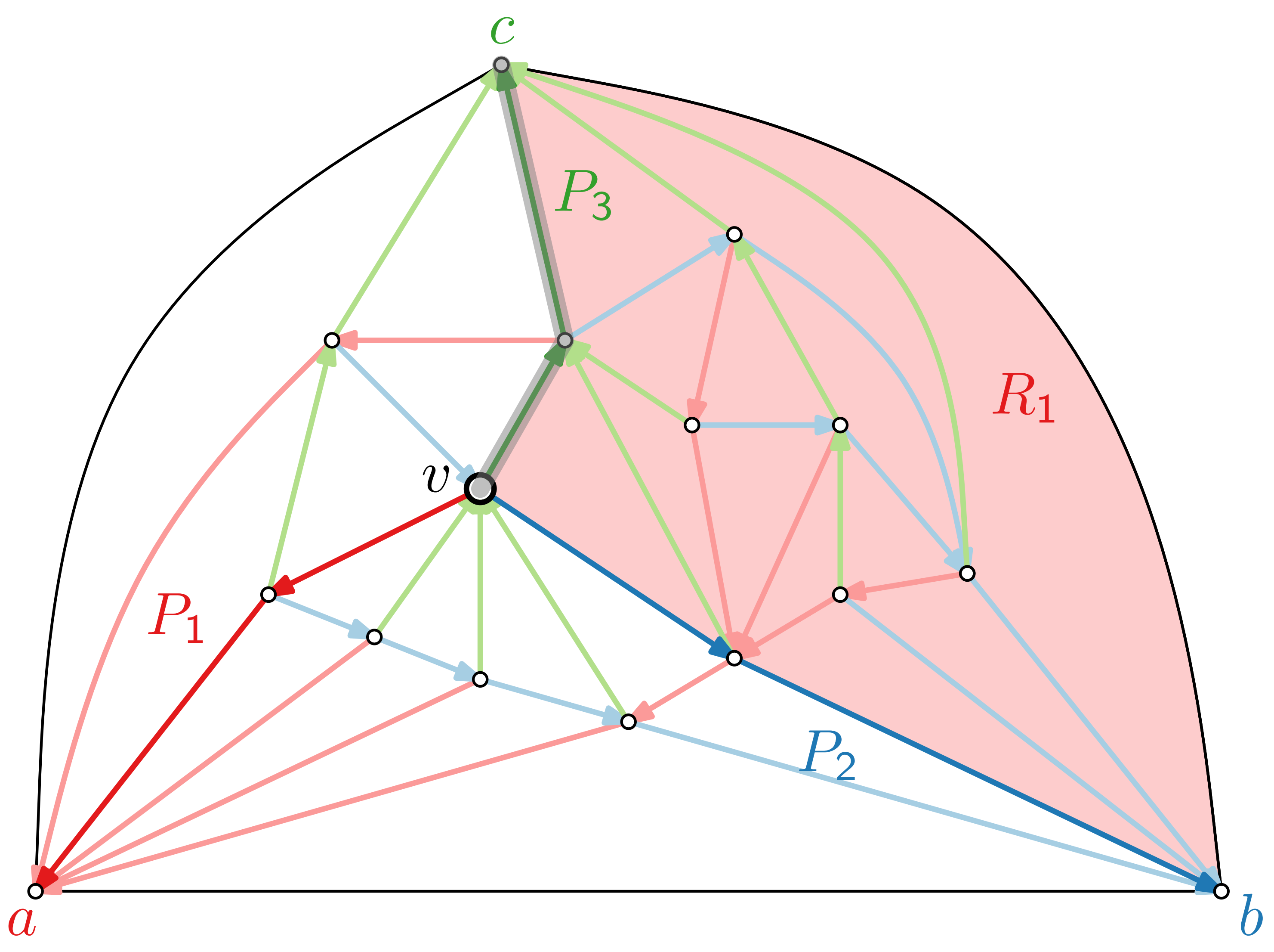

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$v_1 = 10 - 3 = 7$

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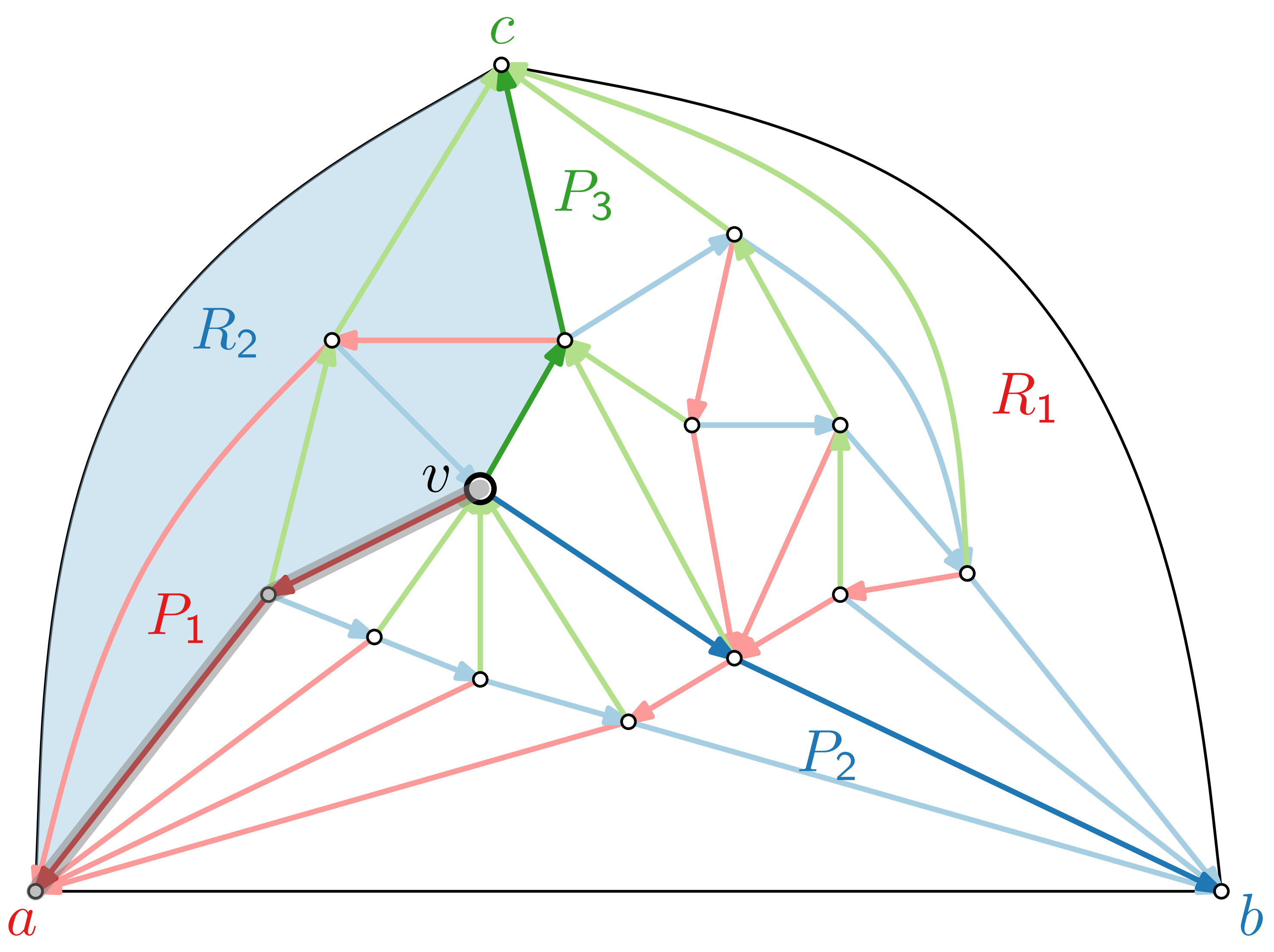

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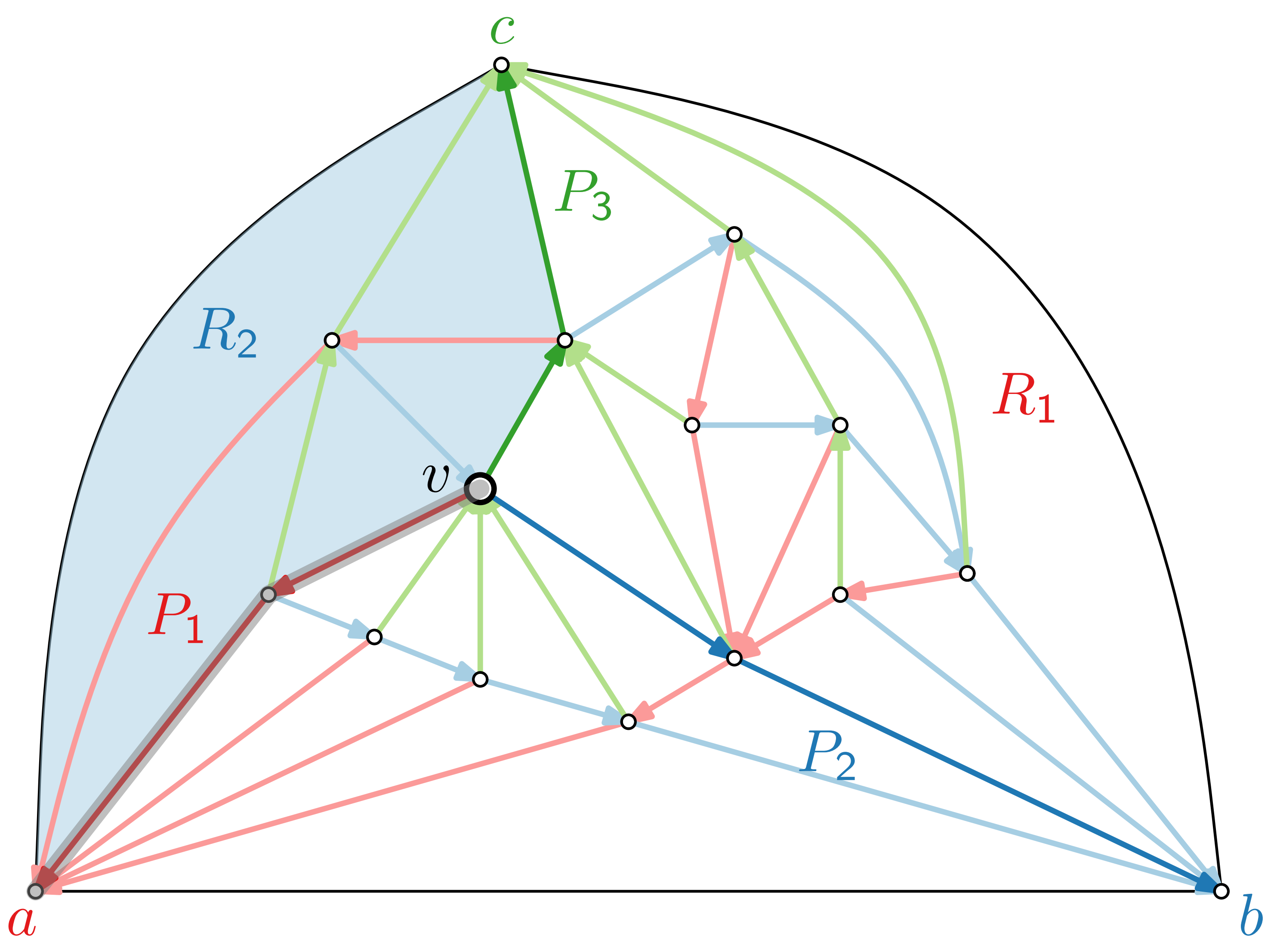

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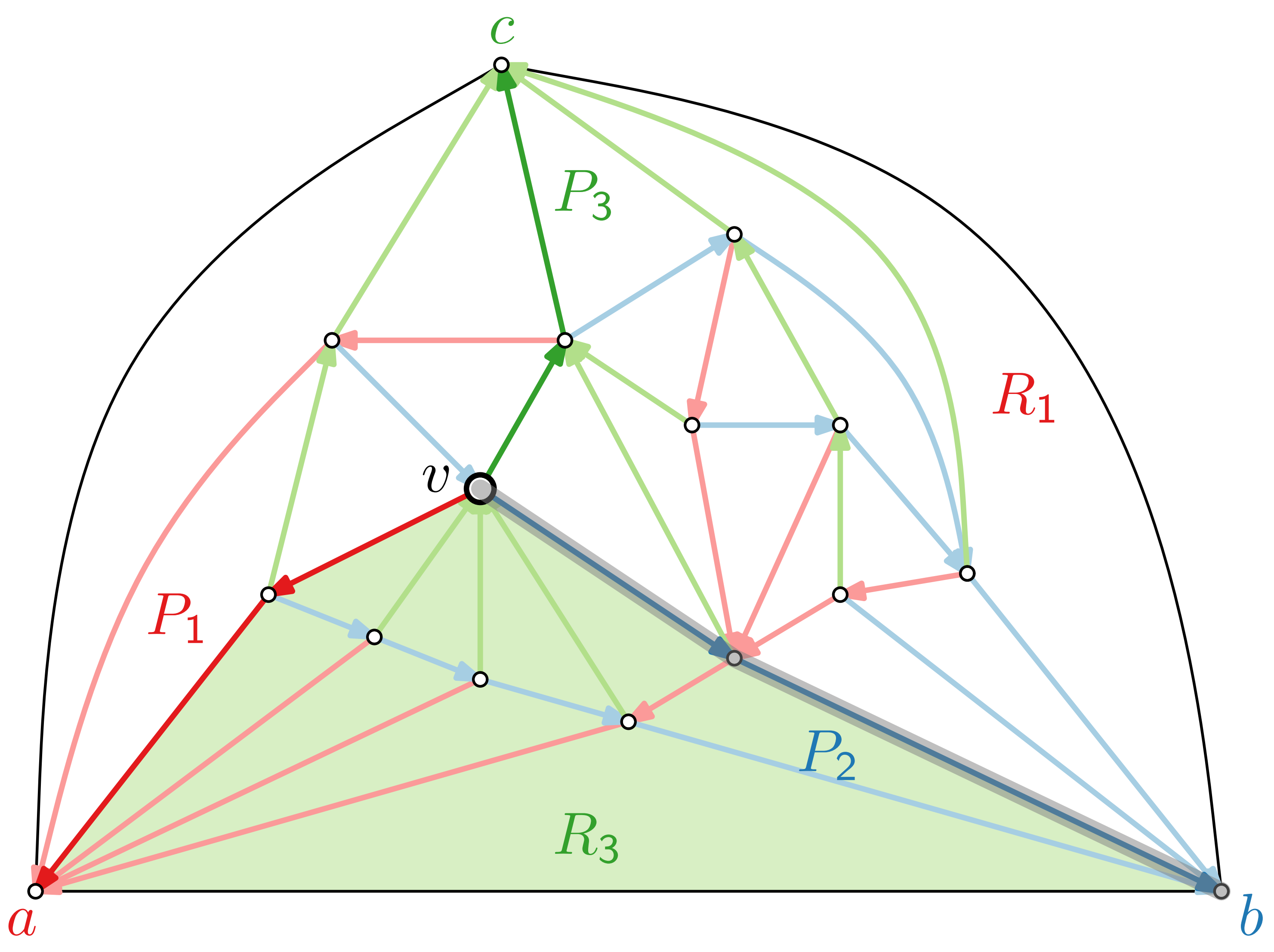

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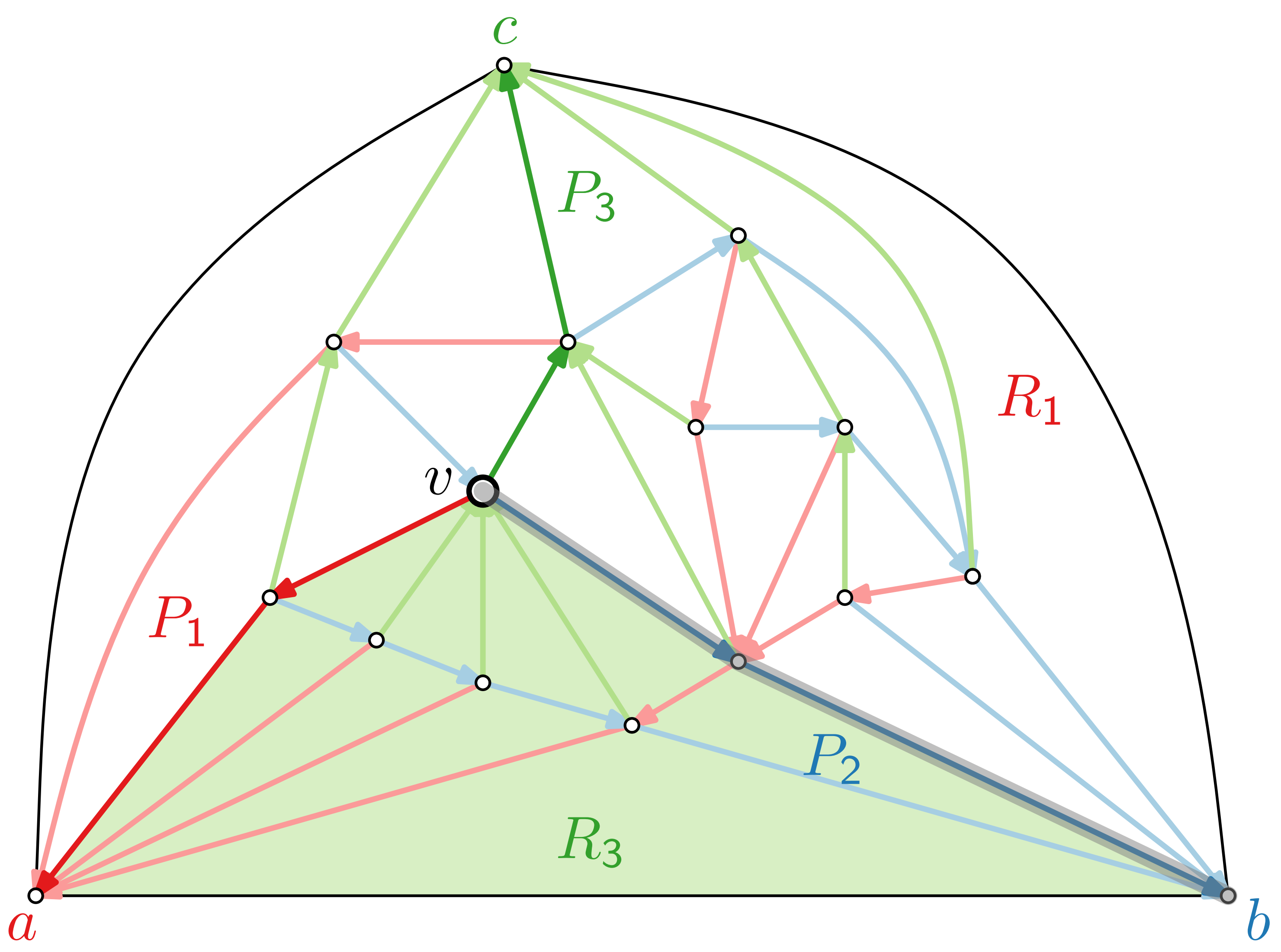

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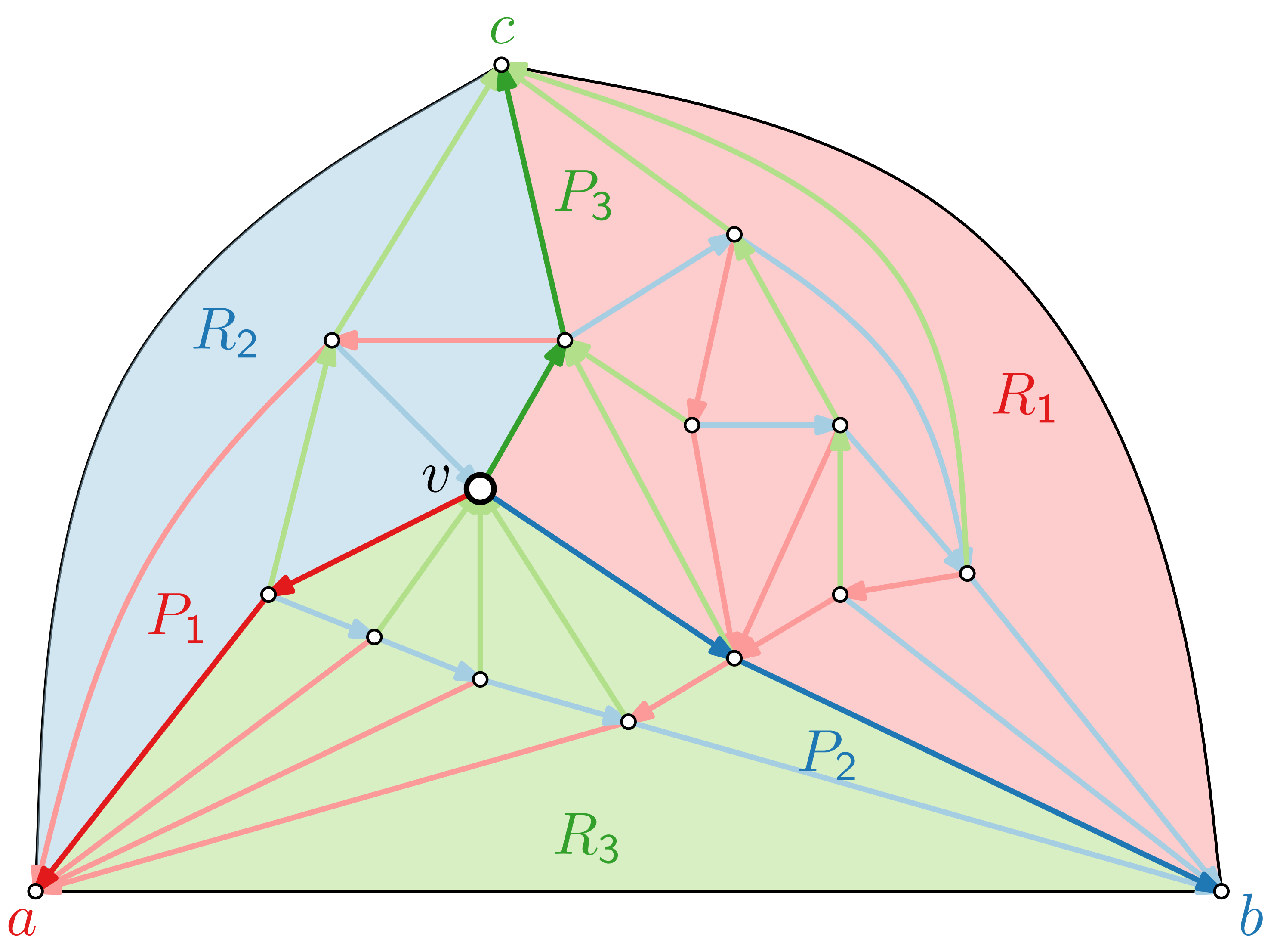

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- For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

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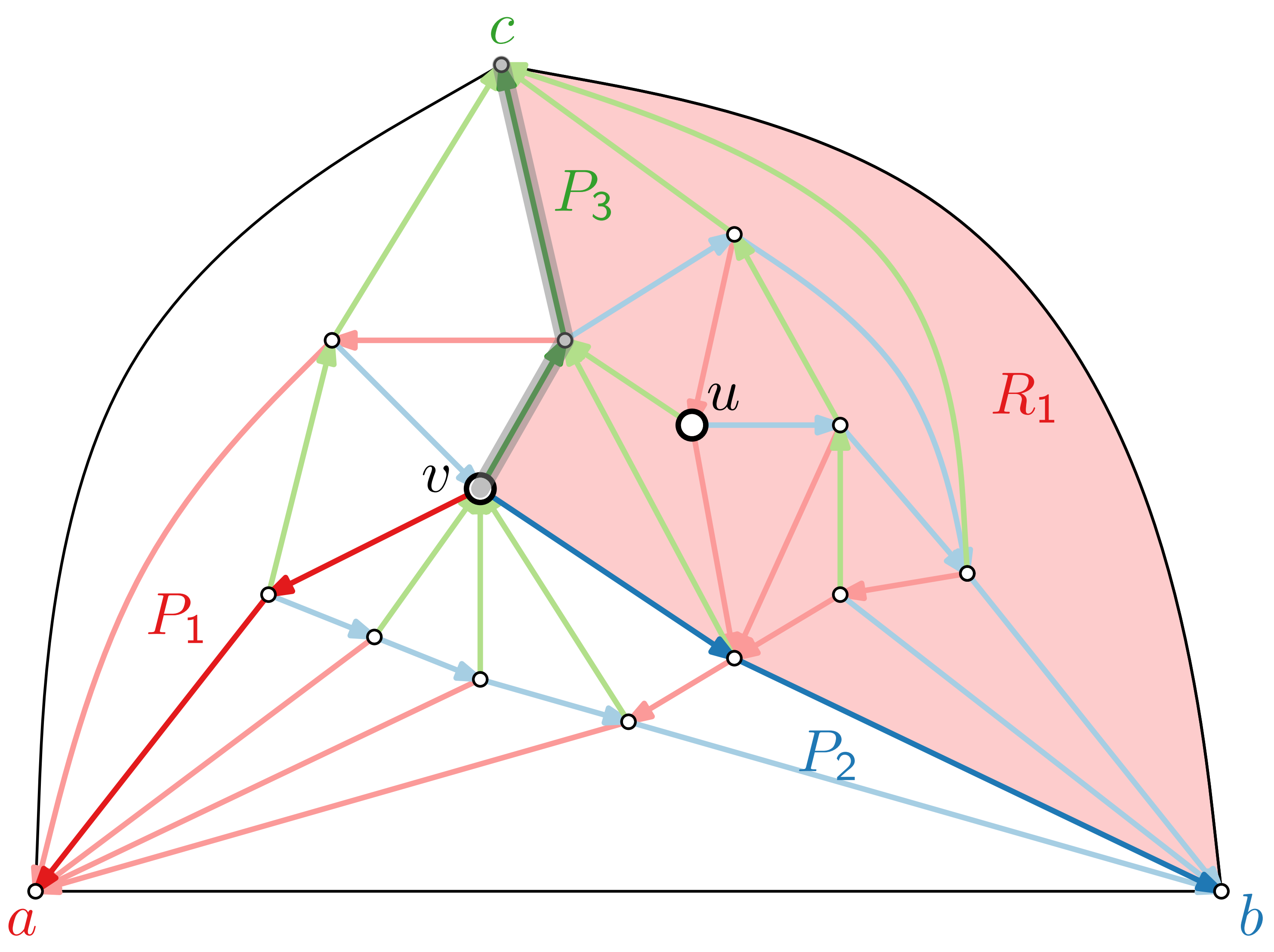

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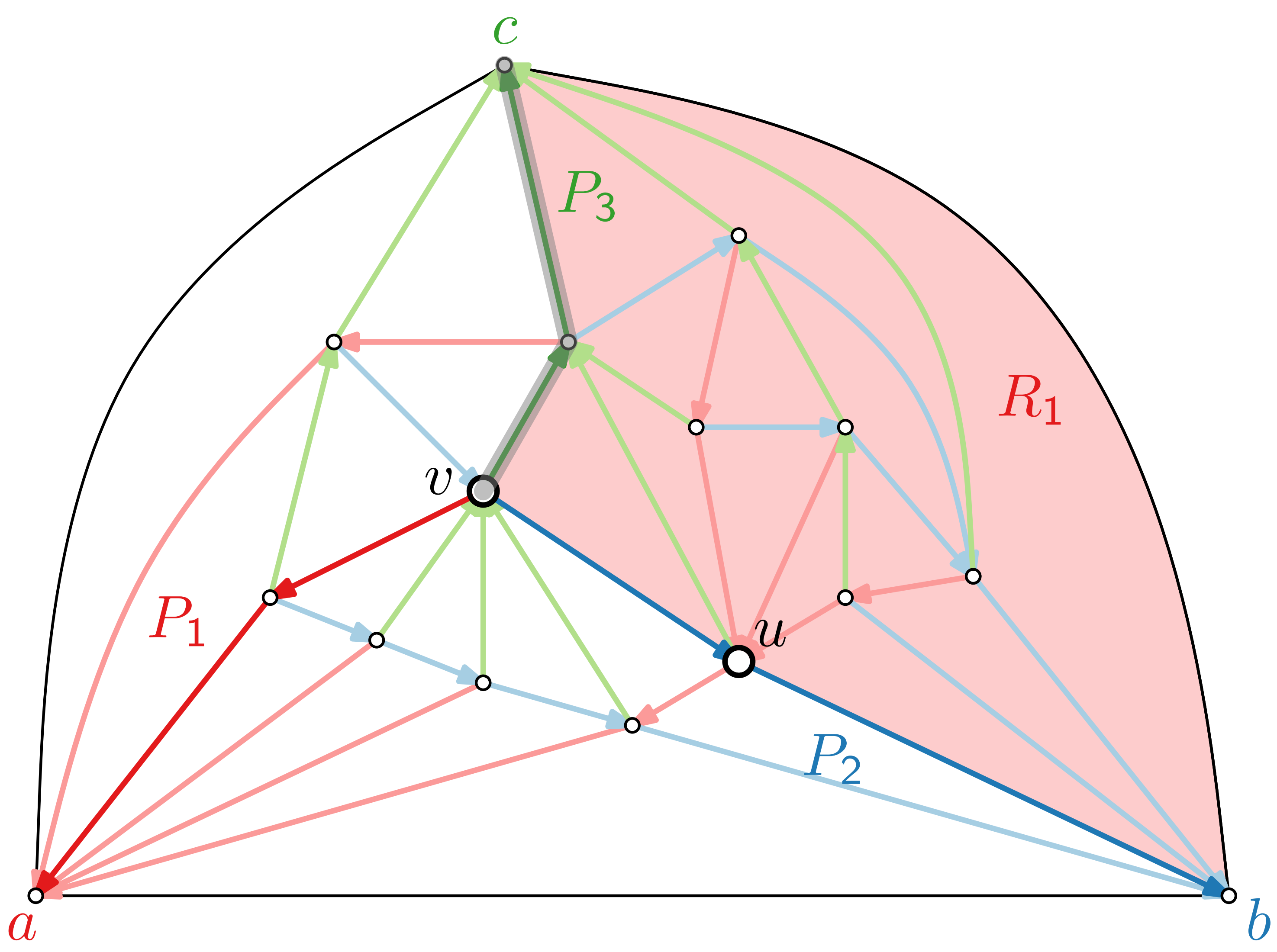

