Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

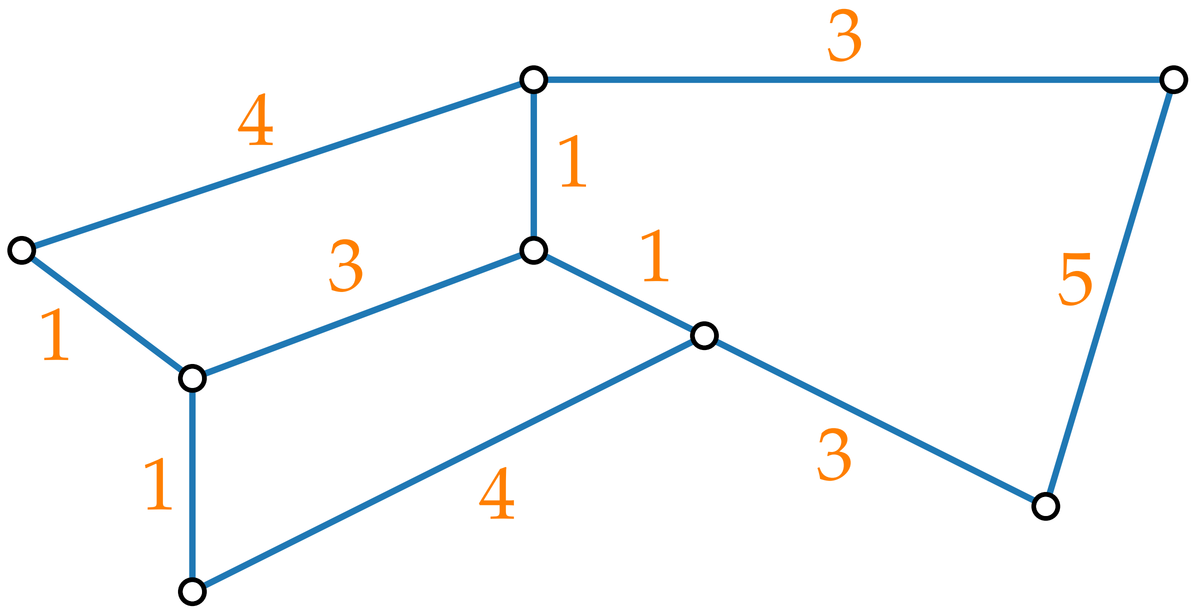
Part I:
SteinerForest

Joachim Spoerhase

Winter 2021/22

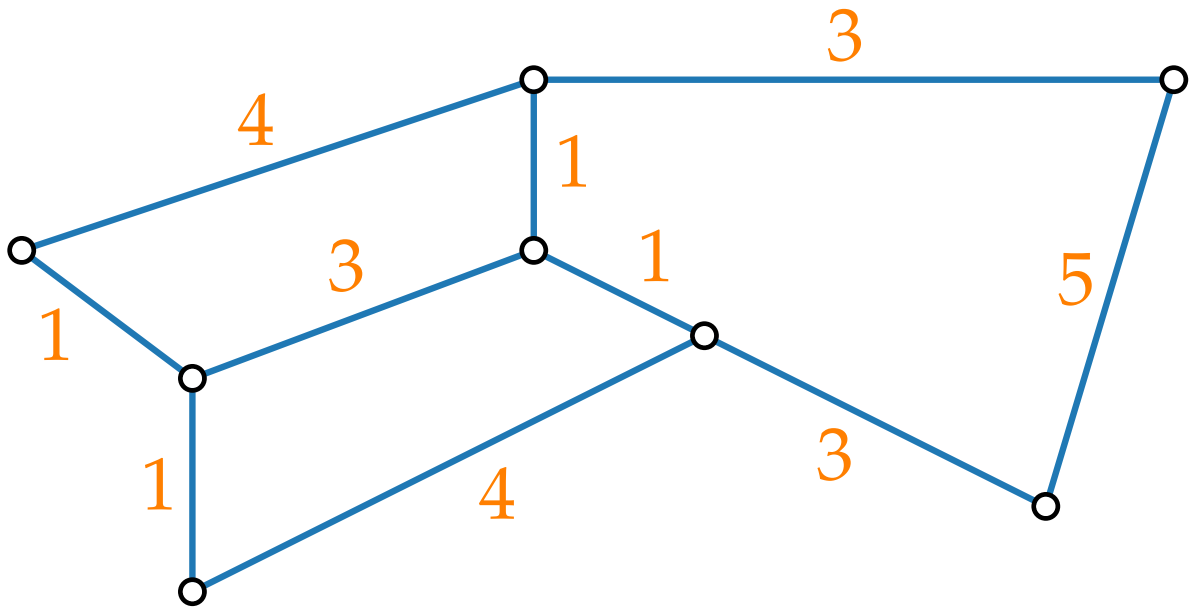
SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$



SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \dots, (s_k, t_k)\}$ of k pairs of vertices



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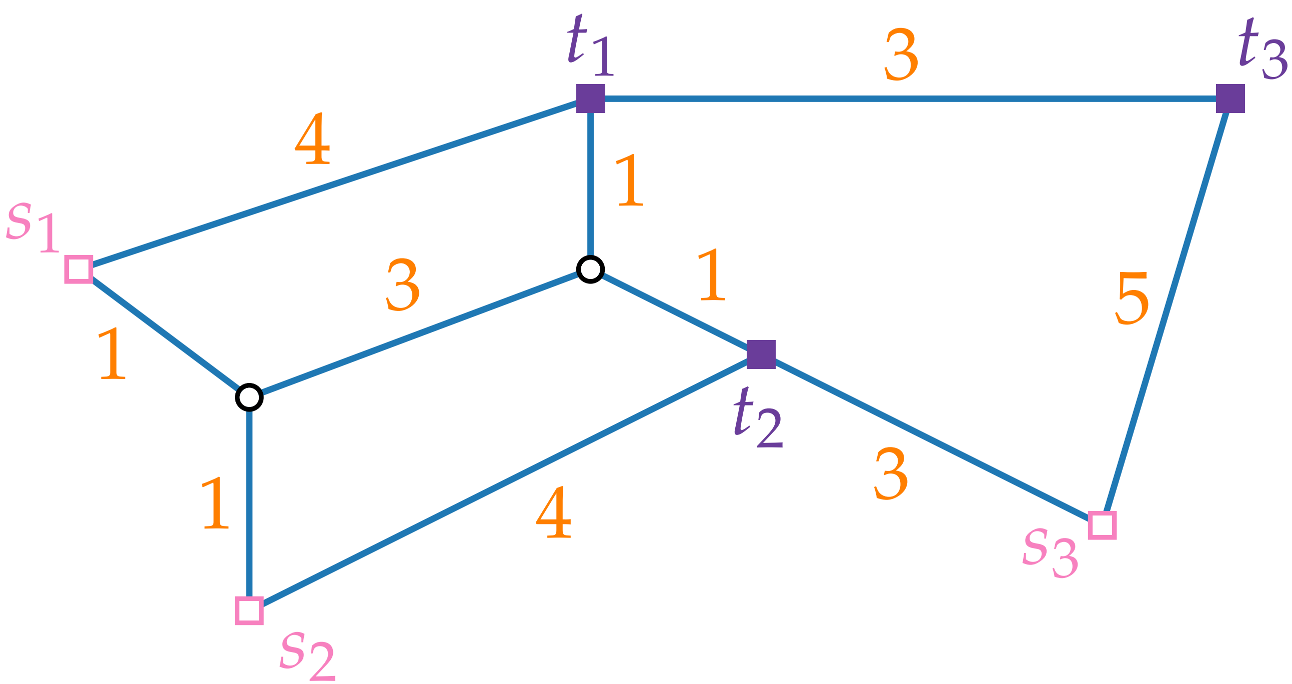

