Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part I:
SteinerForest

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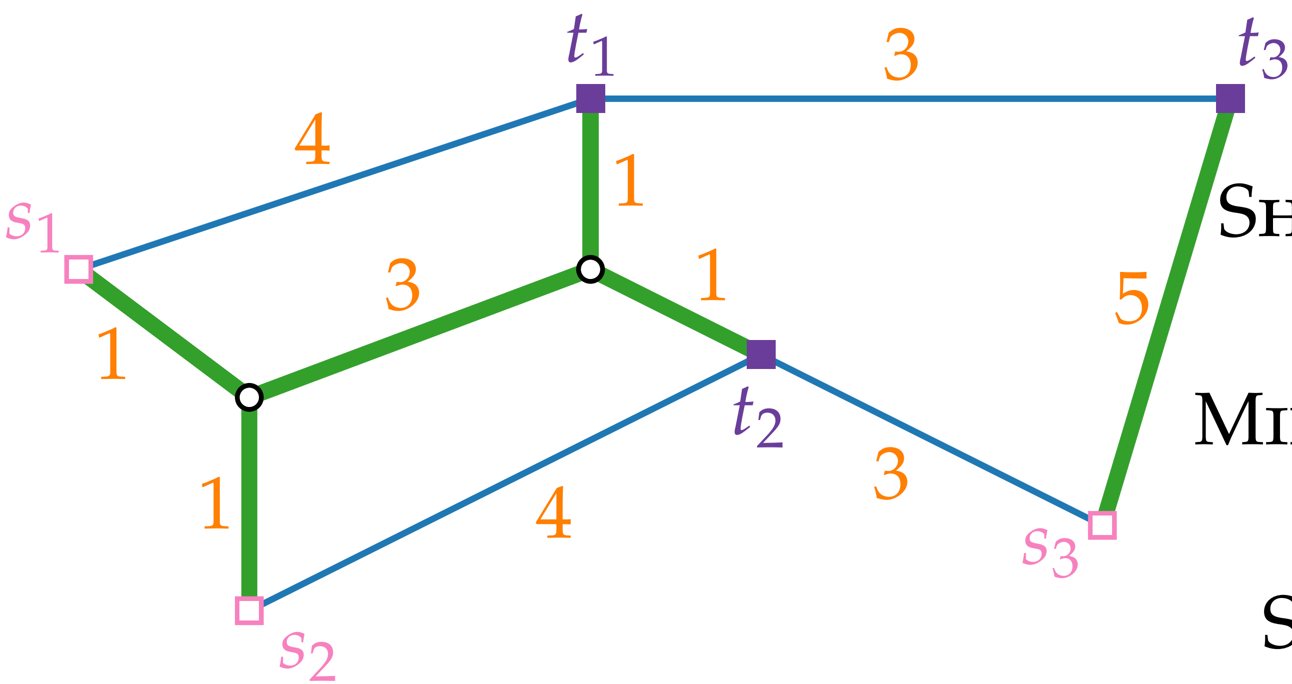
Winter 2021/22

SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \dots, (s_k, t_k)\}$ of k pairs of vertices

Task: Find an edge set $F \subseteq E$ with min. total cost $c(F)$ such that in the subgraph (V, F) each pair (s_i, t_i), $i = 1, \dots, k$ is connected.



Special cases?

ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

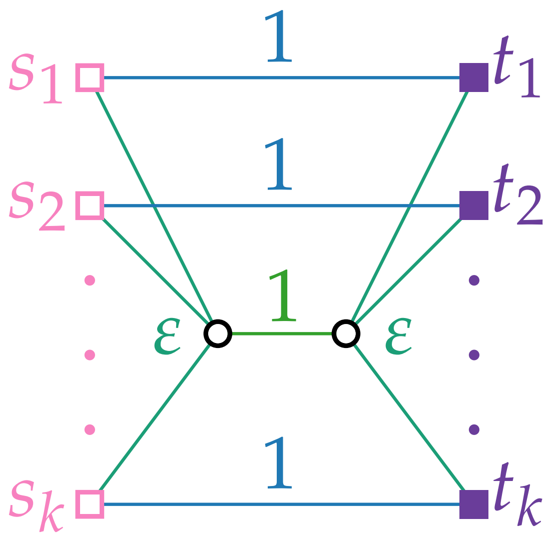
SteinerTree ($R = T \times T$)

