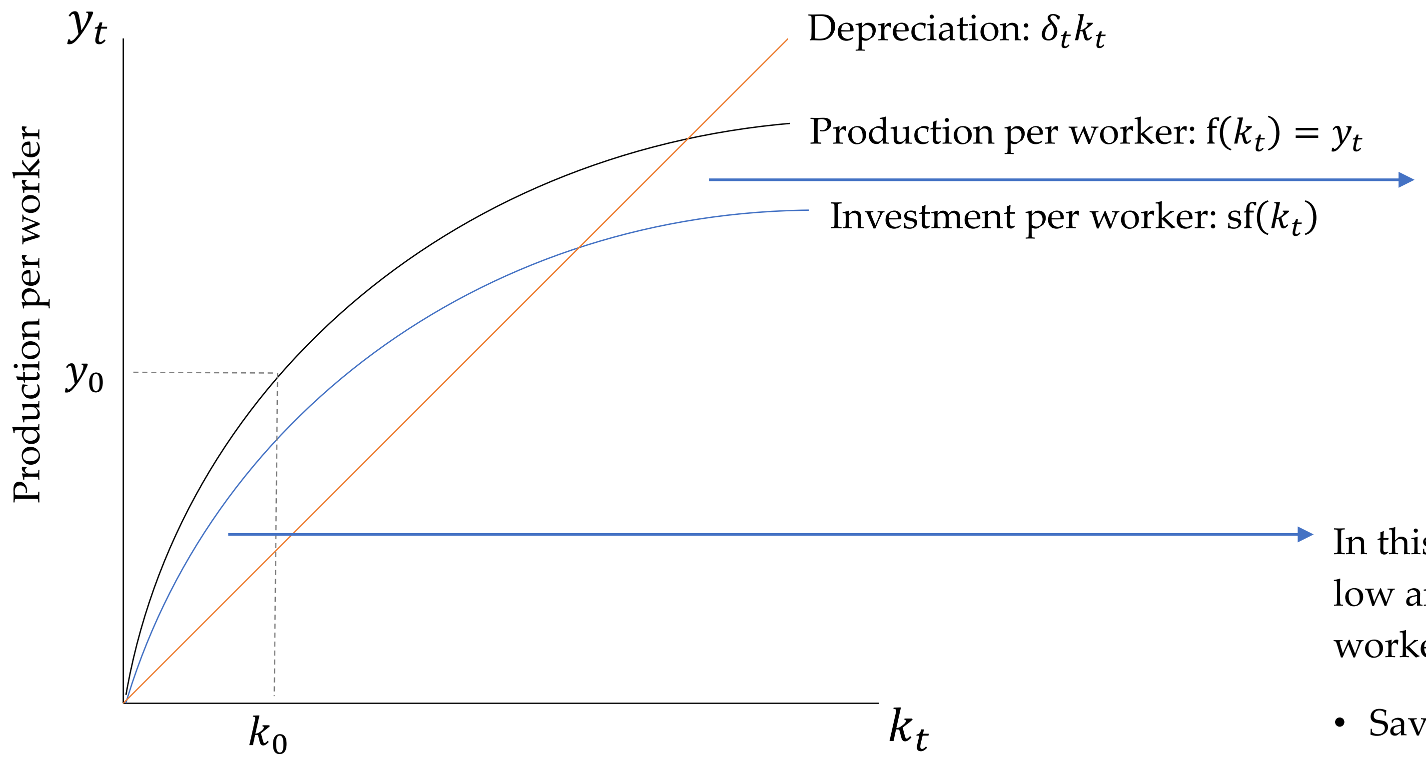
TASK 1: (a) Use the graph below to graphically construct the equilibrium of an economy, assuming constant population and constant technology growth. Clearly indicate the curves and the resulting steady state in the graph.



In this area: capital per worker and production per worker are high and depreciation per worker exceeds investment per worker, hence capital per worker and production decrease → shrinkage

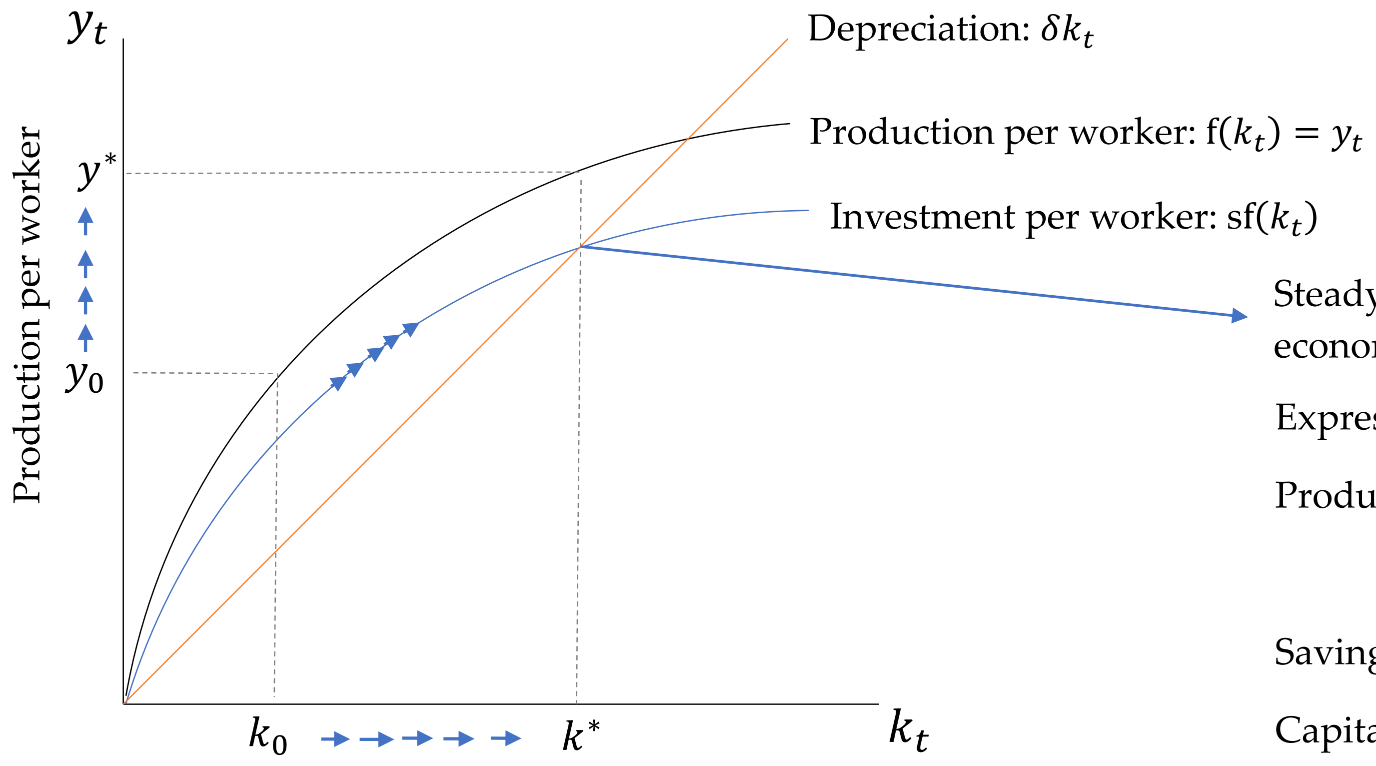
In this area: capital per worker and production per worker are low and investment per worker exceeds depreciation per worker, hence capital accumulates and production increases

