Lehrstuhl für Informatik I
Algorithmen und
Komplexität
Universität Würzburg

Würzburg, 12 May 2022

Prof. Dr. Alexander Wolff
Tim Hegemann
Diana Sieper

Exercise sheet 3

Visualization of Graphs

Exercise 1 – Canonical orders for outerplanar graphs

A graph is *outerplanar* if it has a planar embedding such that all vertices are on the same face, usually the outer face. It is a *maximal outerplanar graph* if it is internally triangulated.

Describe a special canonical order built precisely for maximal outerplanar graphs.

a) Reformulate the conditions (C1)–(C3) for maximal outerplanar graphs. Can we enforce a bound on the degree of v_{k+1}? **2 Points**

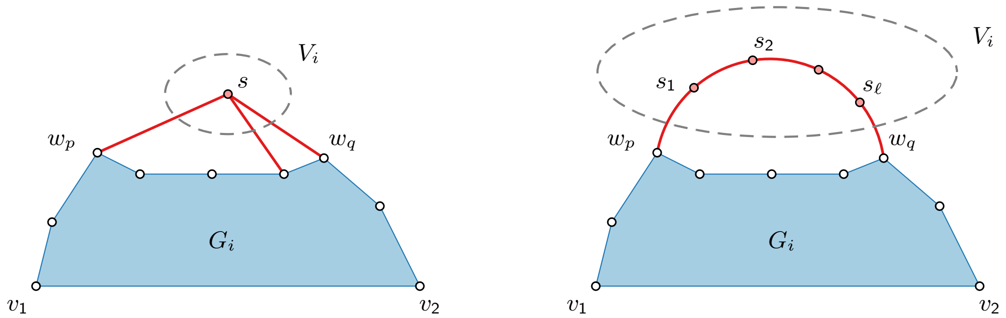
b) How can we use the algorithm for maximal planar graphs to obtain a canonical order for maximal outerplanar graphs? **2 Points**

Exercise 2 – Canonical orders for 3-connected planar graphs

Canonical orders for planar 3-connected graphs are a generalization of canonical orders for plane triangulations. Let G be a 3-connected planar graph. Let $\pi = (V_1, V_2, \dots, V_K)$ be an ordered partition of $V(G)$. That is, $V_1 \cup V_2 \cup \ldots \cup V_K = V(G)$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$. Define G_i to be the planar subgraph of G induced by $V_1 \cup V_2 \cup \cdots \cup V_i$. Let C_i be the subgraph of G induced by the edges on the boundary of the outer face of G_i. As illustrated below, π is a *canonical order* of G if:

- $V_1 = \{v_1, v_2\}$, where v_1 and v_2 lie on the outer face and $v_1v_2 \in E(G)$.
- $V_K = \{v_n\}$, where v_n lies on the outer face, $v_1v_n \in E(G)$, and $v_n \neq v_2$.
- Each C_i $(i > 1)$ is a cycle containing v_1v_2.
- Each G_i is biconnected and internally 3-connected; that is, removing any two interior vertices of G_i does not disconnect it.
- For each $i \in \{2, 3, \dots, K-1\}$, one of the following conditions holds:
    (i) $V_i = \{s\}$ where s is a vertex of C_i with at least three neighbors in C_{i-1}, and s has at least one neighbor in $G \setminus G_i$.

(ii) $V_i = (s_1, s_2, \ldots, s_\ell)$, $\ell \geq 2$, is a path in C_i, where each vertex in V_i has at least one neighbor in $G \setminus G_i$. Furthermore, s_1 and s_ℓ have one neighbor in C_{i-1}, and these are the only two edges between V_i and G_{i-1}.



a) Suppose that G_i is 3-connected. How can we choose V_i? **1 Point**

b) If G_i contains a vertex of degree two, where is it in G_i? **1 Point**

c) A *separation pair* of a graph G is a pair of vertices $\{a, b\}$ such that $G \setminus \{a, b\}$ consists of two or more components.

Suppose that G_i is 2-connected. Show that for every separation pair $\{a, b\}$ both vertices lie on C_i of G_i. **3 Points**

d) Show that the canonical order described above exists for all planar 3-connected graphs. **5 Points**

Hint: Make a case distinction between whether G_i is 3-connected or 2-connected. In the latter case, consider a minimal separation pair.

Exercise 3 – An Alternative Shift Algorithm

We want to examine an alternative drawing algorithm for planar, embedded, triangulated graphs $G = (V, E)$:

- Let $(v_1, v_2, \ldots, v_n)$ be a canonical order of the vertices.
- Draw v_1 at $(0, 0)$, v_2 at $(2, 0)$, and v_3 at $(1, 1)$.
- Draw the graph incrementally for $k = 4, \ldots, n$. Let $v_1 = w_1, \ldots, w_p, \ldots, w_q, \ldots, w_t = v_2$ be the vertices on the boundary of the outer face of G_{k-1} (in this order), where $w_p, \ldots, w_q$ are the neighbors of v_k in G_{k-1}. As the x-coordinate of v_k, choose an integer value $x(v_k)$ with $x(w_p) < x(v_k) < x(w_q)$. If no such value exists, first shift the right part of the drawing to the right by 1; i.e. for $q \leq i \leq t$ move each $L(w_i)$ to the right by 1. Now choose the smallest positive integer y-coordinate for which the drawing stays planar and v_k lies on the outer face.

a) Argue why this algorithm always yields a planar drawing. Why does in step 3 always a suitable y-coordinate exist? **4 Points**

b) Find a good lower bound for the maximum area requirement of the resulting drawing: find an infinite family of graphs where making bad choices for the x-coordinate in step 3 gives huge y-coordinates. **2 Points**

This assignment is due at the beginning of the next lecture, that is, on May 19th at 10 am. Please submit your solutions via WueCampus. The exercises will be discussed in the tutorial session on May 23 at 16:00.