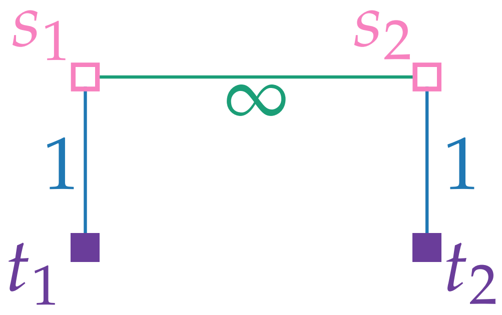
Approaches?

- Merge k shortest s_i-t_i-paths
- SteinerTree on the set of terminals

Above approaches perform poorly :-(

Difficulty: which terminals belong to the same tree of the forest?





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Part II:
Primal and Dual LP

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An ILP

minimize $\sum_{e \in E} c_e x_e$

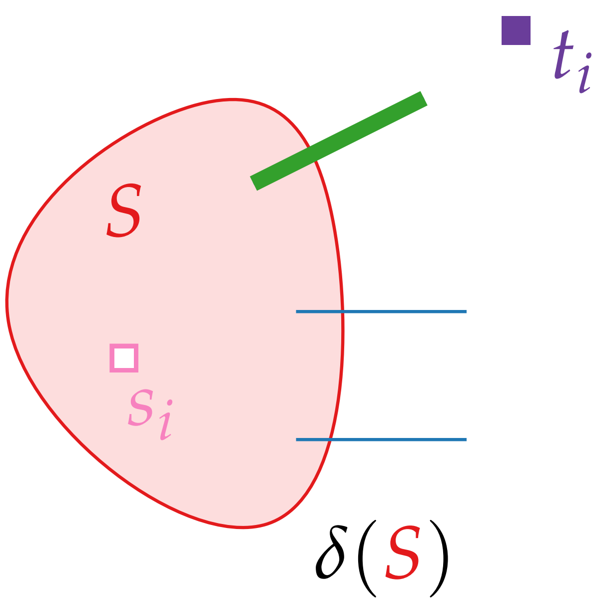
subject to $\sum_{e \in \delta(S)} x_e \geq 1 \qquad S \in \mathcal{S}_i, i = 1, \ldots, k$

