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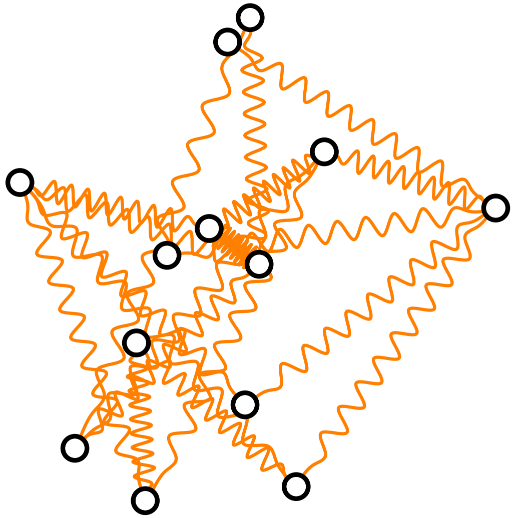
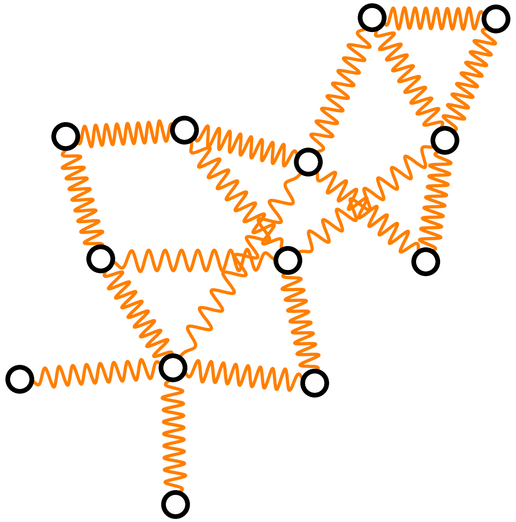
Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part I:
Algorithmic Framework

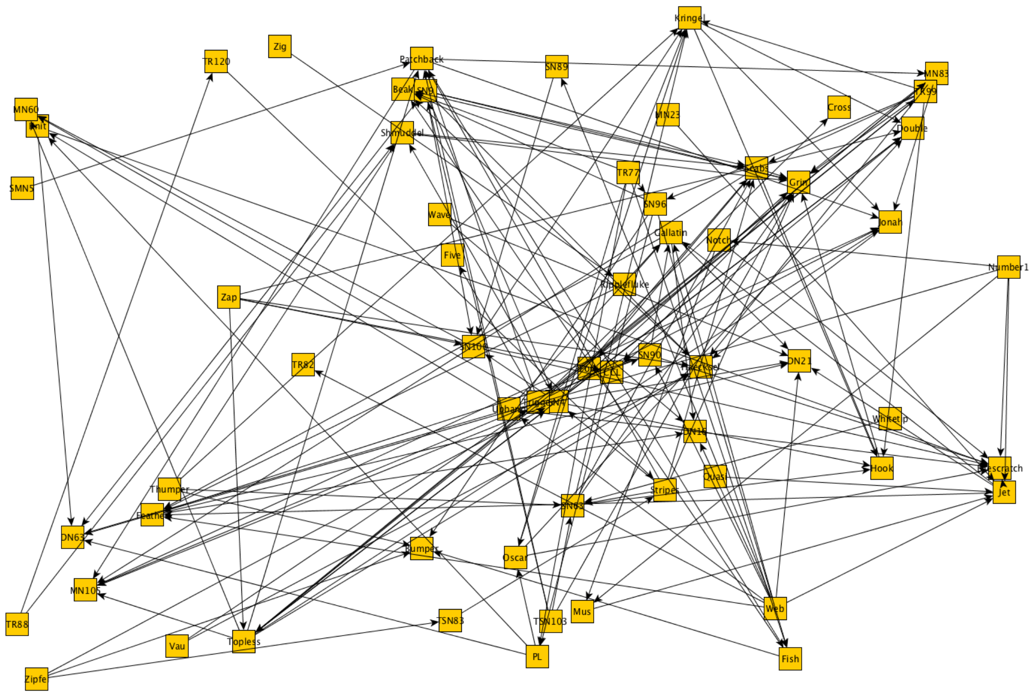


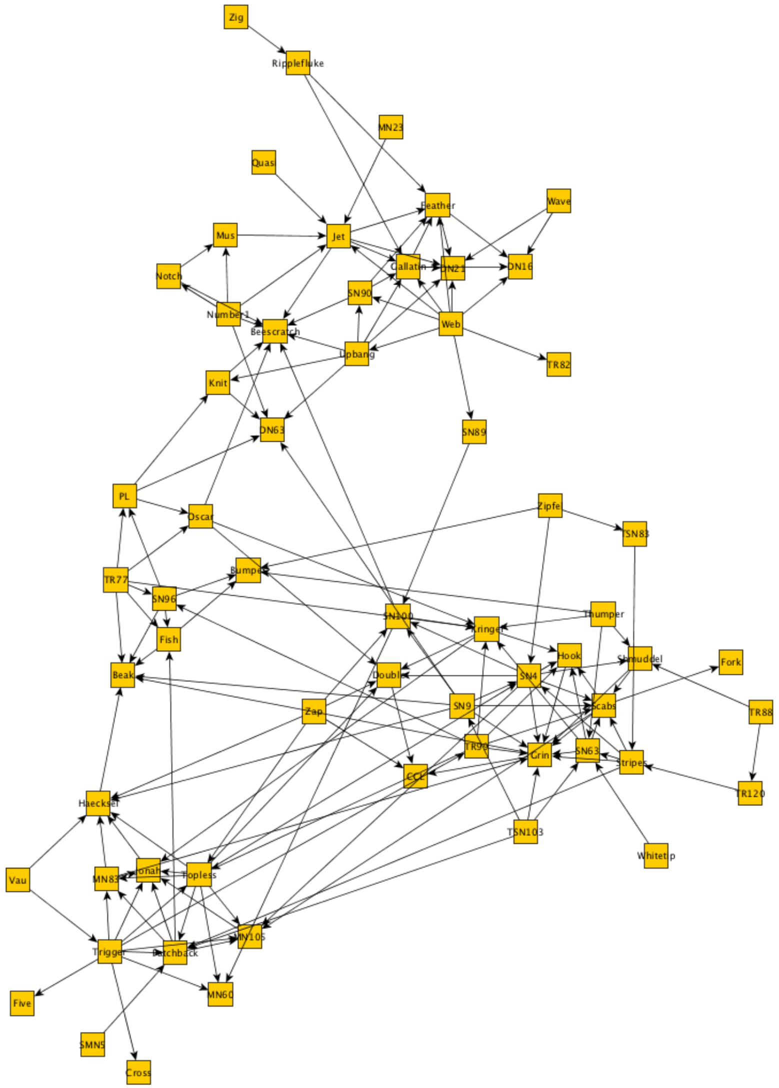
Alexander Wolff

General Layout Problem

Input: Graph G

Output: Clear and readable straight-line drawing of G





General Layout Problem

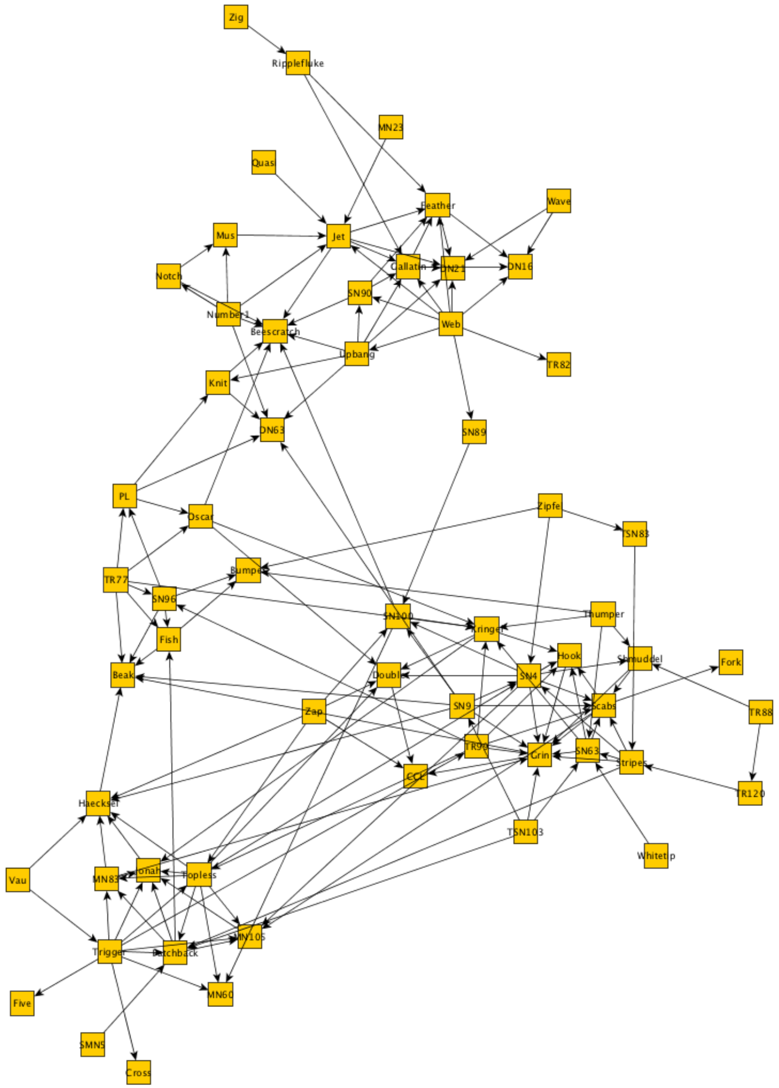
Input: Graph G

Output: Clear and readable straight-line drawing of G

Drawing aesthetics:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**
- densely connected parts (clusters) form communities
- as few crossings as possible
- nodes distributed evenly

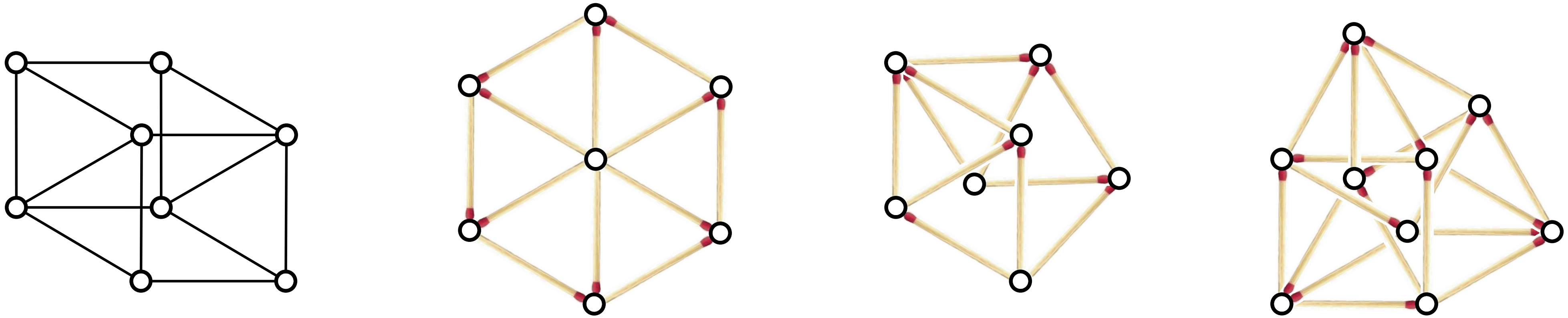
Optimization criteria partially contradict each other.



Fixed Edge Lengths?

Input: Graph G, required edge length $\ell(e)$ for each $e \in E(G)$.

Output: Drawing of G that realizes the given edge lengths.



NP-hard for

Fixed Edge Lengths?