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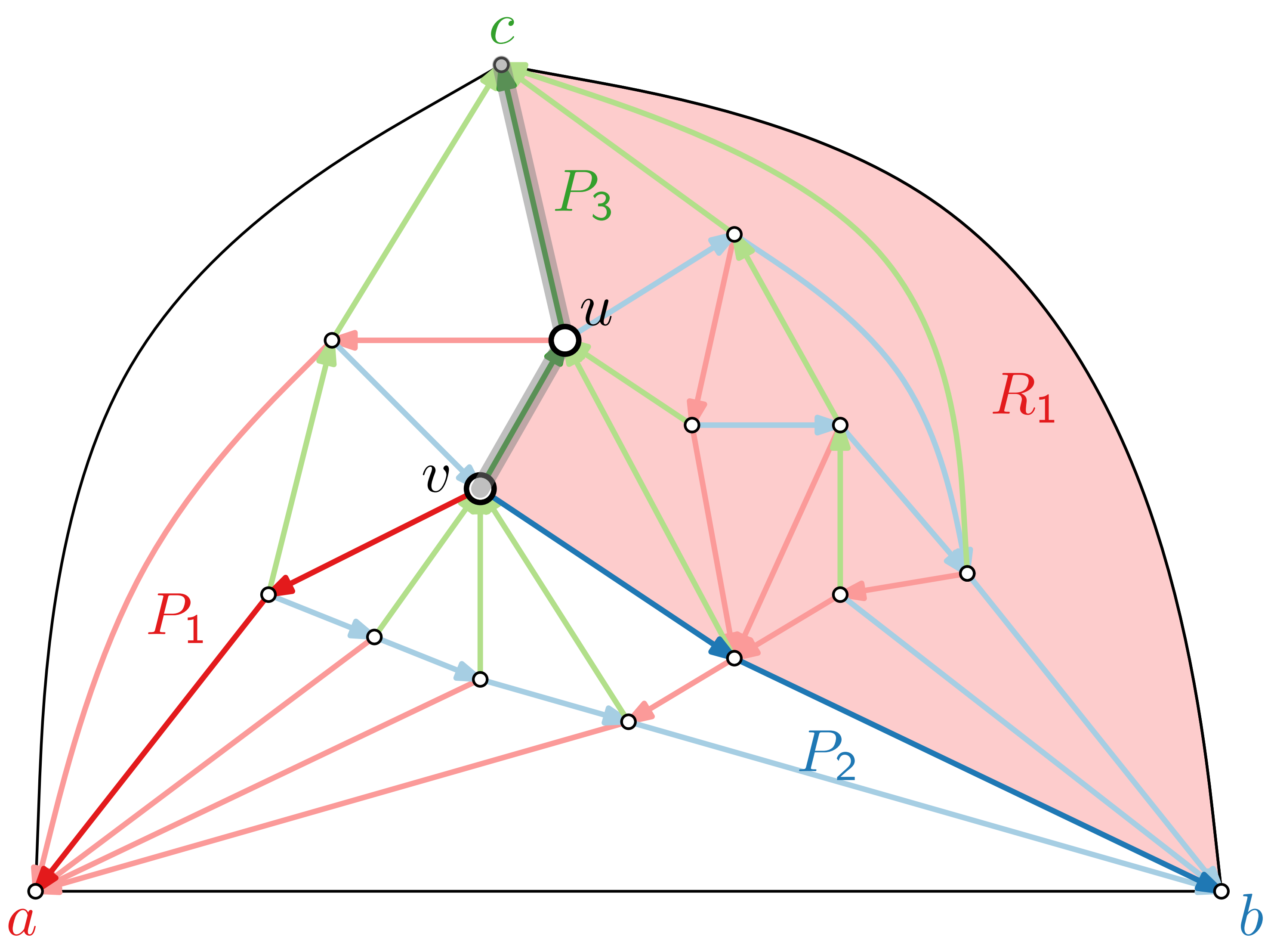

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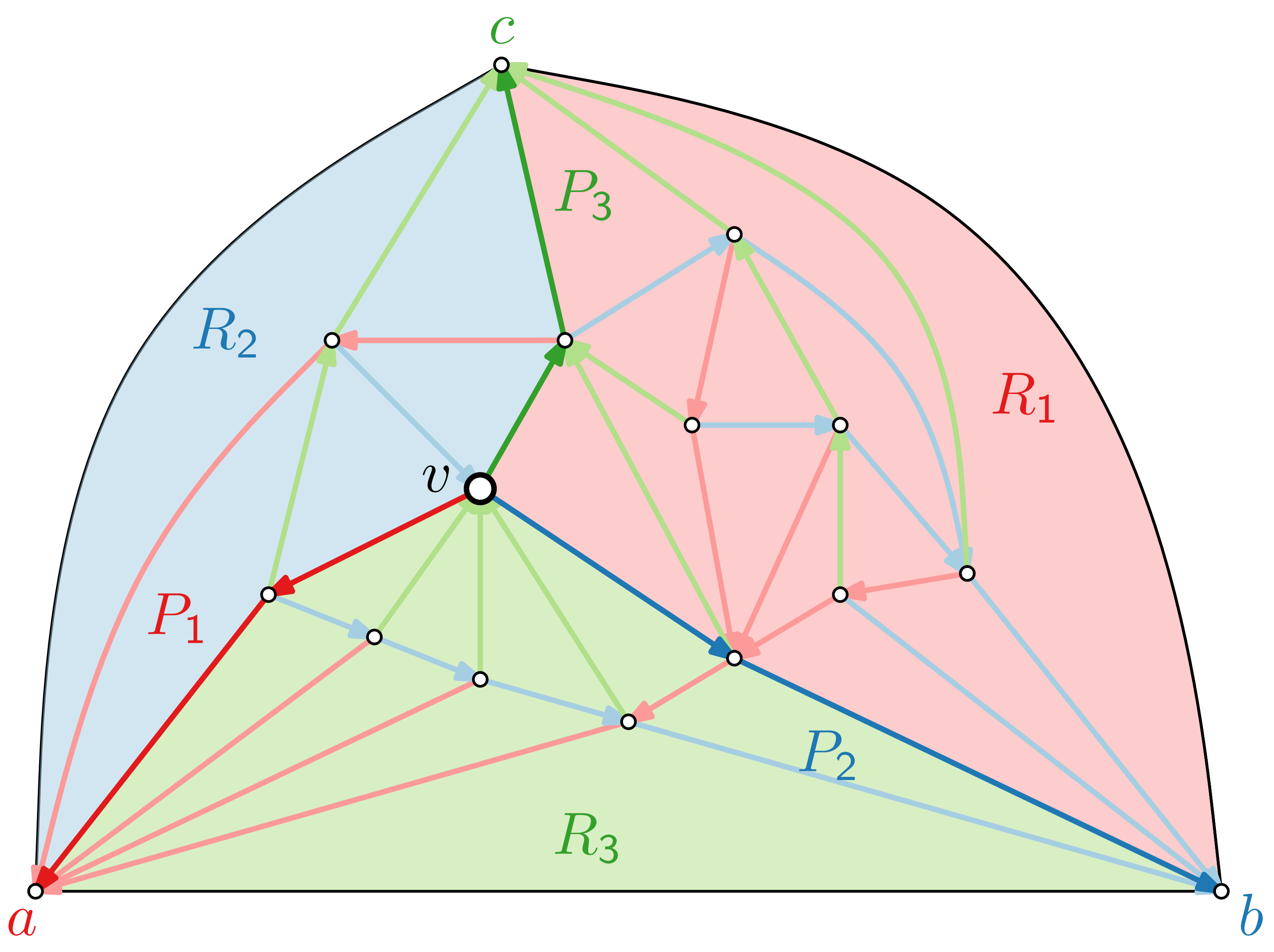

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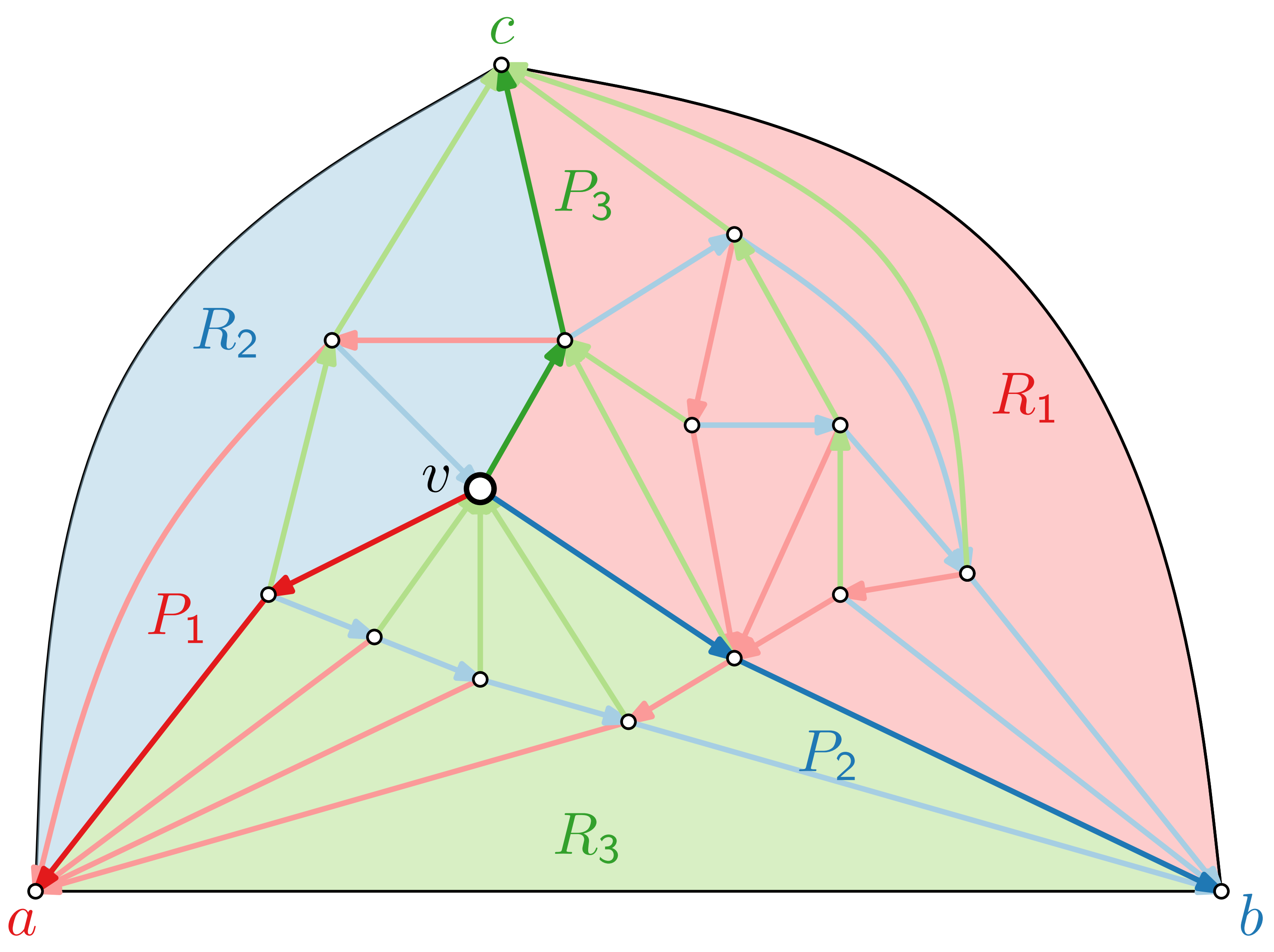

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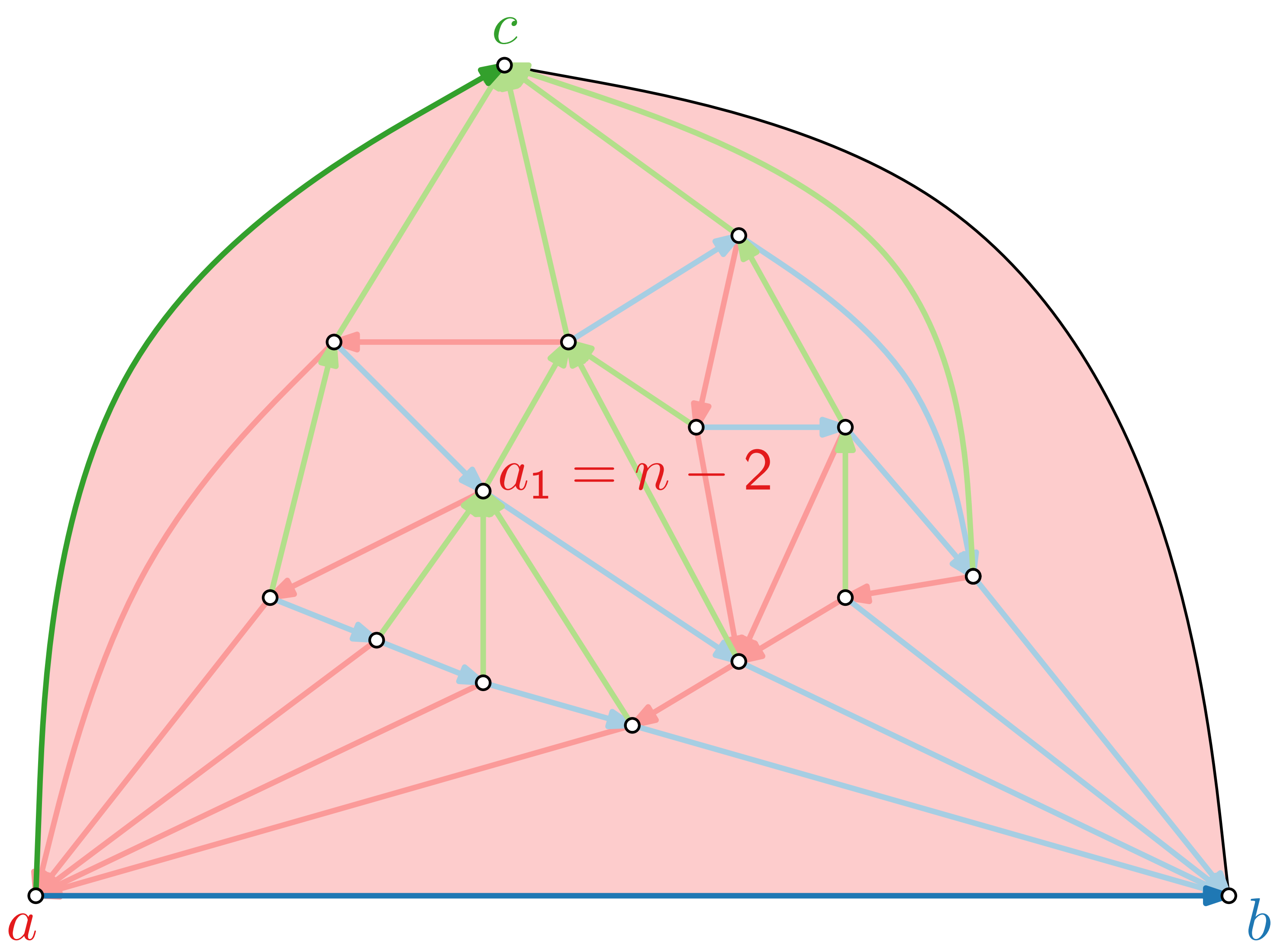

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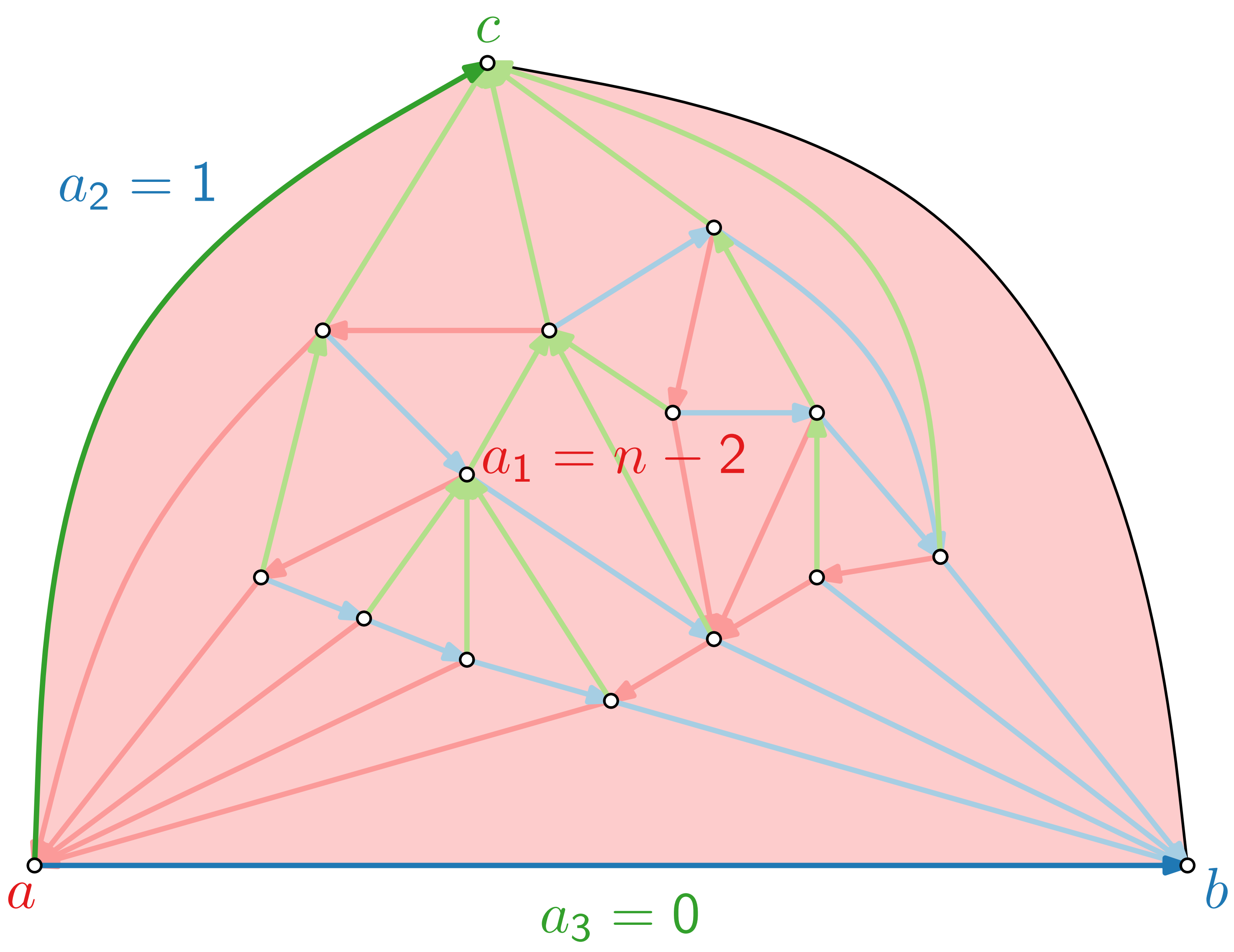

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Schnyder Drawing*

Set $A = (0,0)$, $B = (n-1, 0)$, and $C = (0, n-1)$.

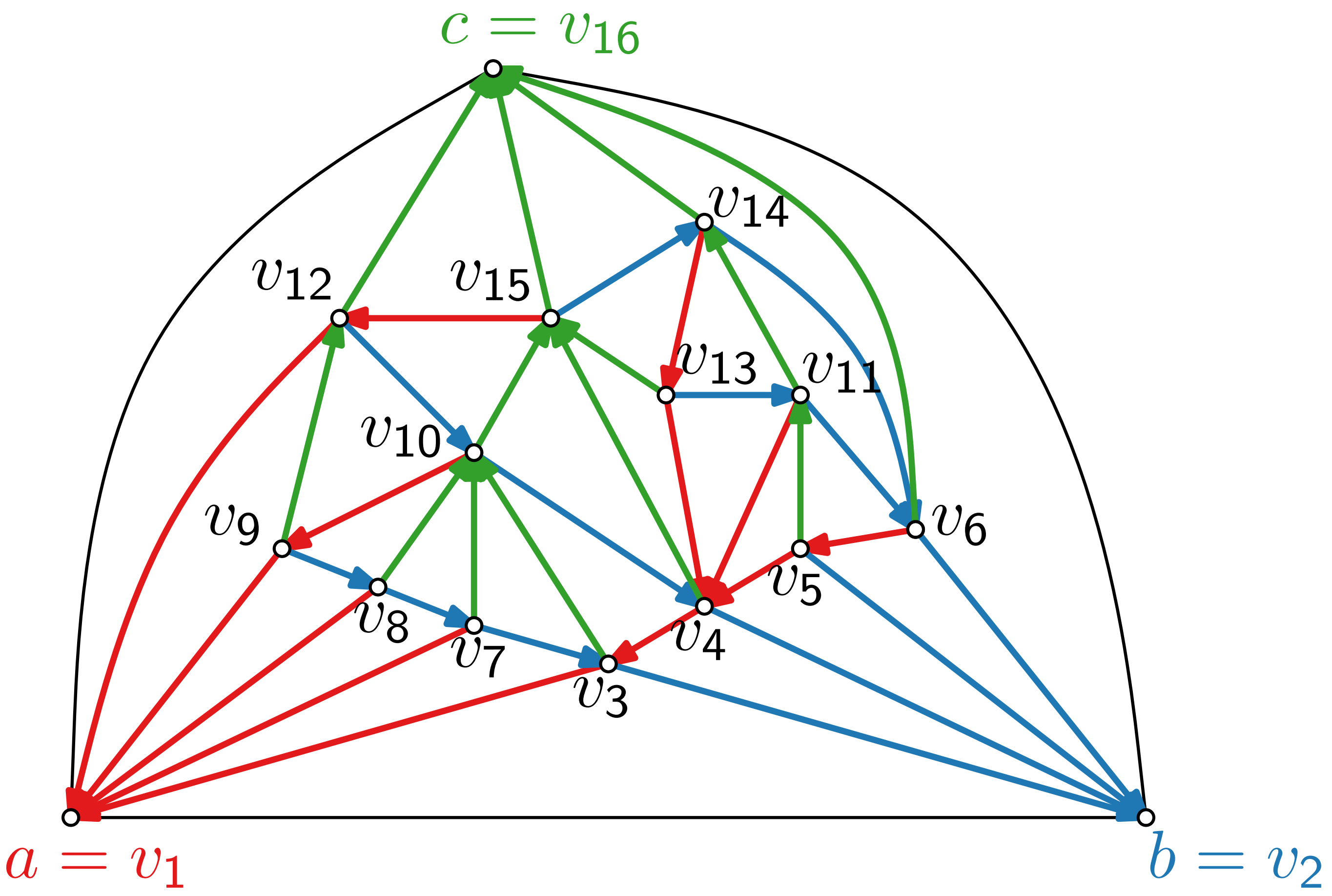
Theorem. [Schnyder '90]

For a plane triangulation G, the mapping

$$f \colon v \mapsto \frac{1}{n-1}(v_1, v_2, v_3)$$

is a barycentric representation of G (and thus yields a planar straight-line drawing of G on the $(n-2) \times (n-2)$ grid).

Schnyder Drawing* – Example



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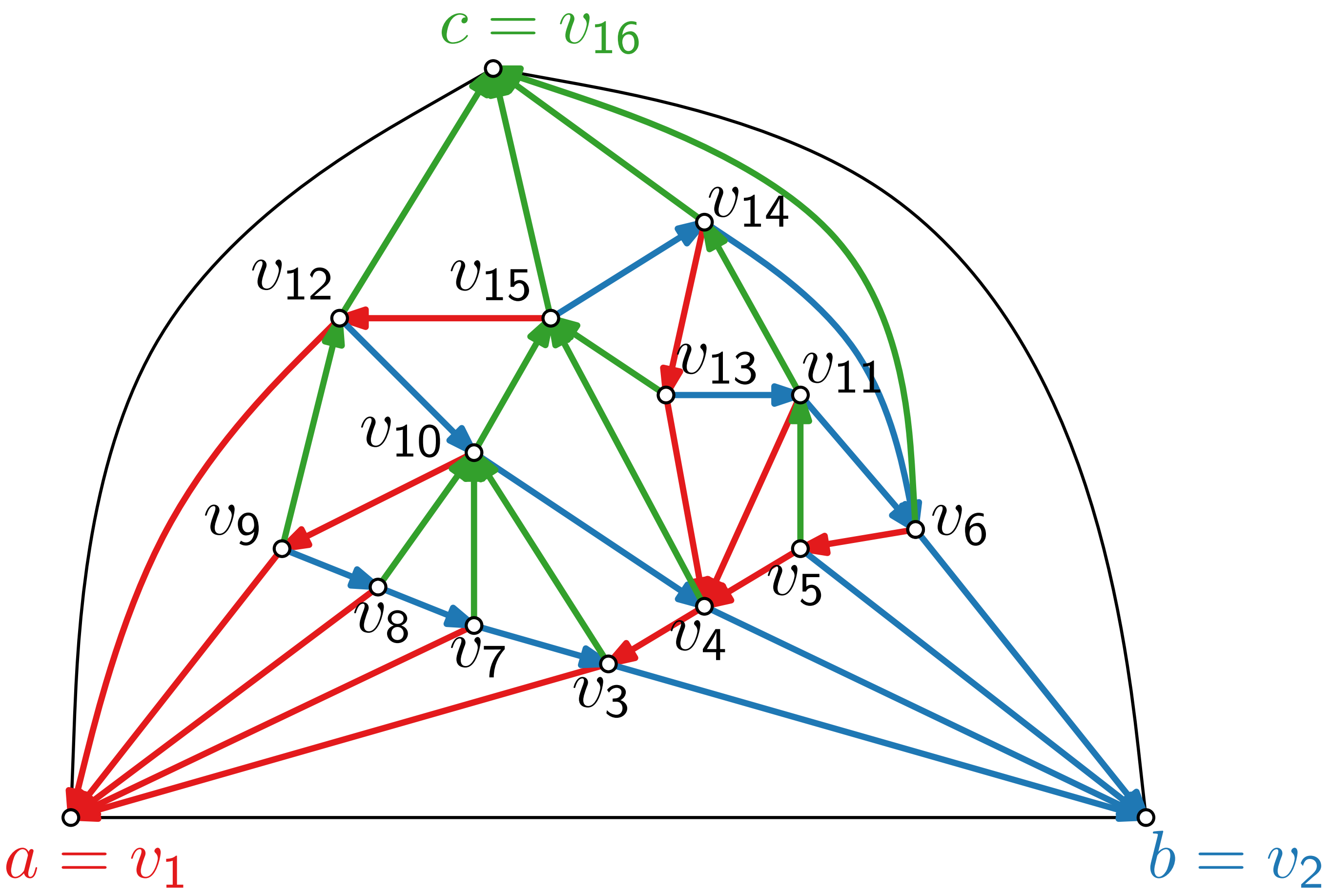

$n = 16,\ n - 2 = 14$

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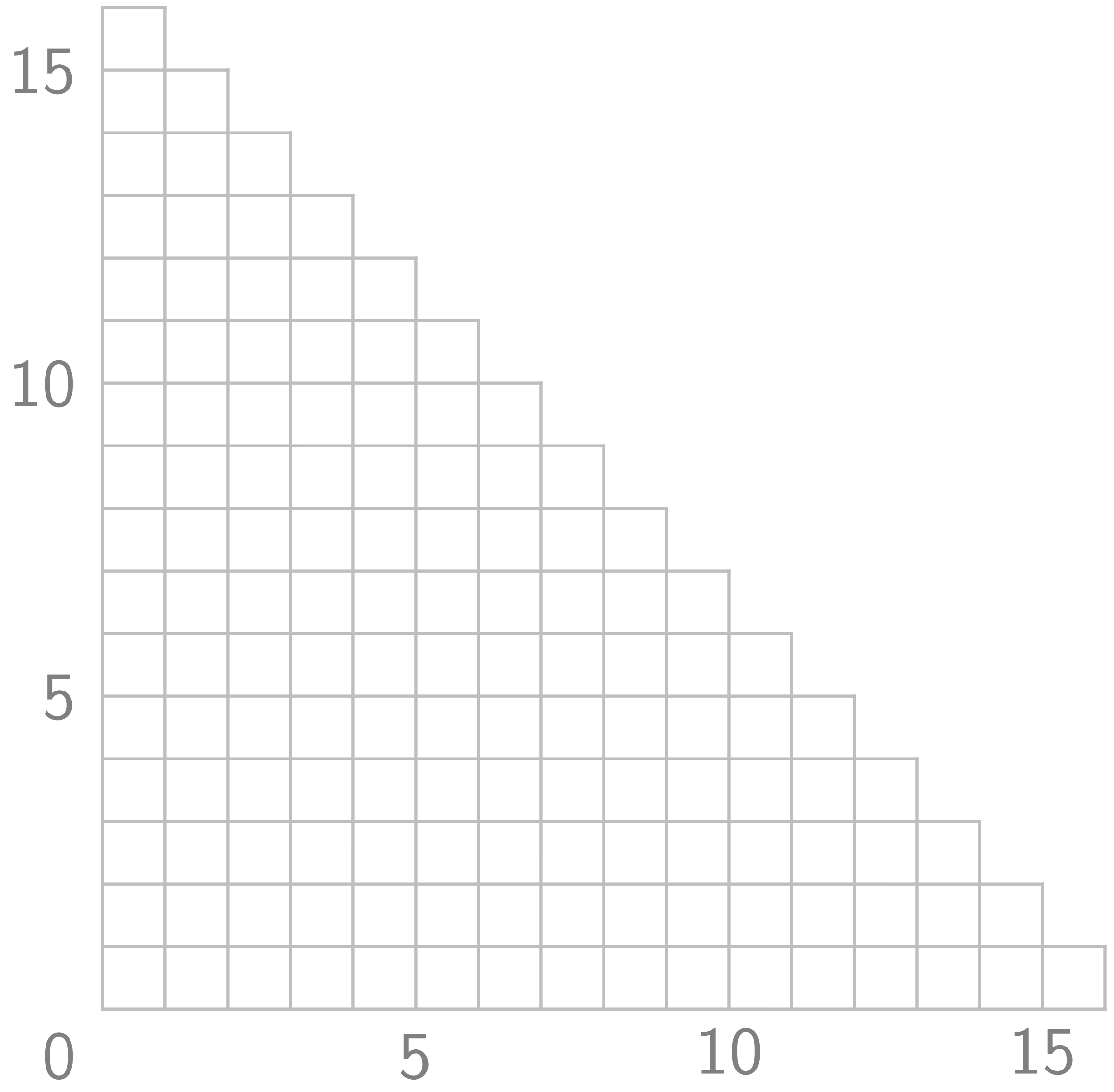