$x_e \in \{0, 1\} \qquad e \in E$

where $\mathcal{S}_i := \{S \subseteq V : |S \cap \{s_i, t_i\}| = 1\}$

and $\delta(S) := \{(u, v) \in E : u \in S \text{ and } v \notin S\}$

$\rightsquigarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

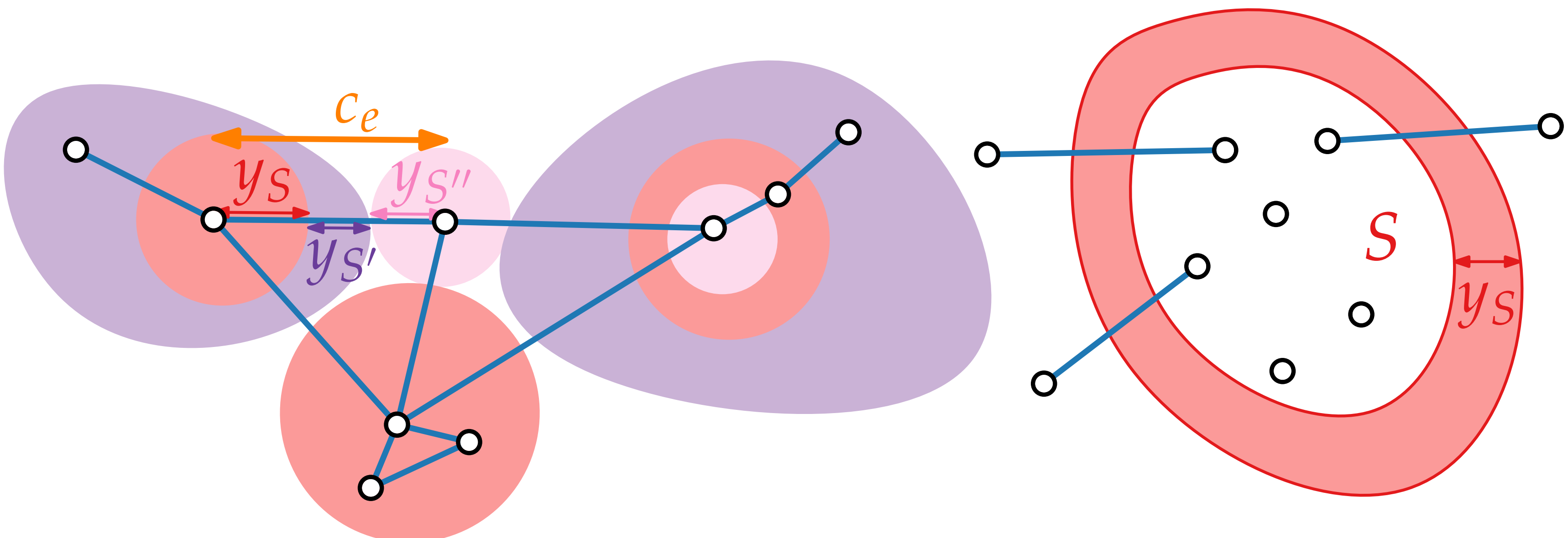
$$\begin{array}{lll} \textbf{minimize} & \sum_{e \in E} c_e x_e & \\ \textbf{subject to} & \sum_{e \in \delta(S)} x_e \geq 1 & S \in \mathcal{S}_i, i = 1, \ldots, k \quad (y_S) \\ & x_e \geq 0 & e \in E \end{array}$$

$$\begin{array}{lll} \textbf{maximize} & \sum_{\substack{S \in \mathcal{S}_i \\ i=1,\ldots,k}} y_S & \\ \textbf{subject to} & \sum_{S:\, e \in \delta(S)} y_S \leq c_e & e \in E \\ & y_S \geq 0 & S \in \mathcal{S}_i, i = 1, \ldots, k \end{array}$$

Intuition for the Dual

$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{\substack{S\in\mathcal{S}_i\\ i=1,\dots,k}} y_S & \\ \textbf{subject to} & \displaystyle\sum_{S:\, e\in\delta(S)} y_S \le c_e & e \in E \\ & y_S \ge 0 & S \in \mathcal{S}_i, i = 1,\dots,k \end{array}$$

The graph is a network of **bridges**, spanning the **moats**.



$\delta(S)$ = set of edges / bridges over the moat around S

y_S = width of the **moat** around S

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Part III: A First Primal-Dual Approach

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Complementary Slackness (Rep.)

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

A First Primal-Dual Approach

Complementary slackness: $x_e > 0 \quad \Rightarrow \sum_{S:\, e \in \delta(S)} y_S = c_e$.

$\Rightarrow$ pick "critical" edges (and only those)

Idea: iteratively build a feasible integral Primal-Solution.

How to find a violated primal constraint? $\left(\sum_{e \in \delta(S)} x_e < 1\right)$

$\rightsquigarrow$ Consider related connected component C!

How do we iteratively improve the Dual-Solution?

$\rightsquigarrow$ increase y_C! (until some edge in $\delta(C)$ becomes critical)

A First Primal-Dual Approach

PrimalDualSteinerForestNaive(G, c, R)

$y \leftarrow 0, F \leftarrow \emptyset$
while some $(s_i, t_i) \in R$ not connected in (V, F) **do**
  $C \leftarrow$ comp. in (V, F) with $|C \cap \{s_i, t_i\}| = 1$ for some i
  Increase y_C
    until $\sum_{S:\, e' \in \delta(S)} y_S = c_{e'}$ for some $e' \in \delta(C)$.
  $F \leftarrow F \cup \{e'\}$
return F

Running Time?

