Business Cycles

- Exercise 1 -

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Question:
What is your background in macroeconomics?

Multiple Choice:

a) strong ...

b) rather strong ...

c) rather poor ...

d) what is macroeconomics?!

Question:
Express the following equations as log-linear functions, i.e. take logs and simplify as far as possible.

1. $Y = zK^{\alpha}N^{1-\alpha}$

Multiple Choice:

a) $Y = \ln(z) + \alpha \ln(K) + (1 - \alpha) \ln(N)$

b) $\ln(Y) = \ln(z) + \ln(\alpha K) + \ln((1 - \alpha)N)$

c) $\ln(Y) = \ln(z) + \alpha \ln(K) + (1 - \alpha) \ln(N)$

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Question:
Express the following equations as log-linear functions, i.e. take logs and simplify as far as possible.

2) $Z = ce^{rt}\beta^K$

Multiple Choice:

a) $\ln(Z) = \ln(c) + \ln(e^{rt}) + K\ln(\beta)$

b) $\ln(Z) = \ln(c) + r + t + K\ln(\beta)$

c) $\ln(Z) = \ln(c) + rt + \ln(K\beta)$

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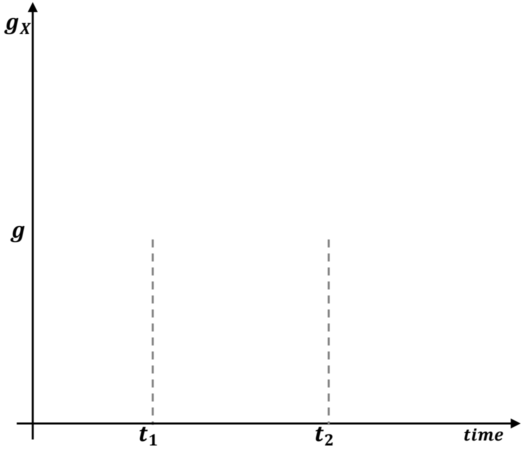
Question:
Suppose that the growth rate of some variable, *X*, is constant and equal to $g > 0$ from time t_0 to time t_1; drops to 0 at time t_1; rises gradually from 0 to g from time t_1 to time t_2; and is constant and equal to g after time t_2.

Question:
Sketch a graph of the growth rate of X as a function of time.



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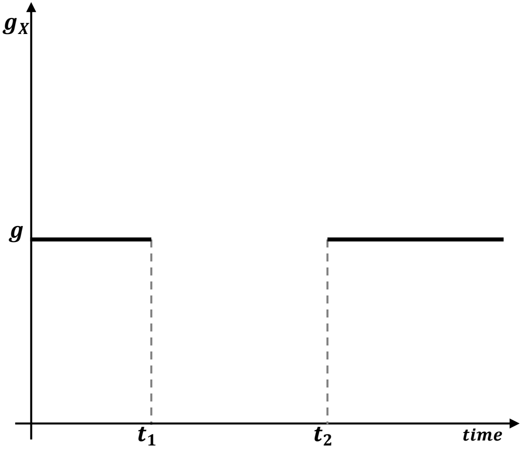

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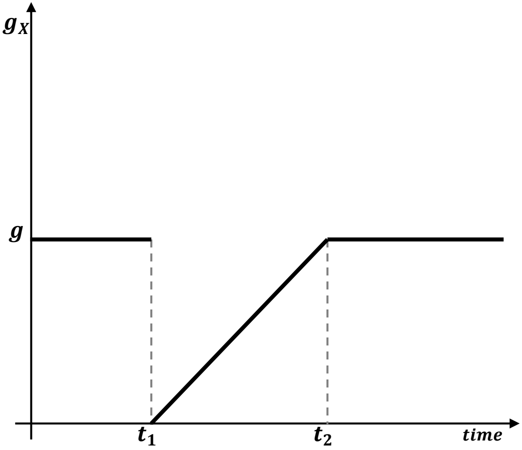
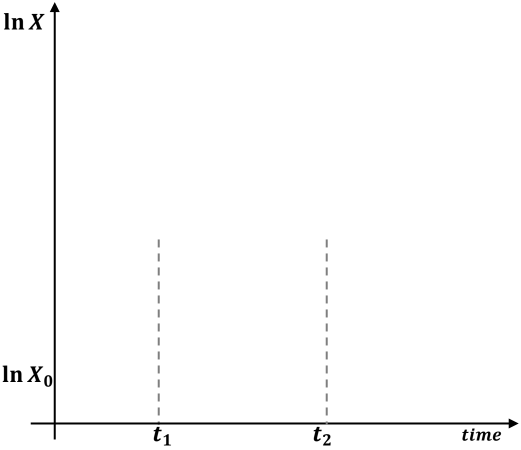


Question:
Sketch a graph of $\ln X$ as a function of time.