- Savings (fraction of production) get invested
- Production increases with diminishing rate

→ growth

Recall: capital stock determines how many goods can be produced & production determines who much can be saved (invested) and therewith capital accumulation

TASK 1: (a) Use the graph below to graphically construct the equilibrium of an economy, assuming constant population and constant technology growth. Clearly indicate the curves and the resulting steady state in the graph.



Steady State: as 'point-of-attraction' in the model to which the economy converges in the long run

Expressing the same as before with a little help of math:

Production function: $Y_t = F(K_t, N)$

$I_t = S_t$

Savings as fraction of production: $S_t = sY_t$

Capital accumulation: $K_{t+1} = (1-\delta)K_t + I_t$

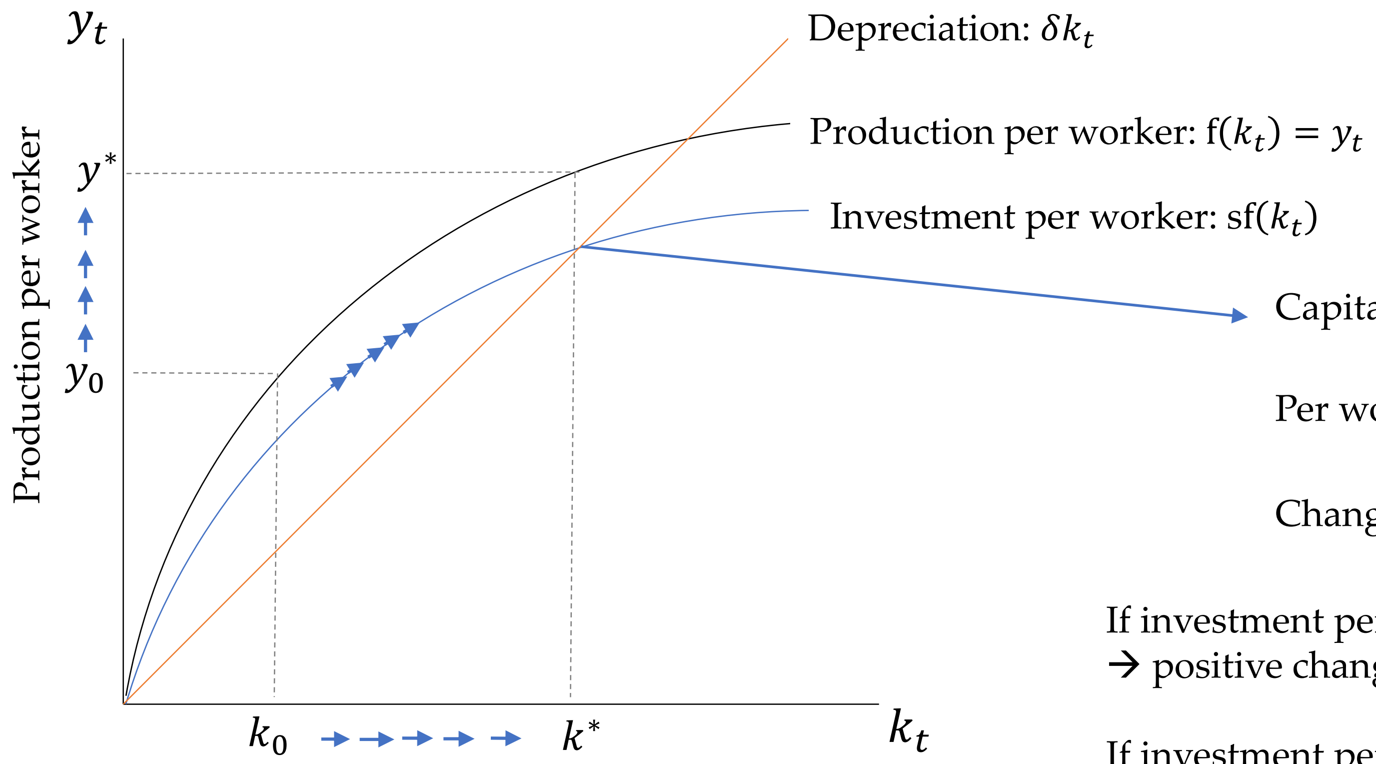
Steady State:

- Point of attraction in the model → model dynamics yield the economy to converge to that point regardless its starting point

- Point of stability for the system in absence of external shocks

- Important for economic analysis not because the economy is always there but because of convergence regardless the initial position
→ concept of long-run equilibrium

TASK 1: (a) Use the graph below to graphically construct the equilibrium of an economy, assuming constant population and constant technology growth. Clearly indicate the curves and the resulting steady state in the graph.