Trick: Handle all y_S with $y_S = 0$ implicitly

Analysis

The cost of the solution F can be written as

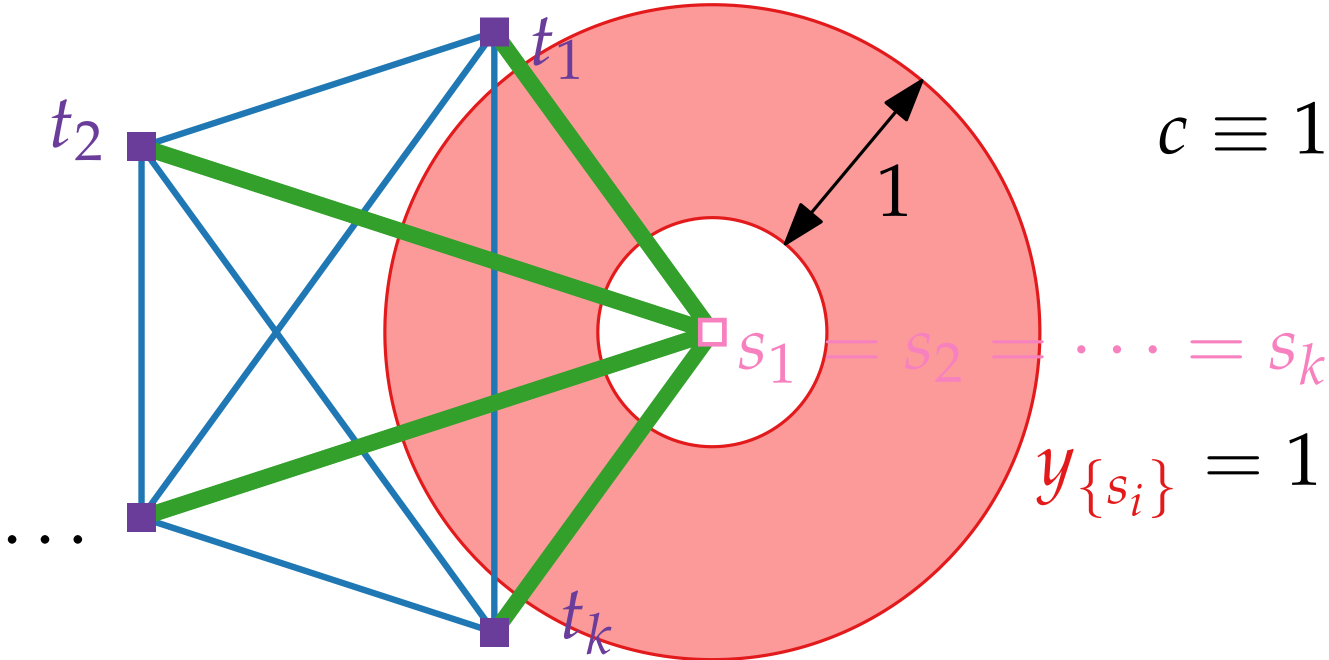
$$\sum_{e \in F} c_e \stackrel{\text{CS}}{=} \sum_{e \in F} \sum_{S:\, e \in \delta(S)} y_S = \sum_S |\delta(S) \cap F| \cdot y_S.$$

Compare to the value of the dual objective function $\sum_S y_S$

There are examples with $|\delta(S) \cap F| = k$ for each $y_S > 0$:

But: Average degree of component is 2!

$\Rightarrow$ Increase y_C for all components C simultaneously!



Approximation Algorithms

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Part IV:
Primal-Dual with Synchronized Increases