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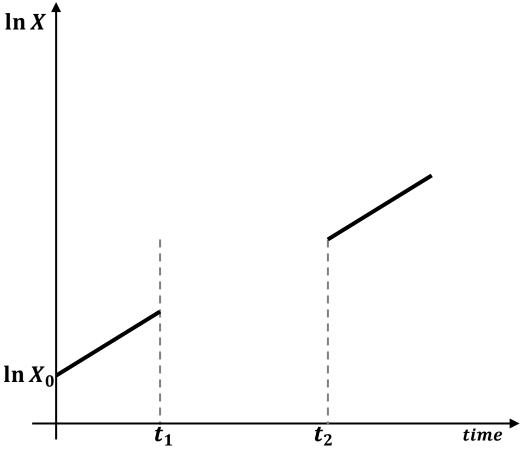

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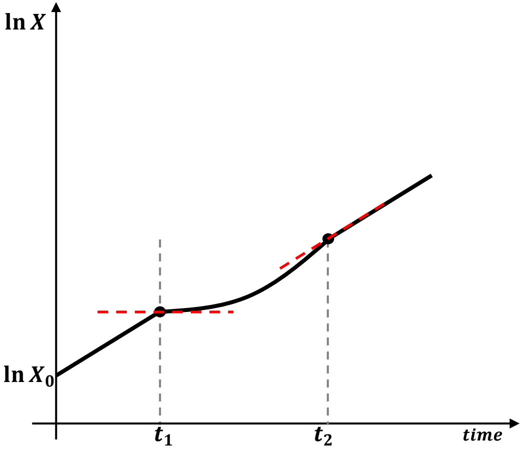

Question:
Show that the growth rate of the ratio of two variables is approximately the difference of their growth rates.

$$g_X \approx \ln(X_t) - \ln(X_{t-1}) = \Delta \ln(X_t)$$

Proof 1: $g_Y \approx g_X - g_Z$

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$$g_X \approx \ln(X_t) - \ln(X_{t-1}) = \Delta \ln(X_t)$$

Proof 1: $g_Y \approx g_X - g_Z$

$$g_Y \approx g_{\frac{X}{Z}} = \Delta \ln\left(\frac{X_t}{Z_t}\right) = \ln\left(\frac{X_t}{Z_t}\right) - \ln\left(\frac{X_{t-1}}{Z_{t-1}}\right)$$

$$= \underbrace{\ln(X_t) - \ln(X_{t-1})}_{=g_X} - \underbrace{(\ln(Z_t) - \ln(Z_{t-1}))}_{g_Z}$$

$$g_Y \approx g_X - g_Z$$

Proof 2: $g_Y \approx g_X - g_Z$

$$g_Y = g_{\frac{X}{Z}} = \frac{\frac{X_t}{Z_t} - \frac{X_{t-1}}{Z_{t-1}}}{\frac{X_{t-1}}{Z_{t-1}}} = \frac{\frac{X_t}{Z_t}}{\frac{X_{t-1}}{Z_{t-1}}} - 1 = \frac{\frac{X_t}{X_{t-1}}}{\frac{Z_t}{Z_{t-1}}} - 1$$

$$= \frac{g_X + 1}{g_Z + 1} - 1 = \frac{g_X + 1 - (g_Z + 1)}{g_Z + 1} = \underbrace{\frac{g_X - g_Z}{1 + g_Z}}_{1+g_Z \approx 1} \approx g_X - g_Z$$

Question:
The real GDP of Germany, measured in year 2010 prices, rose from EUR 2,038,505 million in 1991 to EUR 2,843,226 million in 2016. What was the average annual growth rate?

Multiple Choice:

a) $g \approx 1.39$

b) $g \approx 0.013$

c) $g = 0.016$

d) $g \approx 0.056$

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Question:

Calculate all the first, second, and cross derivatives of the following function.

1) $F(K, N) = zK^{\alpha}N^{1-\alpha}$

Multiple Choice:

a) $\dfrac{\partial F}{\partial K} = z\alpha\left(\dfrac{N}{K}\right)^{1-\alpha}$

b) $\dfrac{\partial F}{\partial K} = z\alpha K^{1-\alpha}N^{1-\alpha}$

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d) $\frac{\partial F}{\partial K} = z\left(\frac{N}{K}\right)^{1-\alpha}$

Question:
Calculate all the first, second, and cross derivatives of the following function.