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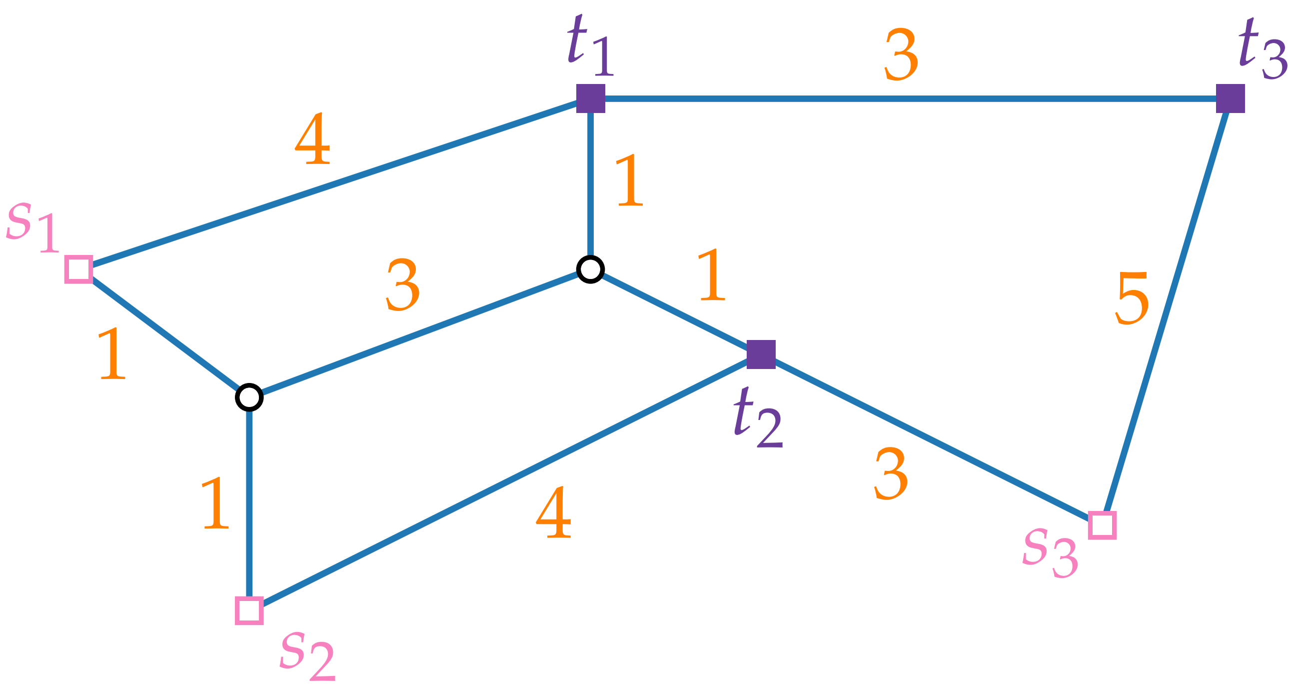
Task: Find an edge set $F \subseteq E$ with min. total cost $c(F)$



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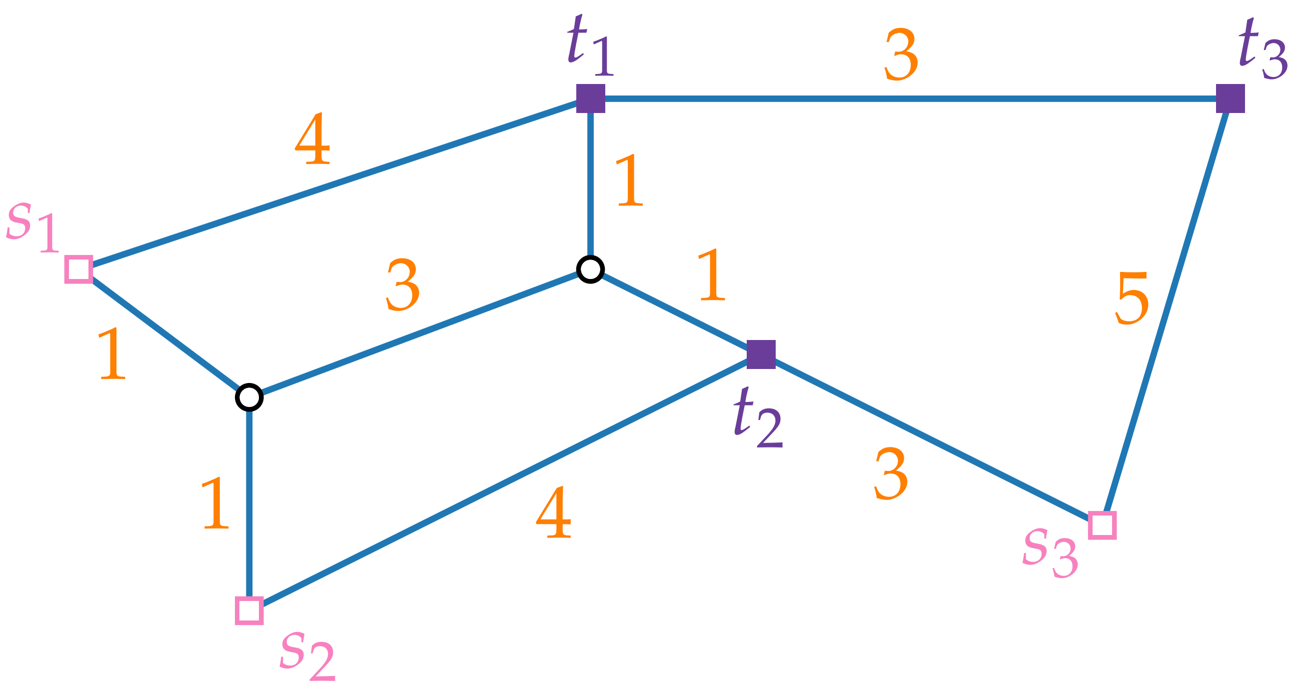
Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \dots, (s_k, t_k)\}$ of k pairs of vertices