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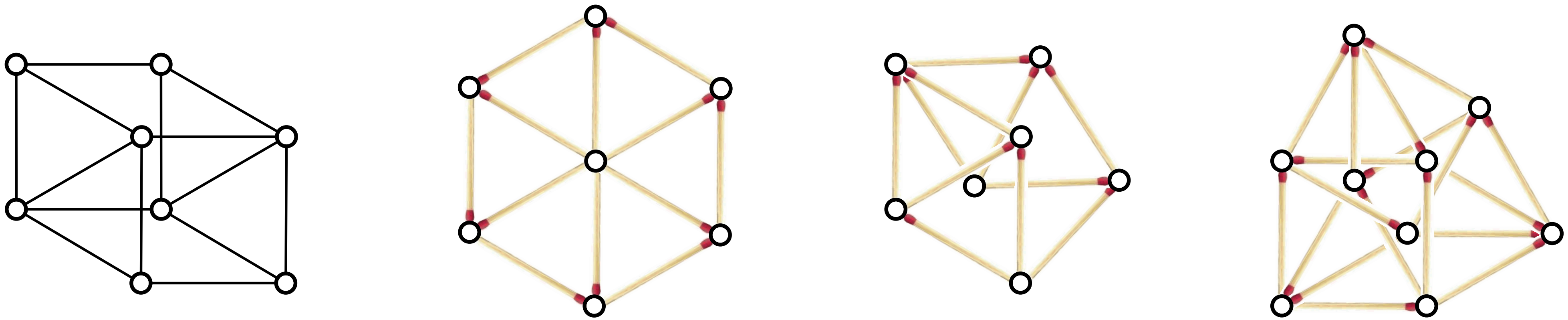


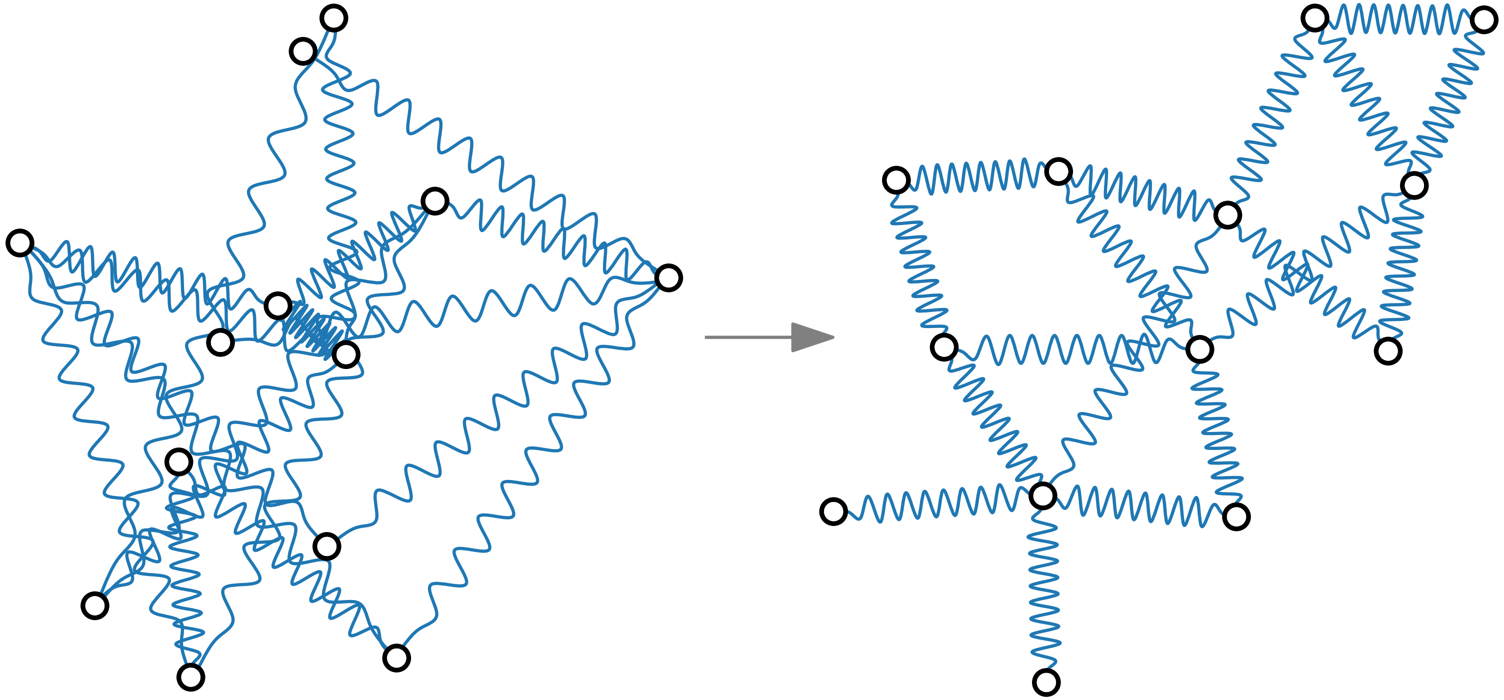
NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades '84]

"To embed a graph we replace the vertices by steel rings and replace each edge with a **spring** to form a mechanical system . . . The vertices are placed in some initial layout and let go so that the spring forces on the rings move the system to a minimal energy state."



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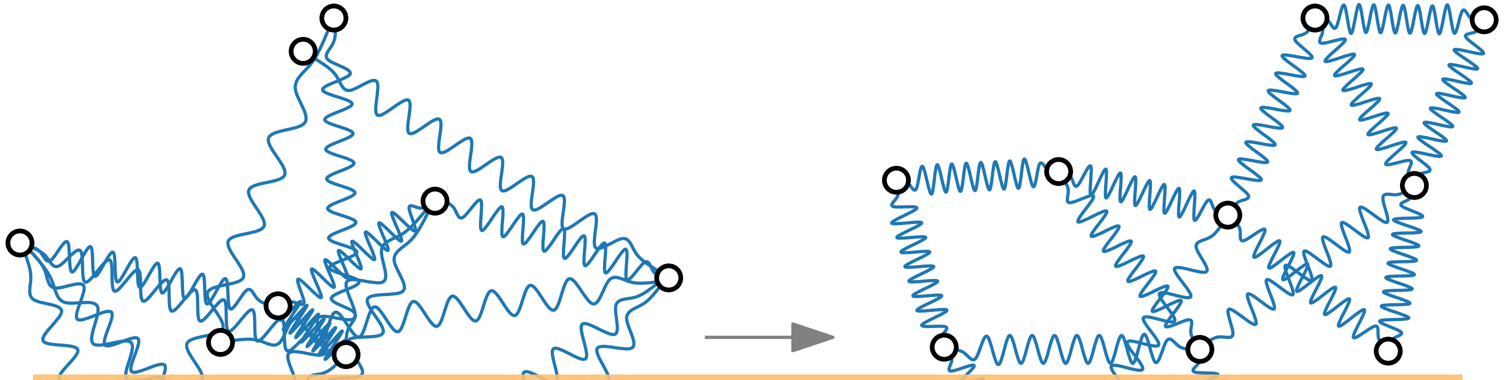


So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

Attractive forces.

pairs $\{u, v\}$ of adjacent vertices:

Repulsive forces.

any pair $\{x, y\}$ of vertices:

Force-Directed Algorithms

initial layout

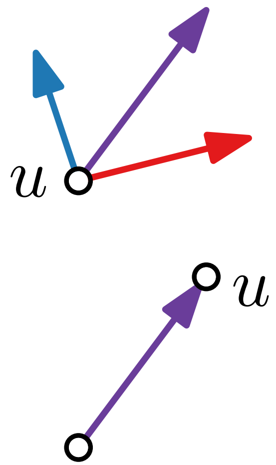
threshold

max # iterations

ForceDirected$(G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N})$
  $t \leftarrow 1$
  while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**
    foreach $u \in V$ **do**
      $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$
    foreach $u \in V$ **do**
      $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

cooling factor

end layout



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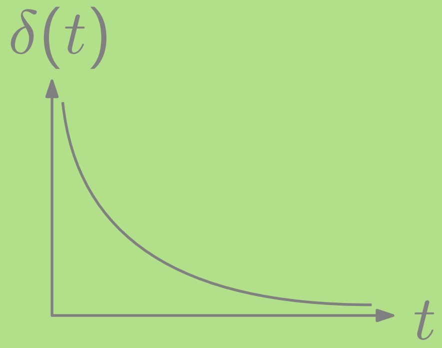
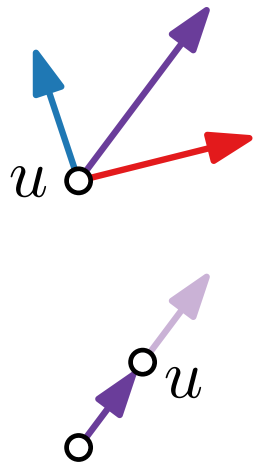
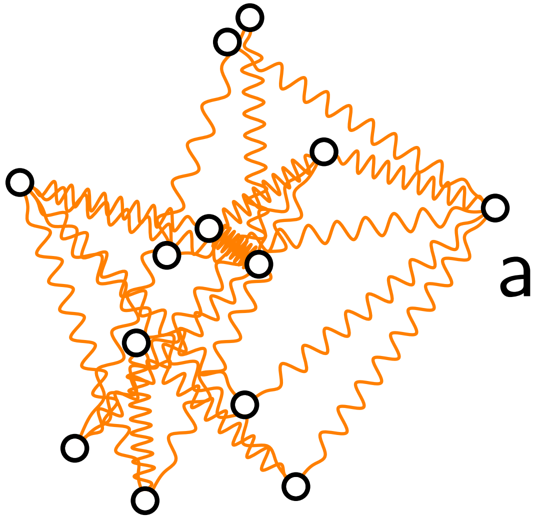


end layout

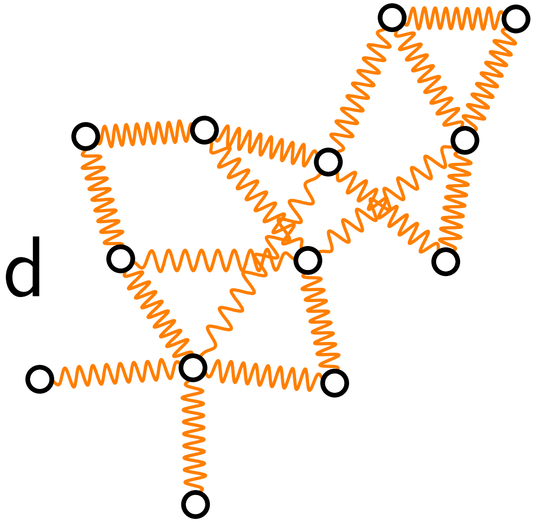


Visualization of Graphs

Lecture 2: Force-Directed Drawing Algorithms



Part II: Spring Embedders by Eades and by Fruchterman & Reingold



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Spring Embedder by Eades – Model

```
ForceDirected(G = (V, E), p = (p_v)_{v∈V}, ε > 0, K ∈ ℕ)
  t ← 1
  while t < K and max_{v∈V} ||F_v(t)|| > ε do
      foreach u ∈ V do
          F_u(t) ← Σ_{v∈V} f_rep(u, v) + Σ_{v∈Adj[u]} f_attr(u, v)
      foreach u ∈ V do
          p_u ← p_u + δ(t) · F_u(t)
      t ← t + 1
  return p
```

- **Repulsive forces**

repulsion constant (e.g., 2.0)

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

spring constant (e.g., 1.0)

$$f_{\text{spring}}(u,v) = c_{\text{spring}} \cdot \log \frac{\|p_v - p_u\|}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{attr}}(u,v) = f_{\text{spring}}(u,v) - f_{\text{rep}}(u,v)$$