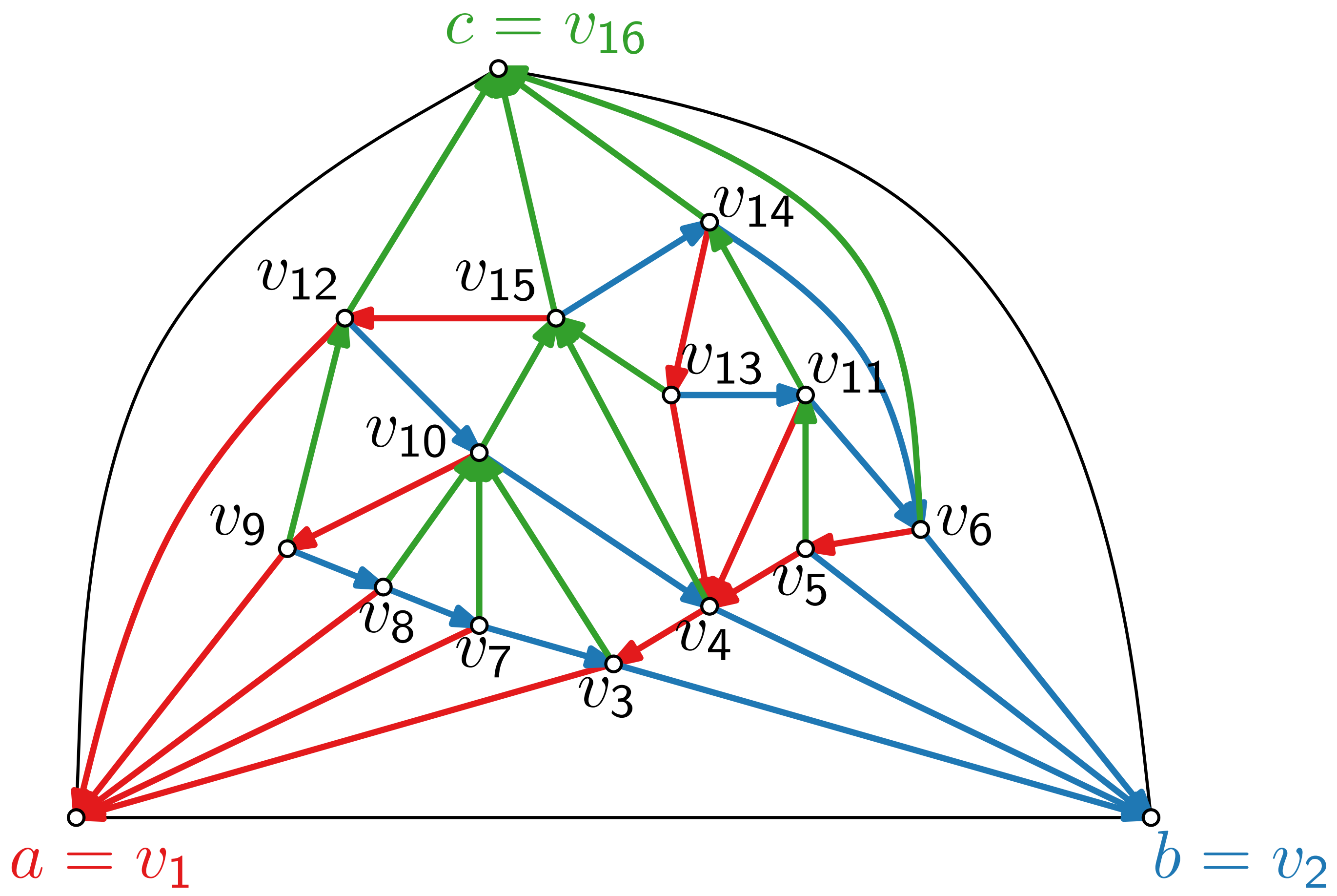

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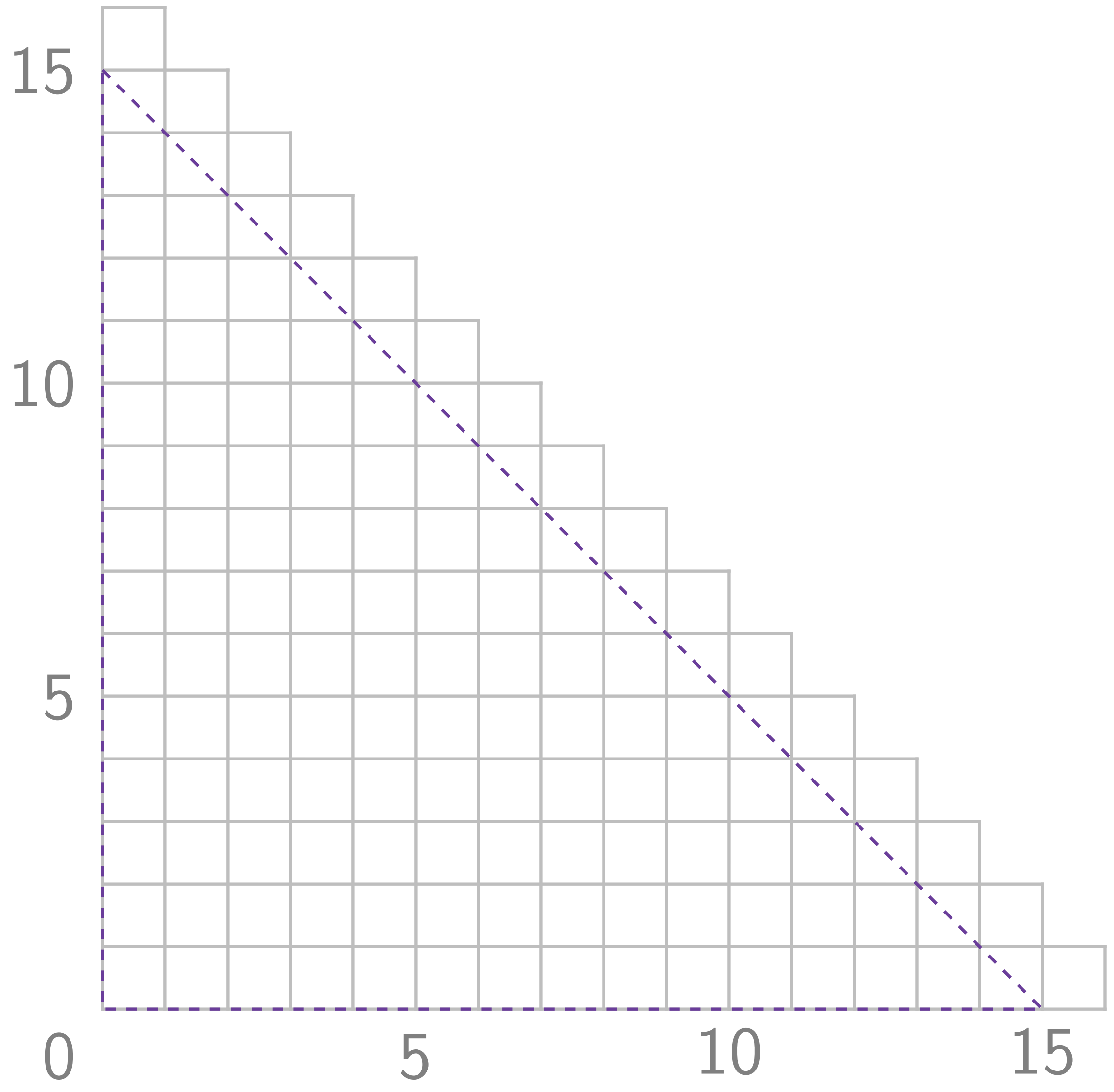
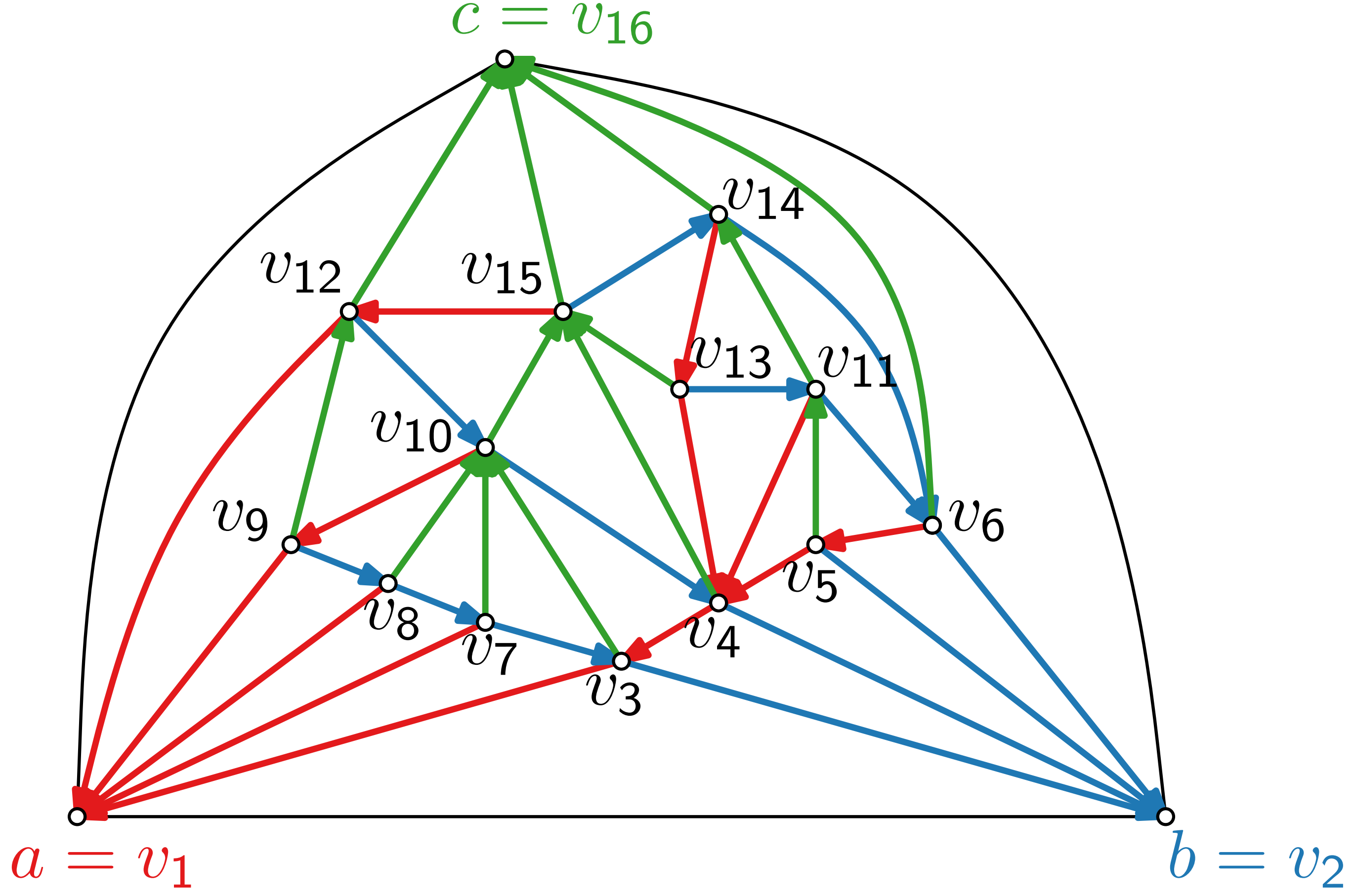
Schnyder Drawing⋆ – Example





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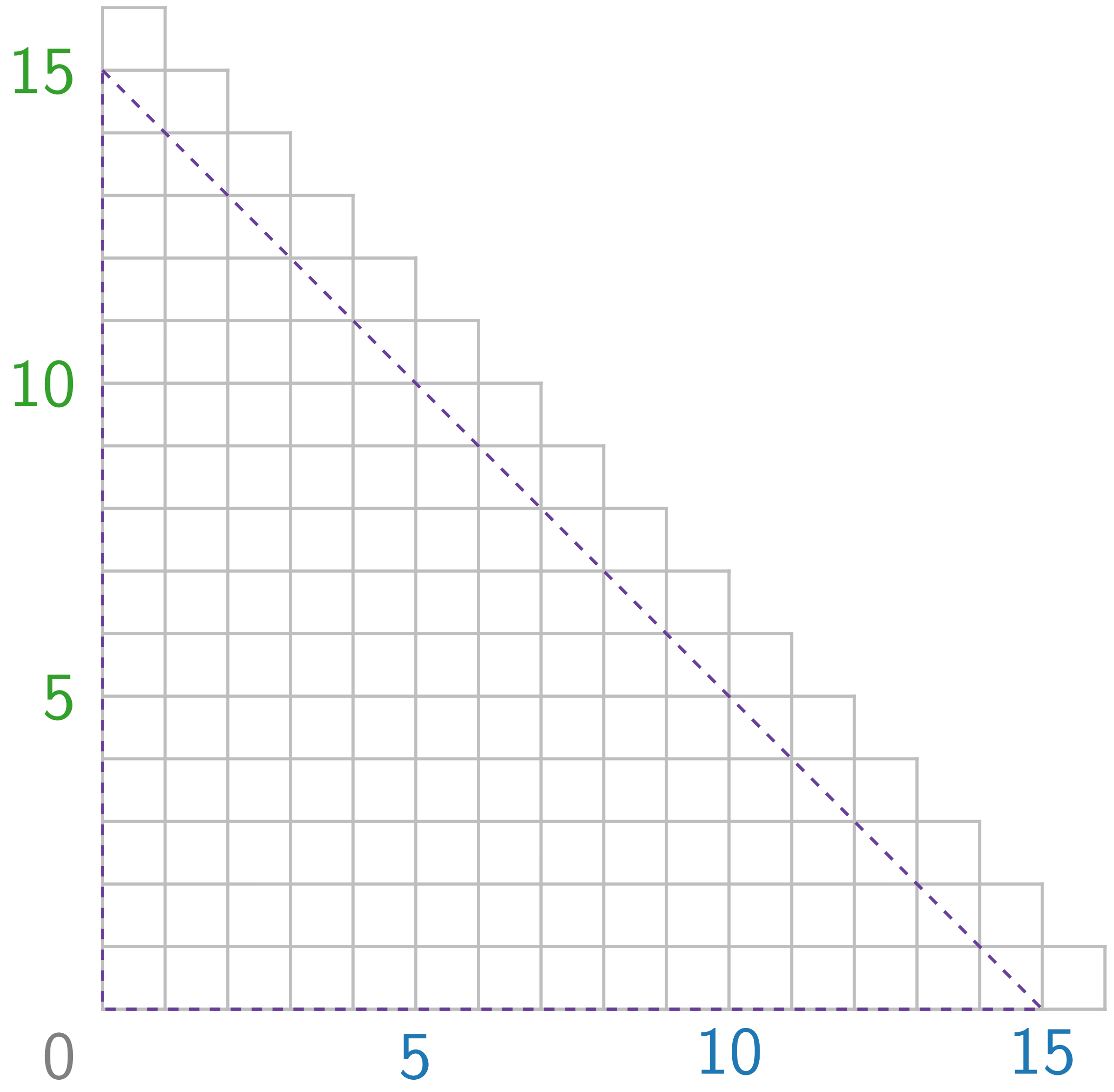

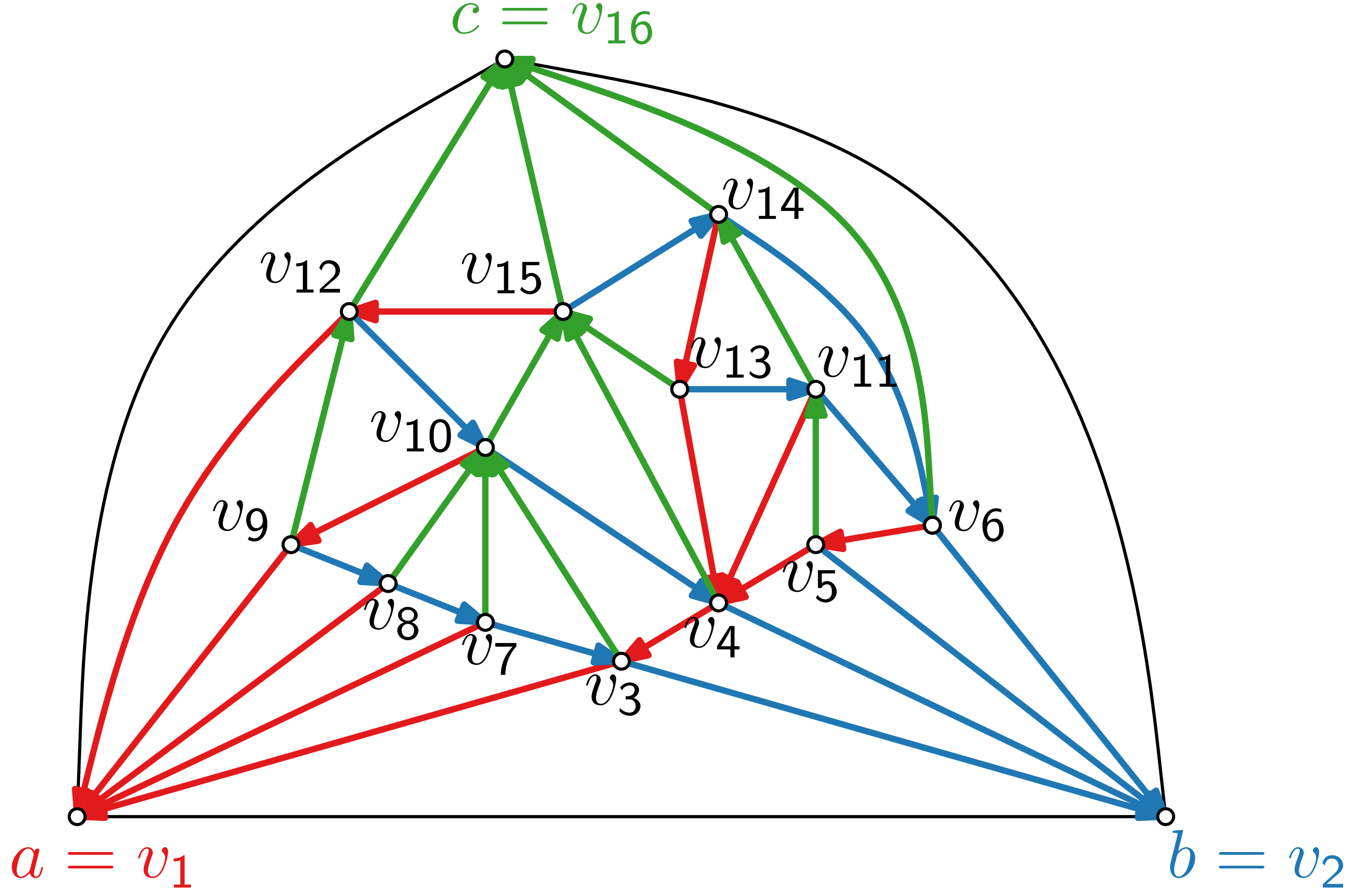

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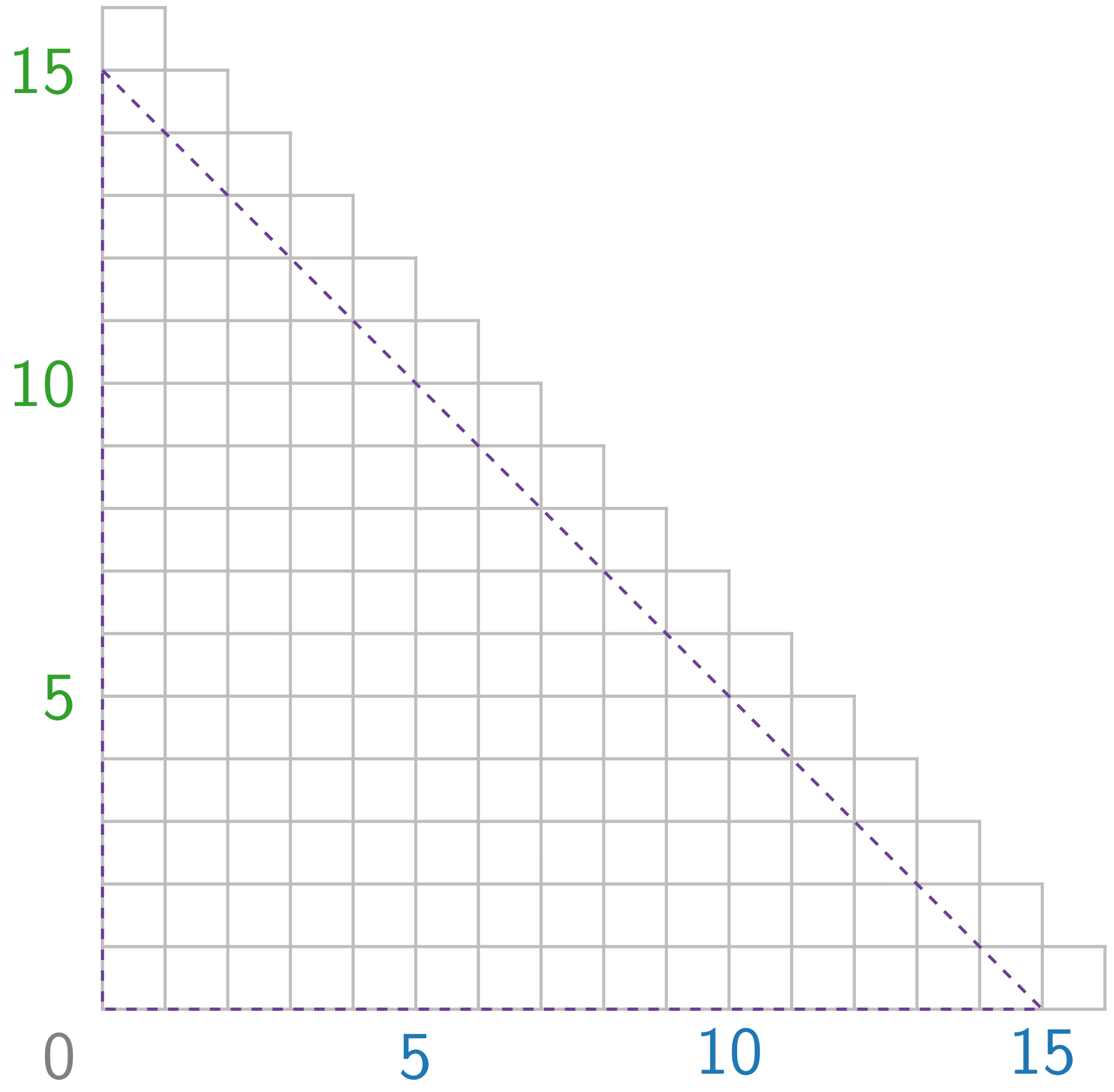

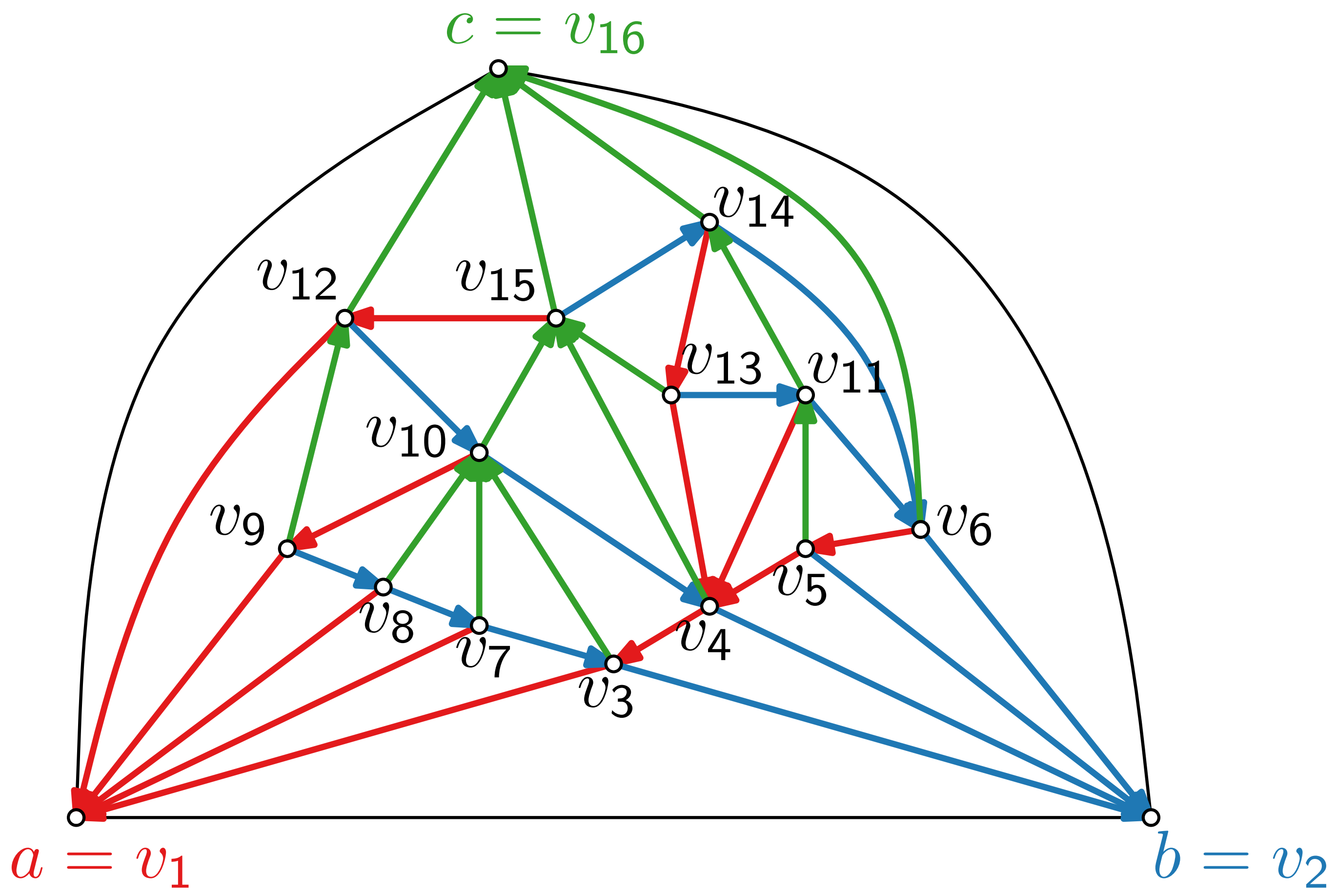

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$f(a) = (n-2, 1, 0)$

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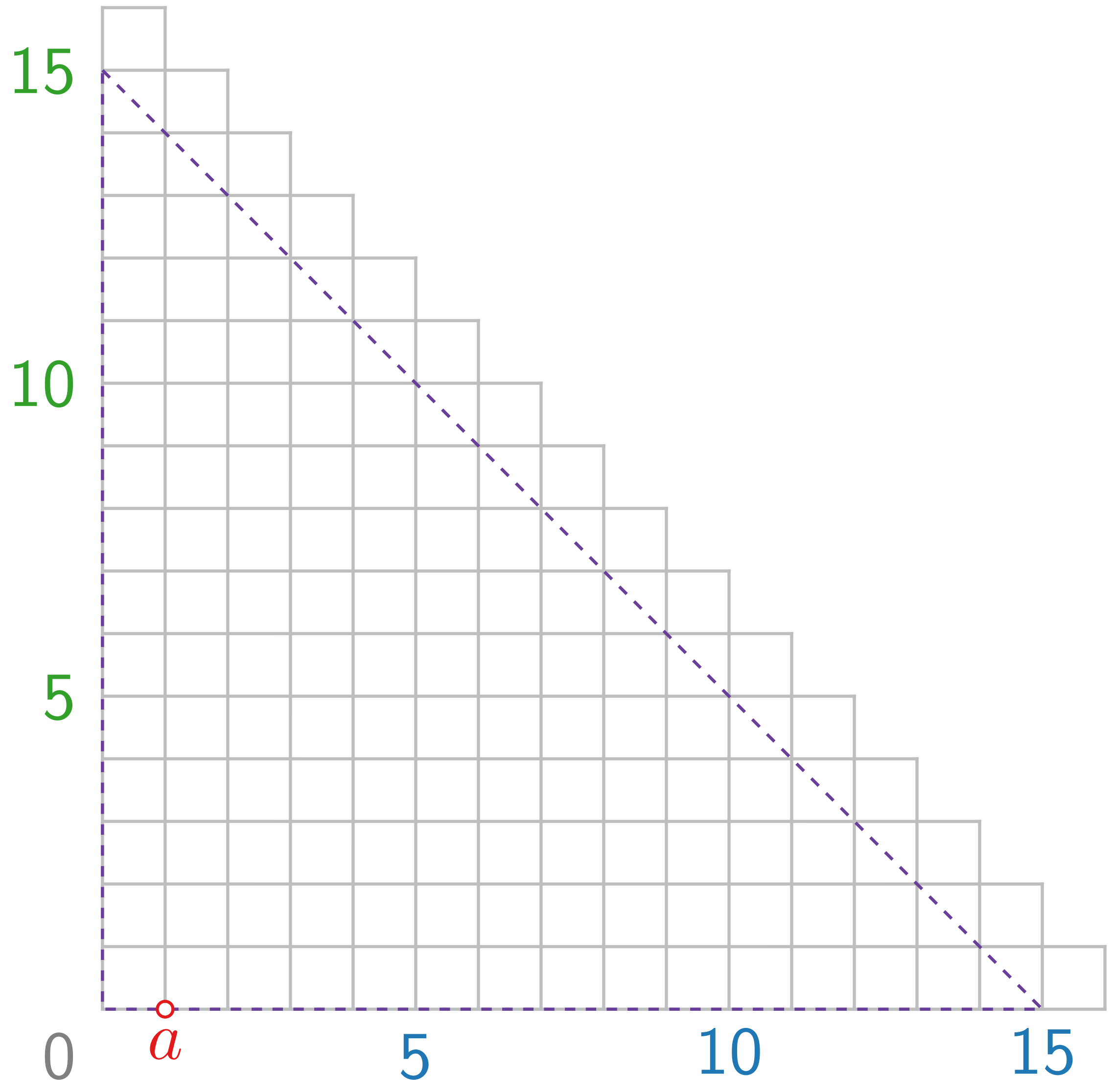