Task: Find an edge set $F \subseteq E$ with min. total cost $c(F)$ such that in the subgraph (V, F) each pair (s_i, t_i), $i = 1, \dots, k$ is connected.



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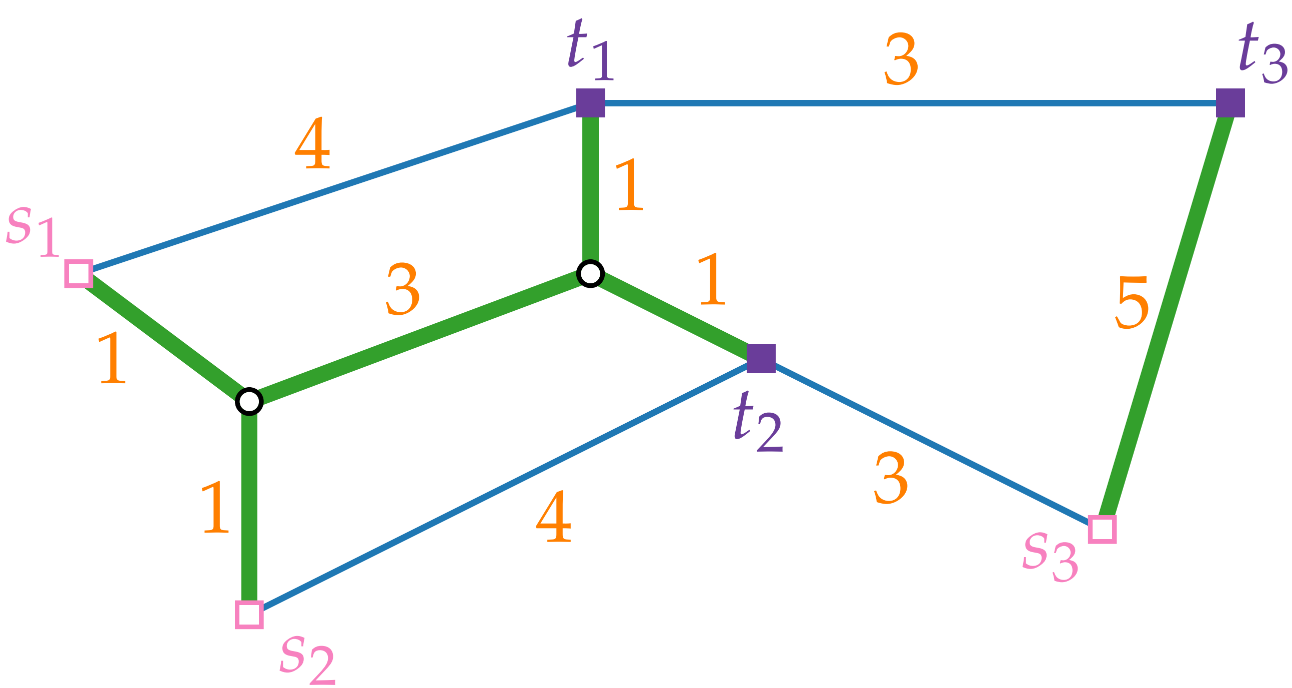