1) $F(K, N) = zK^{\alpha}N^{1-\alpha}$

Remaining Solutions:

$$F_N = z(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$
$$F_{KK} = z\alpha(\alpha-1)K^{(\alpha-2)}N^{(1-\alpha)}$$
$$F_{NN} = -z\alpha(1-\alpha)K^{\alpha}N^{-1-\alpha}$$
$$F_{KN} = z\alpha(1-\alpha)K^{\alpha-1}N^{-\alpha}$$

Question:

Calculate all the first, second, and cross derivatives of the following function.

2) $F(K, N) = \ln(z) + \alpha \ln(K) + (1 - \alpha) \ln(N)$

Multiple Choice:

a) $\frac{\partial^2 F}{\partial K^2} = 1$

b) $\frac{\partial^2 F}{\partial K^2} = -\alpha K^{-2} + (1 - \alpha) \ln(N)$

c) $\frac{\partial^2 F}{\partial K^2} = \alpha \ln(K^{-2})$

d) $\frac{\partial^2 F}{\partial K^2} = -\alpha \frac{1}{K^2}$

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2) $F(K,N) = \ln(z) + \alpha \ln(K) + (1-\alpha)\ln(N)$

Remaining Solutions:

$$F_K = \alpha \frac{1}{K}$$
$$F_N = (1-\alpha)\frac{1}{N}$$
$$F_{NN} = -(1-\alpha)\frac{1}{N^2}$$
$$F_{KN} = 0$$

Question:

Calculate all the first, second, and cross derivatives of the following function.

3) $U(C,L) = \dfrac{C^{1-\gamma}-1}{1-\gamma} + L$

Multiple Choice:

a) $\dfrac{\partial^2 U}{\partial C^2} = 1$

b) $\dfrac{\partial^2 U}{\partial C^2} = 0$

c) $\dfrac{\partial^2 U}{\partial C^2} = -\gamma C^{-\gamma-1}$

d) $\dfrac{\partial^2 U}{\partial C^2} = -\gamma \dfrac{C^{-\gamma-1}}{1-\gamma}$

Question:
Calculate all the first, second, and cross derivatives of the following function.

3) $U(C, L) = \dfrac{C^{1-\gamma} - 1}{1 - \gamma} + L$

Multiple Choice:

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Question:
Calculate all the first, second, and cross derivatives of the following function.

3) $U(C, L) = \frac{C^{1-\gamma} - 1}{1-\gamma} + L$

Remaining Solutions:

$$U_C = C^{-\gamma}$$
$$U_L = 1$$
$$U_{LL} = 0$$
$$U_{CL} = 0$$

Question:

Calculate the first derivatives of the following function.

4) $F(K,N) = \left[\alpha K^{\frac{\nu-1}{\nu}} + (1-\alpha)N^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}$

Multiple Choice:

a) $\frac{\partial F}{\partial K} = \alpha^{\frac{\nu}{\nu-1}}$

b) $\frac{\partial F}{\partial K} = \alpha^{\frac{\nu}{\nu-1}} + \alpha(1-\alpha)^{\frac{1}{\nu-1}} \left(\frac{N}{K}\right)^{\frac{1}{\nu}}$

c) $\frac{\partial F}{\partial K} = \alpha \left(\frac{F}{K}\right)^{\frac{1}{\nu}}$

d) $\frac{\partial F}{\partial K} = \frac{\nu}{\nu-1} \left[\alpha K^{\frac{\nu-1}{\nu}} + (1-\alpha)N^{\frac{\nu-1}{\nu}}\right]^{\frac{1}{\nu-1}}$

Question:

Calculate the first derivatives of the following function.

4) $F(K,N) = \left[\alpha K^{\frac{\nu-1}{\nu}} + (1-\alpha)N^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}$

Multiple Choice:

a) $\frac{\partial F}{\partial K} = \alpha^{\frac{\nu}{\nu-1}}$

b) $\frac{\partial F}{\partial K} = \alpha^{\frac{\nu}{\nu-1}} + \alpha(1-\alpha)^{\frac{1}{\nu-1}} \left(\frac{N}{K}\right)^{\frac{1}{\nu}}$

c) $\frac{\partial F}{\partial K} = \alpha \left(\frac{F}{K}\right)^{\frac{1}{\nu}}$

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Calculate the first derivatives of the following function.