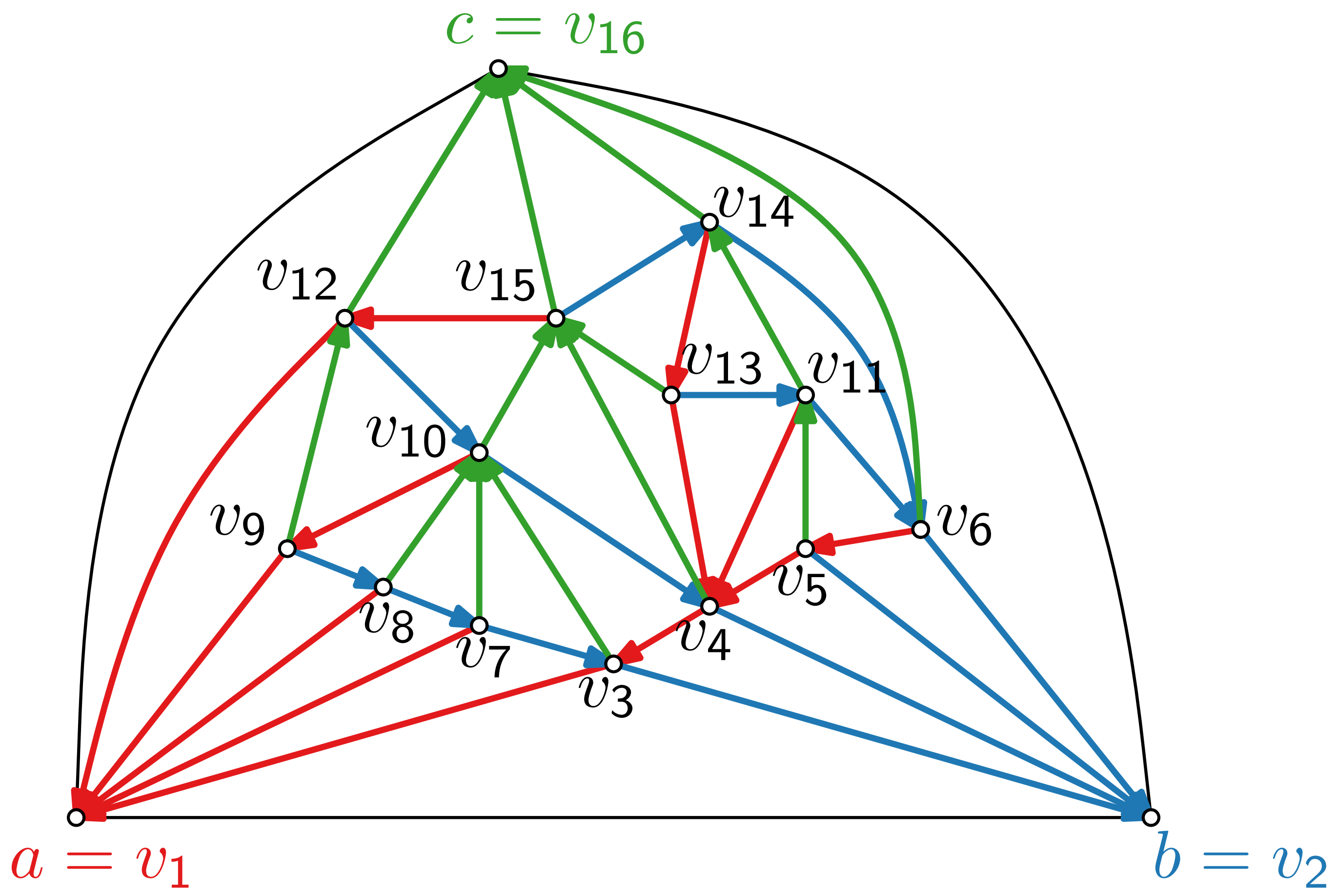

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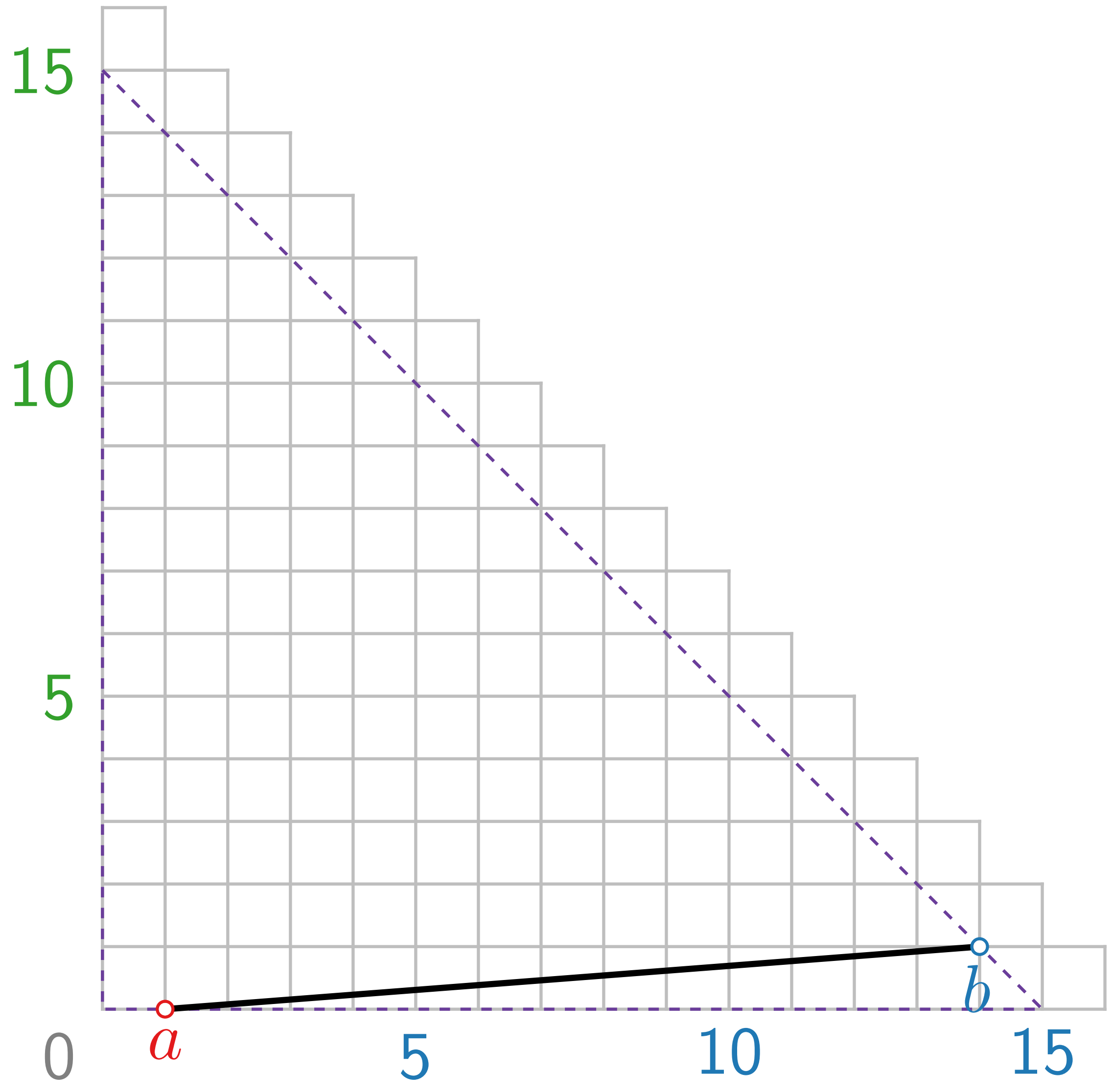

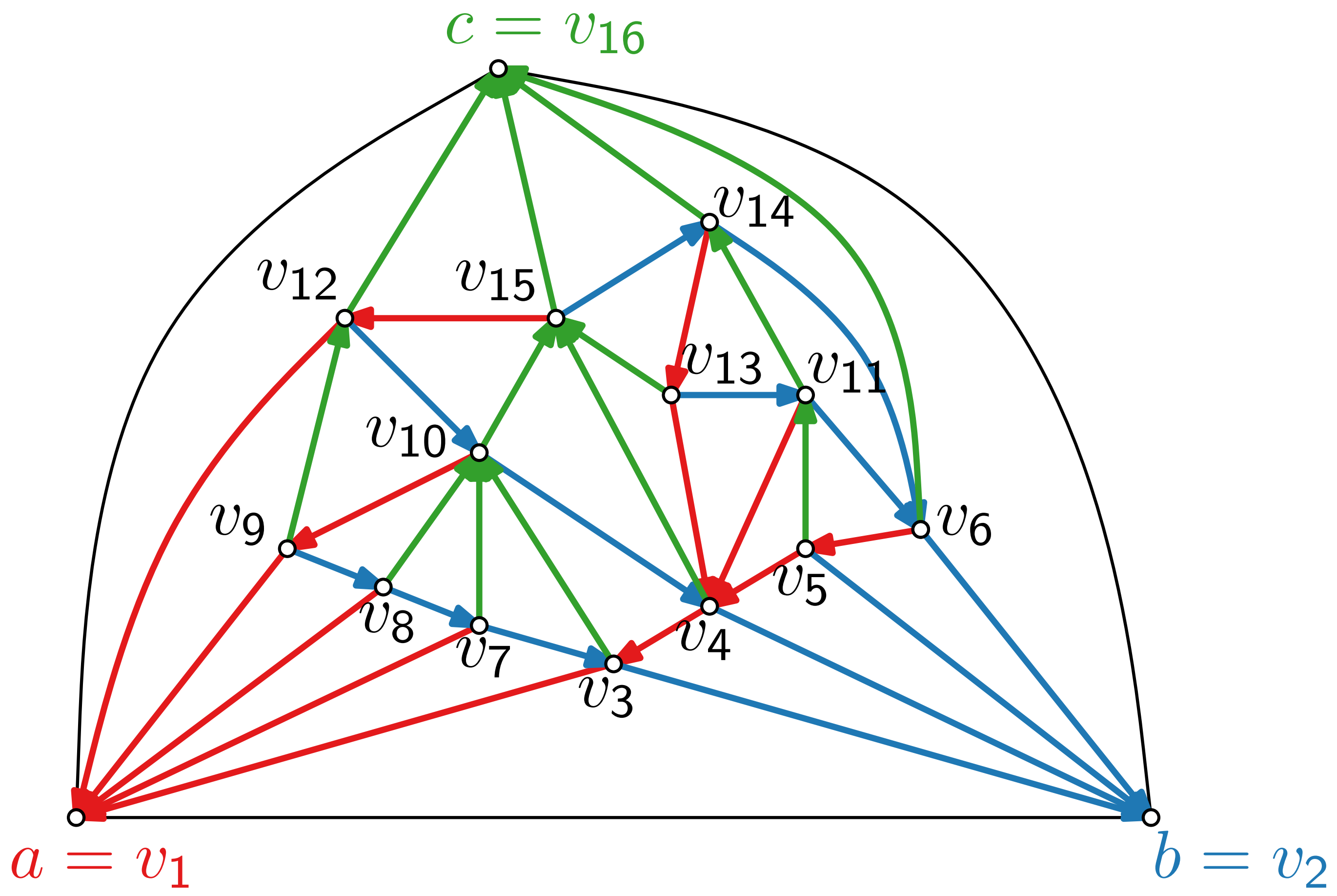

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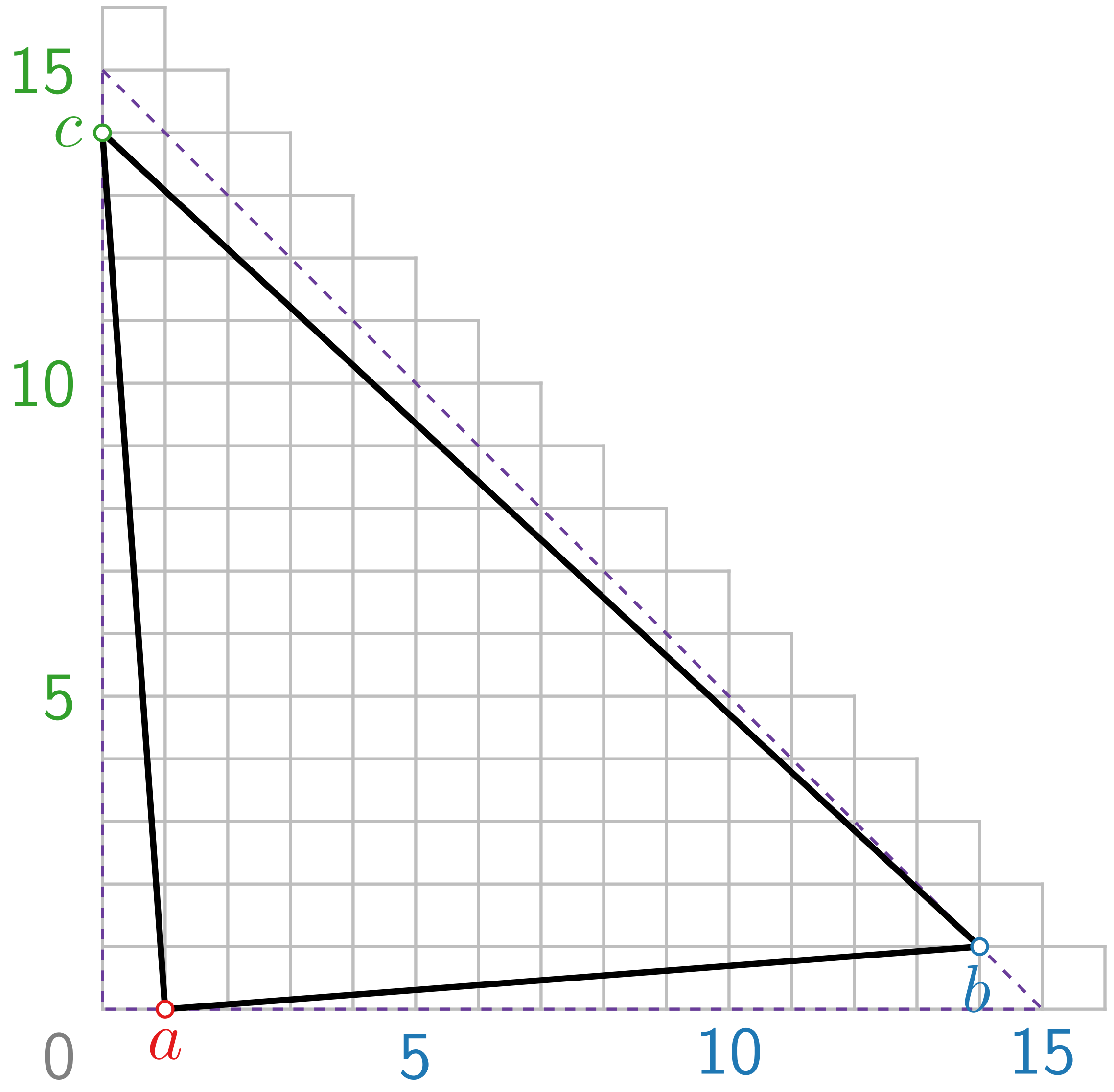

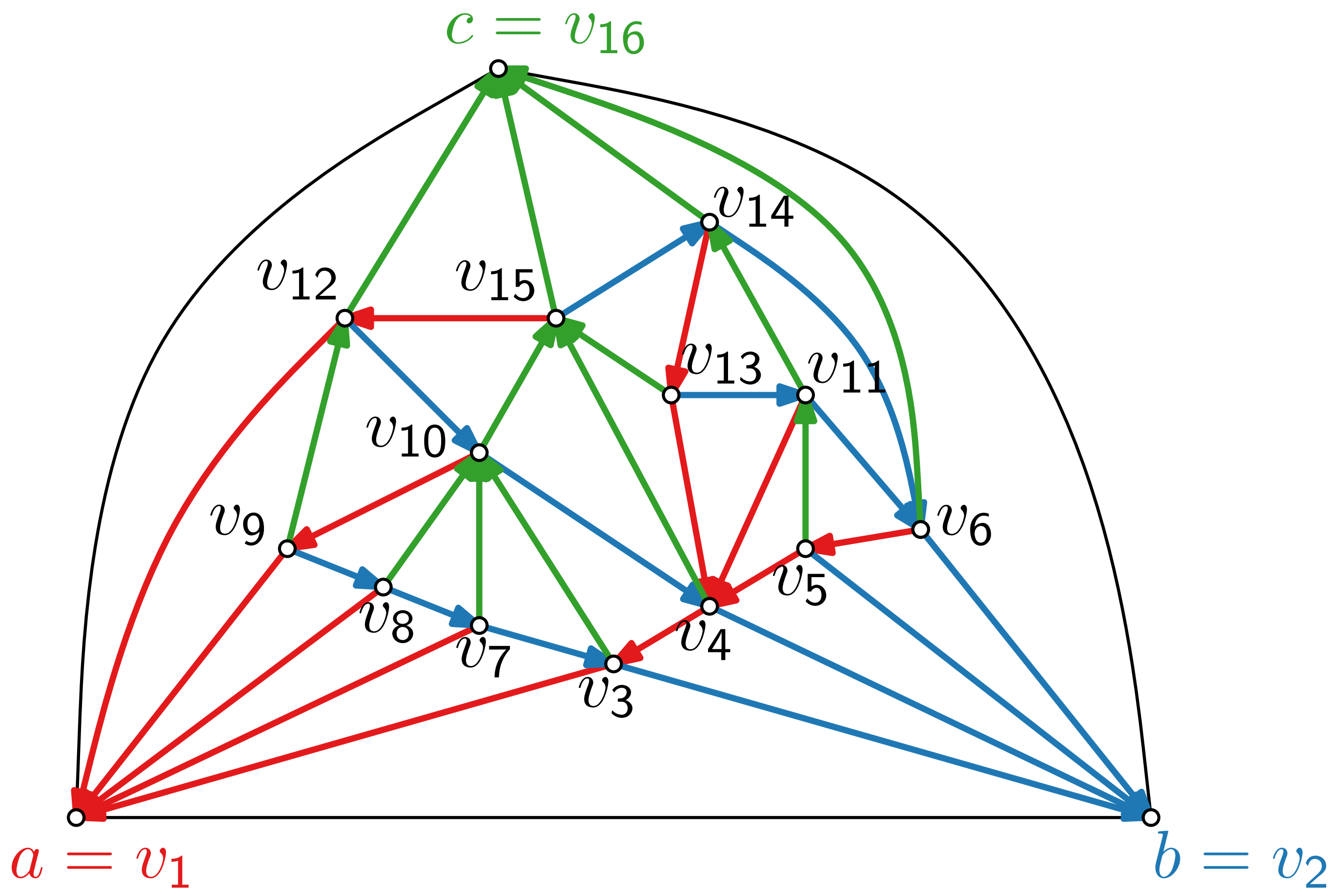

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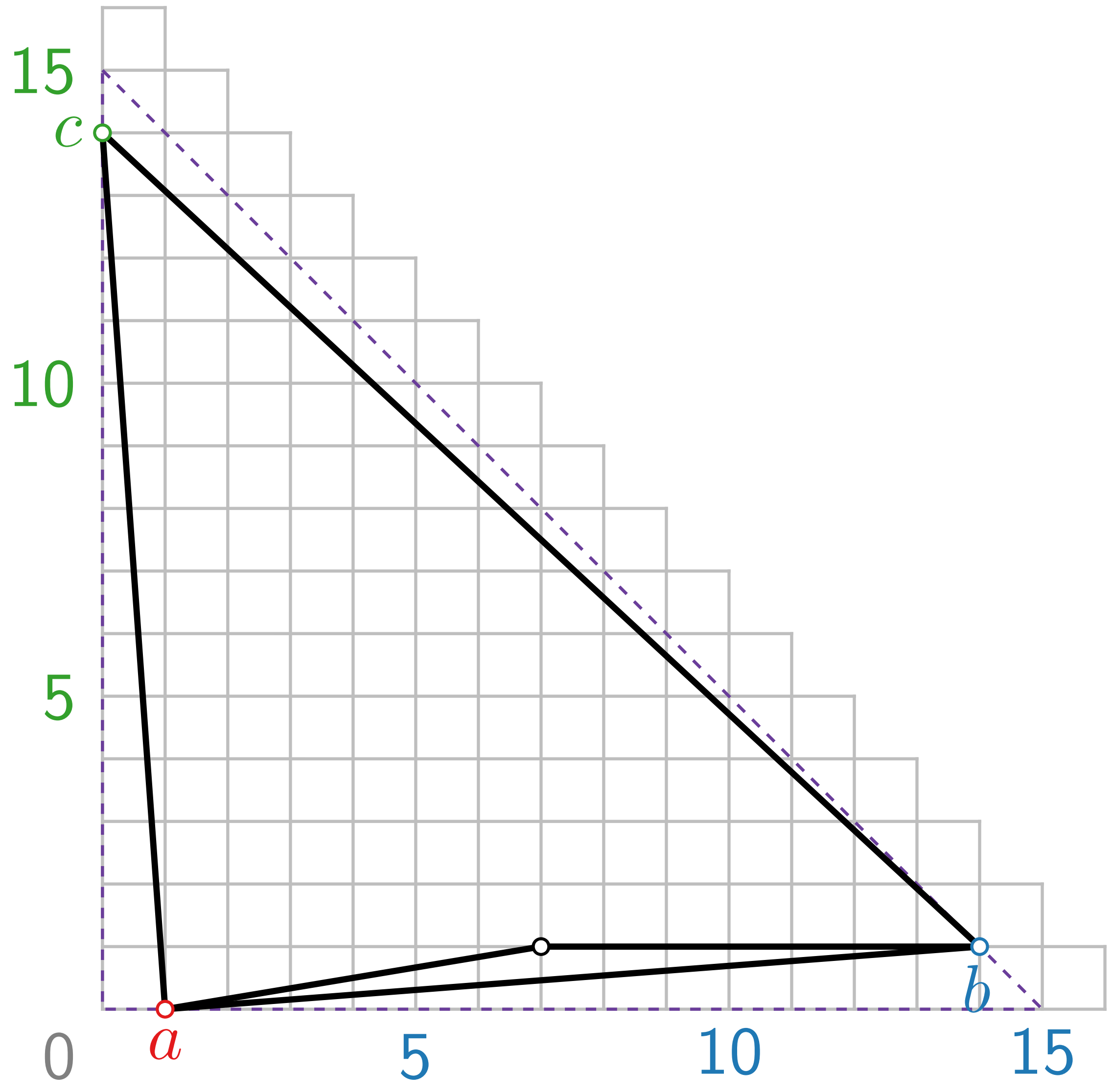

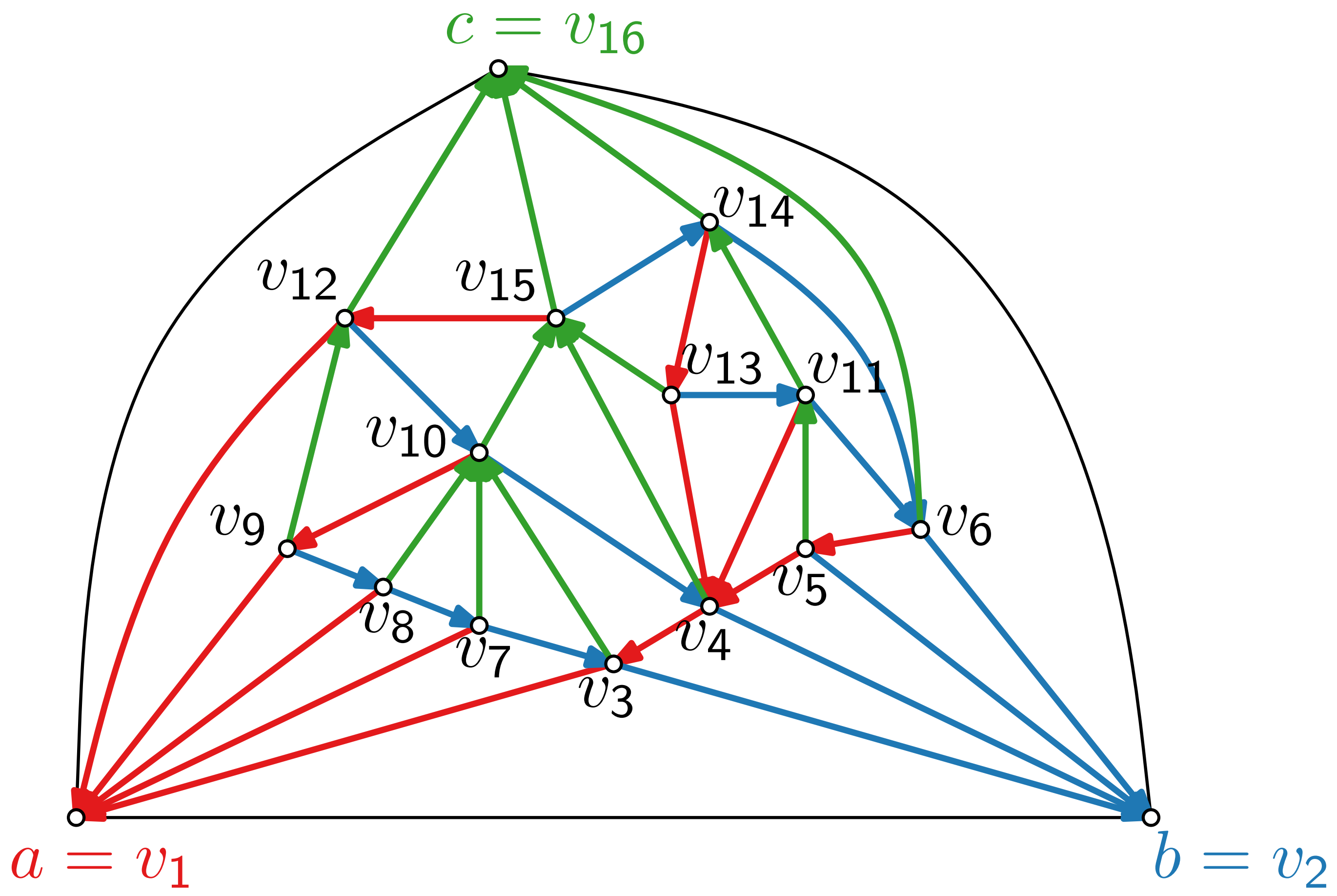

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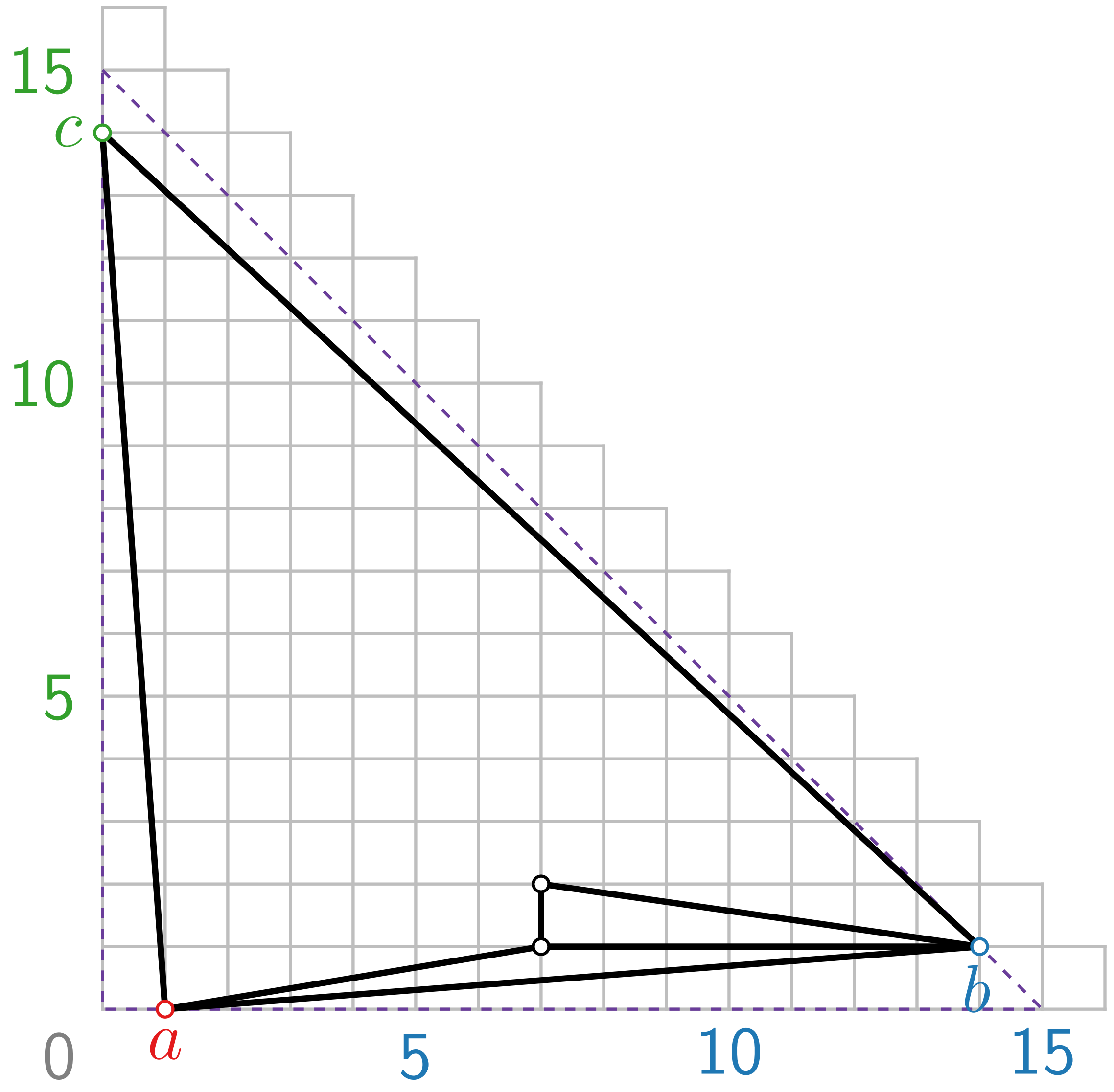