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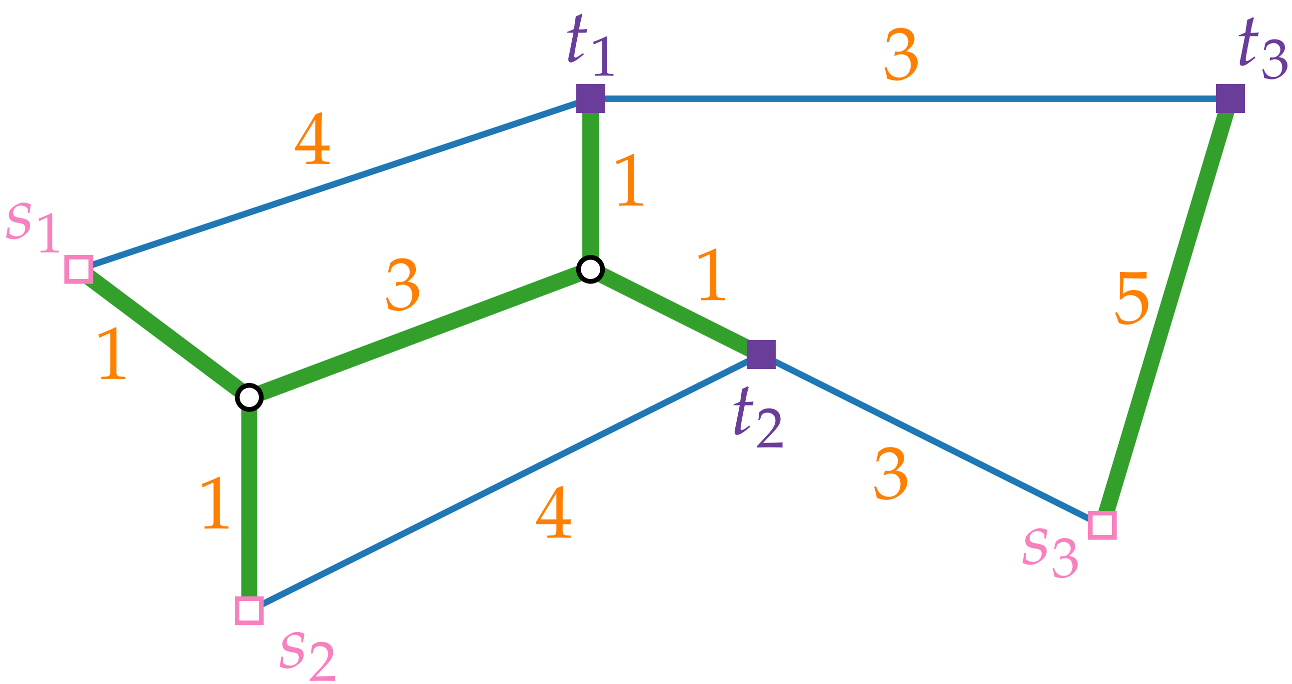

Special cases?

SteinerForest

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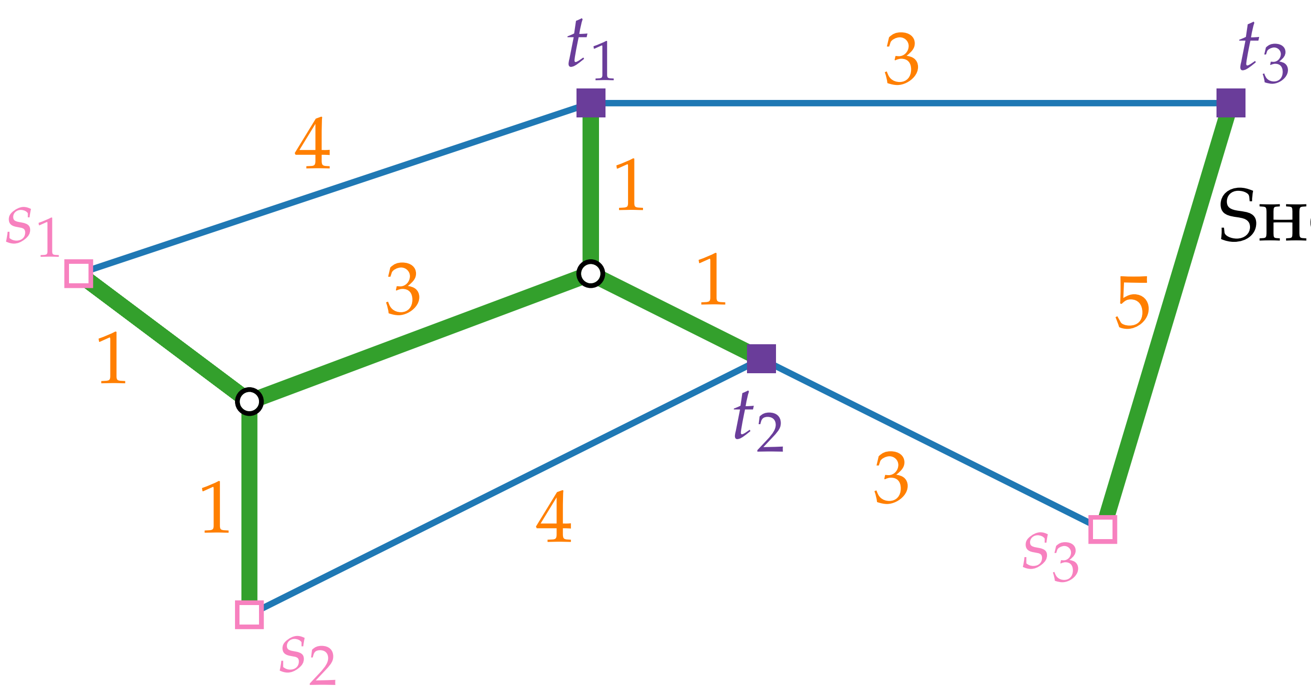