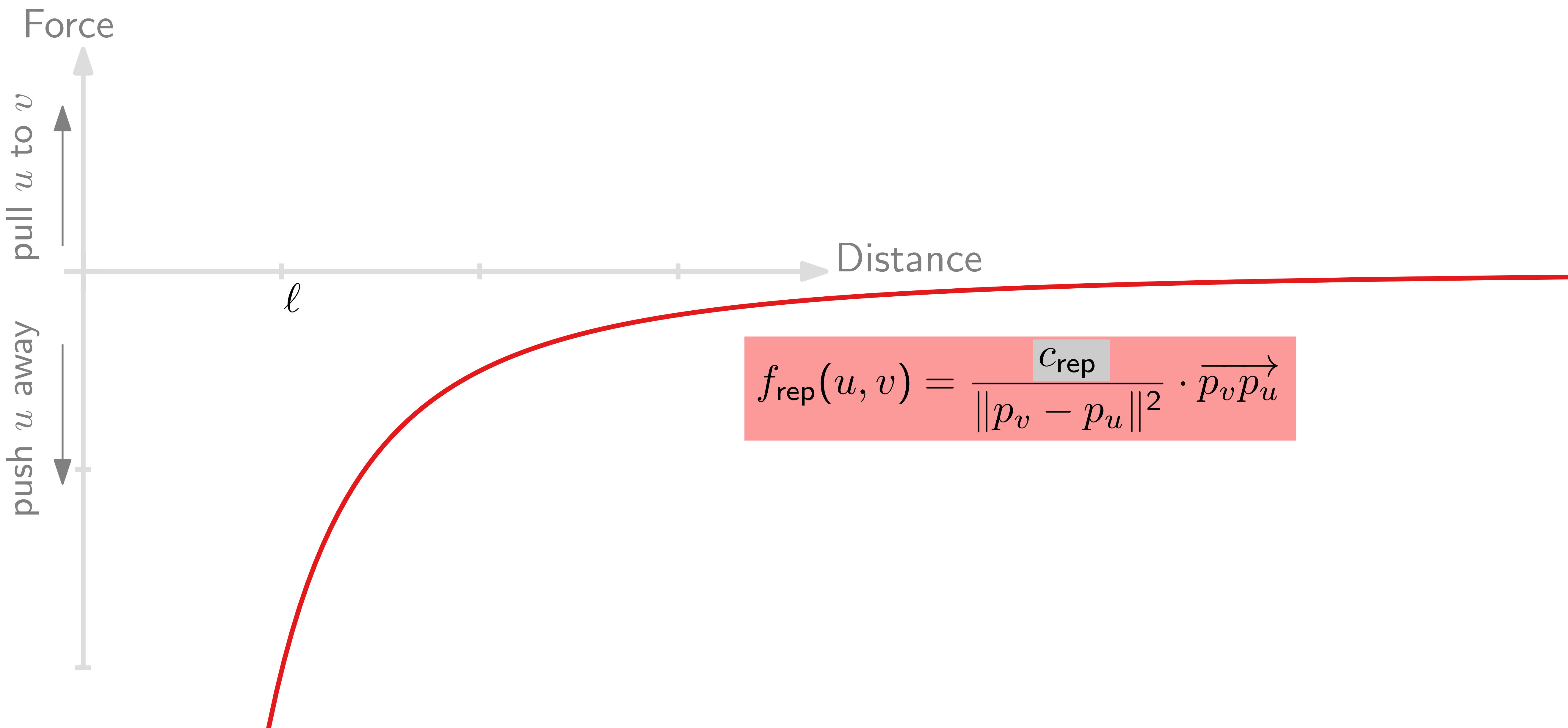
- **Resulting displacement vector**

$$F_u = \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(u,v)$$

Notation.

- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- $\|p_u - p_v\|$ = Euclidean distance between u and v
- ℓ = ideal spring length for edges

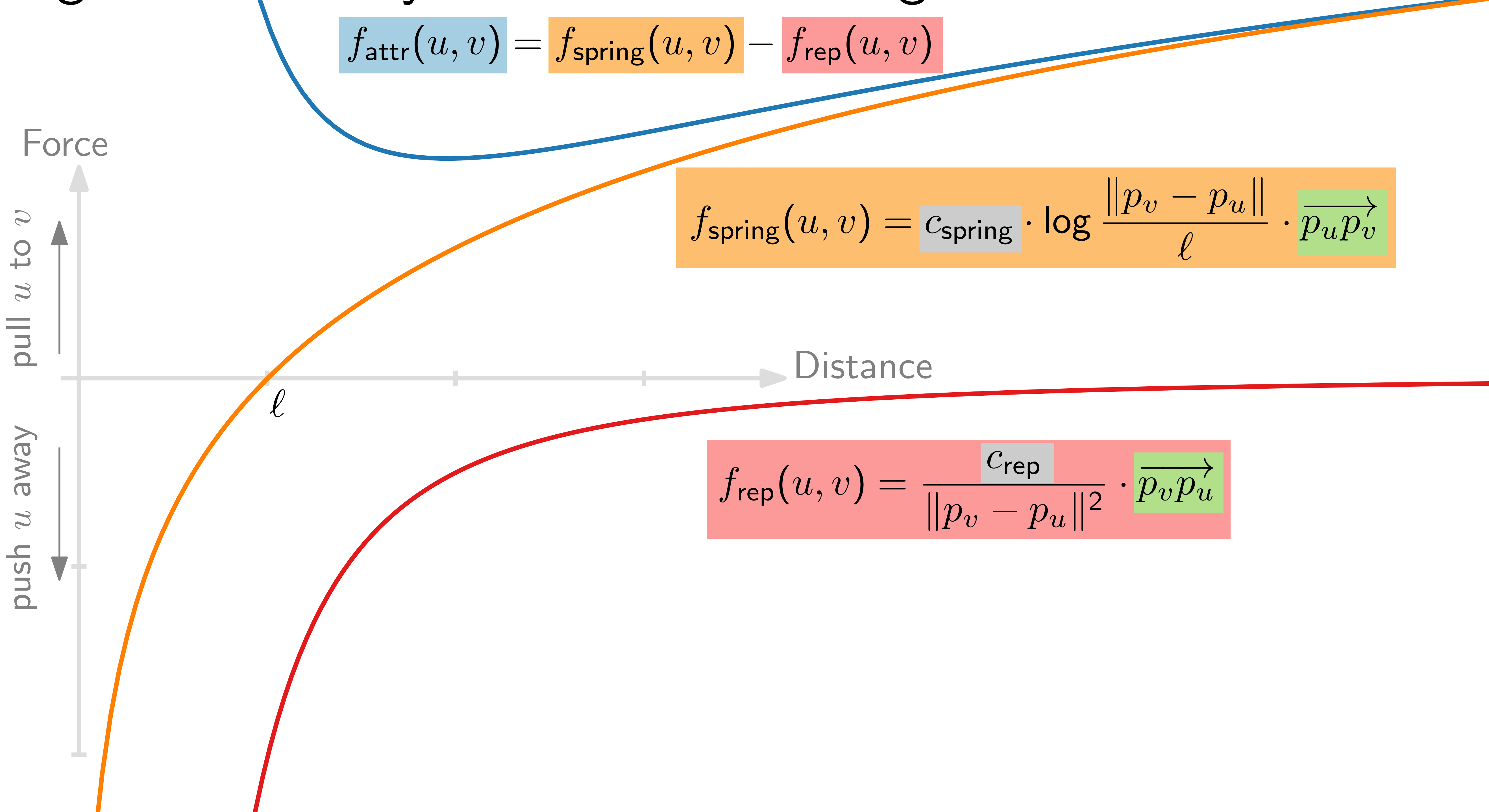
Spring Embedder by Eades – Force Diagram



$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_v p_u}$$

Spring Embedder by Eades – Force Diagram

$$f_{\text{attr}}(u,v) = f_{\text{spring}}(u,v) - f_{\text{rep}}(u,v)$$



$$f_{\text{spring}}(u,v) = c_{\text{spring}} \cdot \log \frac{\|p_v - p_u\|}{\ell} \cdot \overrightarrow{p_u p_v}$$

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Spring Embedder by Eades – Discussion

Advantages.

Spring Embedder by Eades – Discussion

Advantages.

- very simple algorithm
- good results for small and medium-sized graphs
- empirically good representation of symmetry and structure

Disadvantages.

- System may not be stable at the end.
- Converging to local minima.
- Computation of f_{spring} in $\mathcal{O}(|E(G)|)$ time and f_{rep} in $\mathcal{O}(|V(G)|^2)$ time.

Influence.

- original paper by Peter Eades [Eades '84] got $\sim$ 2000 citations
- basis for many further ideas