Capital accumulation: $K_{t+1} = (1-\delta)K_t + I_t$

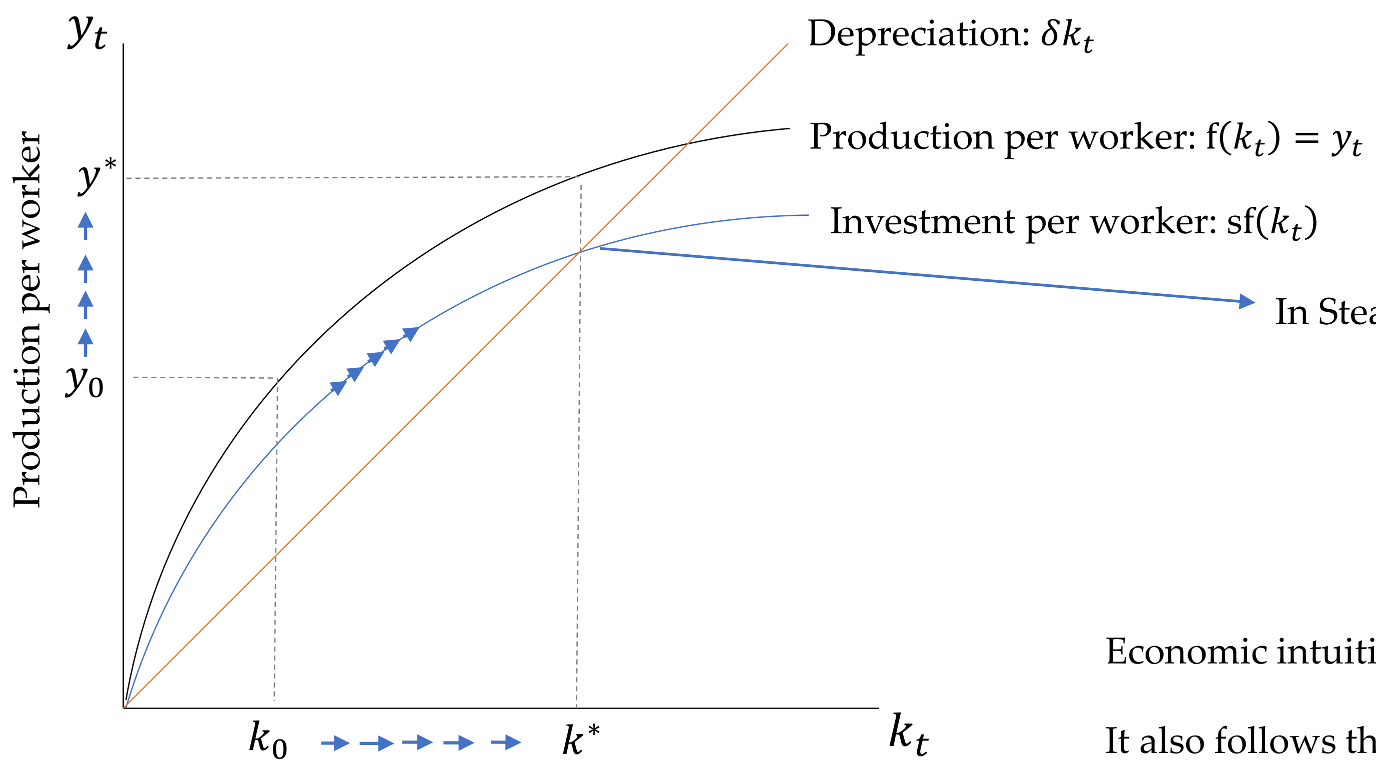
Per worker terms: $\frac{K_{t+1}}{L} = (1-\delta)\frac{K_t}{L} + s\frac{Y_t}{L}$

Change from t to $t+1$: $\frac{K_{t+1}}{L} - \frac{K_t}{L} = sf\left(\frac{K_t}{L}\right) - \delta\frac{K_t}{L}$

If investment per worker exceeds depreciation per worker
→ positive change in capital per worker

If investment per worker is smaller than depreciation per worker
→ negative change in capital per worker

TASK 1: (a) Use the graph below to graphically construct the equilibrium of an economy, assuming constant population and constant technology growth. Clearly indicate the curves and the resulting steady state in the graph.



In Steady State, it holds that: $K_{t+1} = K_t = K^*$

$$\mathbf{0} = sf\left(\frac{K_t}{L}\right) - \delta\frac{K_t}{L}$$

$$\Rightarrow \; sf\left(\frac{K_t}{L}\right) = \delta\frac{K_t}{L}$$

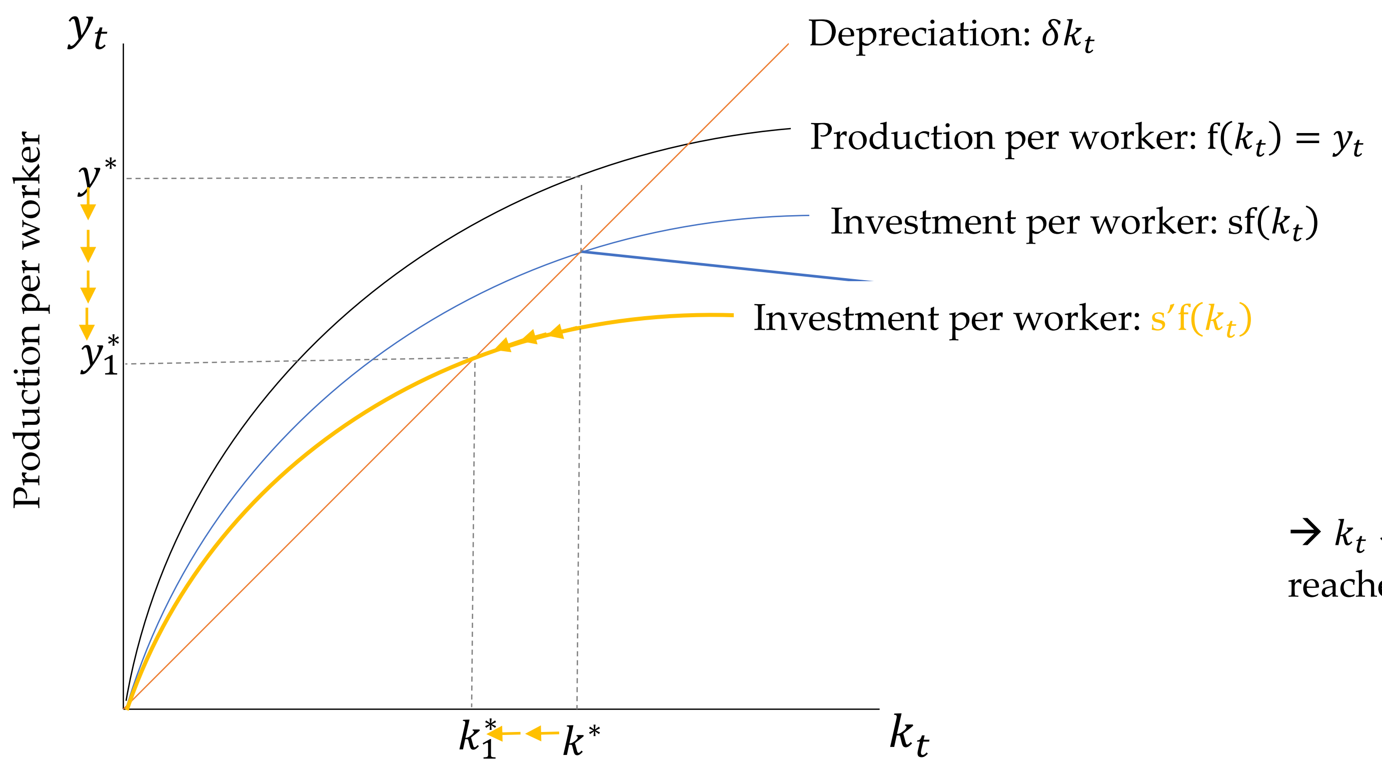
Economic intuition: investment is just as high as depreciation