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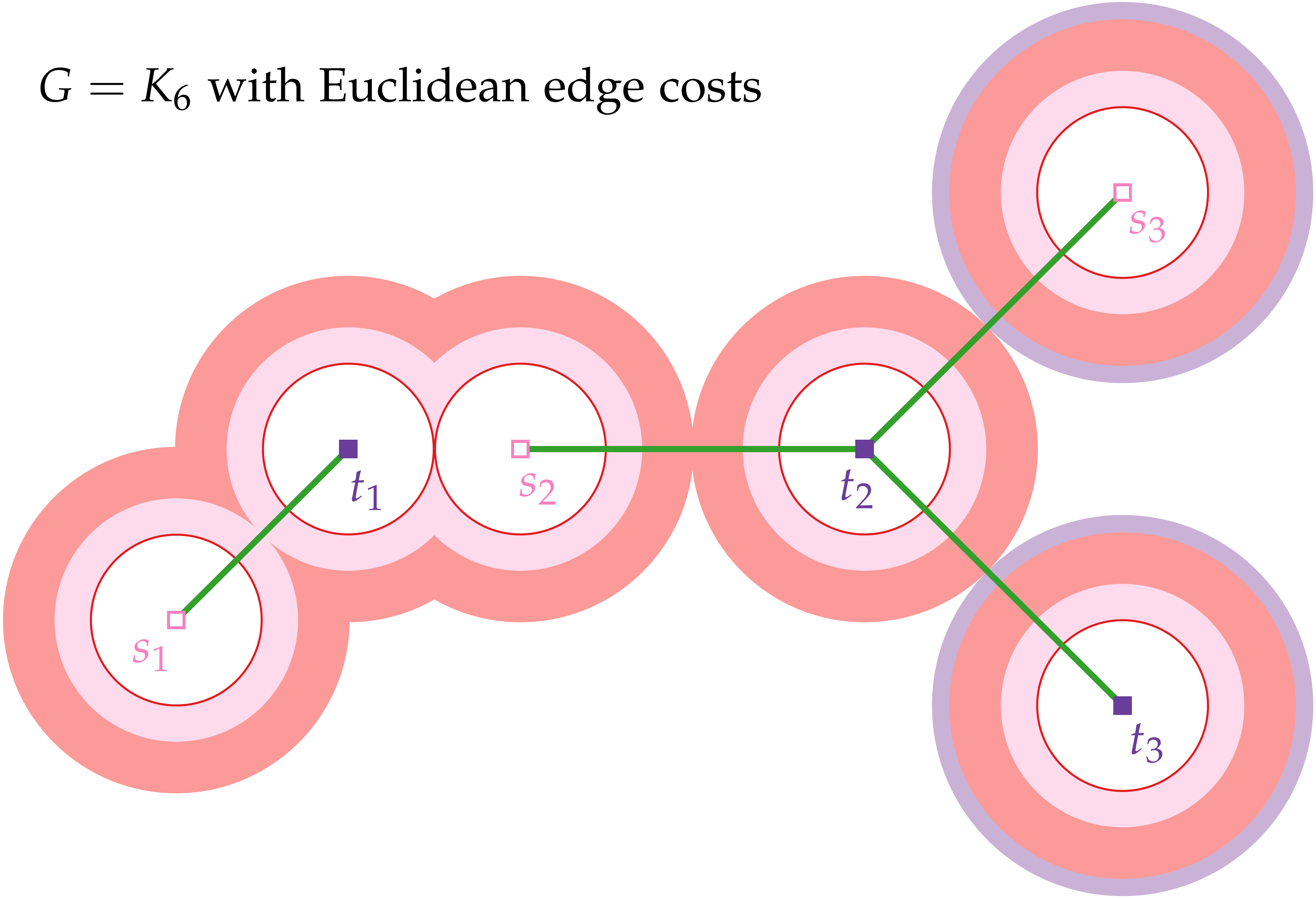
Primal-Dual with Synchronized Increases

PrimalDualSteinerForest(G, c, R)

$y \leftarrow 0, F \leftarrow \emptyset, \ell \leftarrow 0$
while some $(s_i, t_i) \in R$ not connected in (V, F) **do**
    $\ell \leftarrow \ell + 1$
    $\mathcal{C} \leftarrow \{$comp. C in (V, F) with $|C \cap \{s_i, t_i\}| = 1$ for some $i\}$
    Increase y_C for all $C \in \mathcal{C}$ simultaneously
        until $\sum_{S:\, e_\ell \in \delta(S)} y_S = c_{e_\ell}$ for some $e_\ell \in \delta(C), C \in \mathcal{C}$.
    $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
// Pruning
for $j \leftarrow \ell$ **down to** 1 **do**
    if $F' \setminus \{e_j\}$ is feasible solution **then**
        $F' \leftarrow F' \setminus \{e_j\}$
return F'

Illustration

$G = K_6$ with Euclidean edge costs



Approximation Algorithms

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Part V:
Structure Lemma

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Structure Lemma

Lemma. For each $\mathcal{C}$ of an iteration of the algorithm:

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|\,.$$

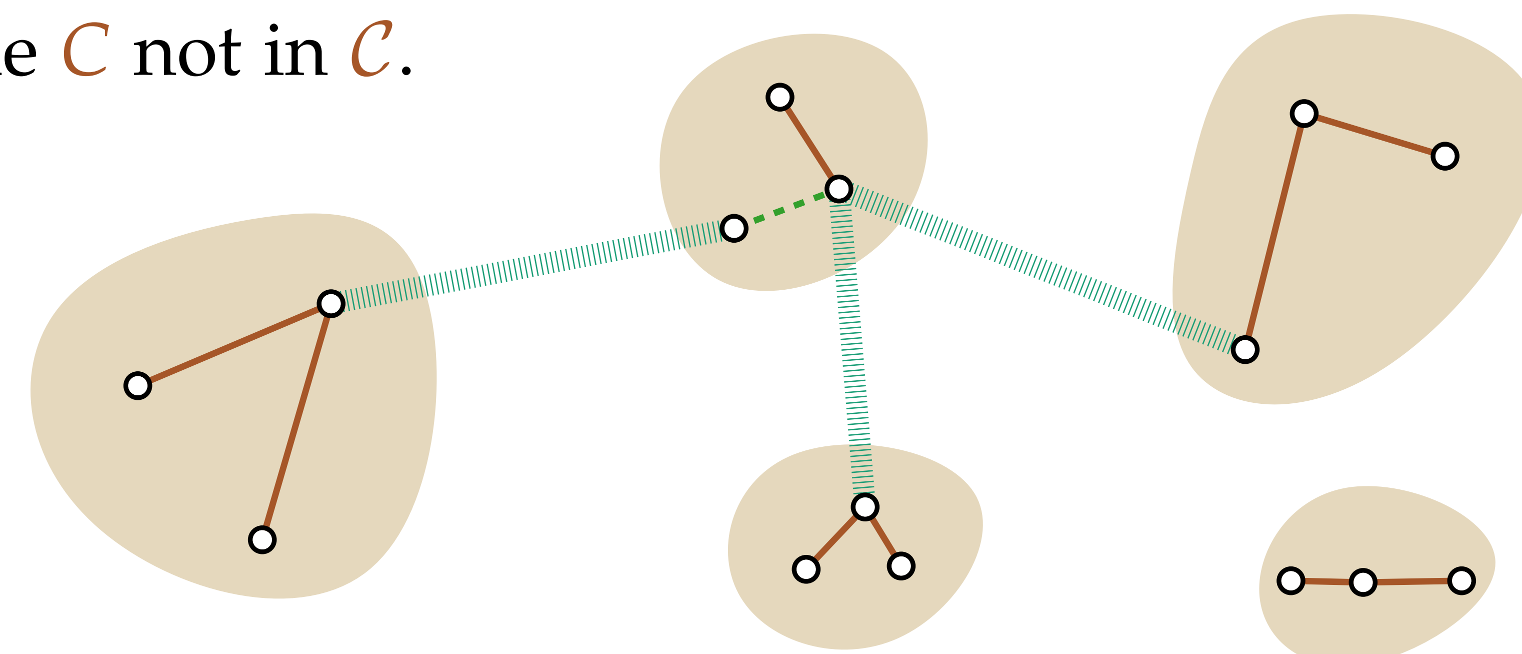
Proof. First the intuition…
each conn. component C of F is a forest in F'
$\rightsquigarrow$ avg. degree ≤ 2

Difficulty: Some C not in $\mathcal{C}$.

$\delta(C) \cap F'$

$F' \cap C$

$F - F'$



Proof of Structure Lemma

Lemma. For each $\mathcal{C}$ of an iteration of the algorithm:

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|\,.$$

Proof.

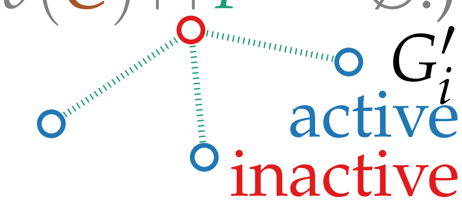
Consider i-th iteration after e_i was added to F, $i = 0, \ldots, \ell$

Let $F_i = \{e_1, \ldots, e_i\}$, $G_i = (V, F_i)$, and $G_i^* = (V, F_i \cup F')$.

Contract each comp. C of G_i in G_i^* to a single vertex $\rightsquigarrow G_i'$.