Variant by Fruchterman & Reingold

```
ForceDirected(G = (V, E), p = (p_v)_{v∈V}, ε > 0, K ∈ ℕ)
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```

- **Repulsive forces**

repulsion constant (e.g. 2.0)

$$f_{\text{rep}}(u,v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_v p_u}$$

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spring constant (e.g. 1.0)

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- $\|p_u - p_v\|$ = Euclidean distance between u and v
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```

- **Repulsive forces**

$$f_{\text{rep}}(u,v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\text{attr}}(u,v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

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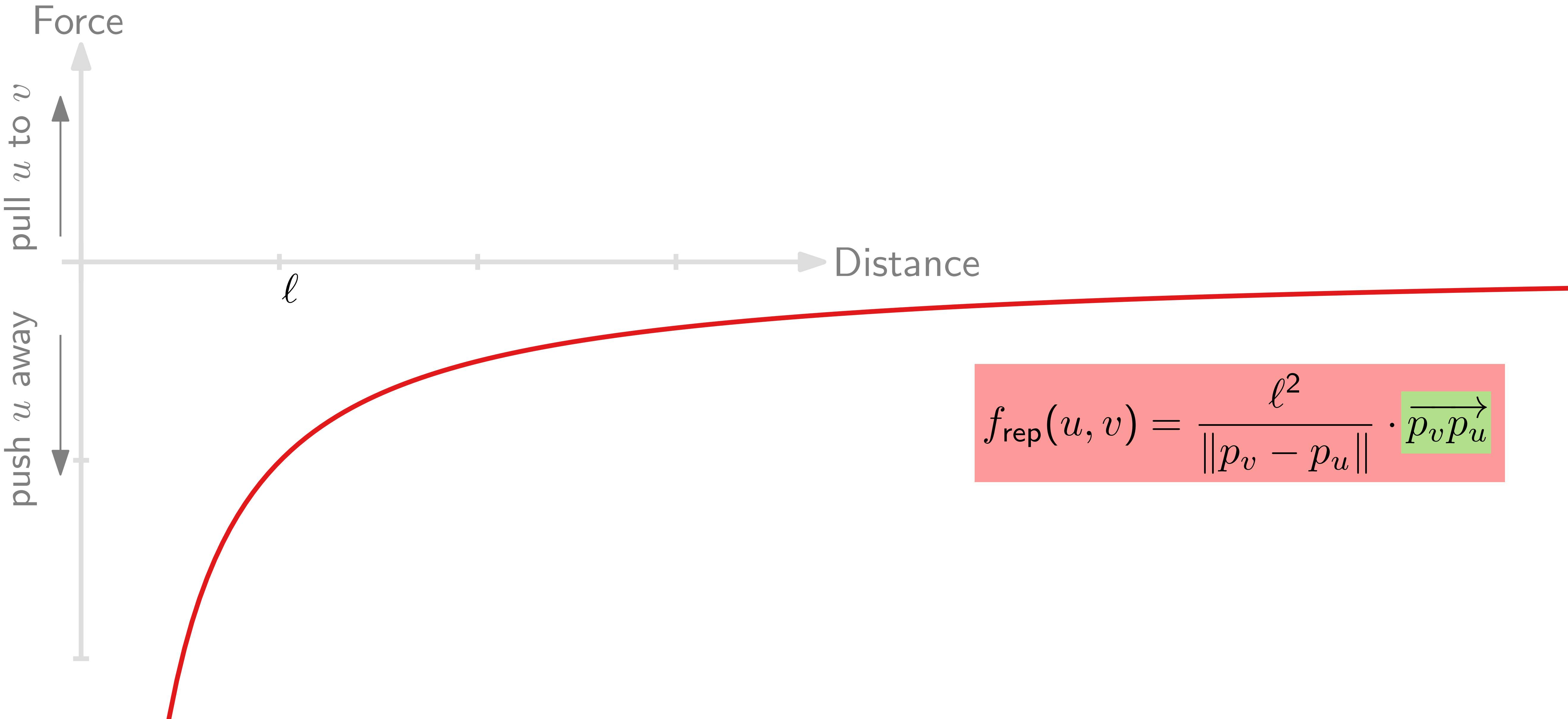
Fruchterman & Reingold – Force Diagram

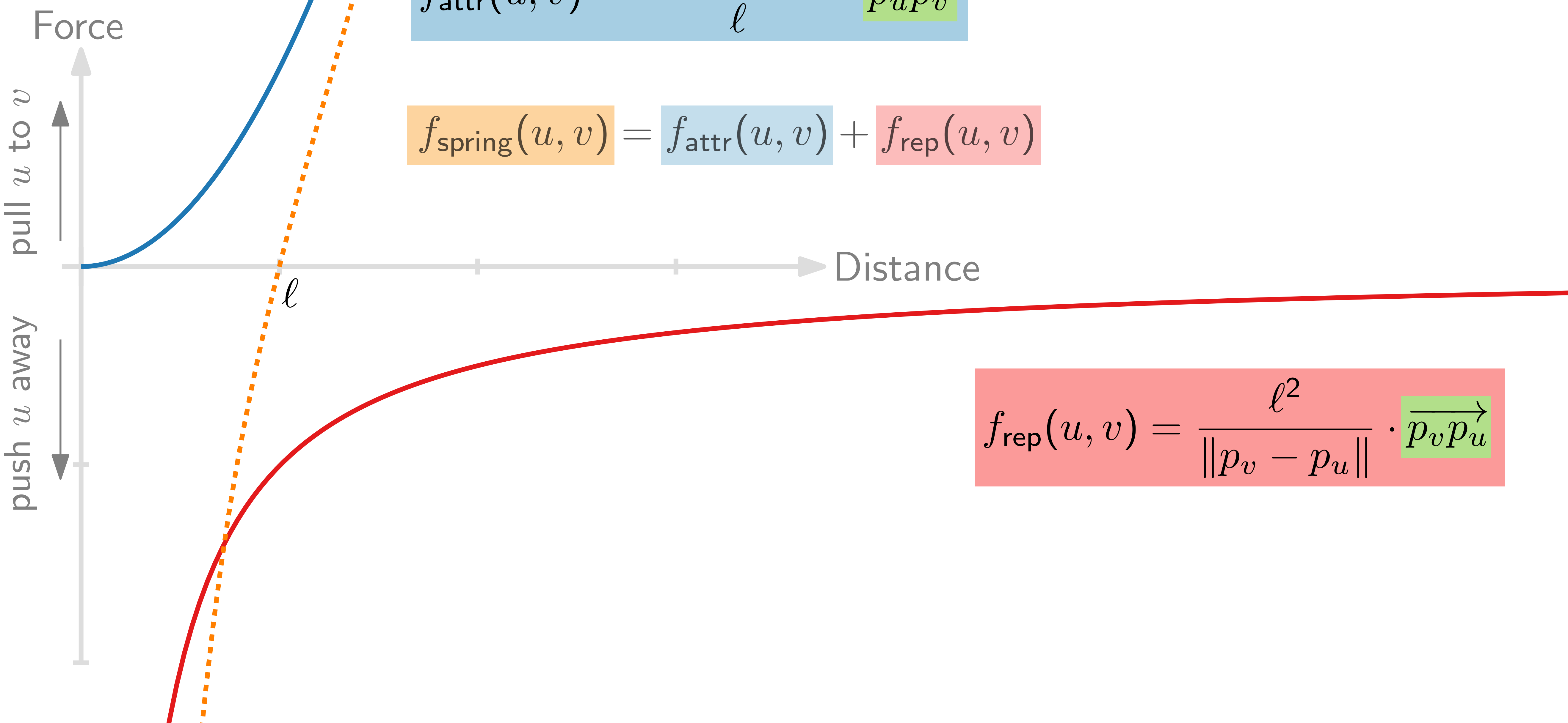


$$f_{\mathsf{rep}}(u,v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(u,v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{spring}}(u,v) = f_{\text{attr}}(u,v) + f_{\text{rep}}(u,v)$$



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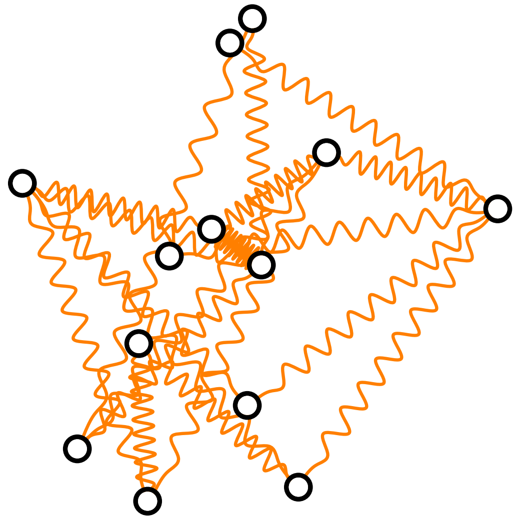
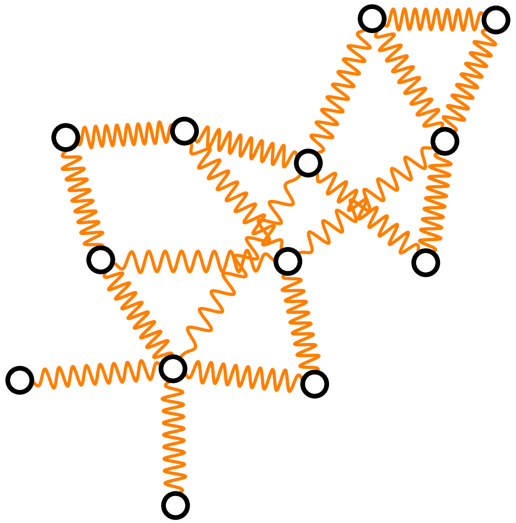
Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part III:
Variants & Improvements



Alexander Wolff

Adaptability

Inertia.

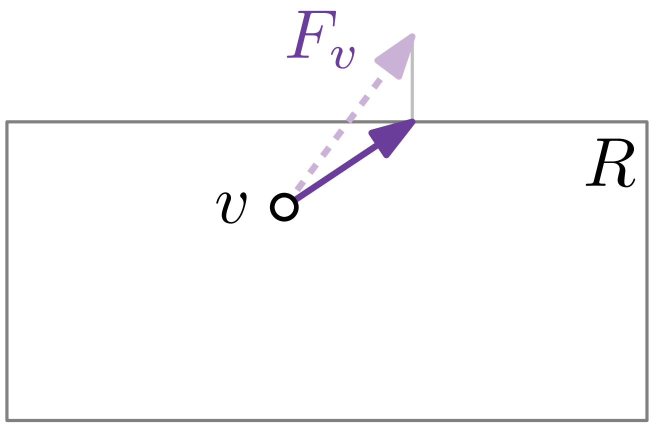
- Define vertex mass $\Phi(v) = 1 + \deg(v)/2$
- Set $f_{\text{attr}}(p_u, p_v) \leftarrow f_{\text{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

Gravitation.

- Define centroid $p_{\text{bary}} = 1/|V| \cdot \sum_{v \in V} p_v$
- Add force $f_{\text{grav}}(p_v) = c_{\text{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v p_{\text{bary}}}$

Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



And many more...

- magnetic orientation of edges [GD Ch. 10.4]
- other energy models
- planarity preserving
- speed-ups

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