It also follows that production per worker is constant

$$\frac{Y^*}{L} = f\left(\frac{K^*}{L}\right)$$

Model prediction: in case of low starting point (low production and capital per worker): faster growth and catching-up (historical example: GER and JPN after WWII)

TASK 1: (b) Use your initial equilibrium to illustrate the changes induced by a ***decrease*** in the savings rate.



$$sf\left(\frac{K_t}{L}\right) < \delta \frac{K_t}{L}$$

→ $k_t \downarrow$ - decumulation of capital until new steady state is reached

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

Kaldor Facts:

1. Output per worker $\left(\frac{Y_t}{N_t}\right)$ grows at an approximately constant rate over long periods of time

2. Capital per worker $\left(\frac{K_t}{N_t}\right)$ grows at an approximately constant rate over long periods of time

3. The capital to output ratio $\left(\frac{K_t}{Y_t}\right)$ is roughly constant over long periods of time

4. Labor's share of income $\left(\frac{w_t N_t}{Y_t}\right)$ is roughly constant over long periods of time (capital share: $R_t K_t Y_t = 1 - w_t N_t Y_t$)

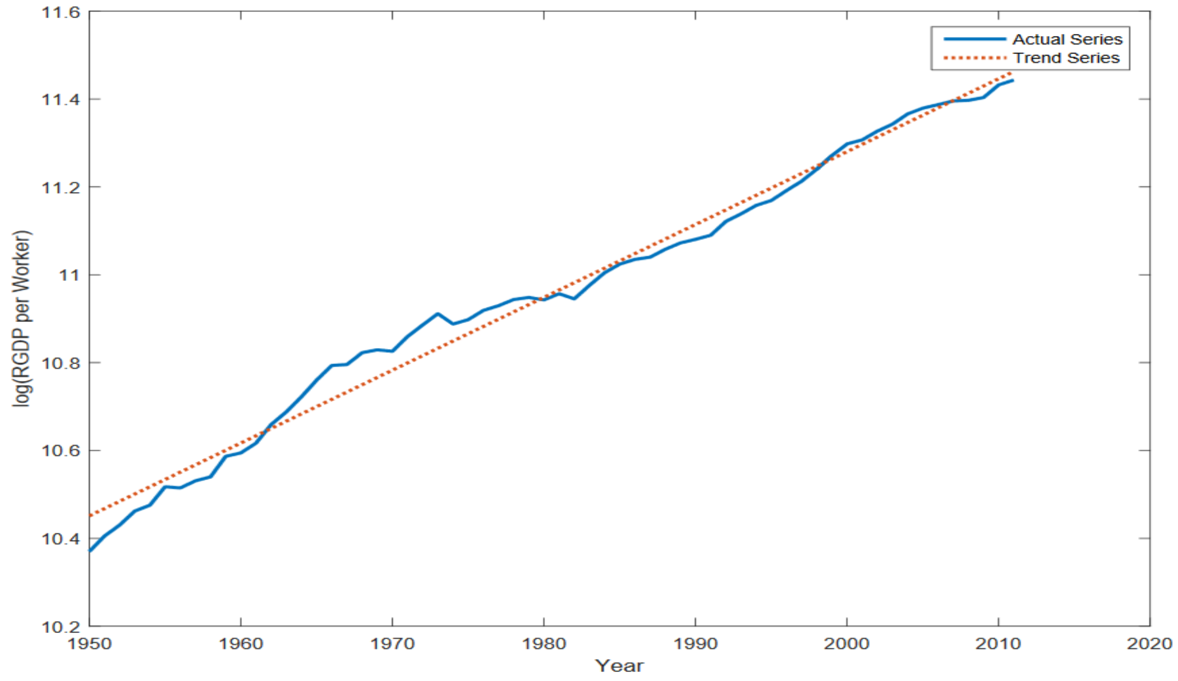
5. The return to capital (r_t) is roughly constant over long periods of time

6. The real wage grows (w_t) at approximately the same rate as output per worker over long periods of time

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

1. Output per worker $\left(\frac{Y_t}{N_t}\right)$ grows at an approximately constant rate over long periods of time:

Figure 4.1: Real GDP per Worker in the US 1950-2011

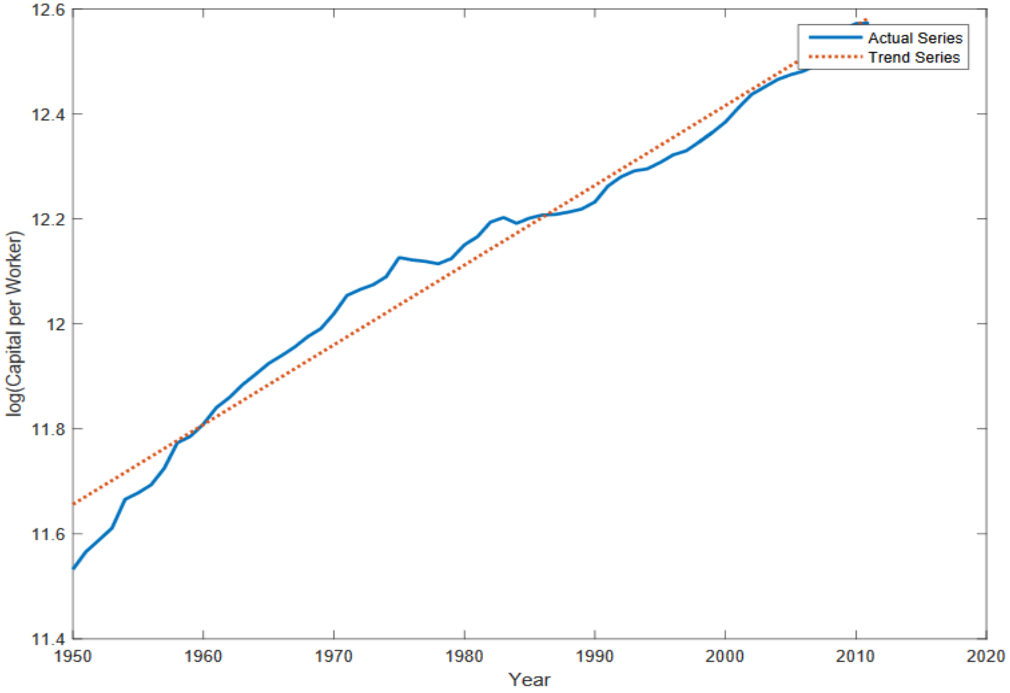


Source: Garín et al. (2020), p. 59.