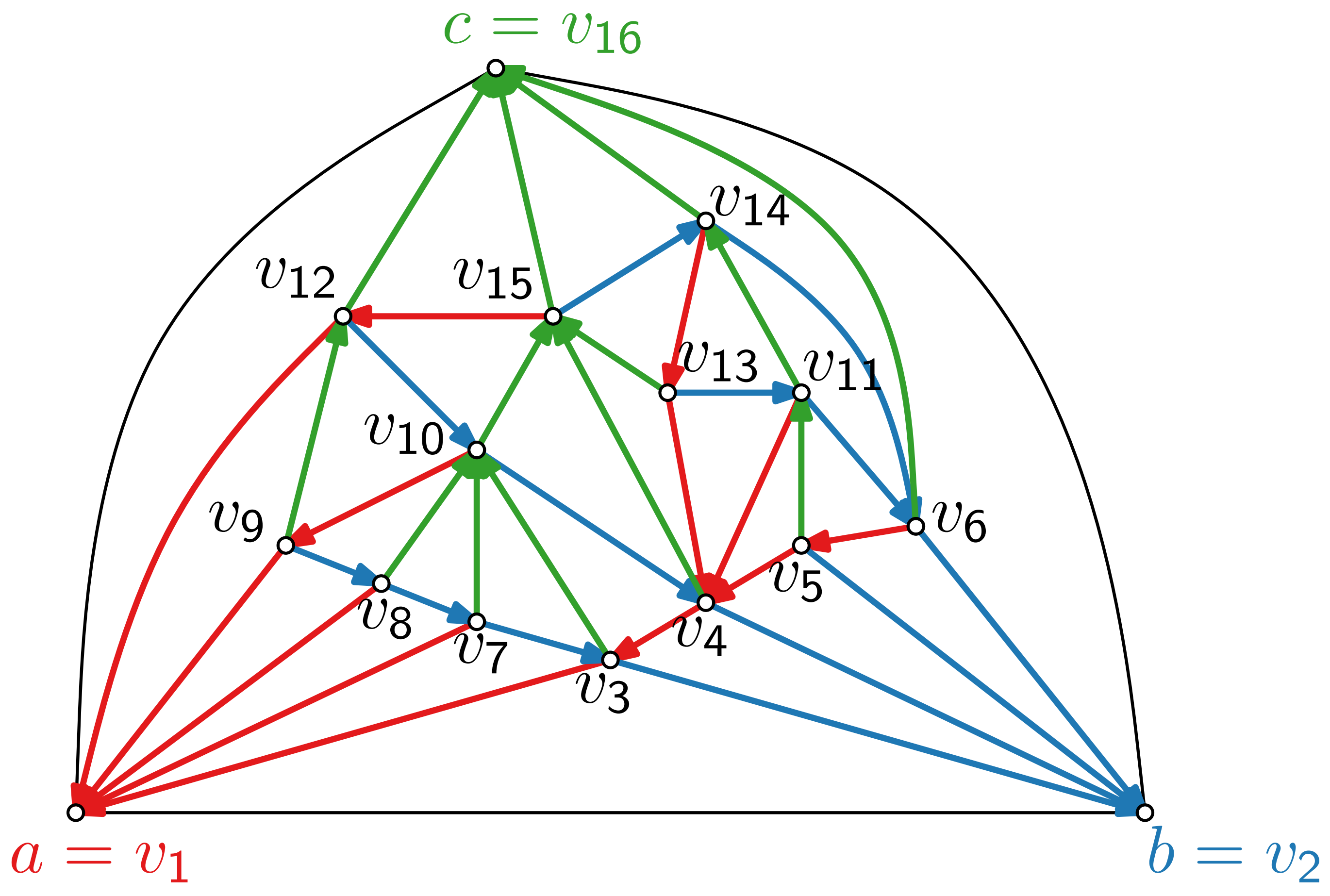

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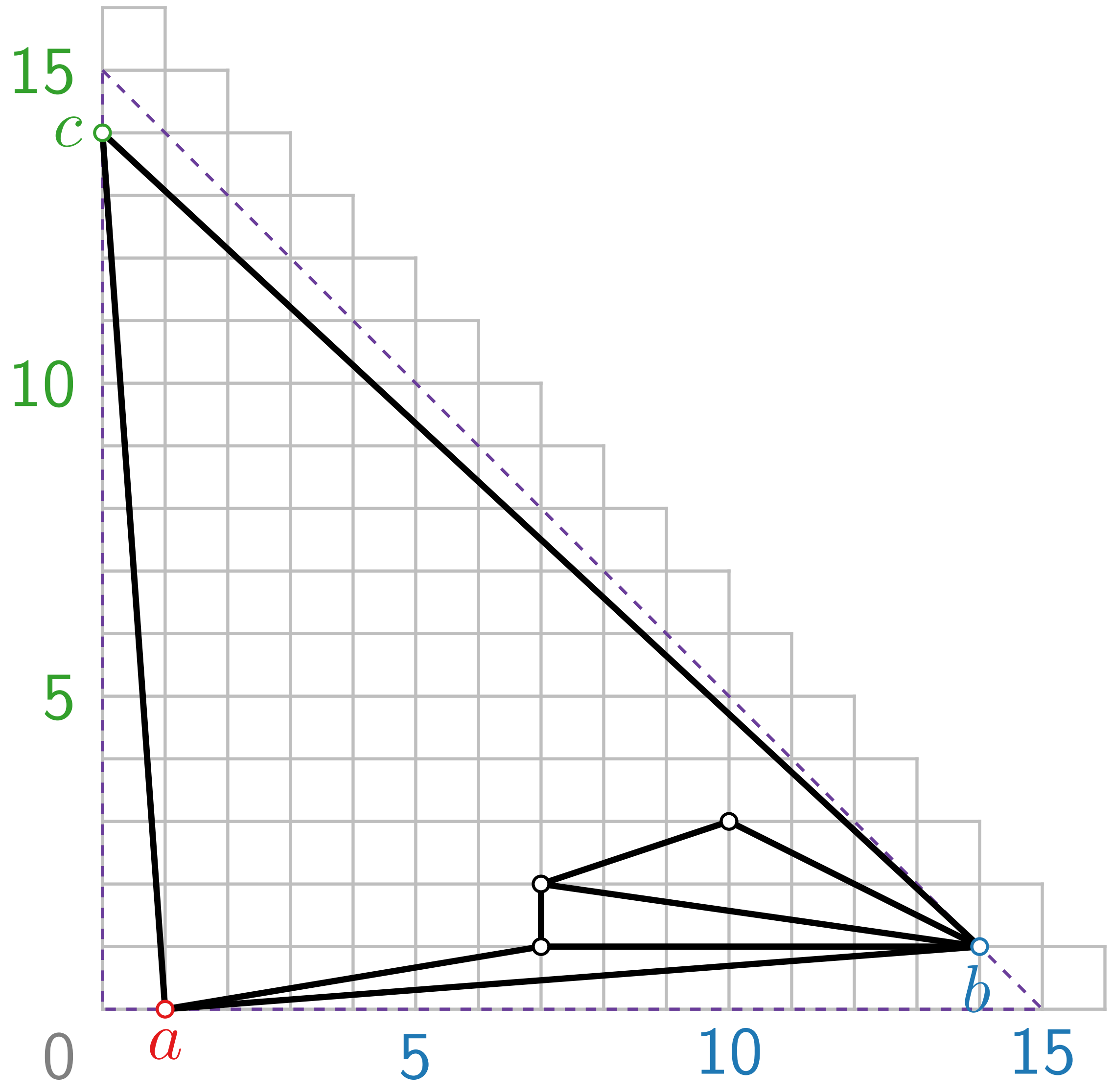

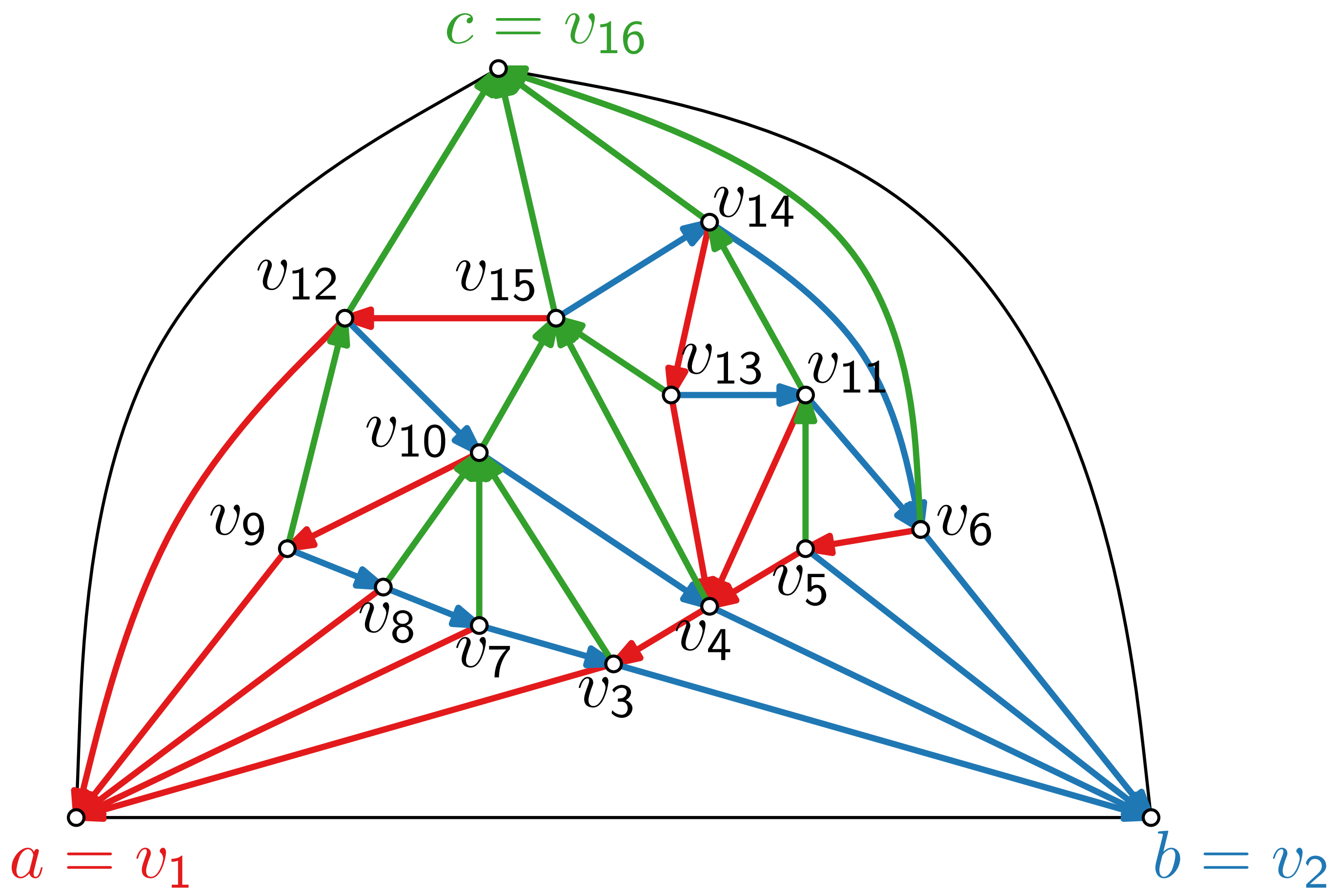

$n = 16,\ n - 2 = 14$

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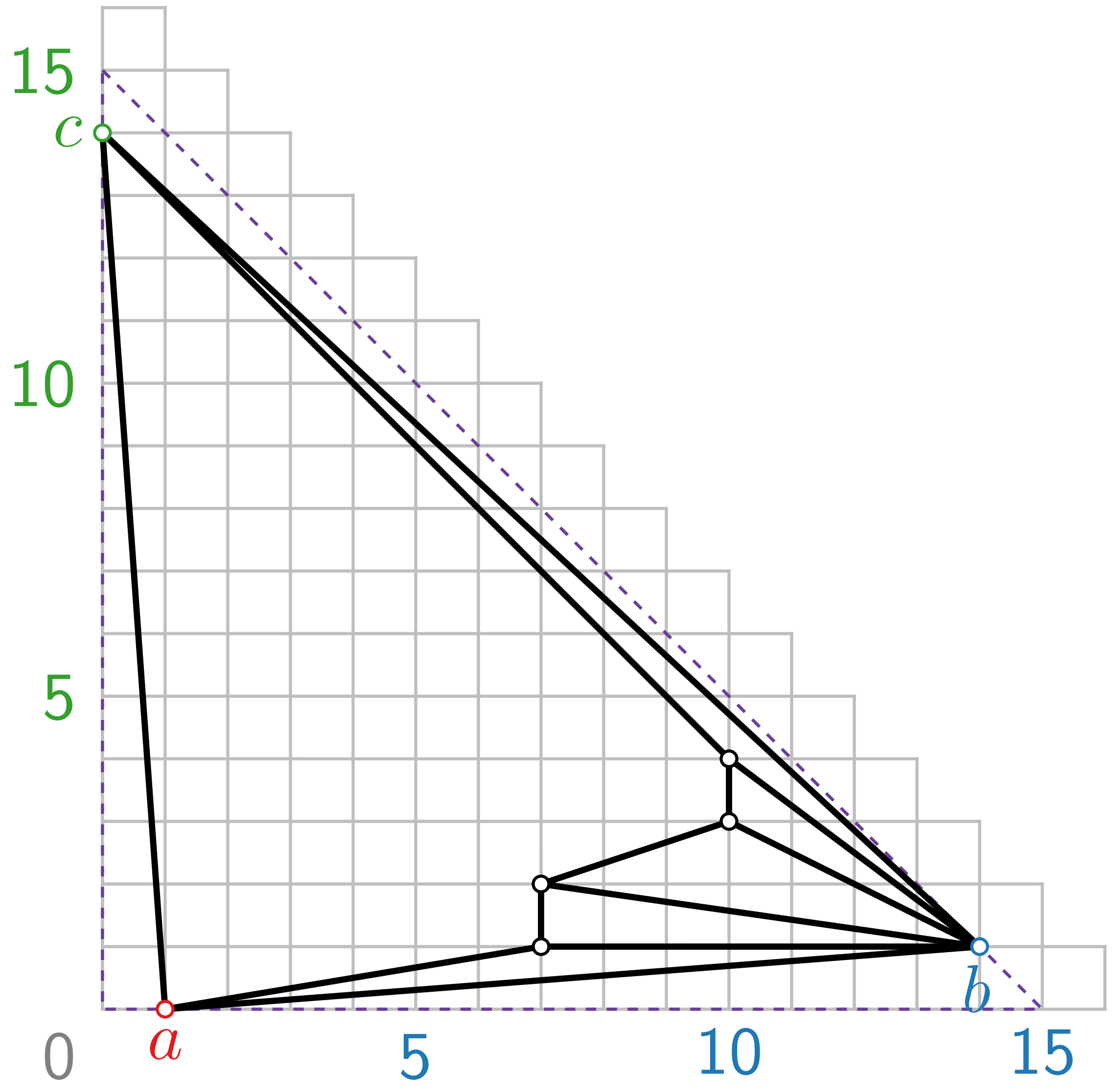

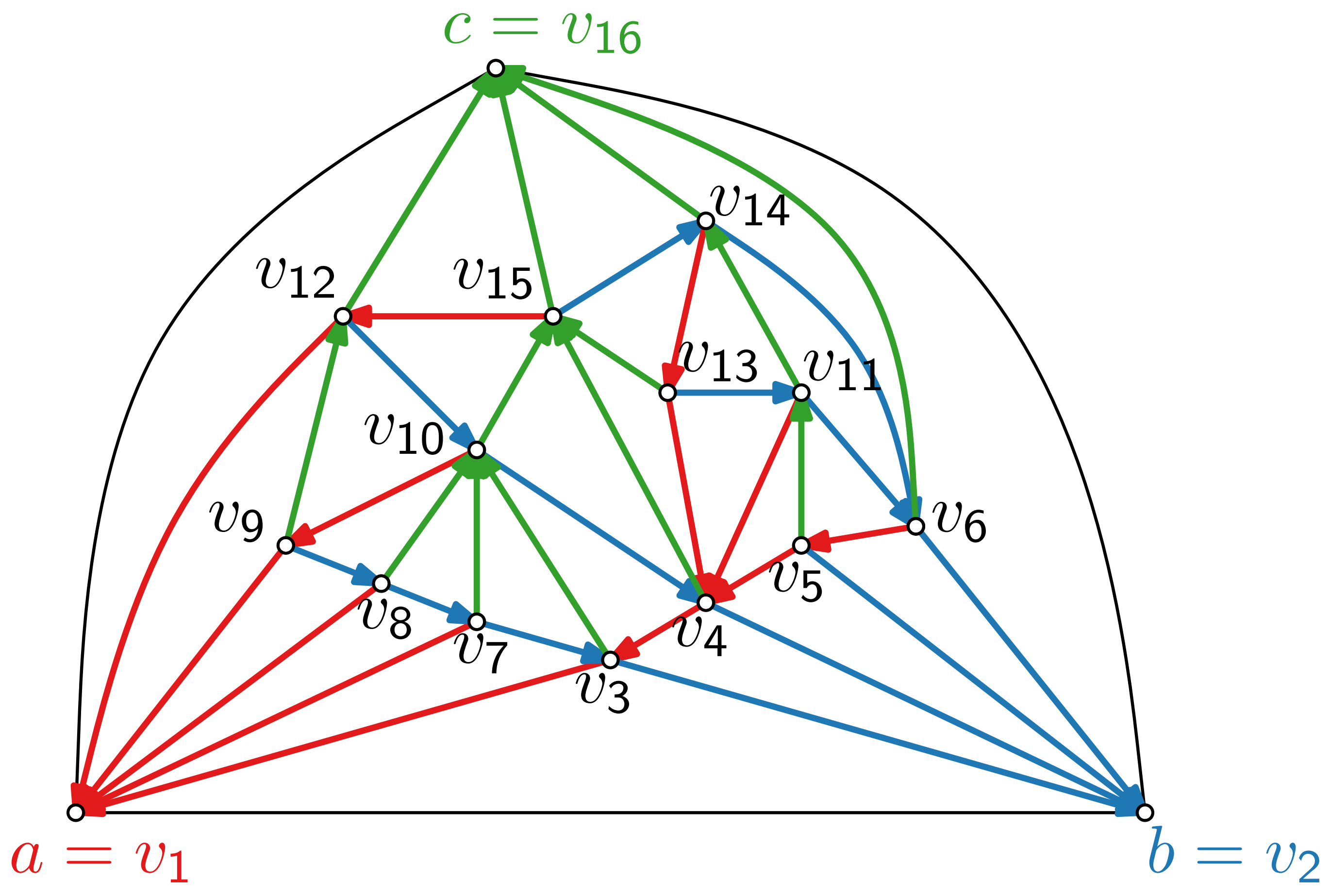

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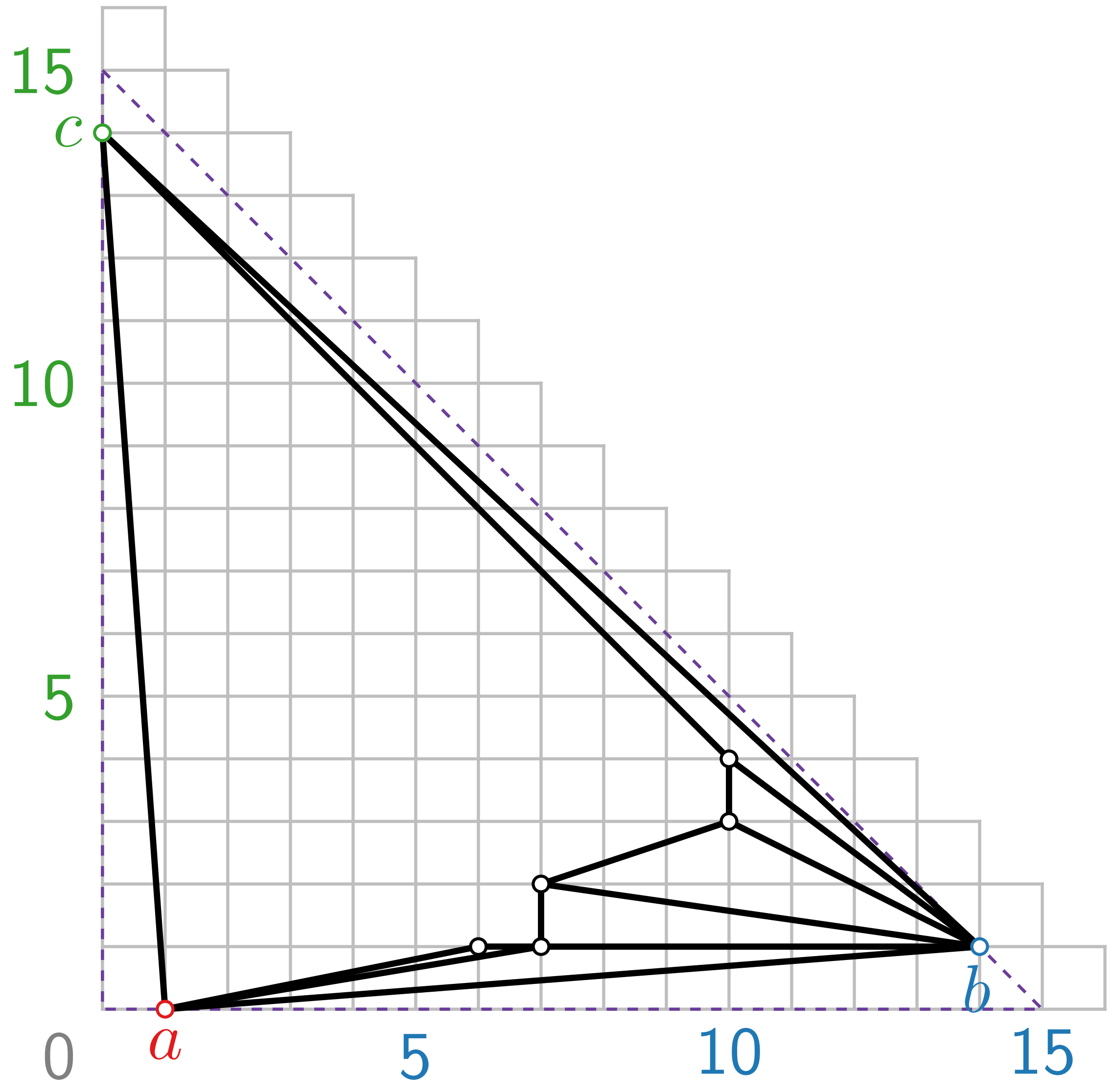

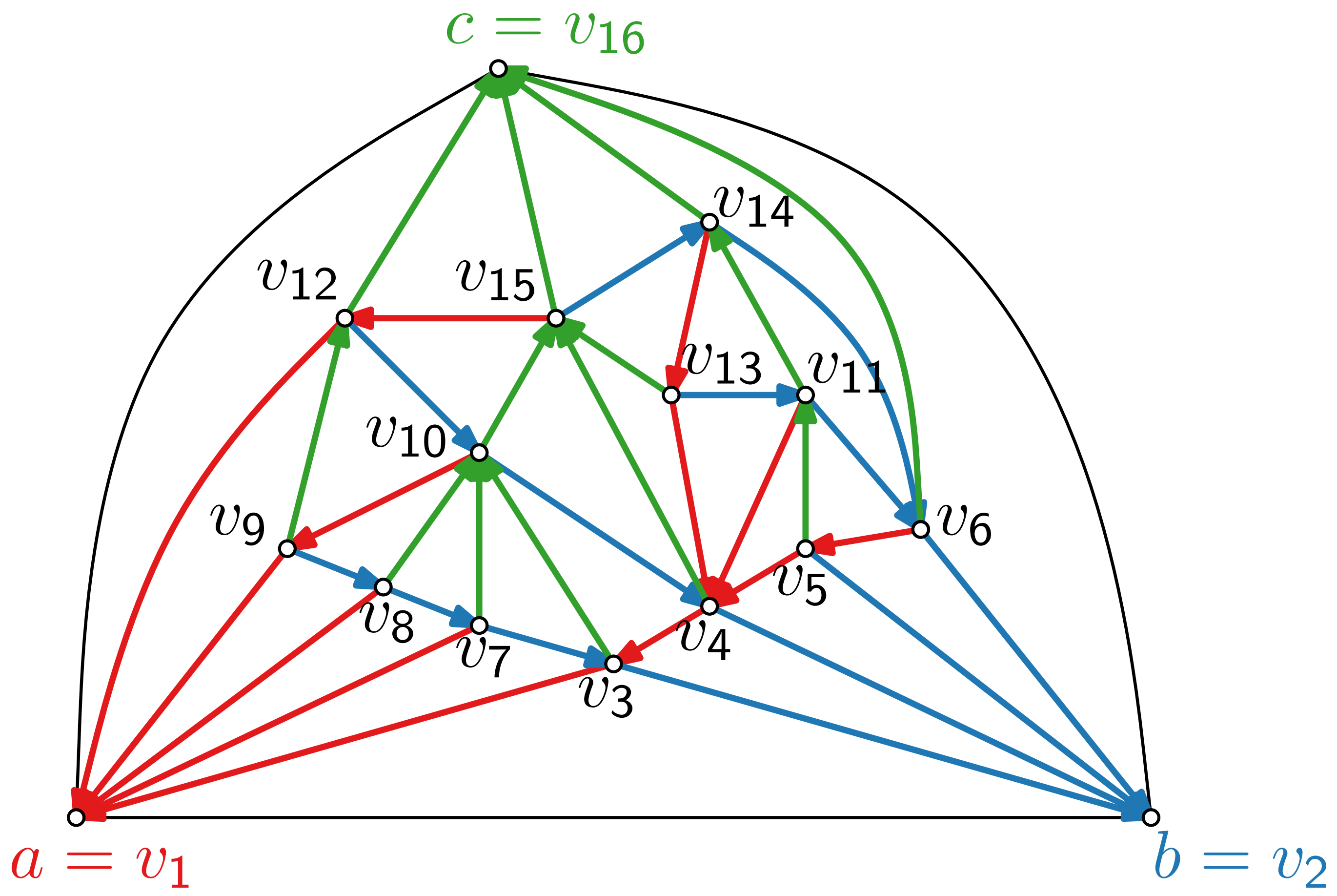

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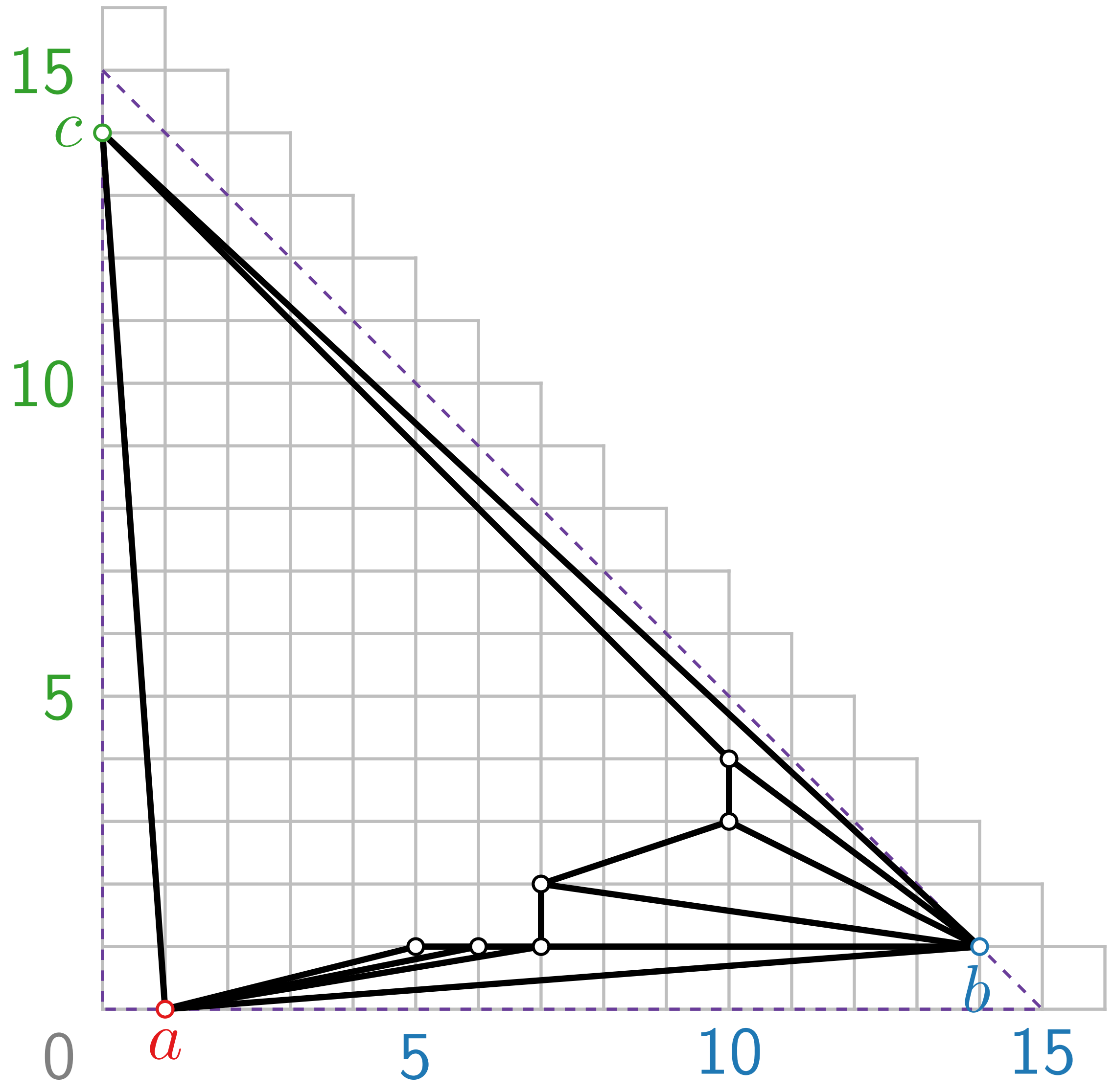