```
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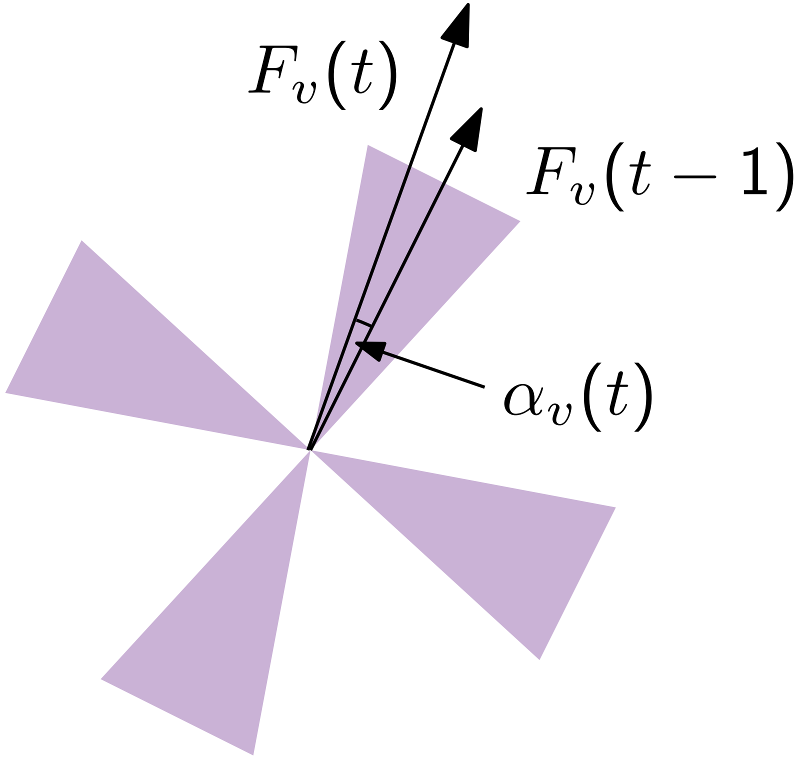
      foreach u ∈ V do
          p_u ← p_u + δ(t) · F_u(t)      [δ(t) struck out, replaced by δ_v(t)]
      t ← t + 1
  return p
```

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

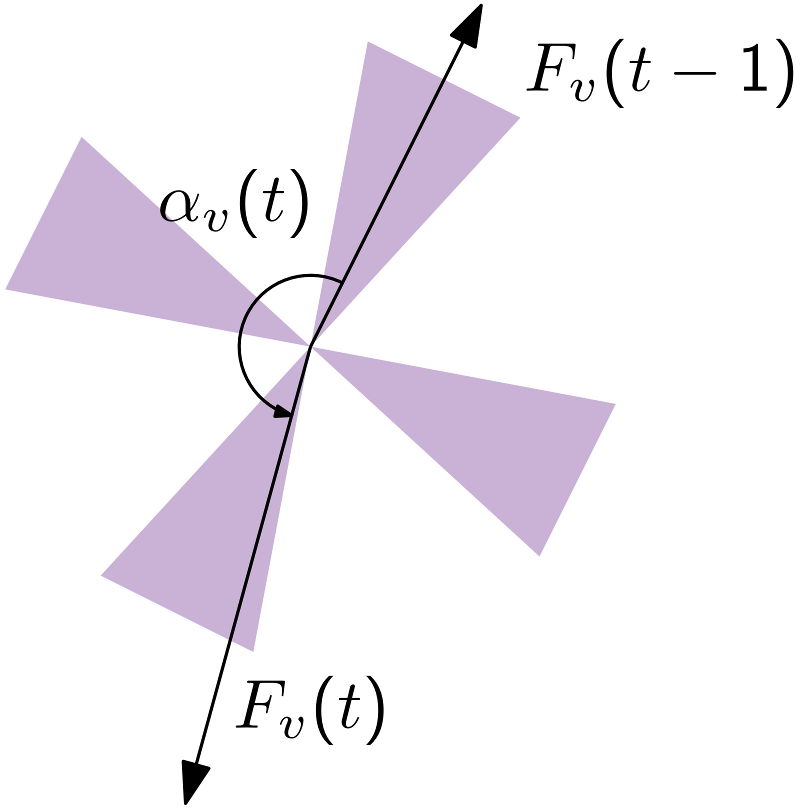


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

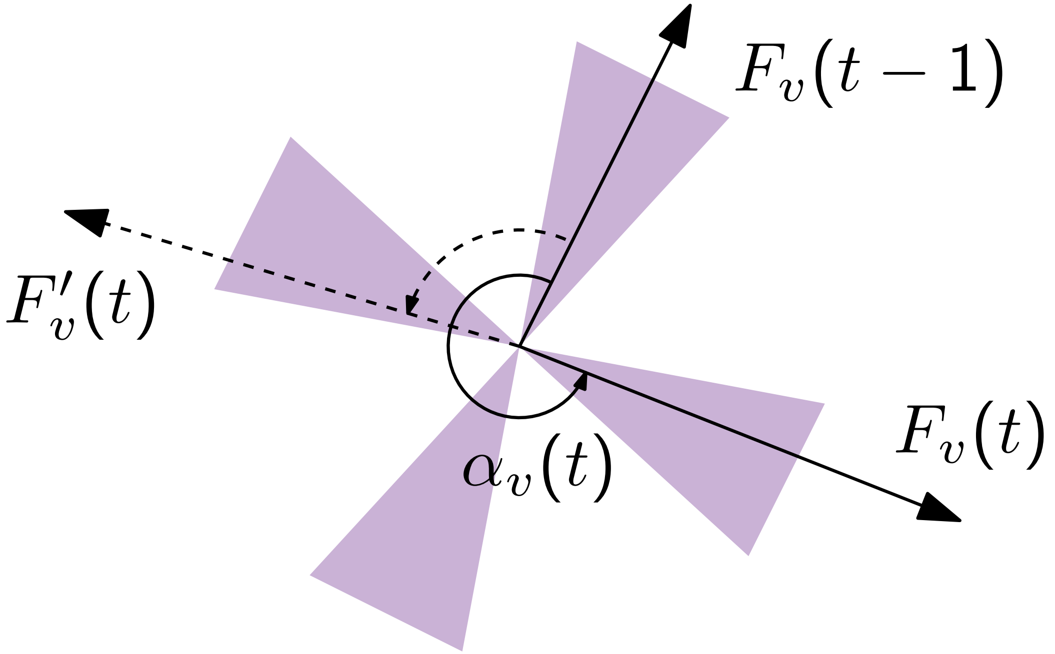


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oszillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Oszillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

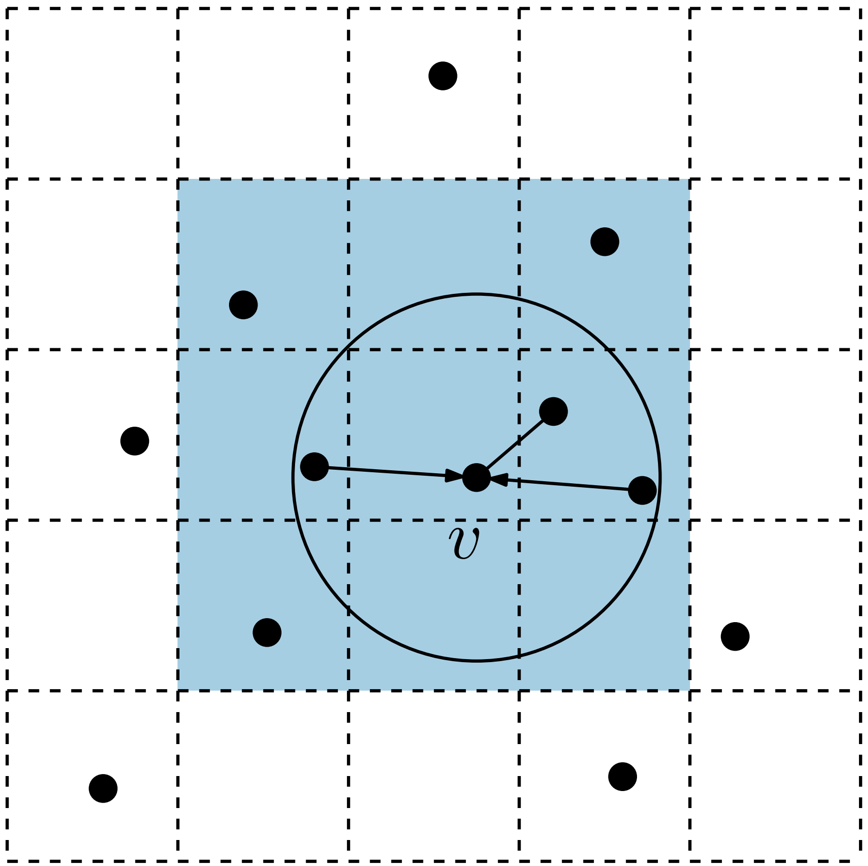
Rotation.

- count rotations
- if applicable

$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



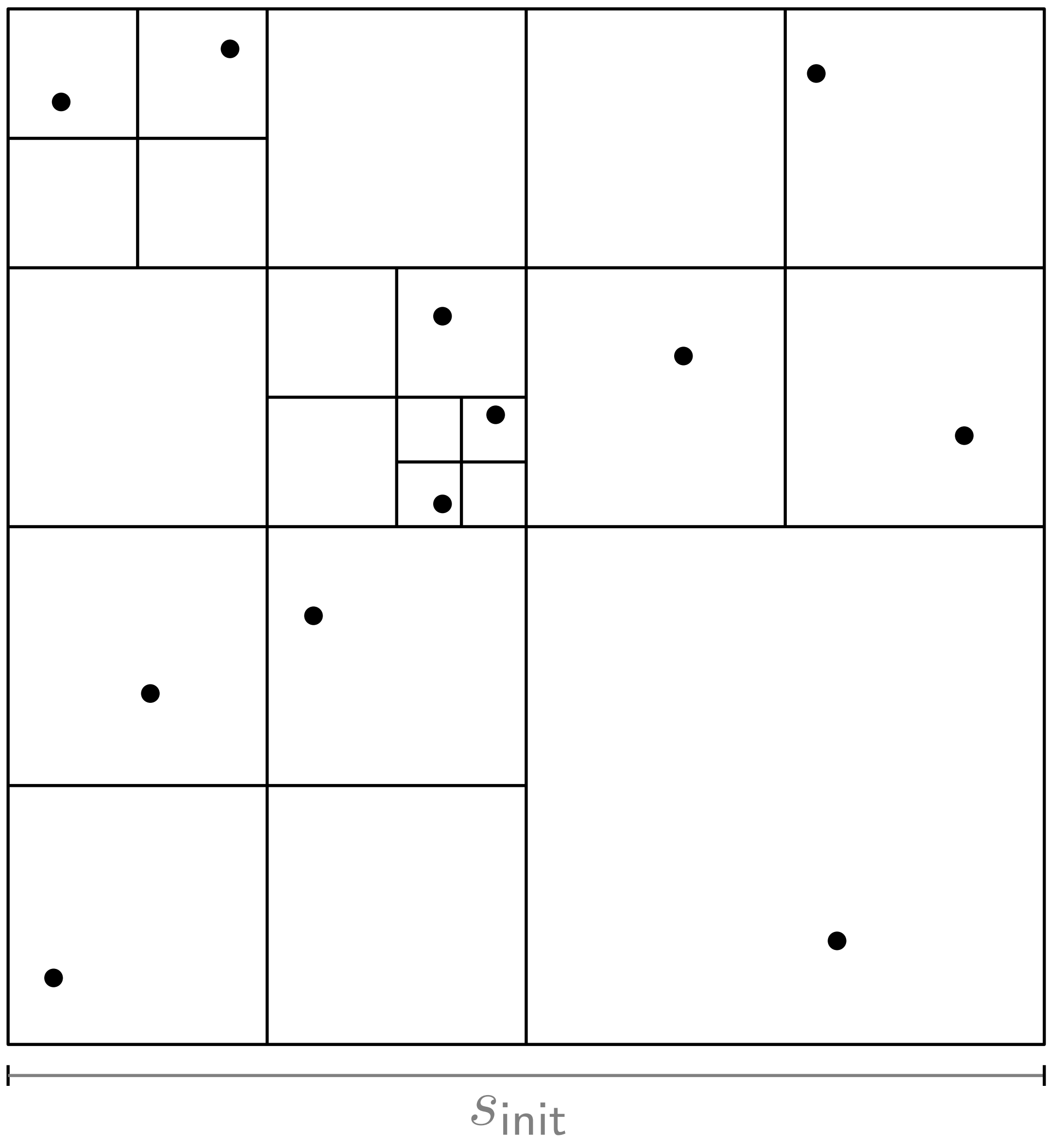
- divide plane into grid
- consider repelling forces only to vertices in neighboring cells
- and only if distance is less than some max distance

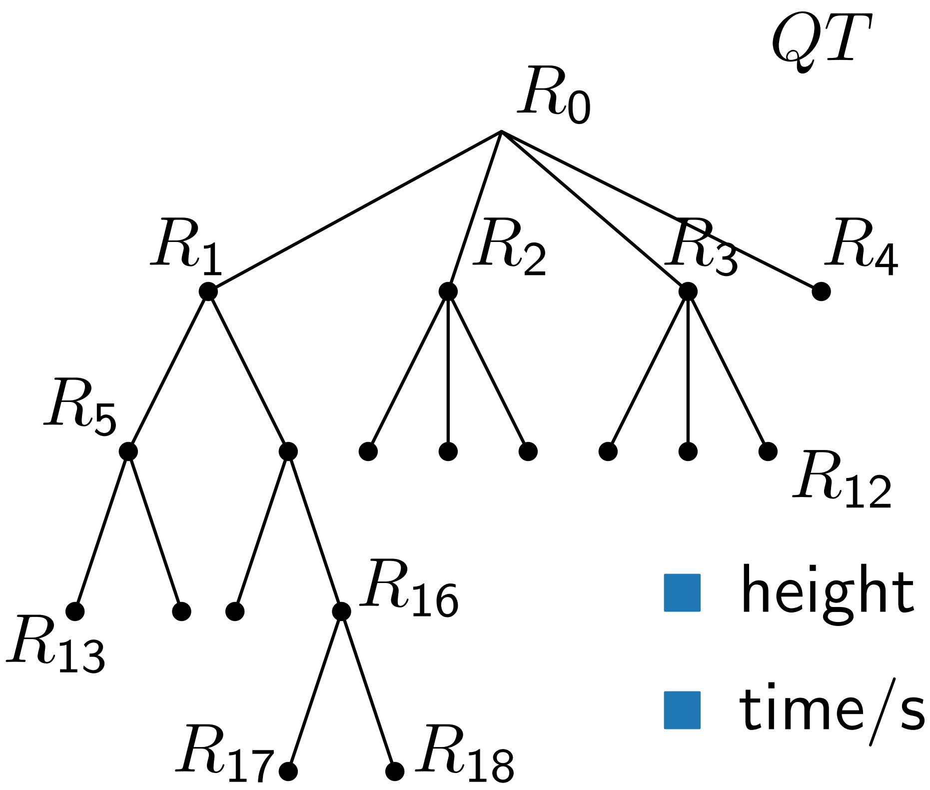
Discussion.

- good idea to improve actual runtime
- asymptotic runtime does not improve
- might introduce oszillation and thus a quality loss

Speeding up with Quad Trees

[Barnes, Hut '86]

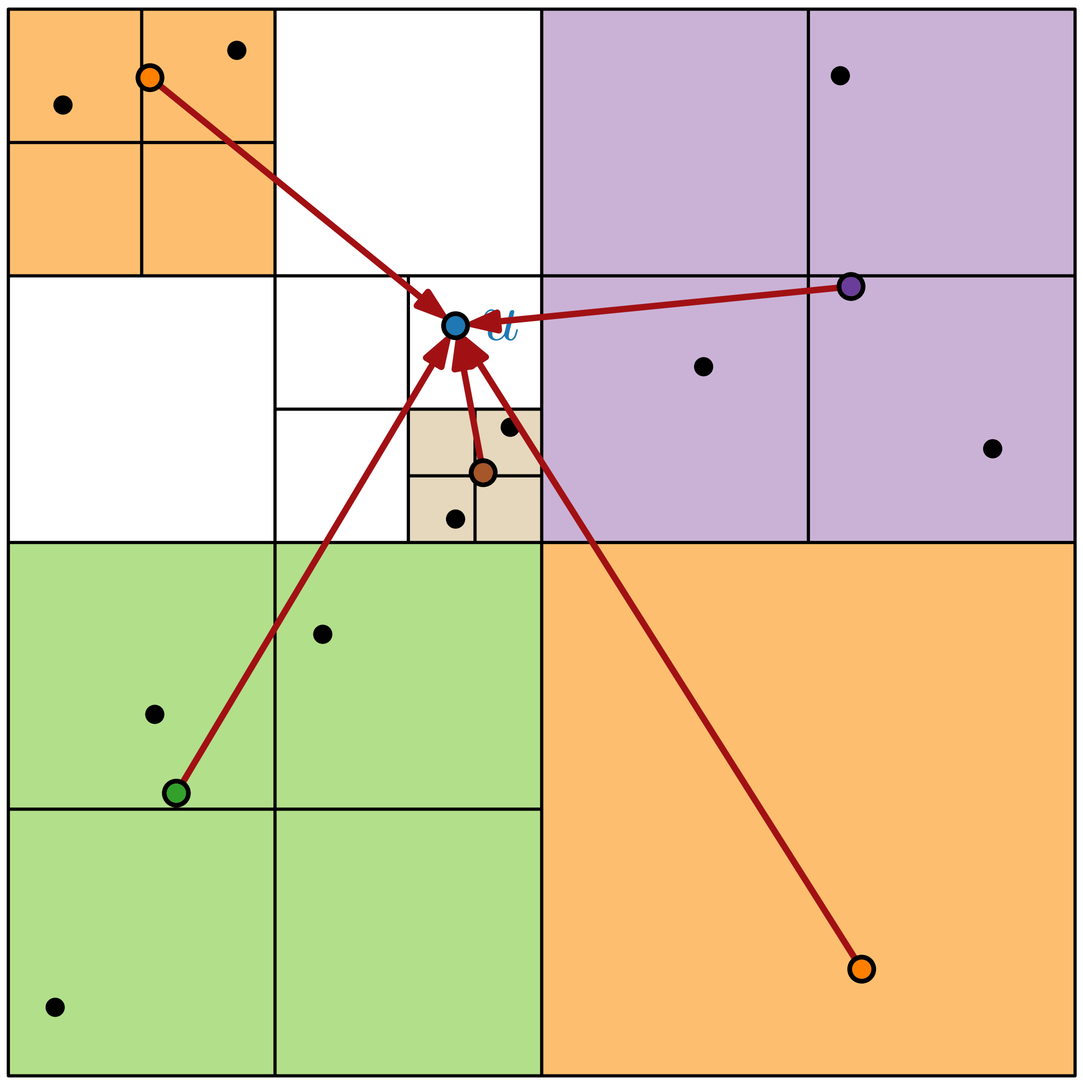


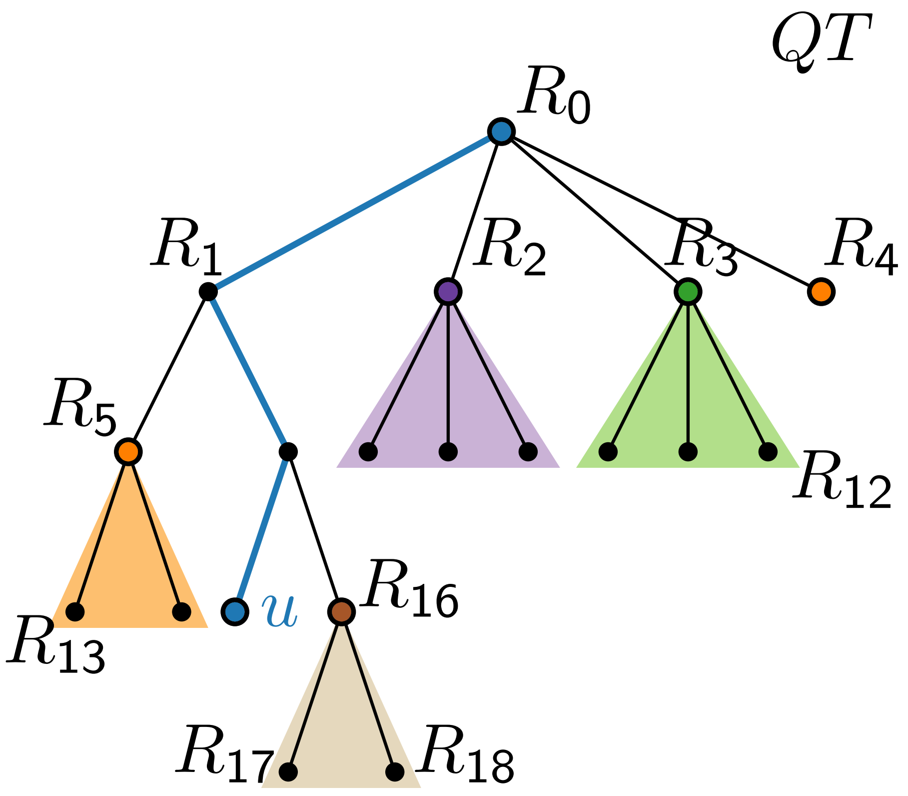


- height $h \leq \log \frac{s_{\text{init}}}{d_{\min}} + \frac{3}{2}$
- time/space in $\mathcal{O}(hn)$
- compressed quad tree can be computed in $\mathcal{O}(n \log n)$ time
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed

Speeding up with Quad Trees

[Barnes, Hut ’86]





$$f_{\text{rep}}(R_i, p_u) = |R_i| \cdot f_{\text{rep}}(\sigma_{R_i}, p_u)$$

for each child R_i of a vertex on path from u to root.

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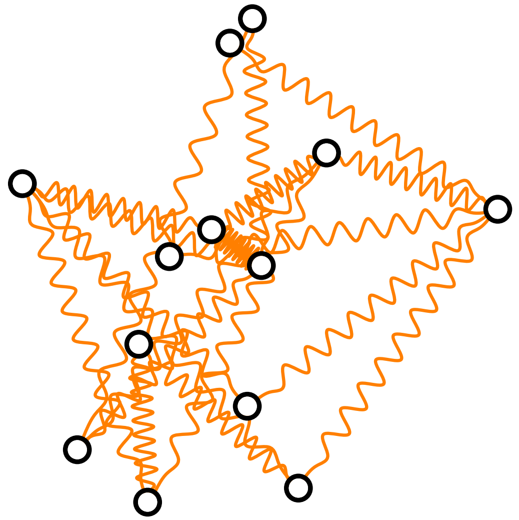
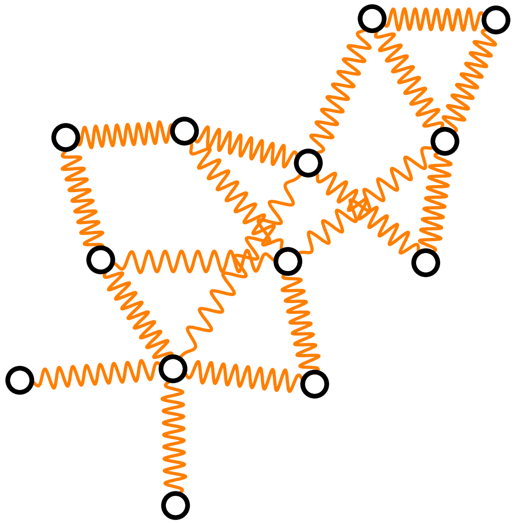


Part IV:
Tutte Embedding



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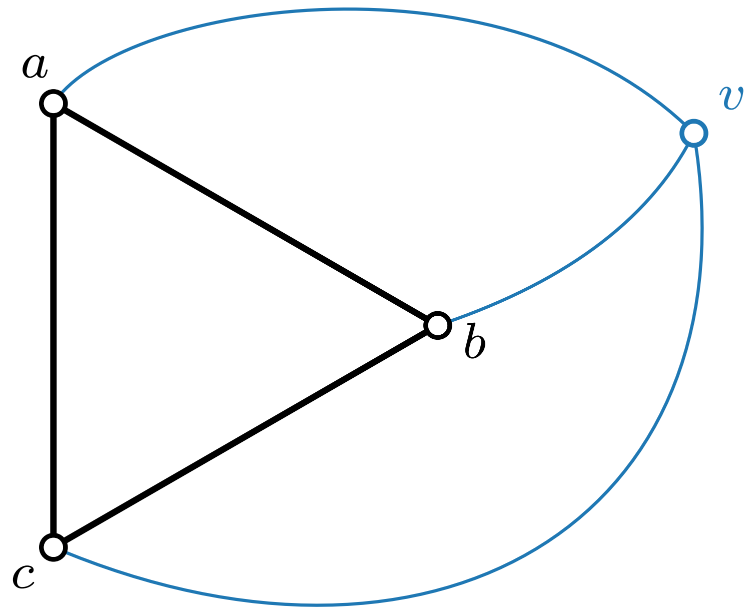
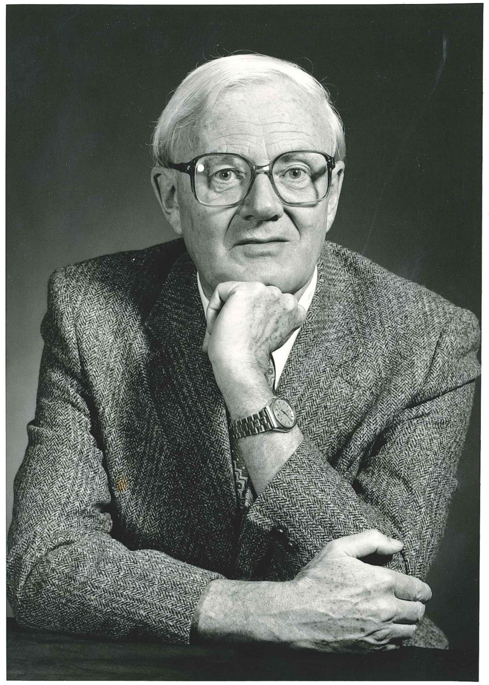
Idea

Consider a fixed triangle (a, b, c) with one common neighbor v

Where would you place v?





William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with one common neighbor v