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

2. Capital per worker $\left(\frac{K_t}{N_t}\right)$ grows at an approximately constant rate over long periods of time

Figure 4.2: Capital per worker in the US 1950-2011.

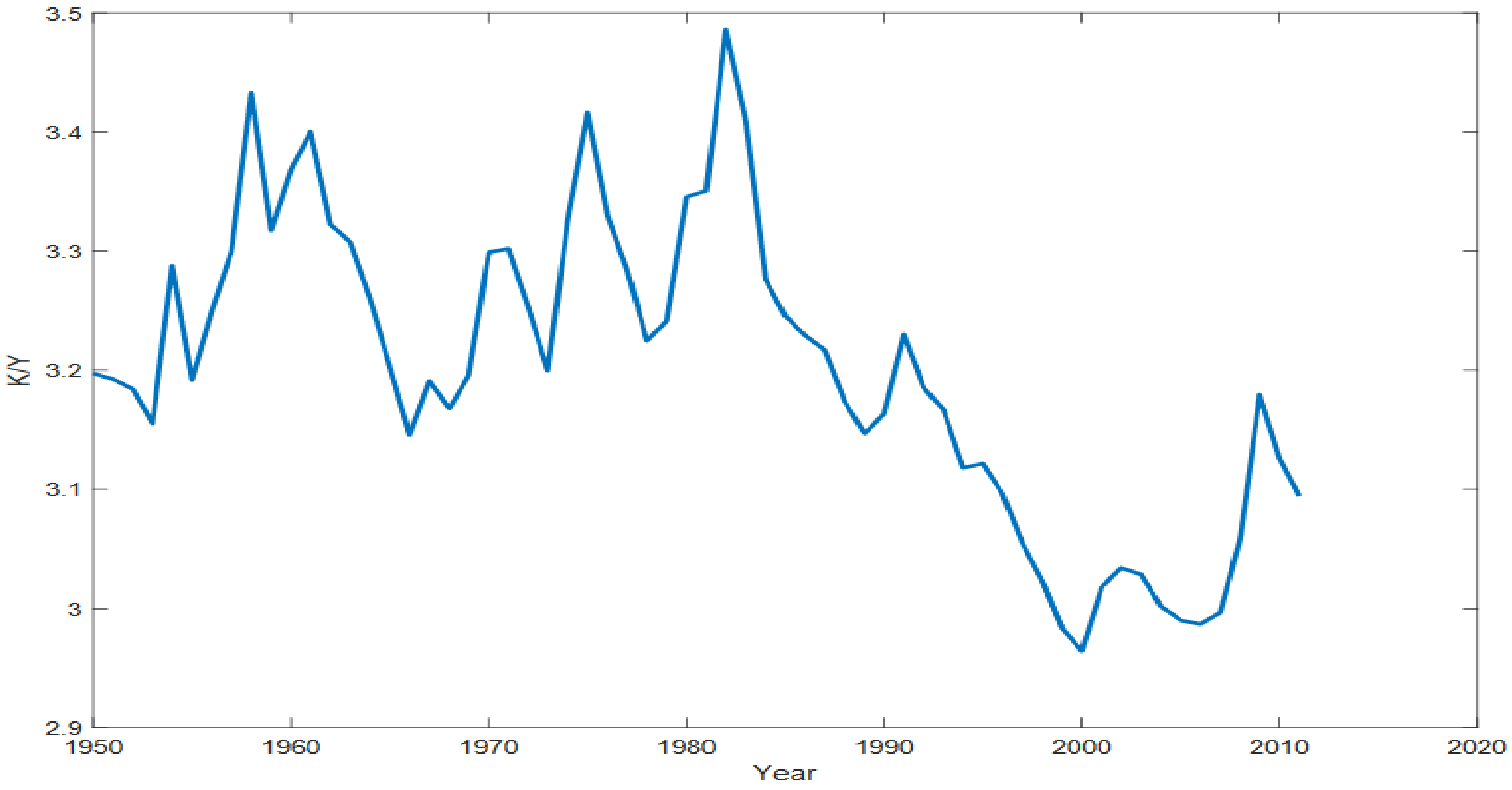


Source: Garín et al. (2020), p. 61.

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

3. The capital to output ratio $\left(\frac{K_t}{Y_t}\right)$ is roughly constant over long periods of time

Figure 4.3: Capital to Output Ratio in the U.S. 1950–2011

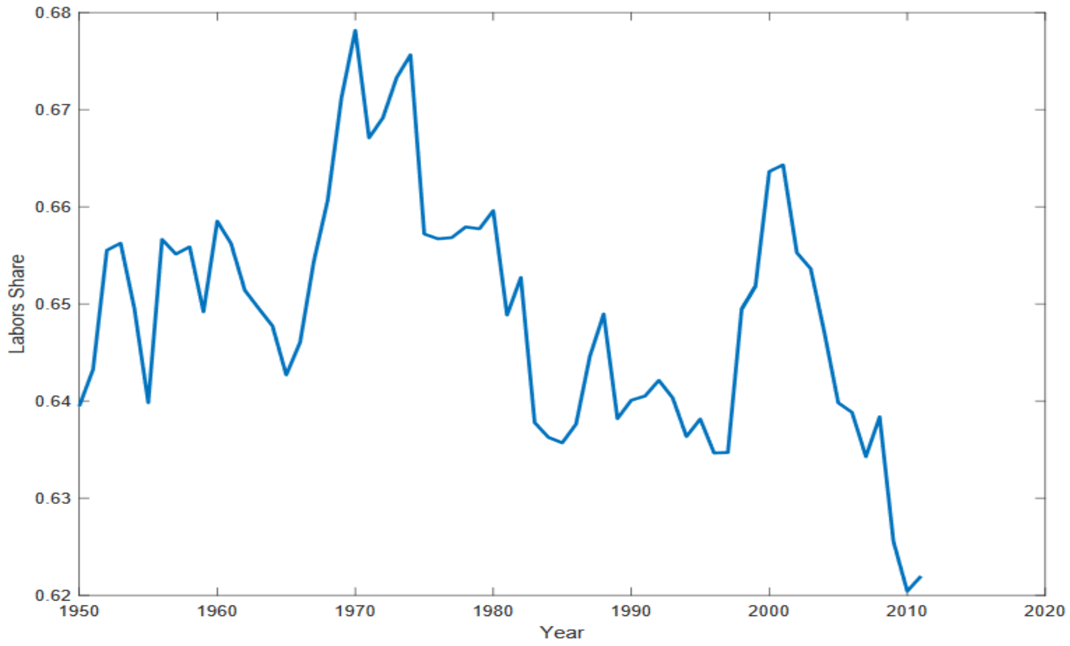


Source: Garín et al. (2020), p. 62.