(Ignore all comp. C with $\delta(C) \cap F' = \emptyset$.)

Claim. G_i' is a forest.



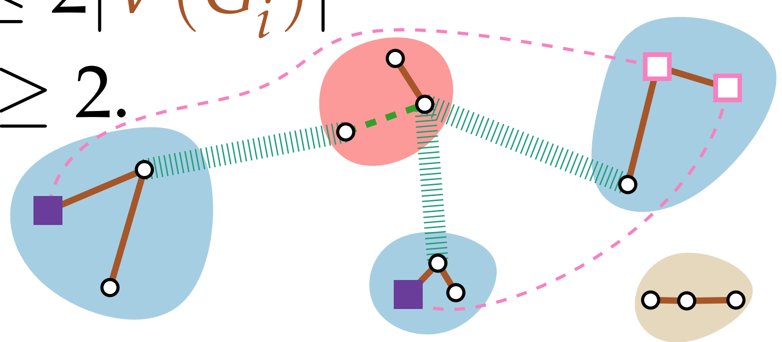
Note: $\sum_{C \text{ comp.}} |\delta(C) \cap F'| = \sum_{v \in V(G_i')} \deg_{G'}(v)$

$= 2|E(G_i')| \leq 2|V(G_i')|$

Claim. Inactive vertices have degree ≥ 2.

Then $\sum_{v \text{ active}} |\deg_{G'}(v)| \leq$

$2 \cdot |V(G')| - 2 \cdot \#(\text{inactive}) = 2|\mathcal{C}|$. □



Approximation Algorithms

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Part VI:
Analysis

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Analysis

Theorem. The Primal-Dual algorithm with synchronized increases gives a 2-approximation for SteinerForest.

Proof.

As before

$$\sum_{e \in F'} c_e \stackrel{\text{CS}}{=} \sum_{e \in F'} \sum_{S \colon e \in \delta(S)} y_S = \sum_S |\delta(S) \cap F'| \cdot y_S.$$

We prove by induction over the number of iterations of the algorithm that

$$\sum_S |\delta(S) \cap F'| \cdot y_S \leq 2 \sum_S y_S. \qquad (*)$$

From that, the claim of the theorem follows.

Analysis

Theorem. The Primal-Dual algorithm with synchronized increases gives a 2-approximation for SteinerForest.

Proof.

$$\sum_{S} |\delta(S) \cap F'| \cdot y_S \leq 2 \sum_{S} y_S. \qquad (*)$$

Base case trivial since we start with $y_S = 0$ for each S.

Assume that $(*)$ holds at the start of each iteration.

In the active iteration, we increase y_C for all $C \in \mathcal{C}$ by the same amount, say $\varepsilon \geq 0$.

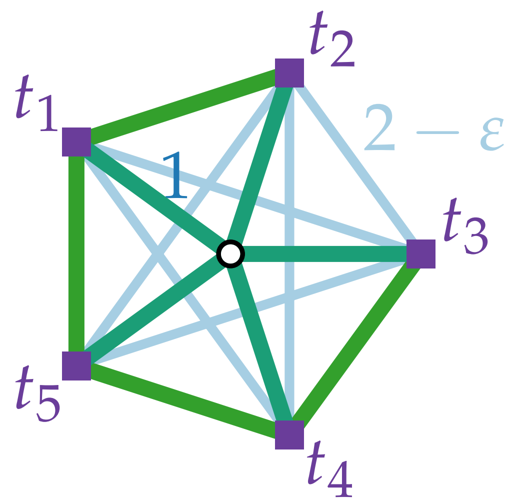
This increases the left side of $(*)$ by $\varepsilon \sum_{C \in \mathcal{C}} |\delta(C) \cap F'|$ and the right side by $2\varepsilon|\mathcal{C}|$.

Thus, by the Structure Lemma, $(*)$ also holds after the active iteration. □

Summary

Theorem. The Primal-Dual algorithm with synchronized increases gives a 2-approximation for SteinerForest.

Analysis tight?



ALG $= (2-\varepsilon)(n-1)$
OPT $= n$

better?
No better approximation factor is known.
The integrality gap is $2 - 1/n$.

SteinerForest (as SteinerTree) cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P=NP) [Chlebik & Chlebikova '08]