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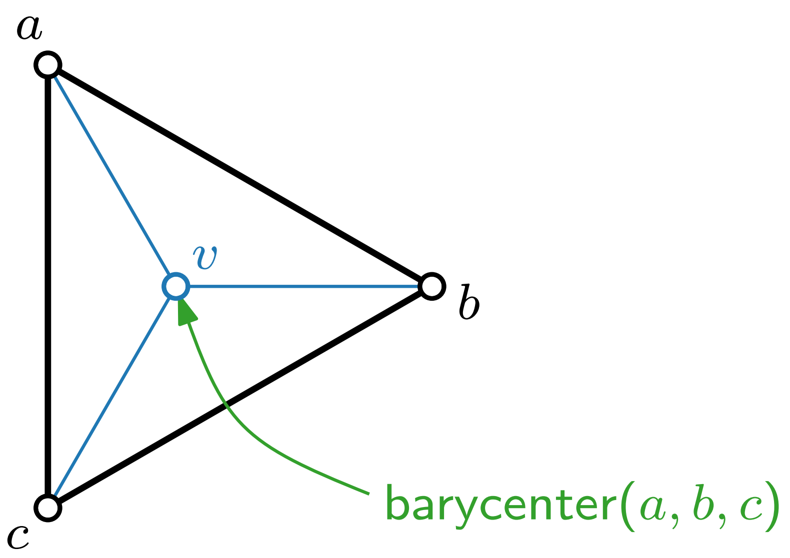
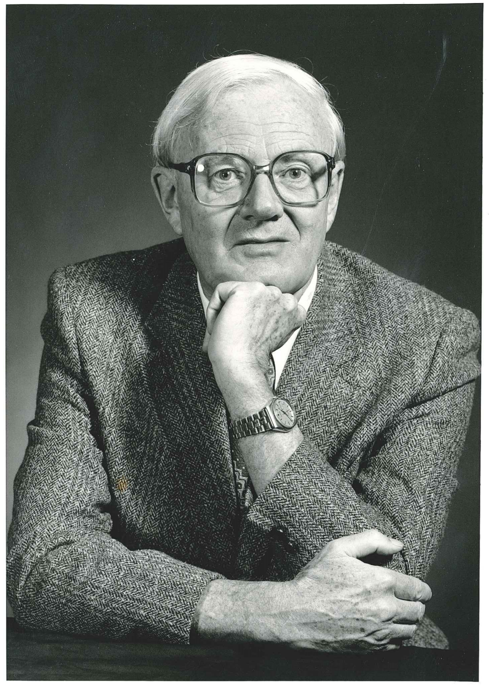


barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Idea.
Repeatedly place every vertex at barycenter of neighbors.



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1917 – 2002

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$
$\quad = \sum_{v \in \text{Adj}(u)} p_v / \deg(u)$

$F_u(t) = \sum_{uv \in E} p_v / \deg(u) - p_u$
$\quad = \sum_{v \in \text{Adj[u]}} (p_v - p_u) / \deg(u)$
$\quad = \sum_{v \in \text{Adj[u]}} \|p_u - p_v\| / \deg(u)$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
- $t \leftarrow 1$
- **while** $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**
  - **foreach** $u \in V$ **do**
    - $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{v \in \text{Adj[u]}} f_{\text{attr}}(u, v)$
  - **foreach** $u \in V$ **do**
    - $p_u \leftarrow p_u +$ ~~$\delta(t)$~~ $1 \cdot F_u(t)$
  - $t \leftarrow t + 1$
- **return** p

barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

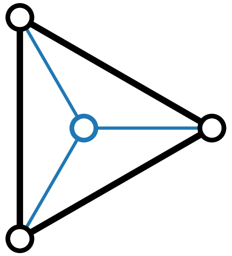
- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

- **Attractive forces**

$$f_{\text{attr}}(u, v) = \begin{cases} 0 & \text{if } u \text{ fixed,} \\ \frac{1}{\deg(u)} \cdot \|p_u - p_v\| & \text{otherwise.} \end{cases}$$

Solution: $p_u = (0, 0)\ \forall u \in V$



Fix coordinates of outer face!

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

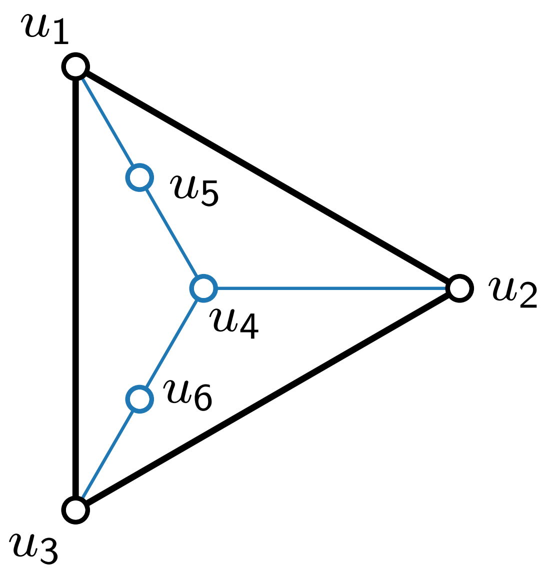
$p_u = \text{barycenter(Adj[u])} = \sum_{v \in \text{Adj[u]}} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two Systems of linear equations:

$x_u = \sum_{v \in \text{Adj[u]}} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj[u]}} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj[u]}} x_v = 0$

$y_u = \sum_{v \in \text{Adj[u]}} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj[u]}} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj[u]}} y_v = 0$



A

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	3	−1	−1	0	−1	0
u_2	−1	3	−1	−1	0	0
u_3	−1	−1	3	0	0	−1
u_4	0	−1	0	3	−1	−1