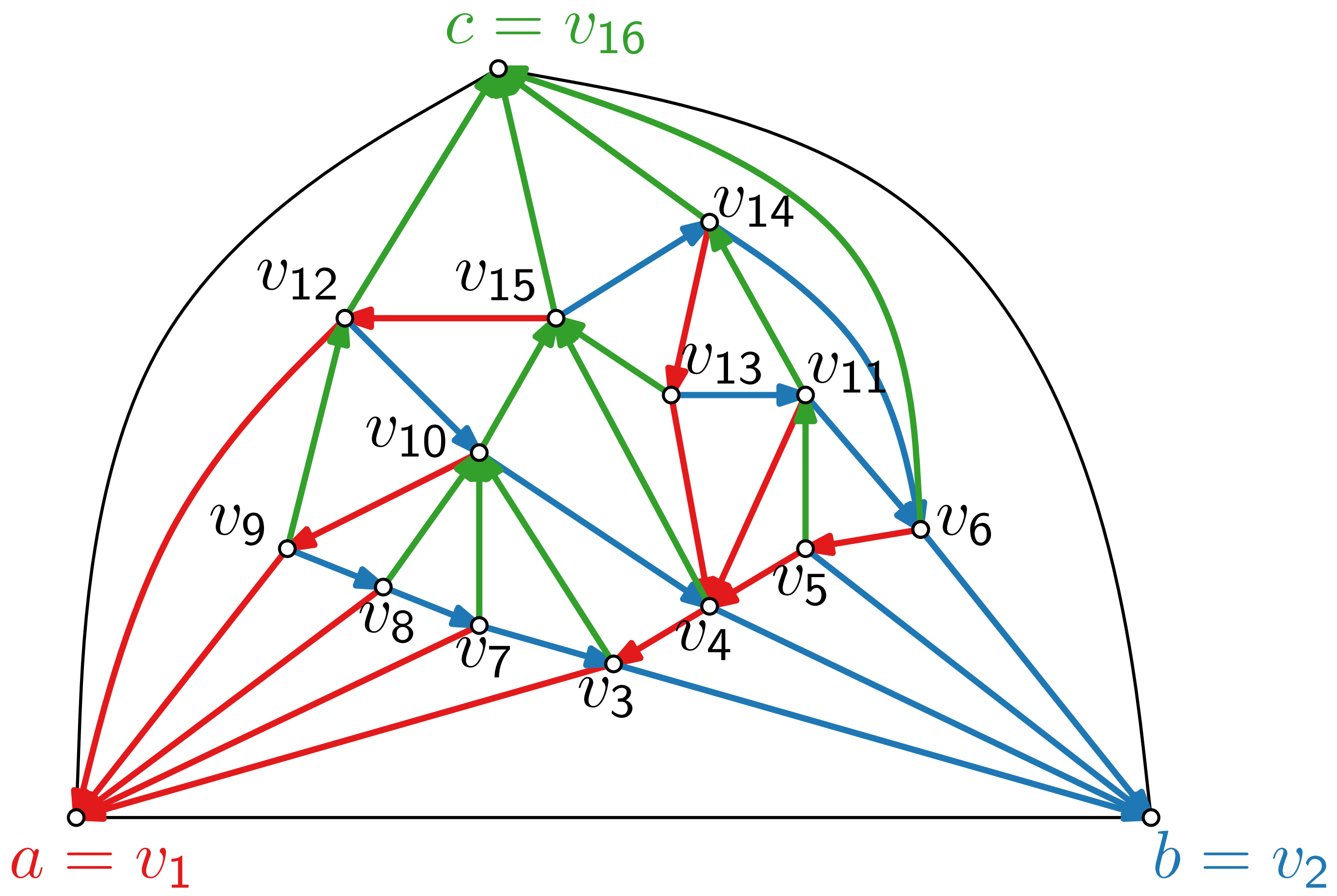

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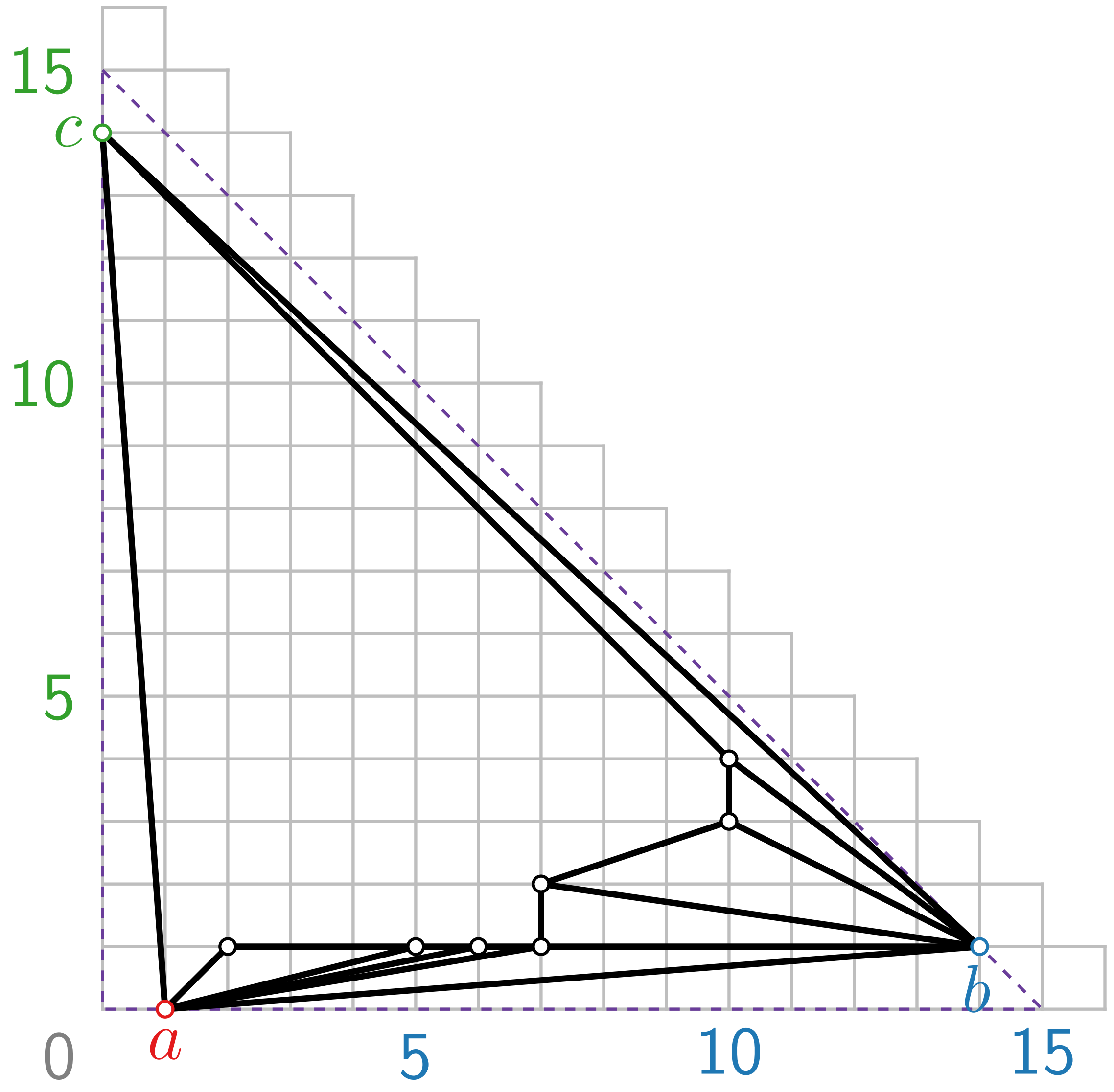

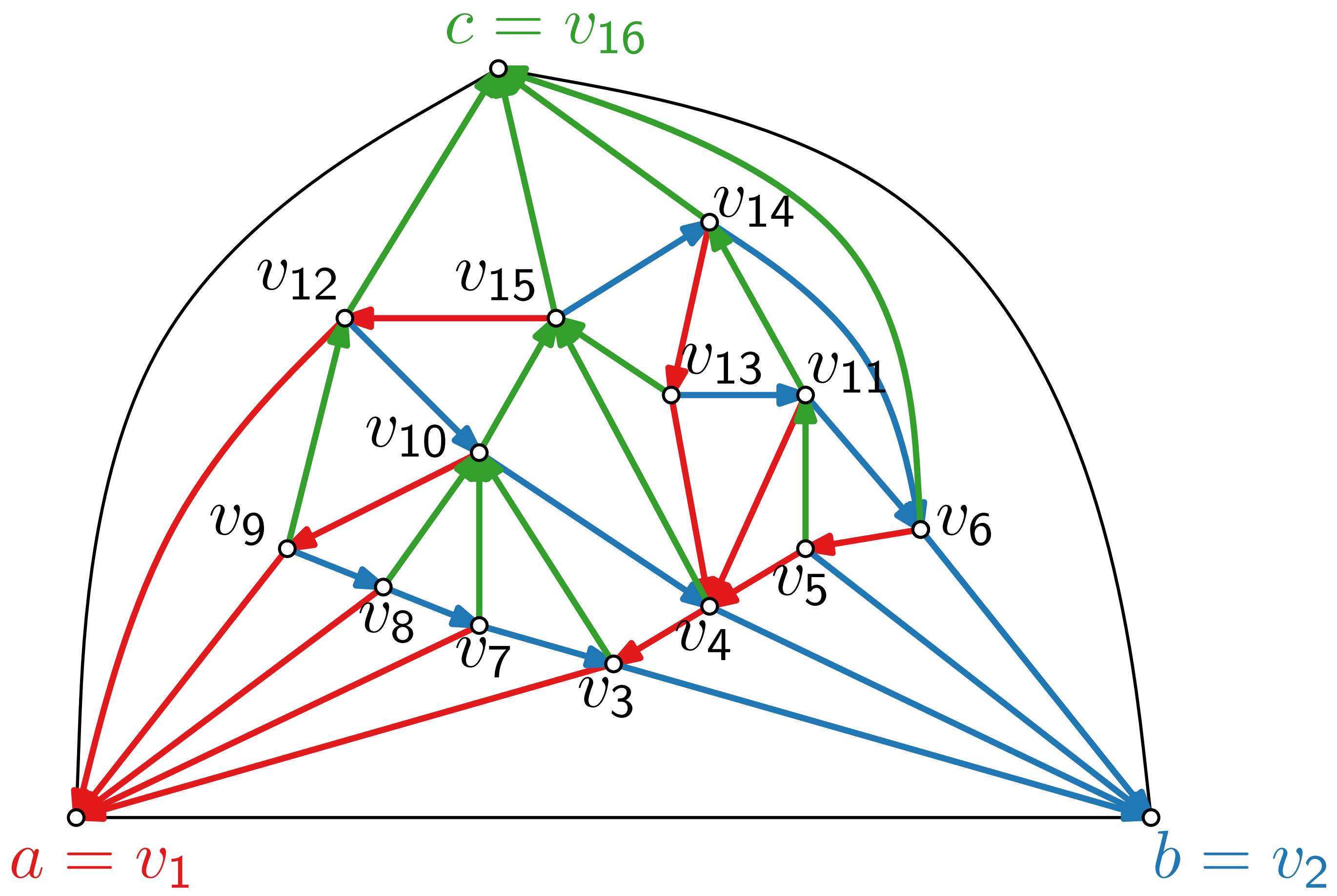

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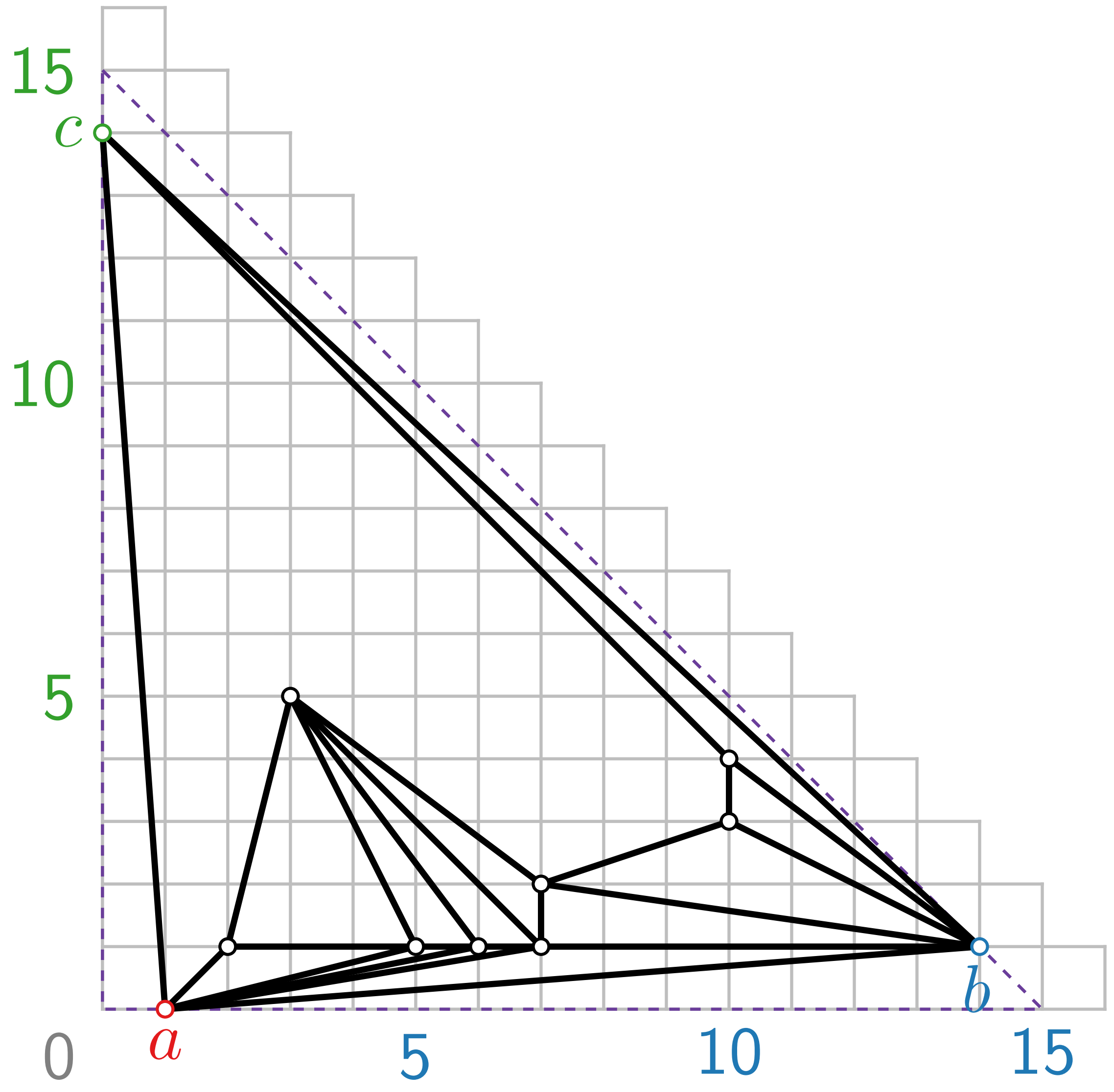

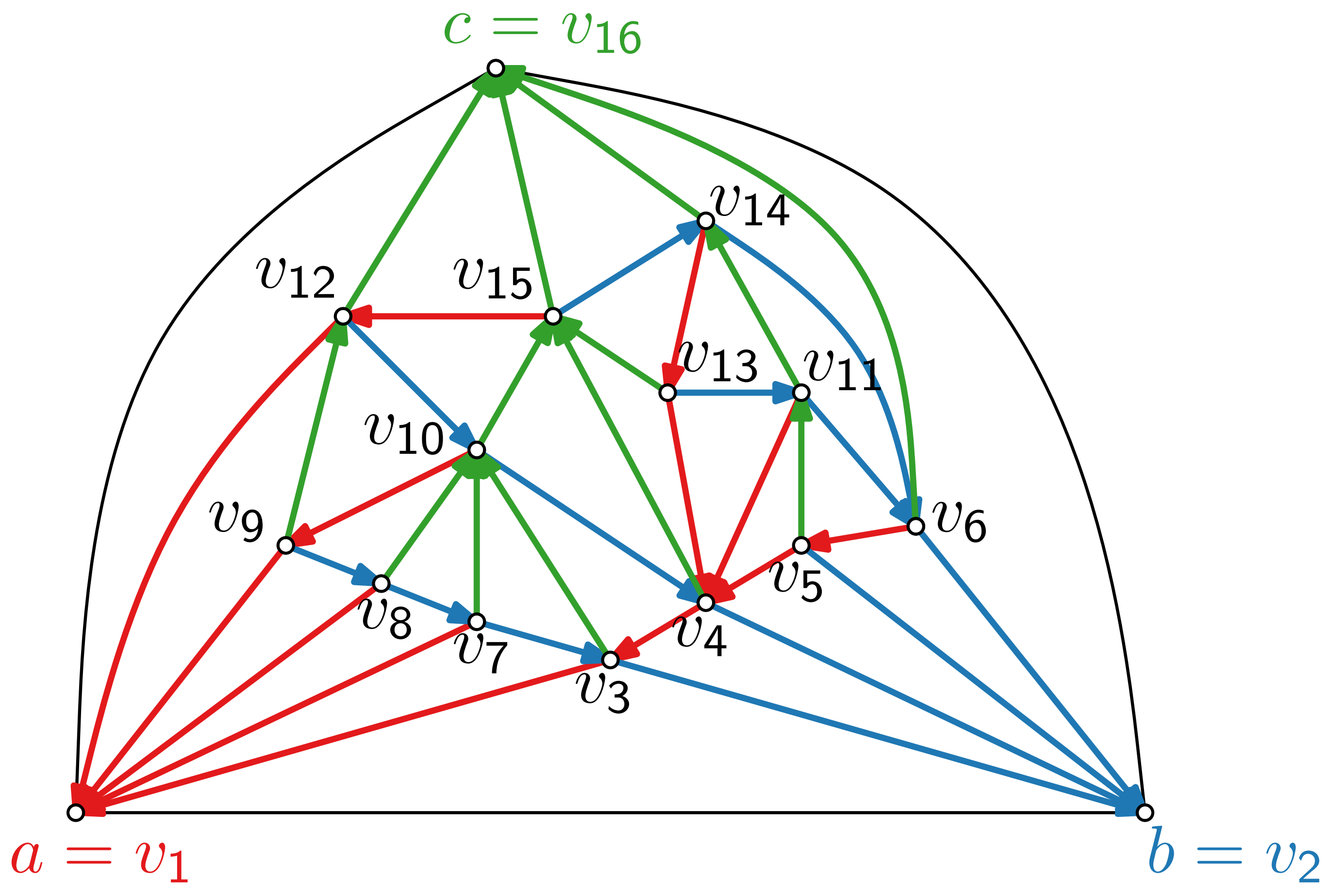

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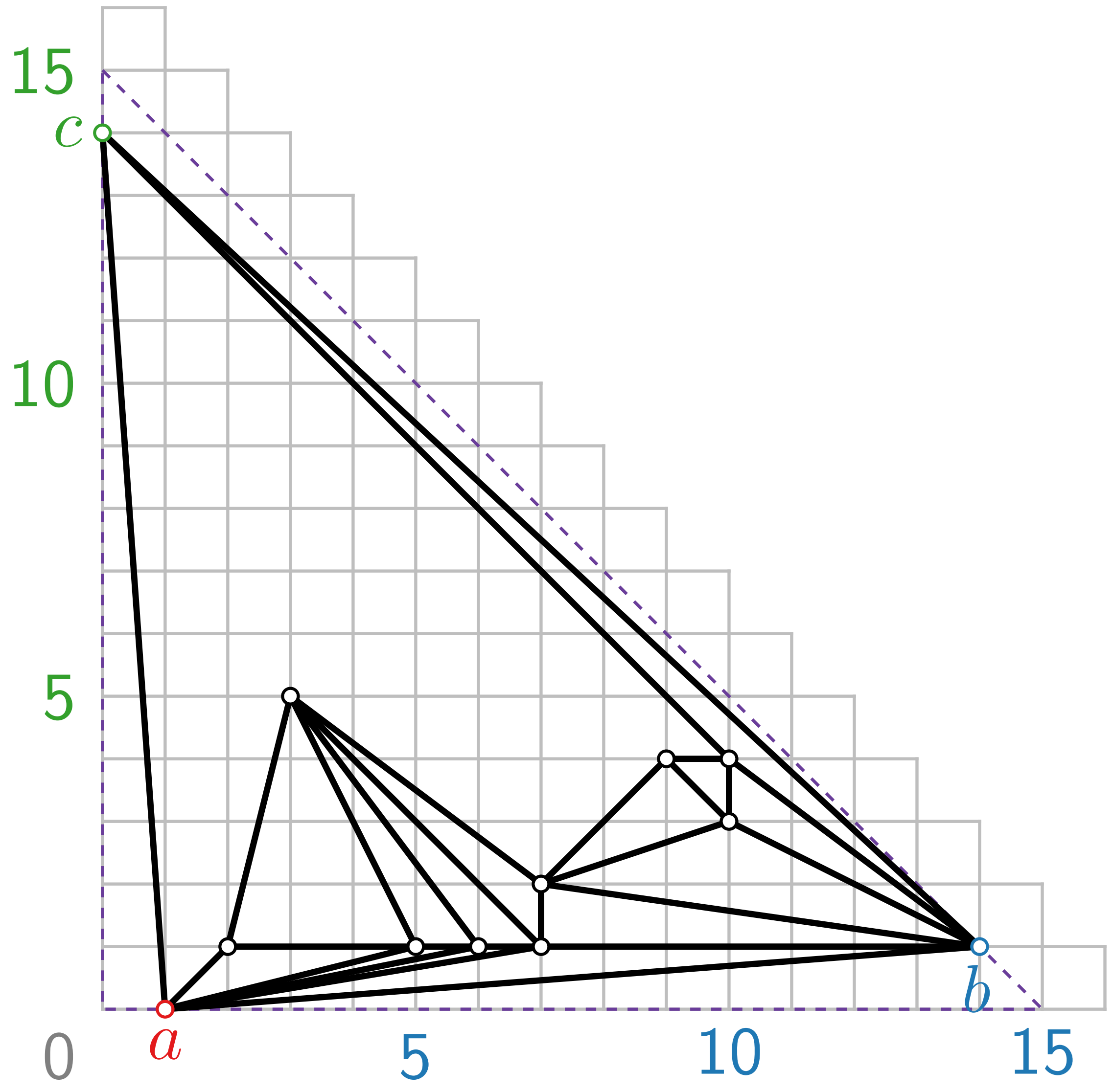

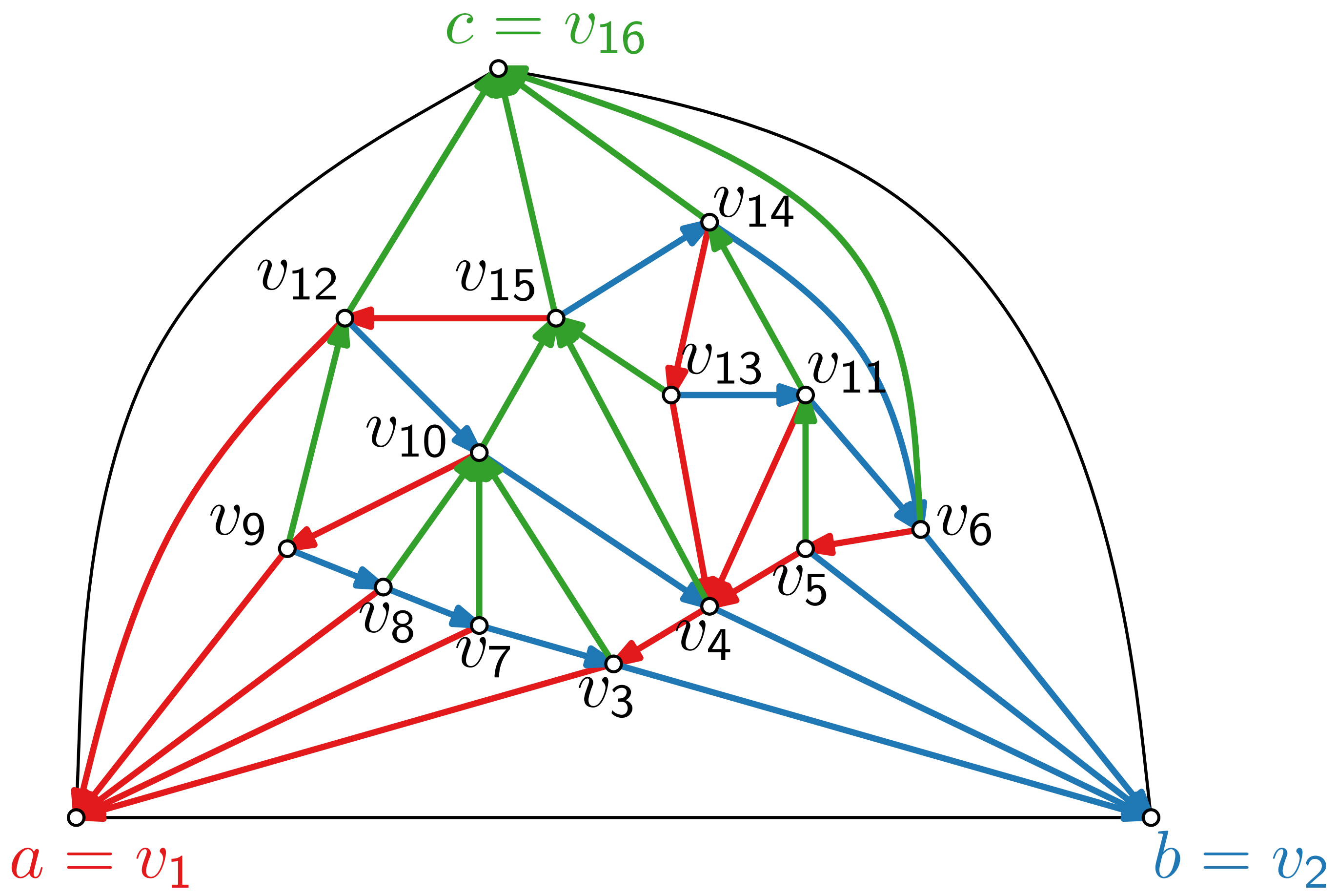

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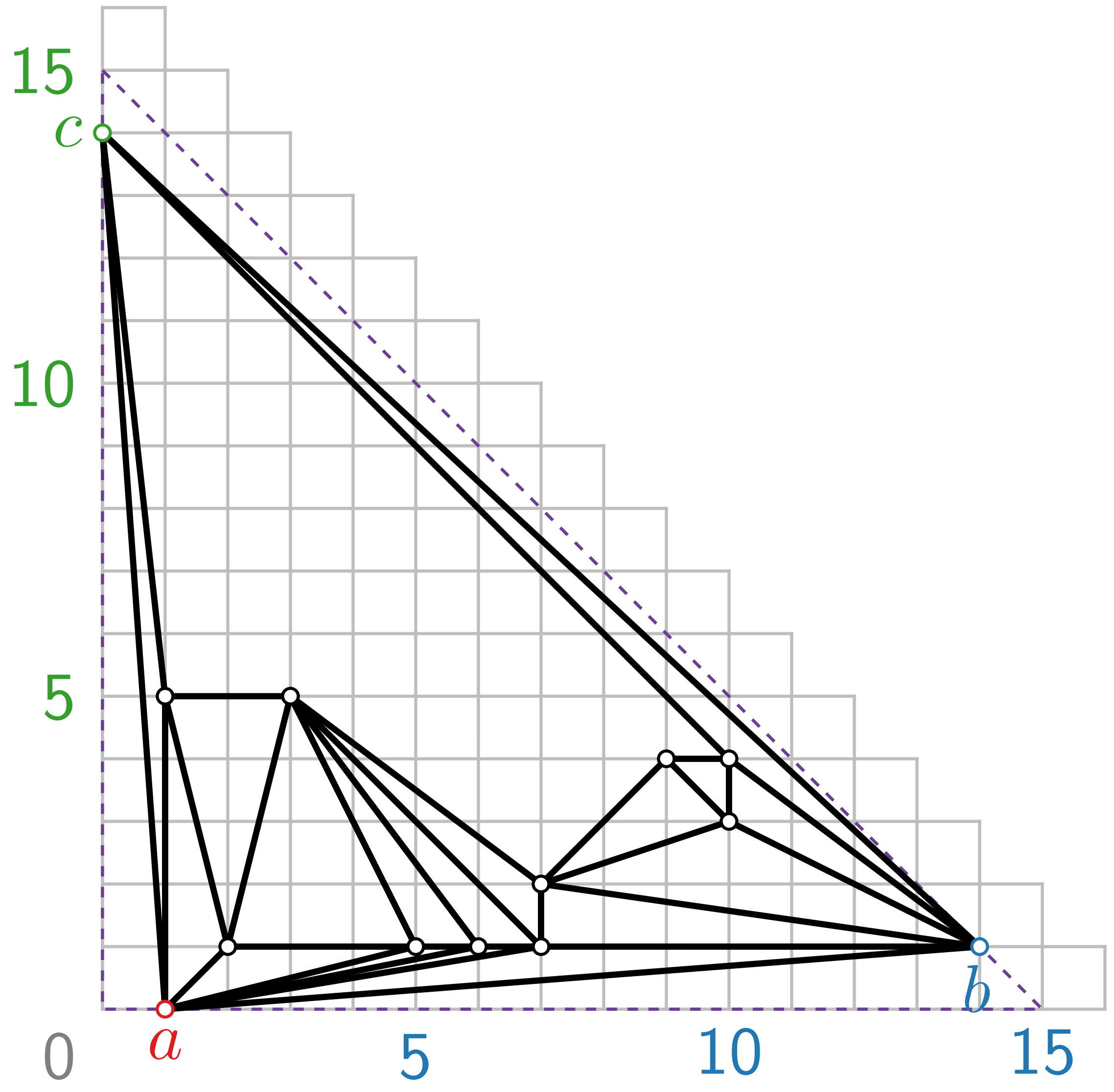