4) $F(K,N) = \left[\alpha K^{\frac{\nu-1}{\nu}} + (1-\alpha) N^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}$

Remaining Solution:

$$F_N = (1-\alpha) N^{-\frac{1}{\nu}} \left[\alpha K^{\frac{\nu-1}{\nu}} + (1-\alpha) N^{\frac{\nu-1}{\nu}}\right]^{\frac{1}{\nu-1}} = (1-\alpha)\left(\frac{F}{N}\right)^{\frac{1}{\nu}}$$

Question:

Solve the following constrained maximization problem using Lagrange multipliers!

$$\max_{x_1,x_2,x_3} U = x_1^{a_1} x_2^{a_2} x_3^{a_3}$$

$$\text{s.t.} \quad w_0 = p_1 x_1 + p_2 x_2 + p_3 x_3$$

Lagrangian:

$$\mathcal{L} = x_1^{a_1} x_2^{a_2} x_3^{a_3} + \lambda[w_0 - p_1 x_1 - p_2 x_2 - p_3 x_3]$$

F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial x_1} : \quad a_1 x_1^{a_1 - 1} x_2^{a_2} x_3^{a_3} = \lambda p_1 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} : \quad a_2 x_1^{a_1} x_2^{a_2 - 1} x_3^{a_3} = \lambda p_2 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} : \quad a_3 x_1^{a_1} x_2^{a_2} x_3^{a_3 - 1} = \lambda p_3 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} : \quad w_0 = p_1 x_1 + p_2 x_2 + p_3 x_3 \tag{4}$$

(1) times $\frac{x_1}{a_1}$, (2) times $\frac{x_2}{a_2}$, and (3) times $\frac{x_3}{a_3}$ yields:

$$x_1^{a_1} x_2^{a_2} x_3^{a_3} = \lambda p_1 \frac{x_1}{a_1} = \lambda p_2 \frac{x_2}{a_2} = \lambda p_3 \frac{x_3}{a_3}$$

(1) times $\frac{x_1}{a_1}$, (2) times $\frac{x_2}{a_2}$, and (3) times $\frac{x_3}{a_3}$ yields:

$$x_1^{a_1} x_2^{a_2} x_3^{a_3} = \lambda p_1 \frac{x_1}{a_1} = \lambda p_2 \frac{x_2}{a_2} = \lambda p_3 \frac{x_3}{a_3}$$

Trivial solution: $\lambda = 0$ (Implication?), if $\lambda \neq 0$:

$$\begin{aligned} p_1 \frac{x_1}{a_1} = p_2 \frac{x_2}{a_2} &= p_3 \frac{x_3}{a_3} \\ \Rightarrow p_2 x_2 &= \frac{a_2 p_1 x_1}{a_1} \\ \Rightarrow p_3 x_3 &= \frac{a_3 p_1 x_1}{a_1} \end{aligned}$$

Thus, (4) becomes:

$$w_0 = p_1 x_1 + \frac{a_2 p_1 x_1}{a_1} + \frac{a_3 p_1 x_1}{a_1} = p_1 x_1 \frac{a_1 + a_2 + a_3}{a_1}$$

Repeat the last to steps for x_2 and x_3:

$$p_1 x_1 = \frac{a_1}{a_1 + a_2 + a_3} w_0$$
$$p_2 x_2 = \frac{a_2}{a_1 + a_2 + a_3} w_0$$
$$p_3 x_3 = \frac{a_3}{a_1 + a_2 + a_3} w_0$$

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- The wealth w_0 is distributed among the commodities using the exponents a_1, a_2, a_3 as weights

Question:

Consider an individual who receives utility from consumption, c, and leisure, l. The individual has $\bar{L}$ time to allocate to work, n, and leisure. The individual's consumption is a function of how much s/he works. In particular, $c = \sqrt{n}$. The individual's maximization problem is

$$\max_{c,l,n} U = \ln(c) + \theta l$$
$$\text{s.t.} \quad c = \sqrt{n}$$
$$\bar{L} = n + l$$

where $\theta > 0$. Solve the maximization problem!

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Lagrangian:

$$\mathcal{L} = \ln(c) + \theta l + \lambda_1[\sqrt{n} - c] + \lambda_2[\bar{L} - n - l]$$

F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial C}: \quad \lambda_1 = \frac{1}{c} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial l}: \quad \lambda_2 = \theta \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial n}: \quad \lambda_2 = 0.5\lambda_1 \frac{1}{\sqrt{n}} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1}: \quad c = \sqrt{n} \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2}: \quad n + l = \bar{L} \tag{9}$$

(5) and (6) in (7):

$$\theta = 0.5\frac{1}{c}\frac{1}{\sqrt{n}} \Rightarrow c = 0.5\frac{1}{\theta}\frac{1}{\sqrt{n}} \tag{10}$$

(10) in (8):

$$
\begin{aligned}
\sqrt{n} &= 0.5\frac{1}{\theta}\frac{1}{\sqrt{n}} \\
n &= 0.5\frac{1}{\theta} \\
c &= \sqrt{0.5\frac{1}{\theta}} = \frac{1}{\sqrt{2}\sqrt{\theta}} \\
l &= \bar{L} - 0.5\frac{1}{\theta}
\end{aligned}
$$

Which economic principle do you see here?