u_5	−1	0	0	−1	2	0
u_6	0	0	−1	−1	0	2

Laplacian matrix of G

$A_{ii} = \deg(u_i)$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

n variables, n constraints, $\det(A) = 0$

$\Rightarrow$ no unique solution ☹

Linear System of Equations

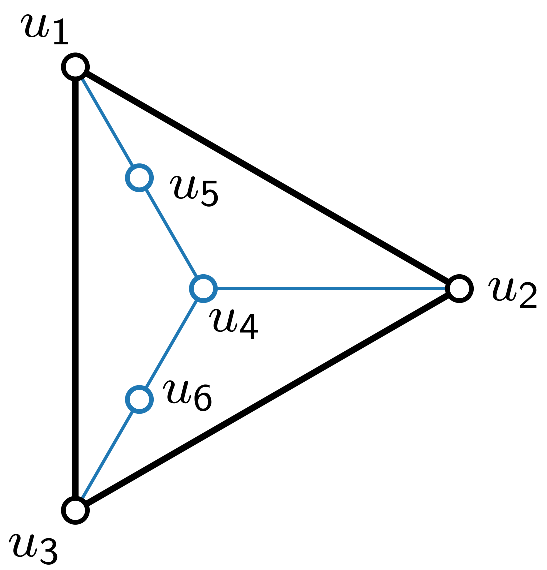
Theorem. **Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$

$p_u =$ barycenter(Adj[u]) $=$

$$x_u = \sum_{v \in \text{Adj[u]}} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj[u]}} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj[u]}} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj[u]}} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj[u]}} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj[u]}} y_v = 0$$



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Laplacian matrix of G

$$A_{ii} = \deg(u_i)$$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

k variables, k constraints, $\det(A) > 0$

$k = \#$free vertices

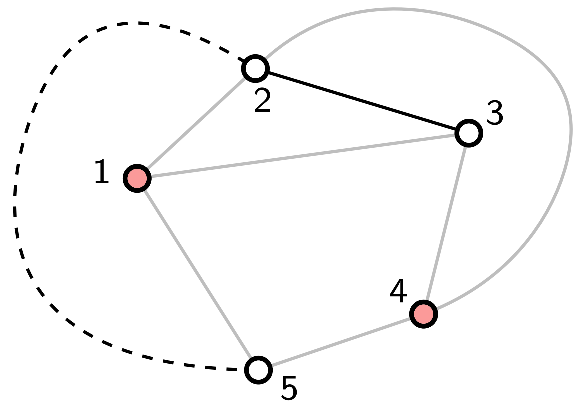
$\Rightarrow$ unique solution 😀

3-Connected Planar Graphs

planar: G can be drawn in such a way that no edges cross each other

connected: $\exists\ u$–v path for every vertex pair $\{u, v\}$.

k-**connected:** $G - \{v_1, \ldots, v_{k-1}\}$ is connected for **any** $k-1$ vertices $v_1 \ldots, v_{k-1}$.



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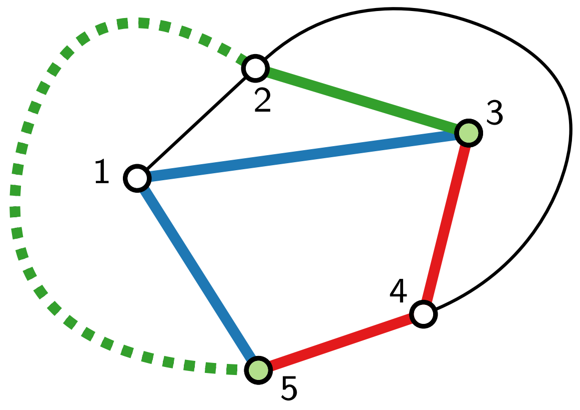
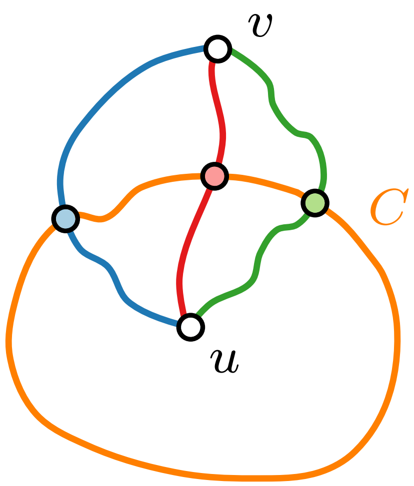
Or (equivalently):
There are at least k vertex-disjoint u–v paths for every vertex pair $\{u, v\}$.

Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.

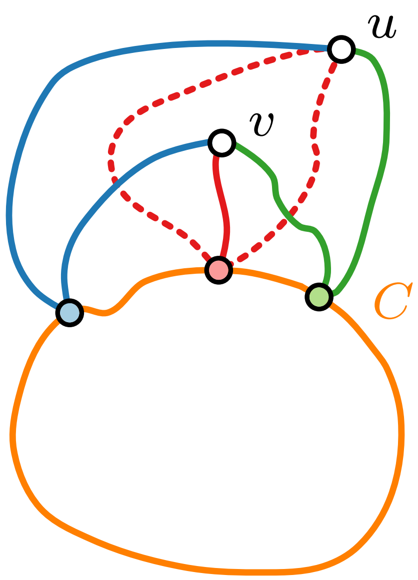
Proof sketch.
Γ_1, Γ_2 embeddings of G.
Let C be a face of Γ_2, but not of Γ_1.
u inside C in Γ_1, v outside C in Γ_1
both on same side in Γ_2



Γ_1



Γ_2



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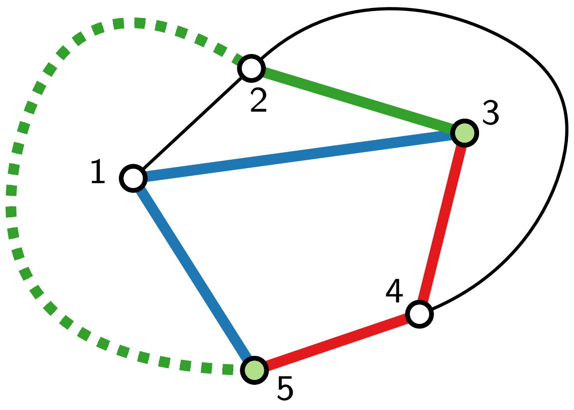

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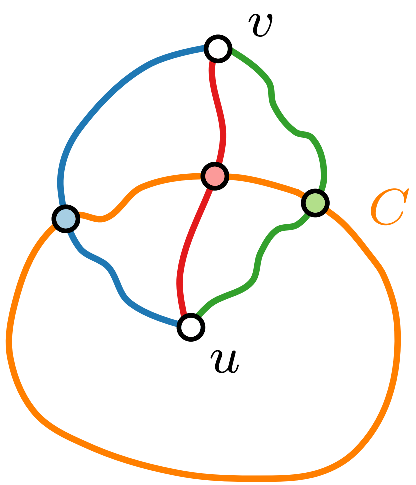
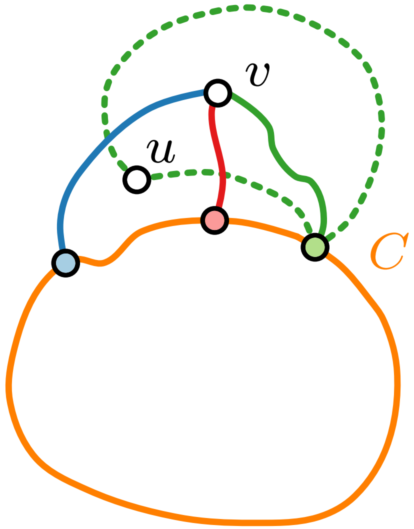


Γ_2



3-Connected Planar Graphs

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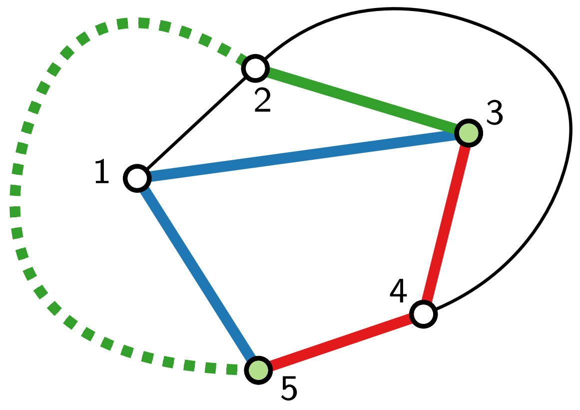