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

4. Labor's share of income $\left(\frac{w_t N_t}{Y_t}\right)$ is roughly constant over long periods of time (capital share: $R_t K_t Y_t = 1 - w_t N_t Y_t$)

Figure 4.4: Labor Share in the US 1950-2011.



Source: Garín et al. (2020), p. 63.

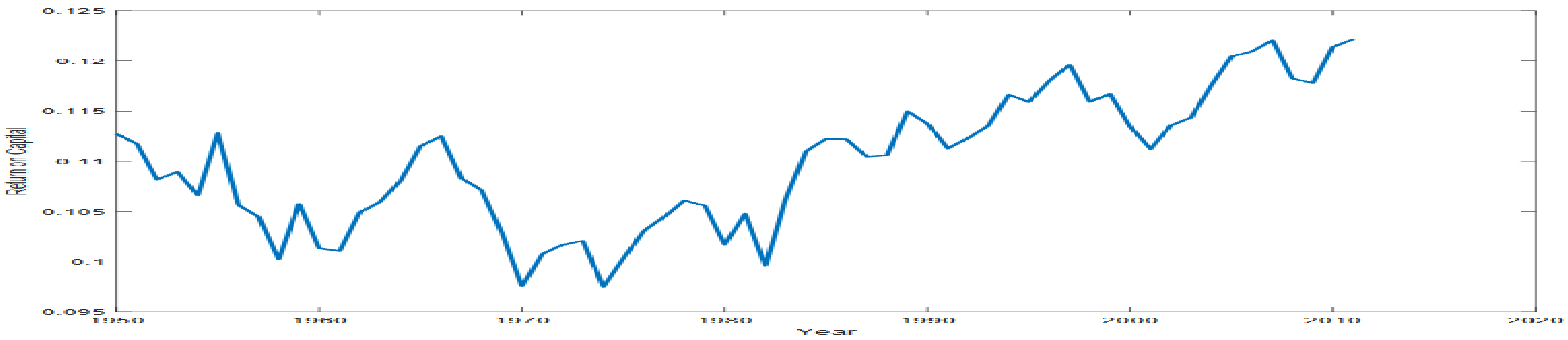
Lately: decline in labor share based on the rise of “superstar firms”

Mechanism: change in economic environment advantages the most productive firms in an industry → product market concentration↑ and labor share ↓ since share of value-added generated by most productive firms (‘superstars’: above-average markups & below-average labor shares, grows (Autor et al. (2019), *QJE*)

→ Highly relevant research debate

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

5. The return to capital (r_t) is roughly constant over long periods of time



Source: Garín et al. (2020), p. 65.

$$r^* = \alpha\left(\frac{\delta + z + n}{s}\right)$$

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

6. The real wage grows (w_t) at approximately the same rate as output per worker over long periods of time

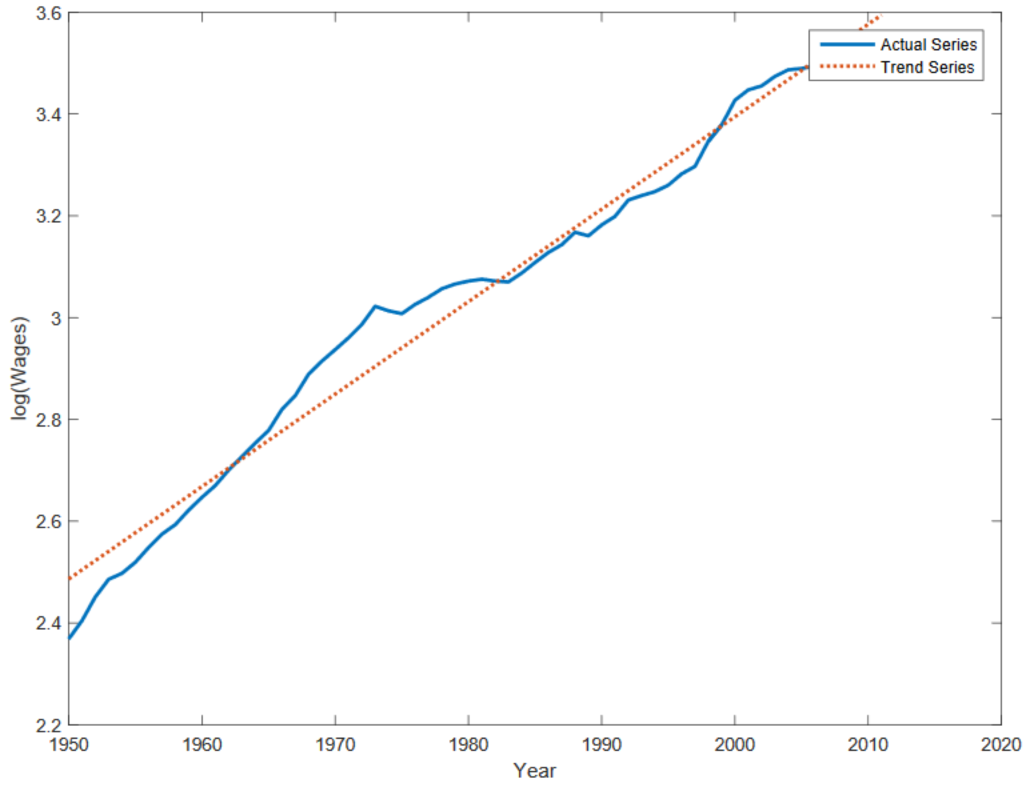


Figure 4.6: Wages in the US 1950-2011.

Source: Garín et al. (2020), p. 66.

For mathematical derivation of the (real) wage rate see the appendix.

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

Assume a standard Cobb-Douglas production function: $Y = AF(K,L) = AK^{\alpha}L^{1-\alpha}$.

a) Derive the first order conditions with respect to the input factors.

$$Y = AF(K,L) = AK^{\alpha}L^{1-\alpha}$$

$$\frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}L^{1-\alpha} = \alpha A\left(\frac{L}{K}\right)^{1-\alpha} = \alpha A\left(\frac{K}{L}\right)^{\alpha-1} = MPK = r$$

$$\frac{dY}{dL} = (1-\alpha)AK^{\alpha}L^{-\alpha} = (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} = MPL = w$$

w denotes the real (not the nominal) wage, r denotes the real return to capital

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

Assume a standard Cobb-Douglas production function: $Y = AF(K, L) = AK^{\alpha}L^{1-\alpha}$.

b) Use your solutions to derive the labor and capital demand functions.

