Chair of Computer Science I
Algorithms, Complexity and
Knowledge-Based Systems
University of Würzburg

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Homework Assignment #5
Approximation Algorithms (Winter Semester 2021/22)

Exercise 1 – Overpacking in the dual LP of SetCover

Consider the dual LP for SetCover:

$$\begin{aligned} \textbf{maximize} \quad & \sum_{u \in U} y_u \\ \textbf{subject to} \quad & \sum_{u \in S} y_u \le c_S, \quad && S \in \mathcal{S} \\ & y_u \ge 0, && u \in U. \end{aligned}$$

Let price(u) be the price of element u as determined by the algorithm GreedySetCover in Lecture 02.

Prove that the solution $y_u = \text{price}(u)$ is in general not feasible for the dual LP. In particular, find an instance with a set S that is overpacked by factor (approximately) $H_{|S|}$, i.e. $\sum_{u \in S} y_u \approx H_{|S|} \cdot c_S$. **[4 points]**

Exercise 2 – Randomized Rounding for SetCover

Consider the following randomized algorithm for SetCover.

Algorithm 1: RandomizedRounding$(U, \mathcal{S}, c)$

```
Compute optimal solution x for LP-Relaxation of SetCover from the lecture
C ← ∅
foreach S ∈ 𝒮 do
    Add S with probability x_S to C
return C
```

a) Prove that the expected cost of $\mathcal{C}$ is exactly OPT. **[4 points]**

b) Let $u \in U$ be an arbitrary element of the ground set. Prove that u is *not* covered by $\mathcal{C}$ with probability at most $1/e$. **[3 points]**

Hint: Use the relation $1 + x \le e^x$ for all $x \in \mathbb{R}$.

Exercise 3 – Randomized Rounding for SETCOVER with Multiple Rounds

Let d be a sufficiently large constant. We now run the algorithm RandomizedRounding from Exercise 2 exactly $d \cdot \log n$ times. Let $\mathcal{C}'$ be the union of all sets selected this way.

a) Prove that every element $u \in U$ is *not* selected by $\mathcal{C}'$ with probability at most $1/(4n)$, as long as d was chosen large enough. **[3 points]**

b) Prove that the cost of the set $\mathcal{C}'$ is greater than $4d \log n \cdot \text{OPT}_{\text{relax}}$ with probability at most $1/4$. **[3 points]**

c) Prove that $\mathcal{C}'$ is a *feasible* solution with cost at most $4d \log n \cdot \text{OPT}_{\text{relax}} = O(\log n) \cdot \text{OPT}_{\text{relax}}$ with probability at least $1/2$. **[3 points]**

Please hand in your solutions via WueCampus until 10:30 on Friday, November 26.