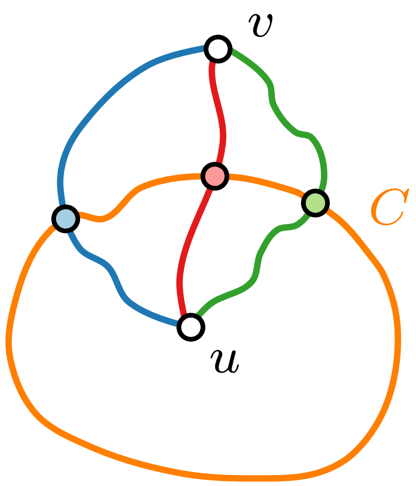
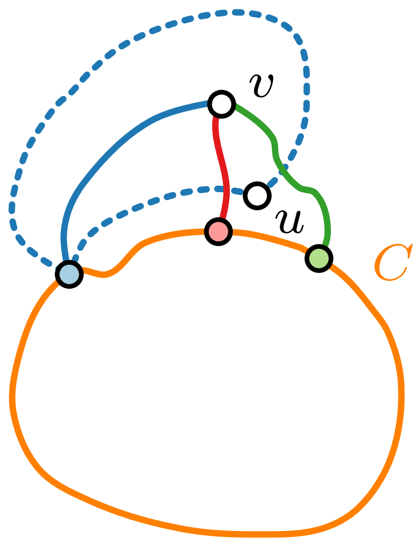


Γ_1

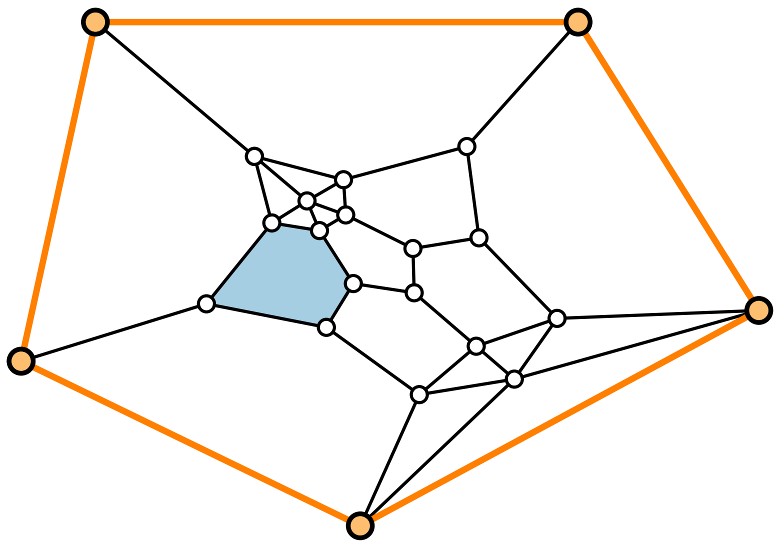


Γ_2



Tutte's Theorem

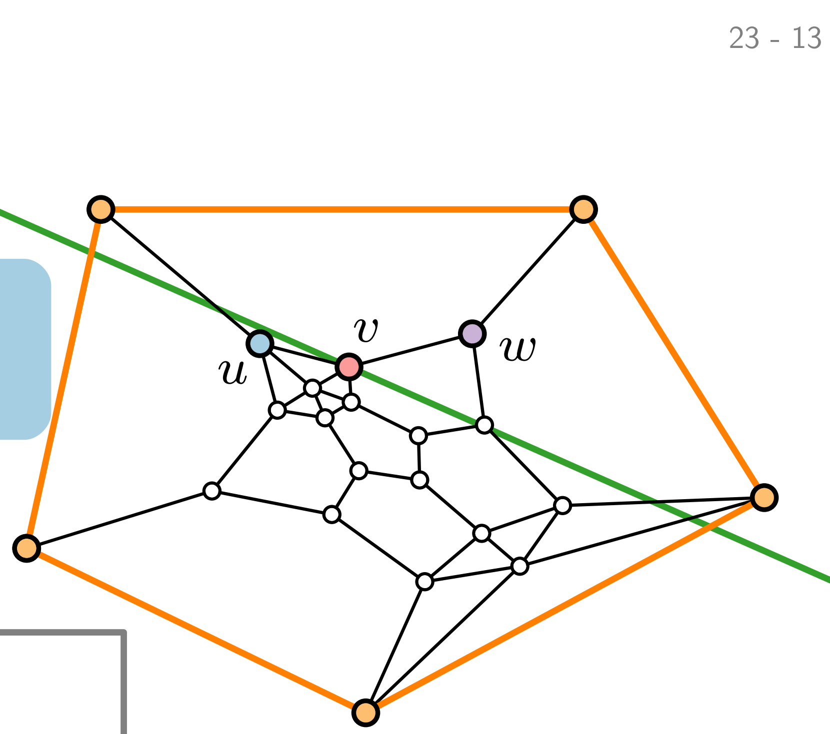
Theorem. [Tutte 1963]
Let G be a 3-connected planar graph, and
let C be a face of its unique embedding.
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar and all its faces are strictly convex.



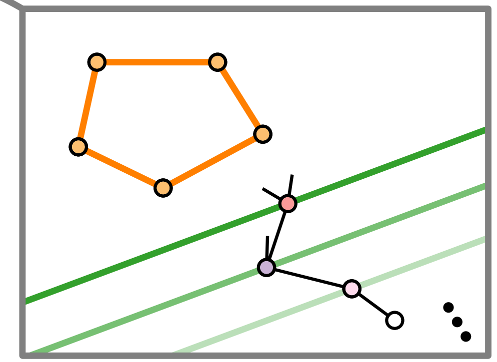
Properties of Tutte Drawings

Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

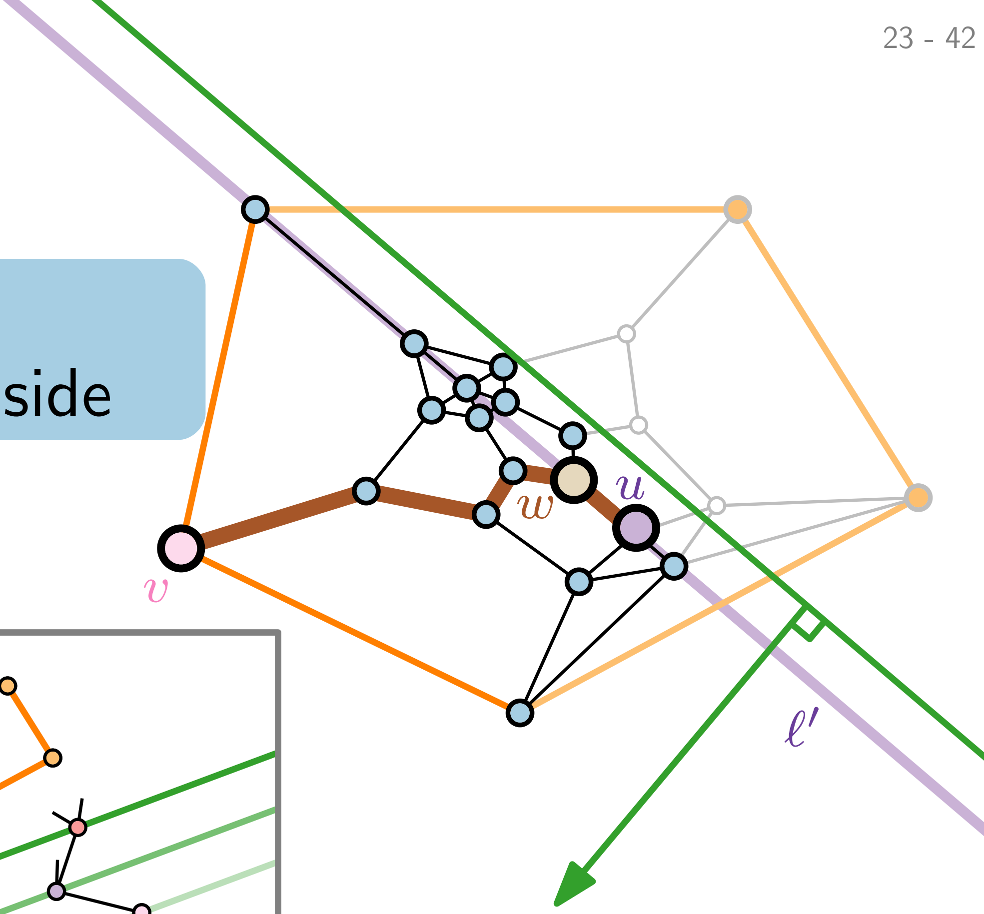
Otherwise, all forces to same side ...



Property 2. All free vertices lie inside C.



Properties of Tutte Drawings



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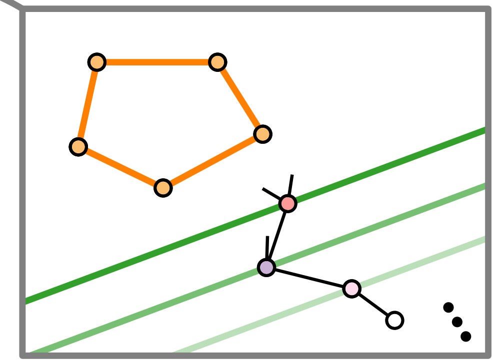
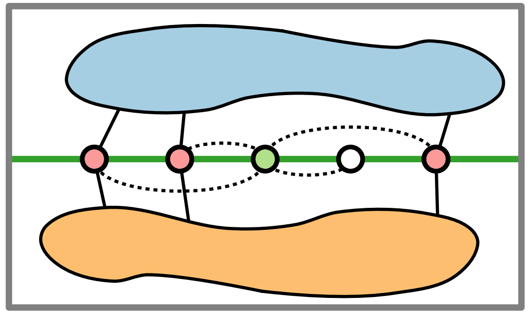


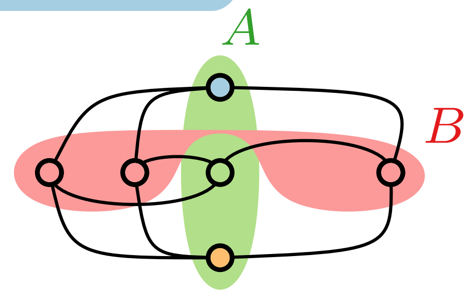
Property 3. Let ℓ be any line.
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ through u
G connected, v not on $\ell' \Rightarrow \exists w$ on ℓ' with neighbor further away from ℓ
$\Rightarrow \exists$ path from u to v

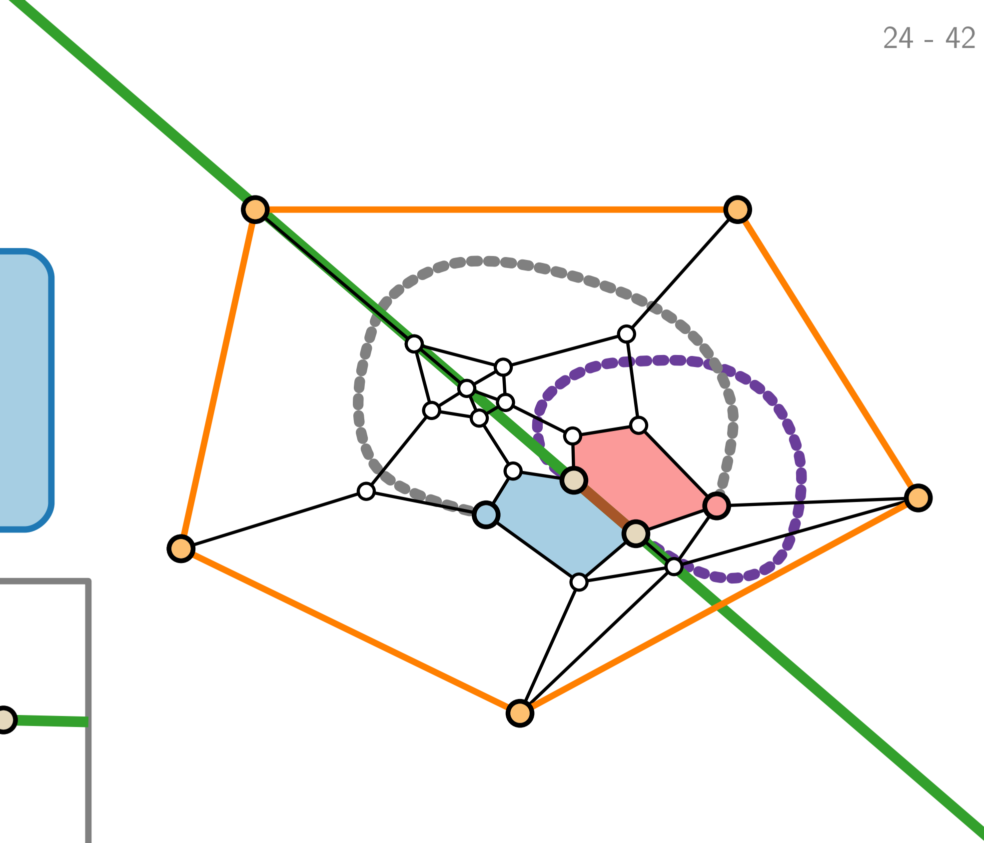
Property 4. No vertex is collinear with all of its neighbors.

Not all vertices collinear
G 3-connected
$\Rightarrow K_{3,3}$ minor





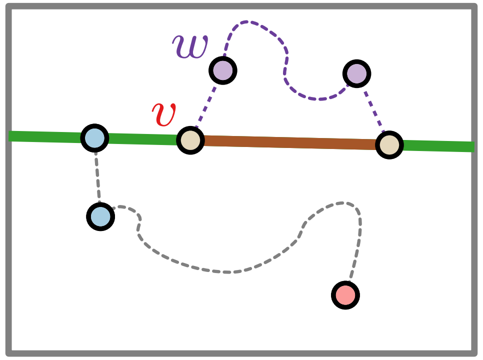
Proof of Tutte's Theorem



Lemma. Let uv be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.

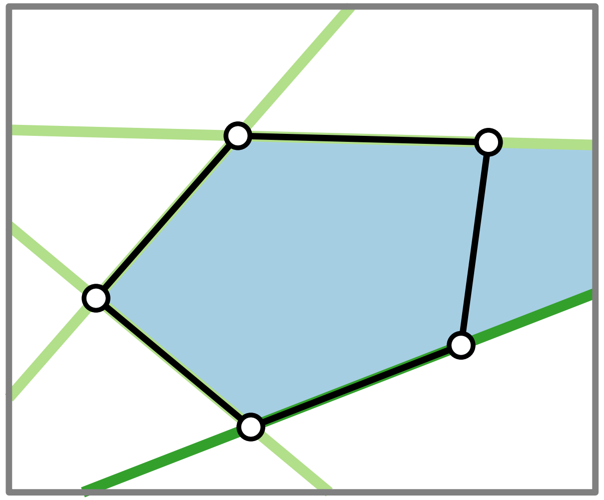
Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Property 3. Let ℓ be any line.
Let V_ℓ be the set of vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.



Property 4. No vertex is collinear with all of its neighbors.

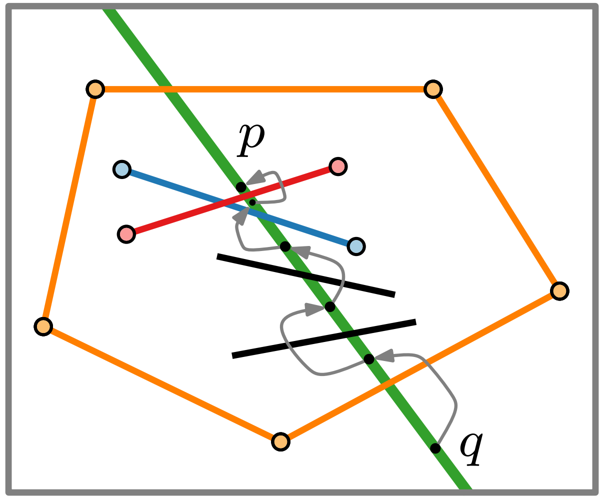
Lemma. All faces are strictly convex.



Lemma. The drawing is planar.

Property 2. All free vertices lie inside C.

p inside two faces
$\Rightarrow q$ in one face
jumping over edge
$\rightarrow$ #faces the same
$\Rightarrow p$ inside one face ↯



Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Original papers:

- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Tutte 1963] How to draw a graph