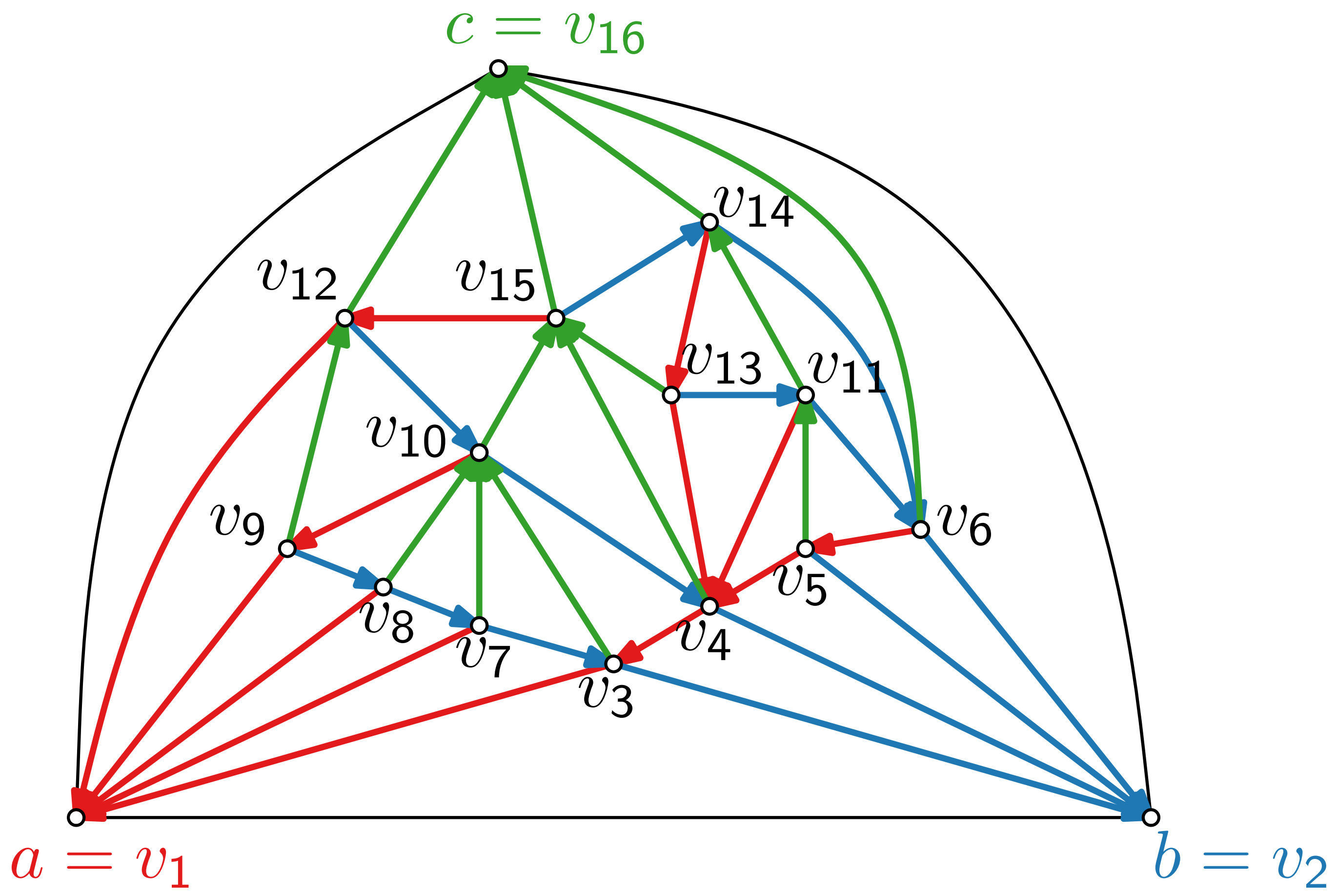

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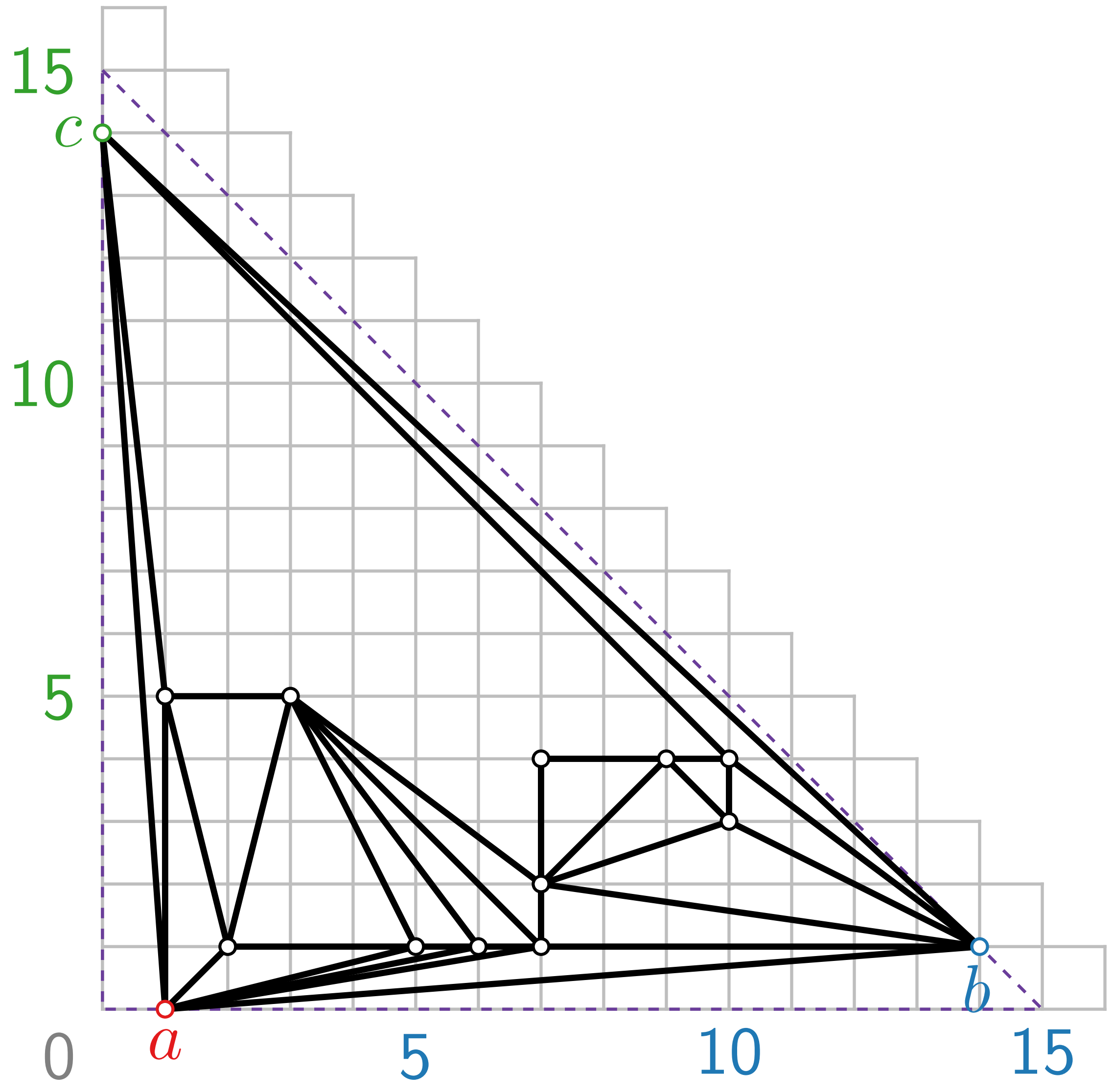

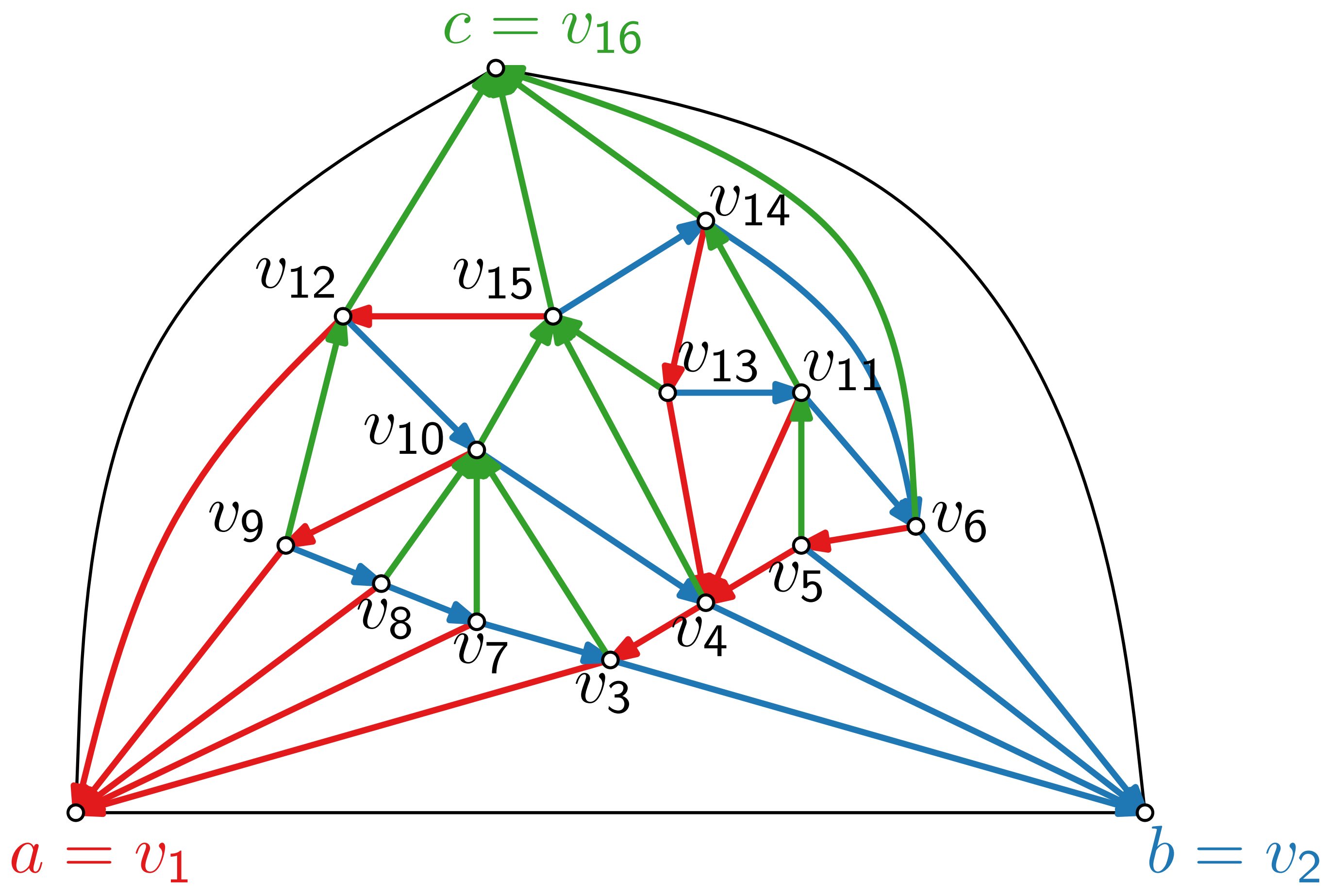

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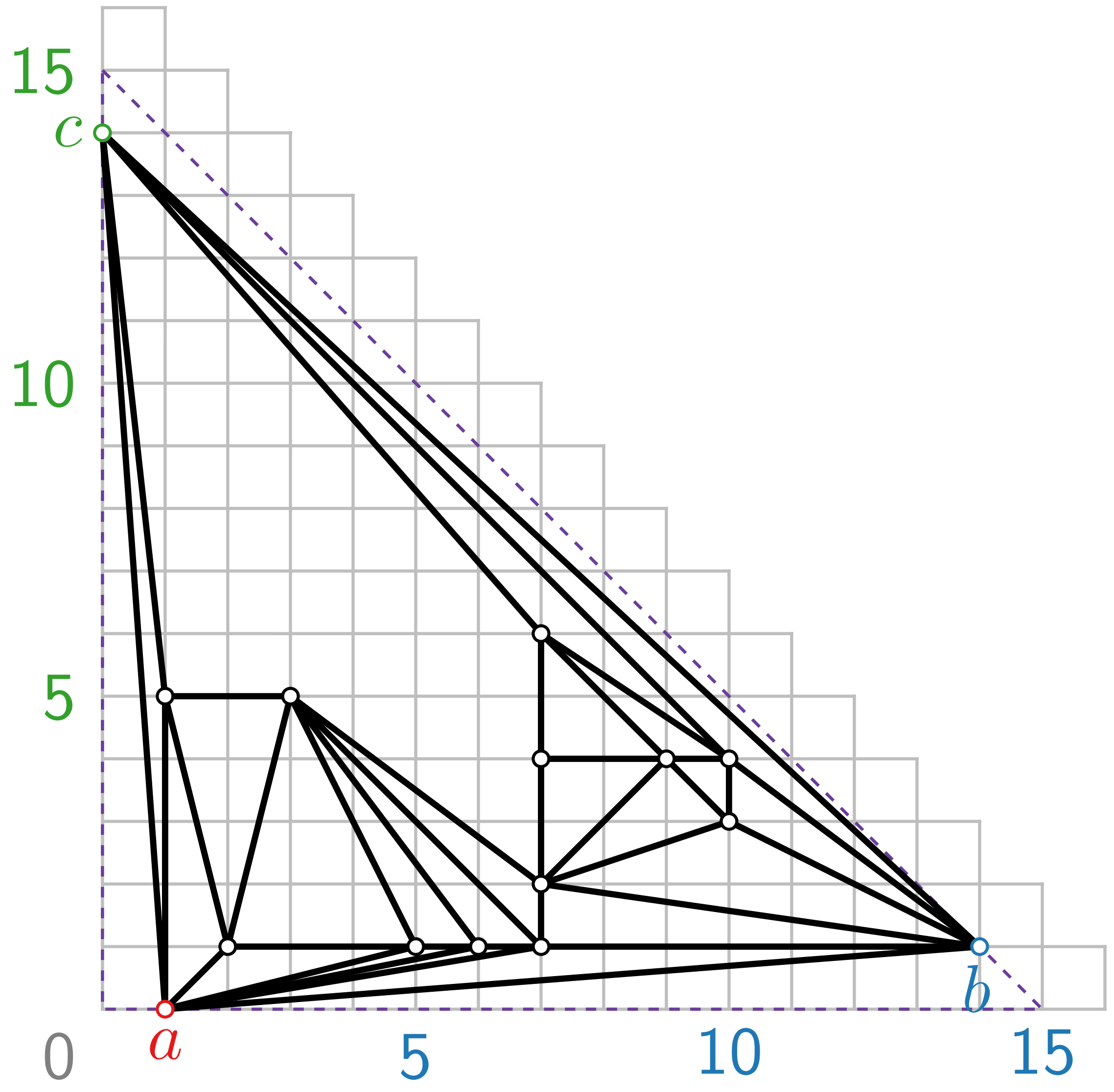

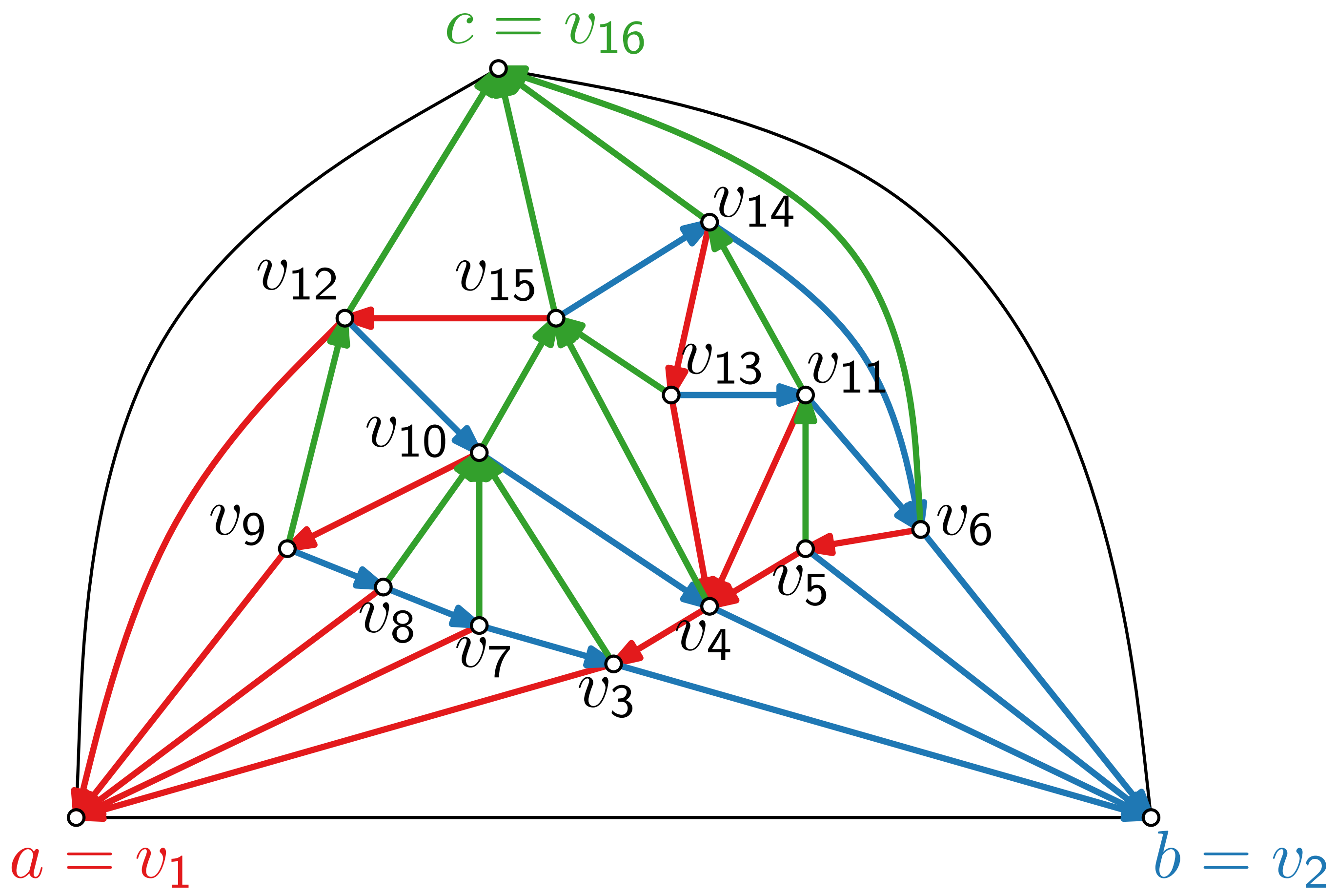

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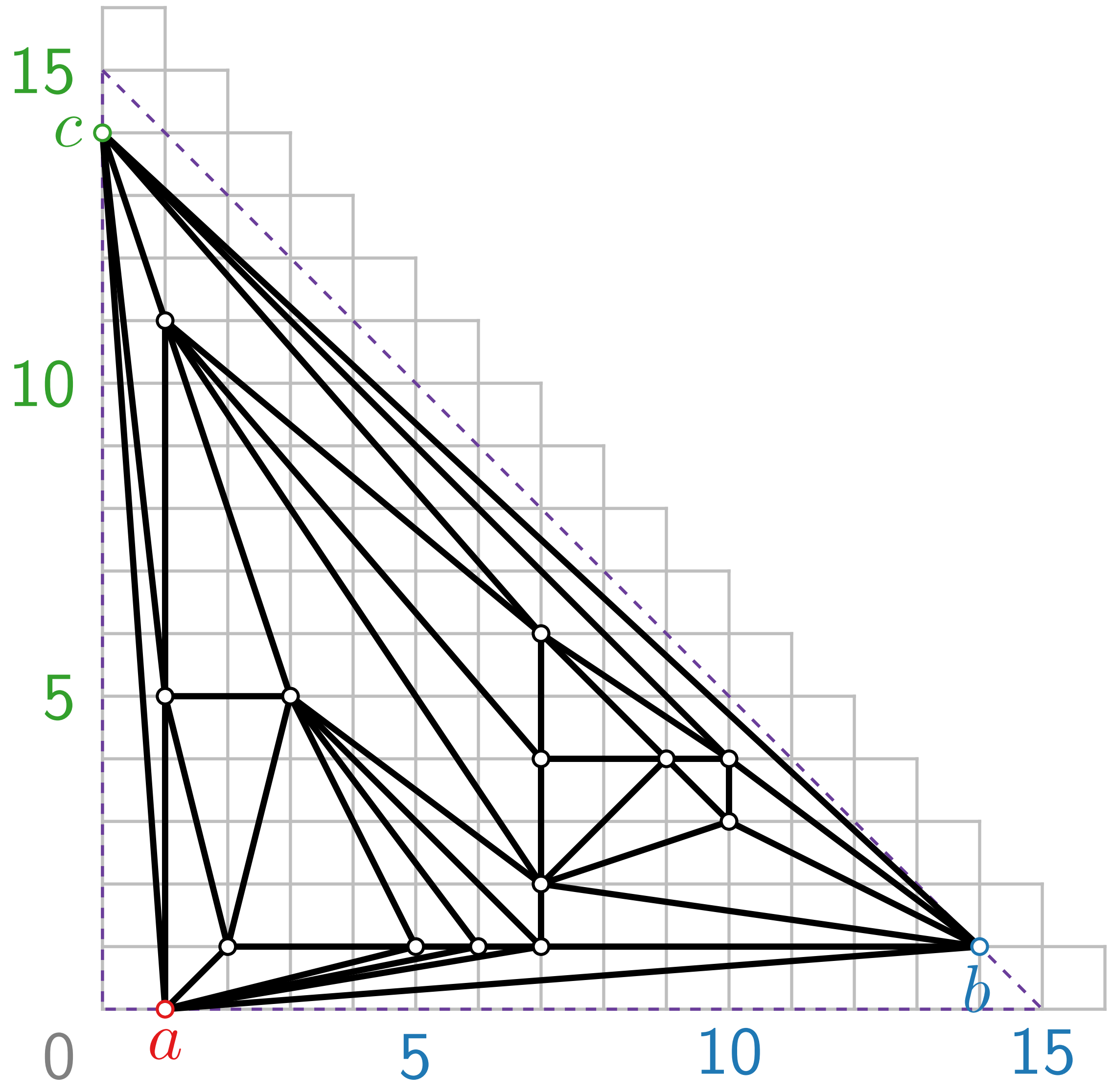

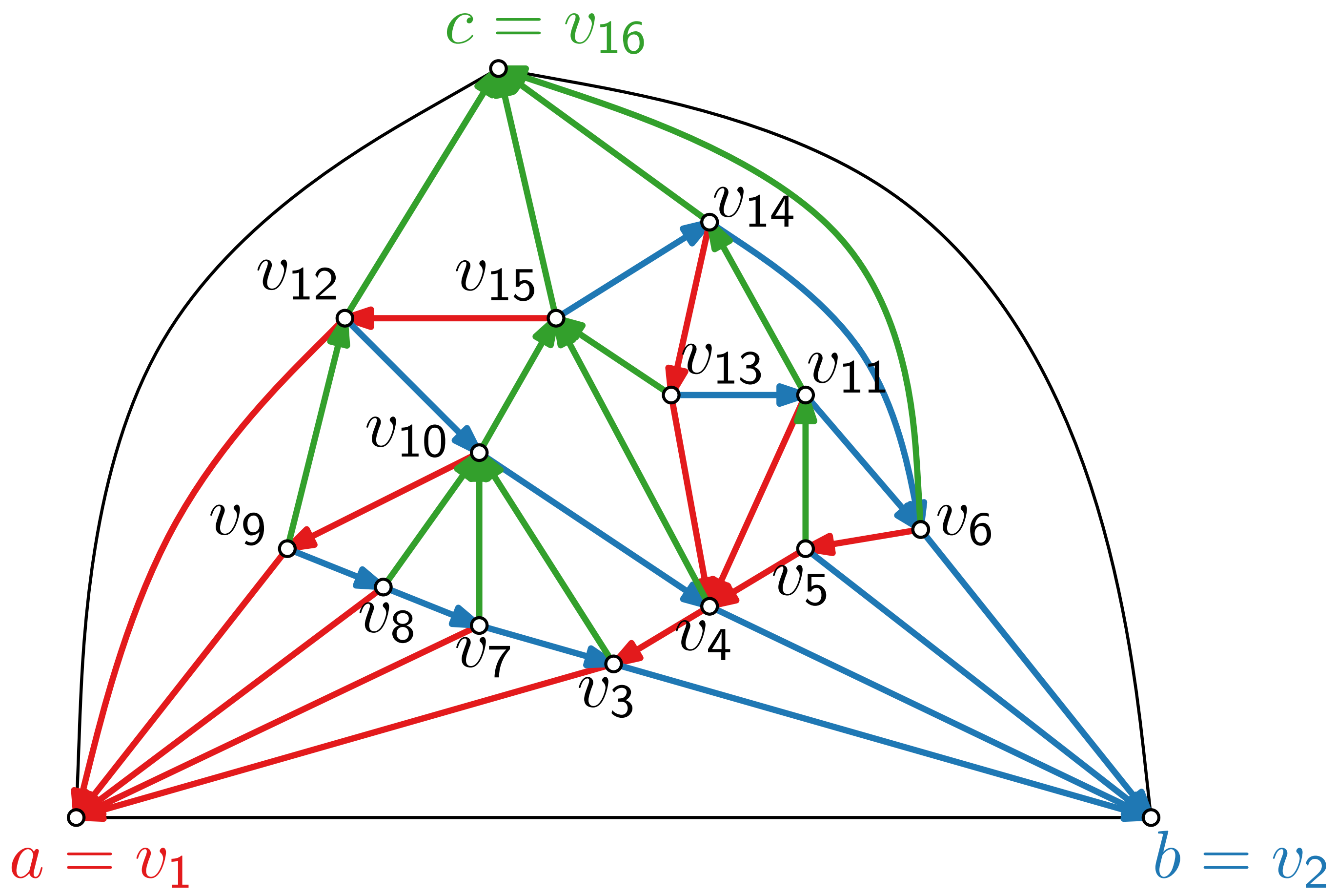

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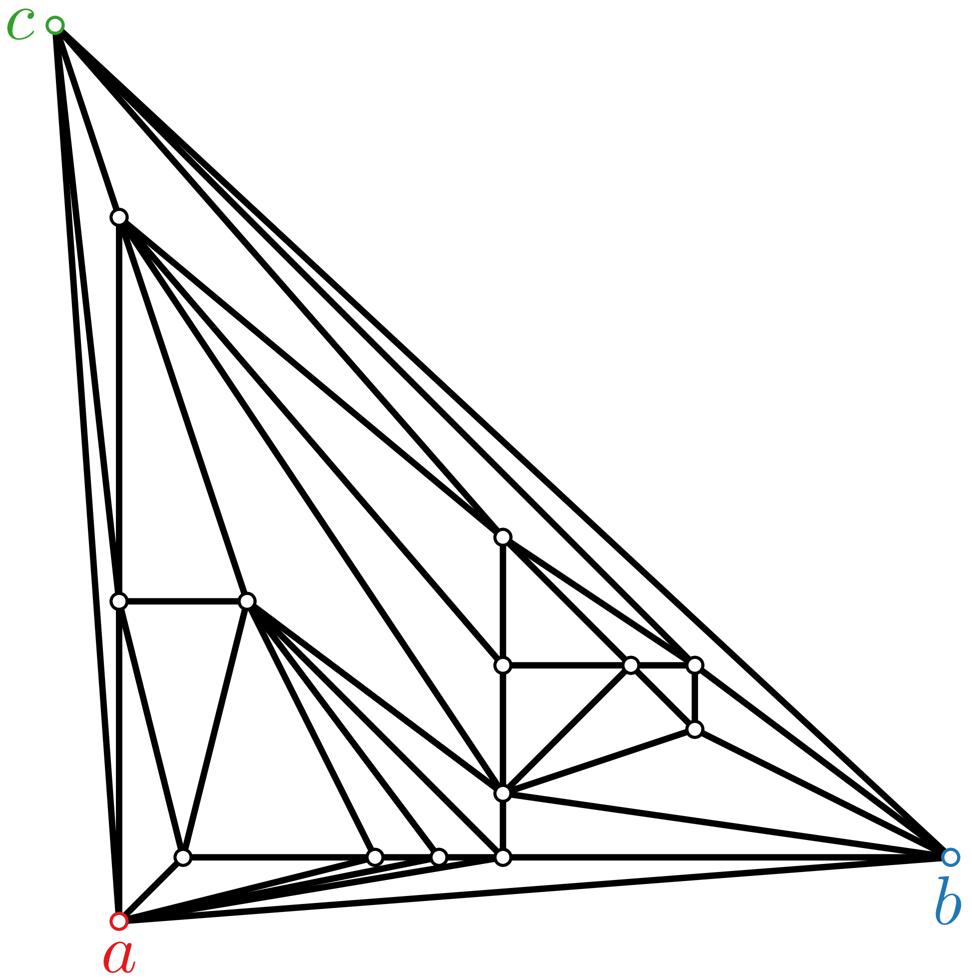

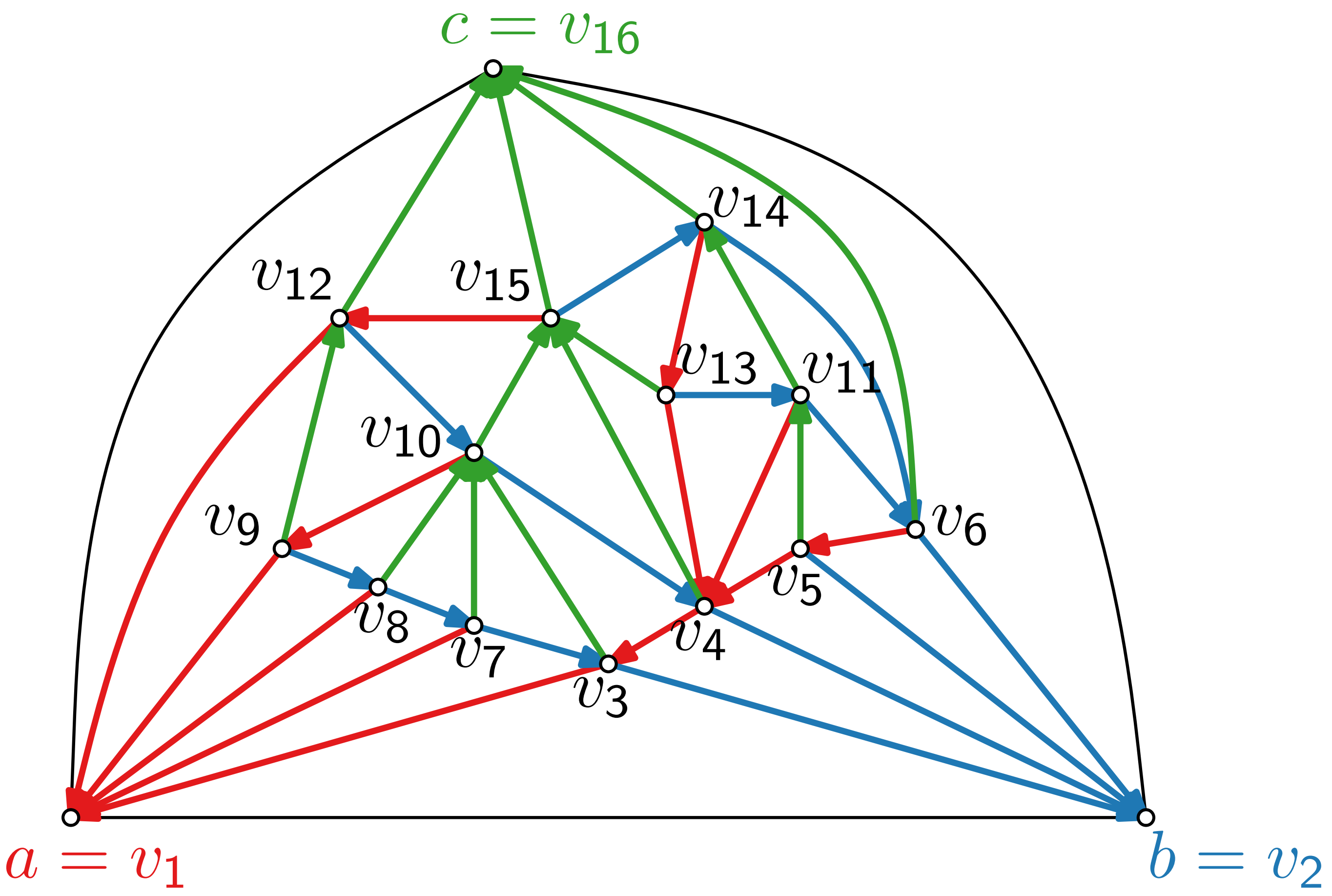

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Results & Variations

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$. Such a drawing can be computed in $O(n)$ time.