Solve FOC for K and L to get the demand functions:

$$\alpha A\left(\frac{L}{K}\right)^{1-\alpha} = r \Leftrightarrow (\alpha A)^{\frac{1}{1-\alpha}}\left(\frac{L}{K}\right) = r^{\frac{1}{1-\alpha}} \Leftrightarrow K^D = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} L$$

$$(1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} = w \Leftrightarrow [(1-\alpha)A]^{\frac{1}{\alpha}}\left(\frac{K}{L}\right) = w^{\frac{1}{\alpha}} \Leftrightarrow L^D = \left(\frac{(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}} K$$

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

Now instead of using the basic Solow model assume the complete general Solow model, with time-varying total factor productivity:

$Y_t = A_t F(K, L) = A_t K_t^{\alpha} L_t^{1-\alpha}$.

c) Derive the steady state values for w^* and r^* of the per worker variables.

Initial step: rewrite production function in terms of labor-augmented technological change (e.g. Sørensen and Whitta-Jacobsen (2010), p. 129):

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

c) Derive the steady state values for w^* and r^* of the per worker variables.
Initial step: rewrite production function in terms of labor-augmented technological change:

Recall that: $k = \frac{K}{ZL}$ (with $Z_t = A_t^{\frac{1}{1-\alpha}}$) (efficiency units)
Intuitively: while A represents total factor productivity, Z represents labor productivity (hence, only concerning the input factor labor)

Following Sørensen and Whitta-Jacobsen (2010), p. 128, we can rewrite the production function in terms of A_t in the following way, as with the Cobb-Douglas production function it does not make a difference if "*we describe technological change by a certain time sequence, (A_t), of total factor productivity, or by an appropriately defined sequence, (Z_t), of the labor productivity variable […]*":

With $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$ and $Z_t \equiv A_t^{\frac{1}{1-\alpha}} \Leftrightarrow A_t \equiv Z_t^{1-\alpha}$ hence,

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \Longrightarrow Y_t = K_t^{\alpha}(Z_t L_t)^{1-\alpha}$$

→ Labor-augmented technological change

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

c) Derive the steady state values for w^* and r^* of the per worker variables.

[For complete derivation of central equation in Solow model variants (dynamic capital accumulation equations, refer, e.g. to Garìn et al. (2018), pp. 78 ff (basic model) or Sørensen and Whitta-Jacobsen (2010), pp. 130-131 general model]

1st step: derive steady state value for k^* (in efficiency units)

$$k_{t+1} - k_t = sk_t^\alpha - (\delta + z + n)k_t$$

In steady state it, holds that: $k_{t+1} = k_t = k^*$

$$\Longrightarrow 0 = s{k^*}^\alpha - (\delta + z + n)k^*$$

$$\Leftrightarrow s{k^*}^\alpha = (\delta + z + n)k^*$$

$$\Leftrightarrow \frac{{k^*}^\alpha}{k^*} = \frac{(\delta + z + n)}{s} = {k^*}^{\alpha - 1}$$

$$\Leftrightarrow k^* = \left(\frac{\delta + z + n}{s}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{s}{\delta + z + n}\right)^{\frac{1}{1 - \alpha}}$$

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

c) Derive the steady state values for w^* and r^* of the per worker variables.

2nd step: insert expression for k^* into FOCs derived from

$Y_t = K_t^{\alpha}(Z_t L_t)^{1-\alpha}$ (labor-augmenting technological change):

(Recall that: $Z = A^{\frac{1}{1-\alpha}} \Leftrightarrow Z^{1-\alpha} = A$)

Results for FOCs: Train how to get here (Sørensen and Whitta-Jacobsen (2010), p. 129):

$$\frac{dY_t}{dK_t} = r_t = \alpha\left(\frac{K_t}{Z_t L_t}\right)^{\alpha-1} \text{ and } \frac{dY_t}{dL_t} = w_t = Z_t(1-\alpha)\left(\frac{K_t}{Z_t L_t}\right)^{\alpha}$$

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

c) Derive the steady state values for w^* and r^* of the per worker variables.

3rd step: use expression for k^* and insert into FOCs (recall: $k = \frac{K}{ZL}$):

$$r^* = \alpha(k^*)^{\alpha-1}$$

$$r^* = \alpha\left(\left(\frac{s}{\delta+z+n}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha-1} = \alpha\left(\left(\frac{\delta+z+n}{s}\right)^{\frac{1}{\alpha-1}}\right)^{\alpha-1}$$

$$r^* = \alpha\left(\frac{\delta+z+n}{s}\right)^{\frac{\alpha-1}{\alpha-1}} = \alpha\left(\frac{\delta+z+n}{s}\right)$$

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

c) Derive the steady state values for w^* and r^* of the per worker variables.

3rd step: use expression for k^* and insert into FOCs (recall: $k = \frac{K}{ZL}$):

$$\mathrm{w}_t^* = Z_t(1-\alpha)(k^*)^\alpha$$

$$w_t^* = Z_t(1-\alpha)\left(\left(\frac{s}{\delta+z+n}\right)^{\frac{1}{1-\alpha}}\right)^\alpha$$

$$w_t^* = Z_t(1-\alpha)\left(\frac{s}{\delta+z+n}\right)^{\frac{\alpha}{1-\alpha}} = (1-\alpha)A_t^{\frac{1}{1-\alpha}}\left(\frac{s}{n+\mathrm{z}+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.
c) Derive the steady state values for w^* and r^* of the per worker variables.

$$r^* = \alpha\left(\frac{\delta + z + n}{s}\right)$$

r^*: closely related to the natural interest rate. The current decrease in the natural interest rate is one of the most important discussions in monetary policy nowadays because it implies that the zero lower bound on nominal interest rates is more often binding than in the past.

r^*: increases with factors that increase the demand for investment: technology growth z, depreciation δ, population growth n, and the capital share α.

It decreases with an increase in the savings rate as this would increase the supply of capital/investment.

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.
c) Derive the steady state values for w^* and r^* of the per worker variables.

$$r^* = \alpha\left(\frac{\delta+z+n}{s}\right) \text{ and } w_t^* = Z_t(1-\alpha)\left(\frac{s}{\delta+z+n}\right)^{\frac{\alpha}{1-\alpha}} = A^{\frac{1}{1-\alpha}}(1-\alpha)\left(\frac{s}{\delta+z+n}\right)^{\frac{\alpha}{1-\alpha}}$$

Through the derivations above, we were able to show that r^* is constant over time, while w^* grows with technology. Both also reflect the empirical evidence (Kaldor facts).