Theorem. [Schnyder '90]
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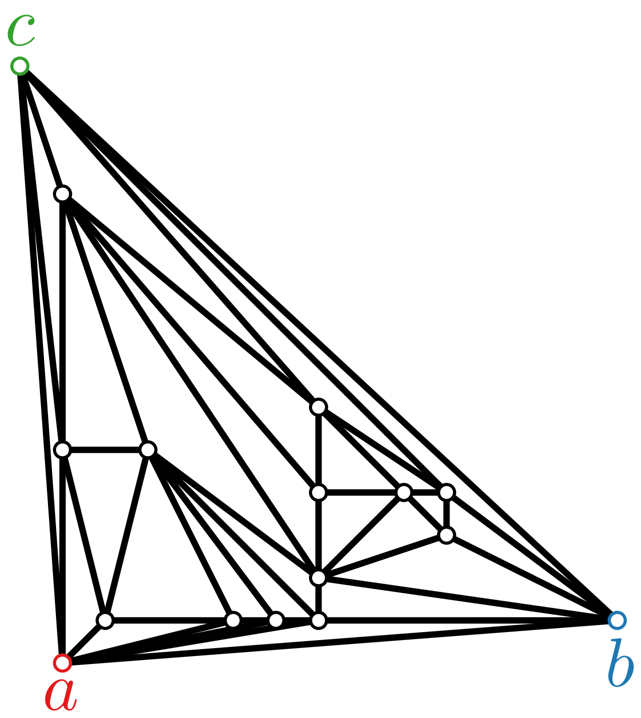

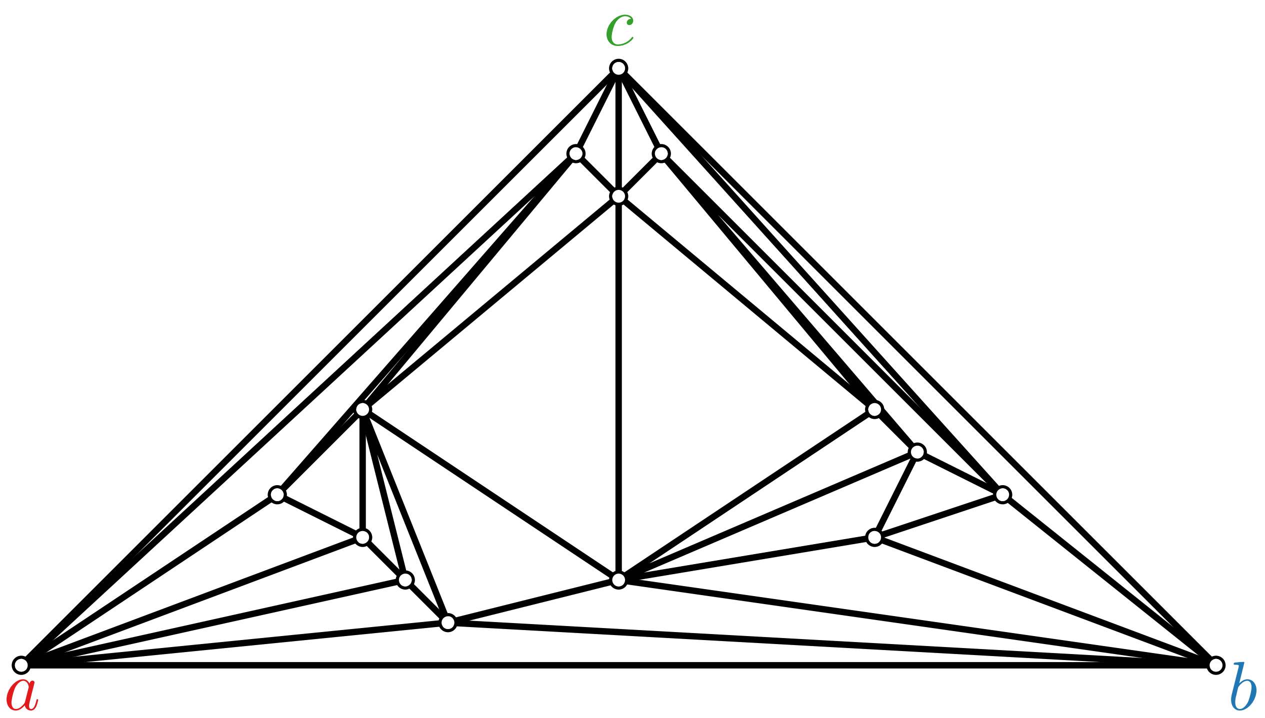

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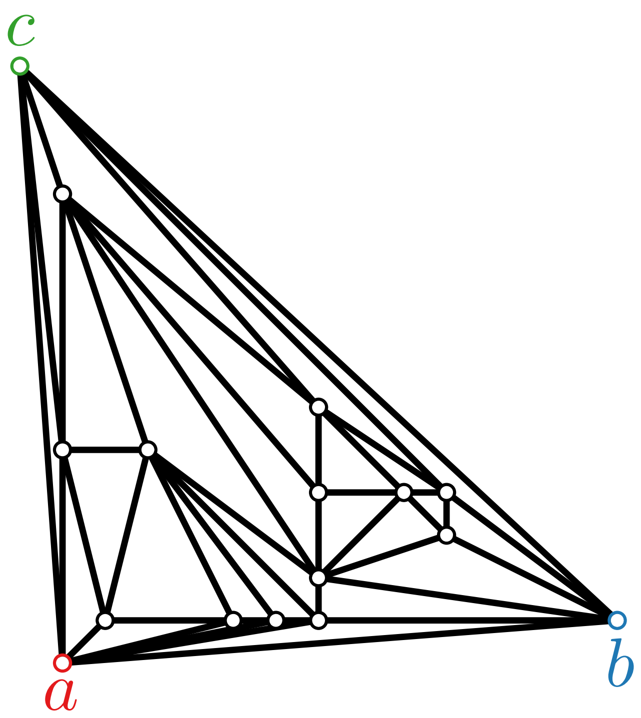

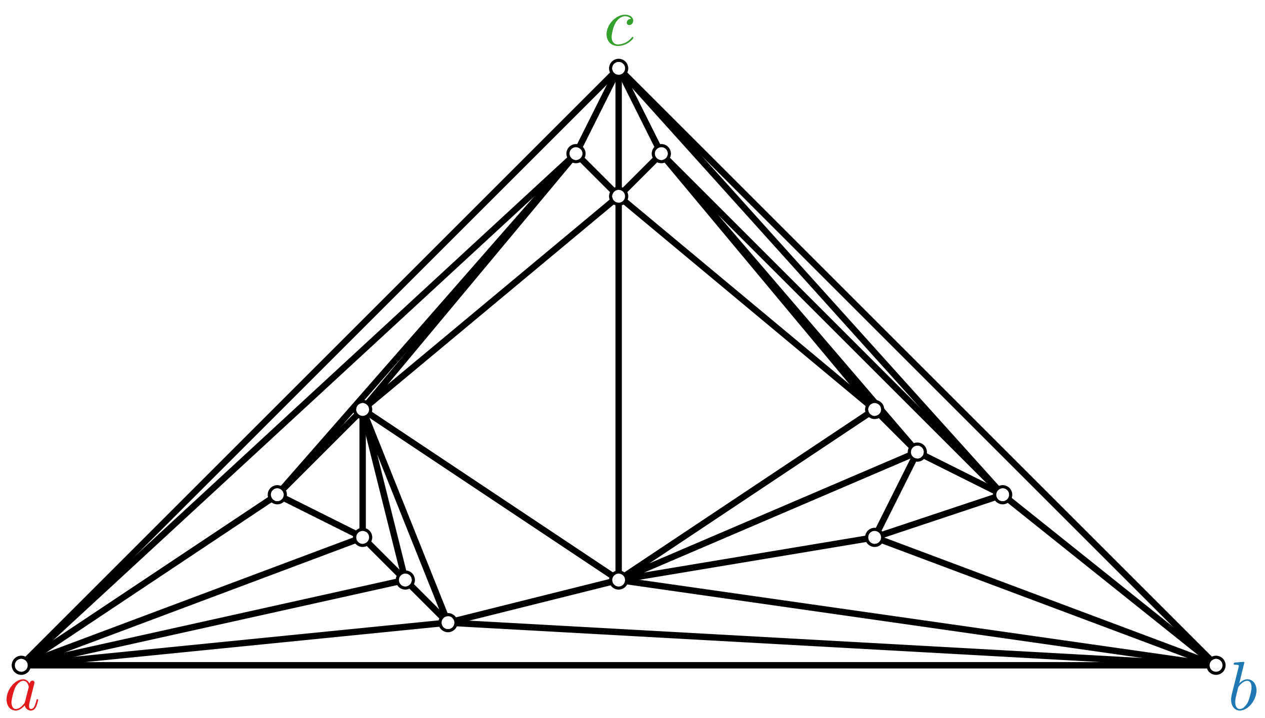

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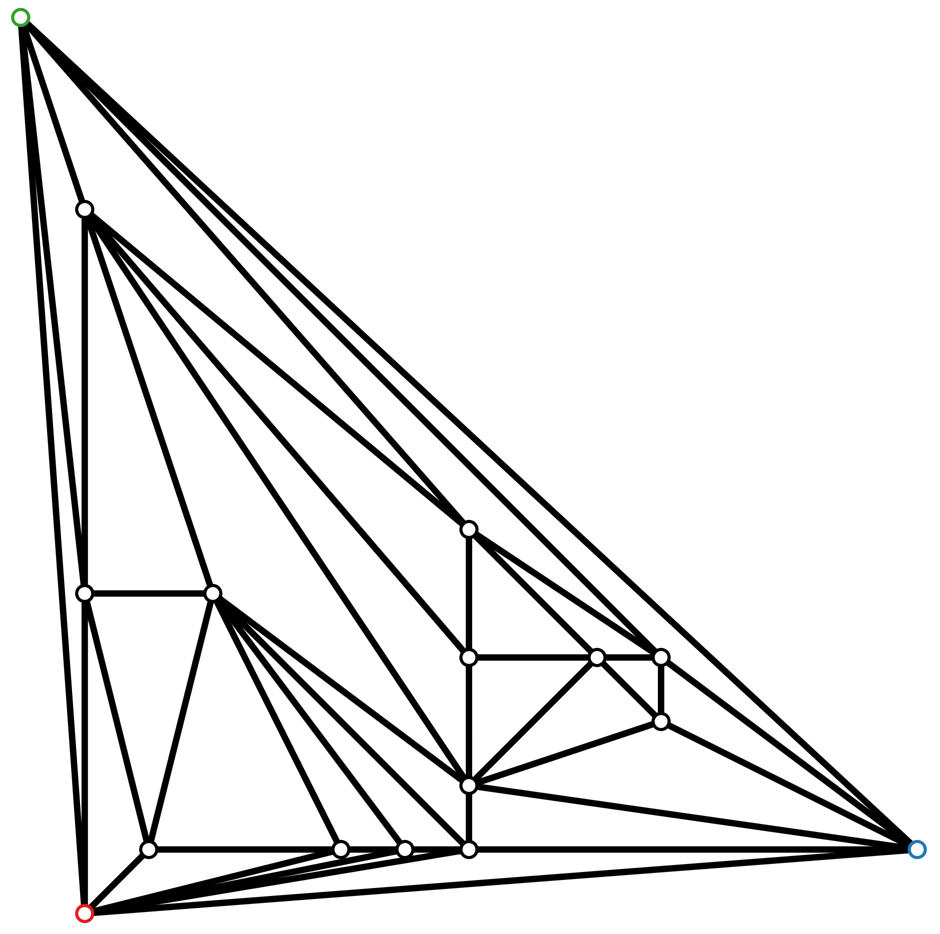
Every n-vertex planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time. ← *Exercise!*

Theorem. [Brandenburg '08]
Every n-vertex planar graph has a planar straight-line drawing of size $\frac{4}{3}n \times \frac{2}{3}n$. Such a drawing can be computed in $O(n)$ time.

Results & Variations



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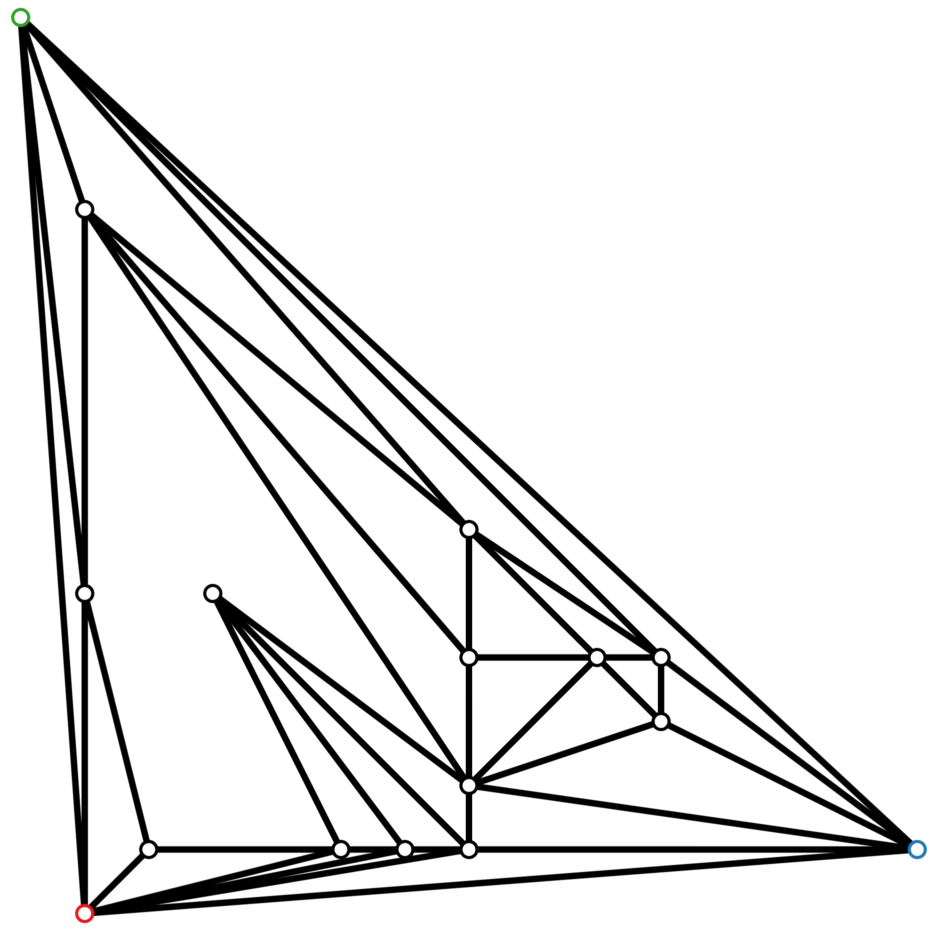

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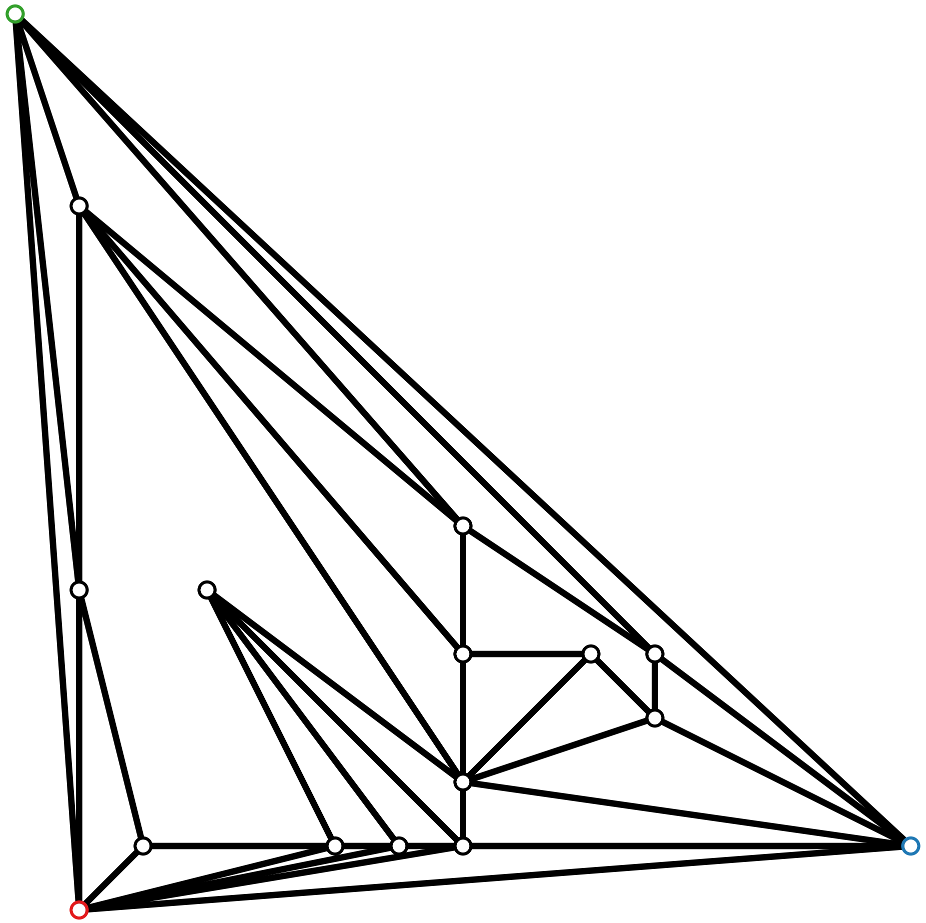

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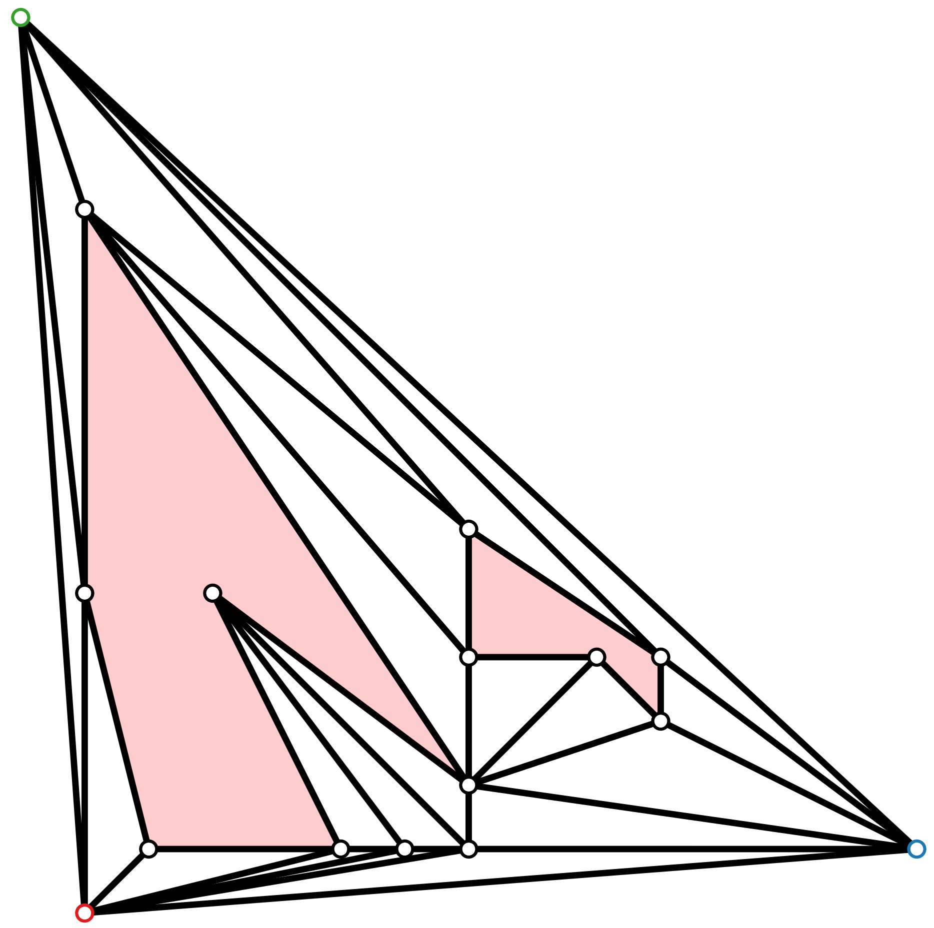

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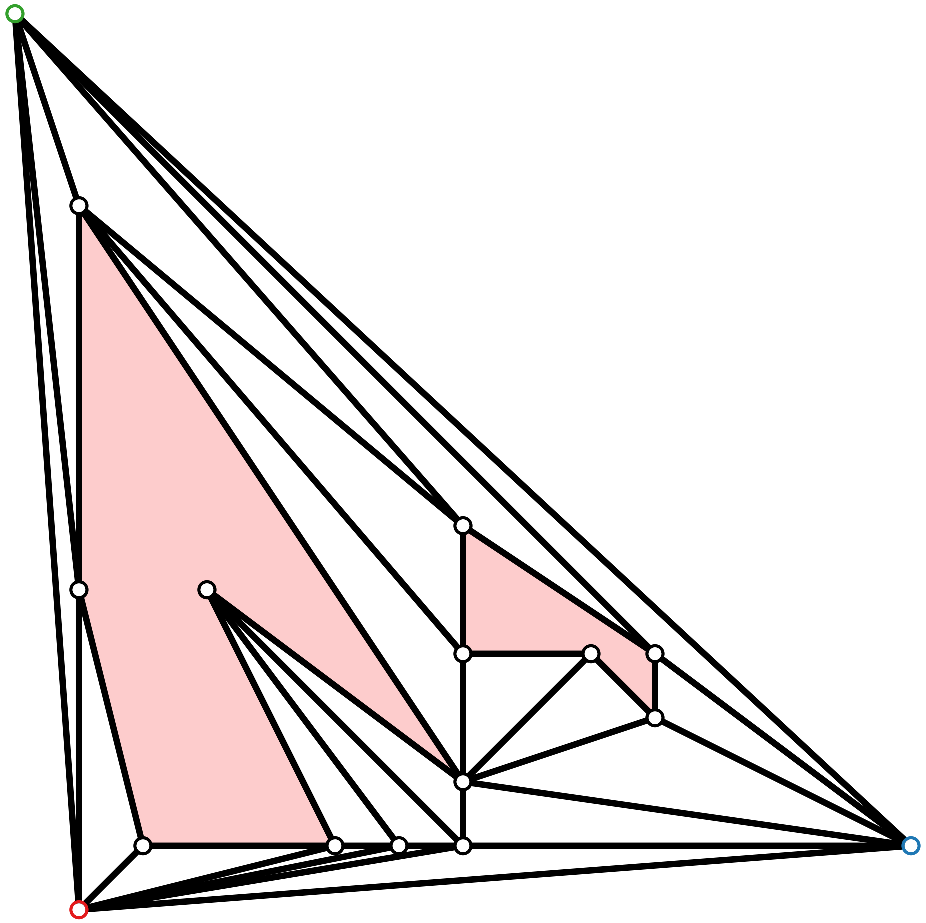

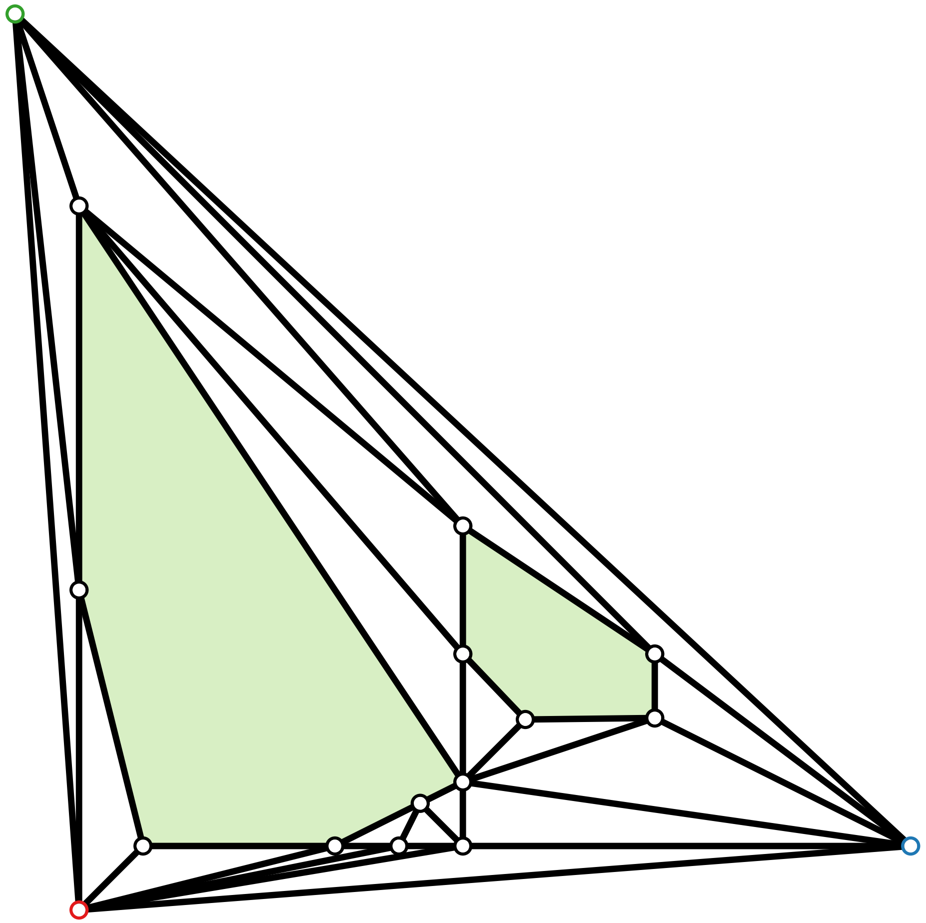

Results & Variations

Theorem. [Kant '96]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Results & Variations

Theorem. [Chrobak & Kant '97]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Results & Variations

Theorem. [Chrobak & Kant '97]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$ where all faces are drawn convex.
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Theorem. [Felsner '01]
Every 3-connected planar graph with f faces has a planar straight-line drawing of size $(f-1) \times (f-1)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Literature

- [PGD Ch. 4.3] for detailed explanation of shift method
- [Sch90] “Embedding planar graphs on the grid”, Walter Schnyder, SoCG 1990 – original paper on Schnyder realizer method.