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

d) You will arrive at the result that in the general Solow growth model, w_t^* is growing with technology, while r^* is not. Provide economic intuition for this result.

$$r^* = \alpha\left(\frac{\delta + z + n}{s}\right)$$

$$w_t^* = Z_t(1-\alpha)\left(\frac{s}{\delta + z + n}\right)^{\frac{\alpha}{1-\alpha}} = A^{\frac{1}{1-\alpha}}(1-\alpha)\left(\frac{s}{\delta + z + n}\right)^{\frac{\alpha}{1-\alpha}}$$

- real wages grow over time, while the real return on capital stays rather constant over time
- Intuition 1: (level) of labor input (N) is rather constant over time (only small changes in the workforce); so with technology growth, the ‚fixed' labor input gets more productive so that its MPL increases → real wage increases
- Intuition 2: capital stock (level) can grow over time (in steady state it does with z+n), hence MPC does not grow leaving r^* to be rather constant over time

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

e) Is the Solow Model discussed in class able to capture the stylized Kaldor facts of long-term economic growth? Discuss this in context of the results of the model, i.e., e.g., the steady state values of the wage rate and the rate of return on capital.

In general, (augmented) Solow model expressed in efficiency units in line with stylized Kaldor facts – growth in per worker variables, constant r and growing w

- Looming change in labor's share of income currently big debate in macroeconomic research

TASK 2: Kaldor Facts and formal Derivation of the Results for, w_t^* and r^*.

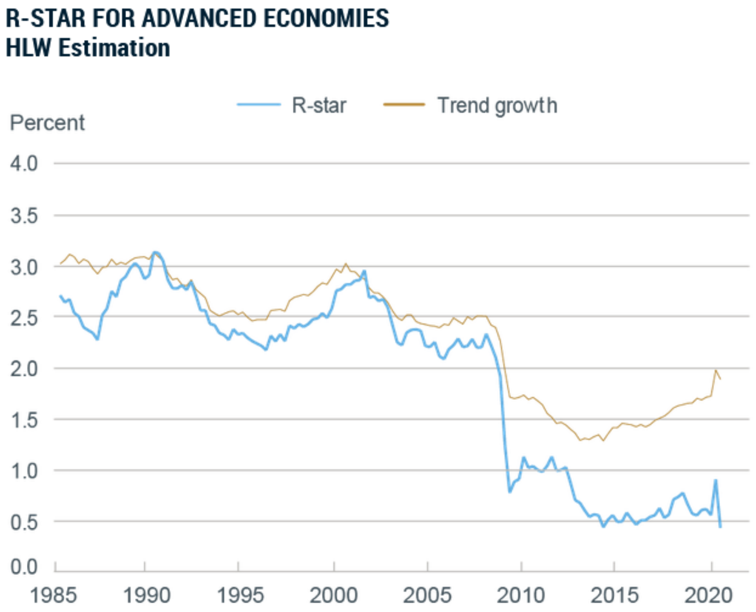
f) How is the rate of return on capital linked to the real natural interest rate?

$$r^* = \alpha\left(\frac{\delta + z + n}{s}\right)$$

$r^* \rightarrow$ Real interest rate: $r^* - \delta$:

closely related to the natural interest rate.

The current decrease in the natural interest rate is one of the most important discussions in monetary policy nowadays because it implies that the ZLB on nominal interest rates is more often binding than in the past.



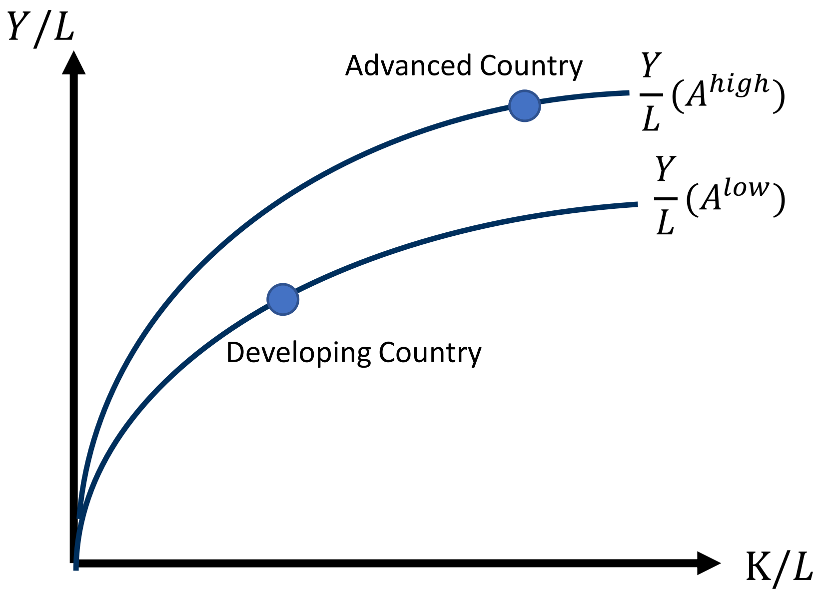
Sources: Holston, Laubach, and Williams (2017); Organisation for Economic Co-operation and Development (OECD).
Notes: Estimates are GDP-weighted averages across the United States, Canada, the Euro Area, and the United Kingdom. We use OECD estimates of GDP at purchasing power parity. For dates prior to 1995, Euro-Area weights are the summed weights of the eleven original Euro-Area countries.

r^*: increases with factors that increase the demand for investment: technology growth z, depreciation δ, population growth n, and the capital share α.

It decreases with an increase in the savings rate as this would increase the supply of capital/investment.

TASK 3: Describe how an emerging economy could catch-up to advanced economies in con-text of the Solow Model. Why might it be easier for relatively open emerging economies to do so?

$$Y = AK^{\alpha}L^{1-\alpha} \Longrightarrow \frac{Y}{L} = A\left(\frac{K}{L}\right)^{\alpha}$$



Y increases either because of $K\uparrow$ or $A\uparrow$

while capital stock relatively high (small potential for additional growth) in advanced countries → potential for growth in A and K in emerging economies (e.g. via raising s, financial integration via capital inflows, increasing labor market participation, raising TFP)

Task 4: Briefly explain the concept of efficiency wages.

Labor market does not clear even in the medium to long run, i.e. wages do not adjust in a way so that labor demand equals labor supply.

Efficiency wages

Effort: Employers pay a higher wage than employees require to take on a job to create a cost to the worker of losing the job, creating an incentive to work conscientiously

Turnover: Via paying wages above the market-clearing level, firms can prevent high turnover of workers.

→ employers offer higher wages, when the probability of the worker getting another job is higher

→ creates wages that move with the business cycle and the probability of becoming unemployed