Special cases?

ShortestPath ($R = \{s, t\}$)

SteinerForest

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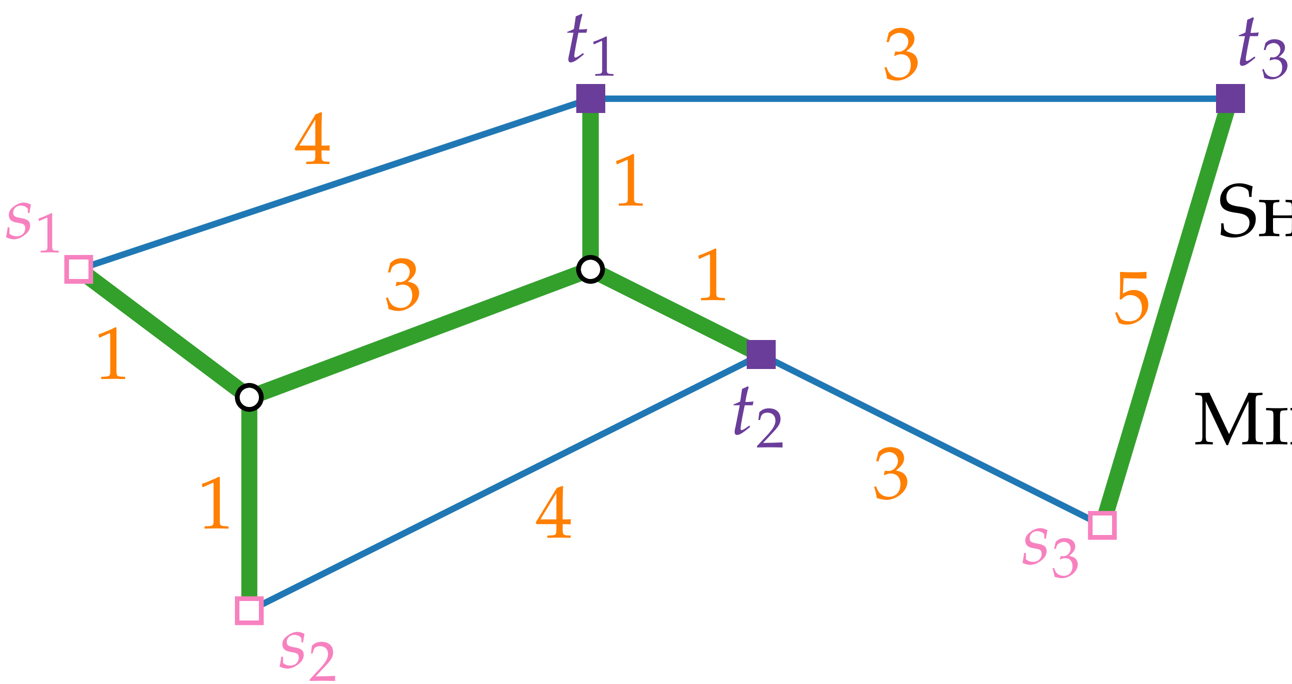

Special cases?

ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

SteinerForest

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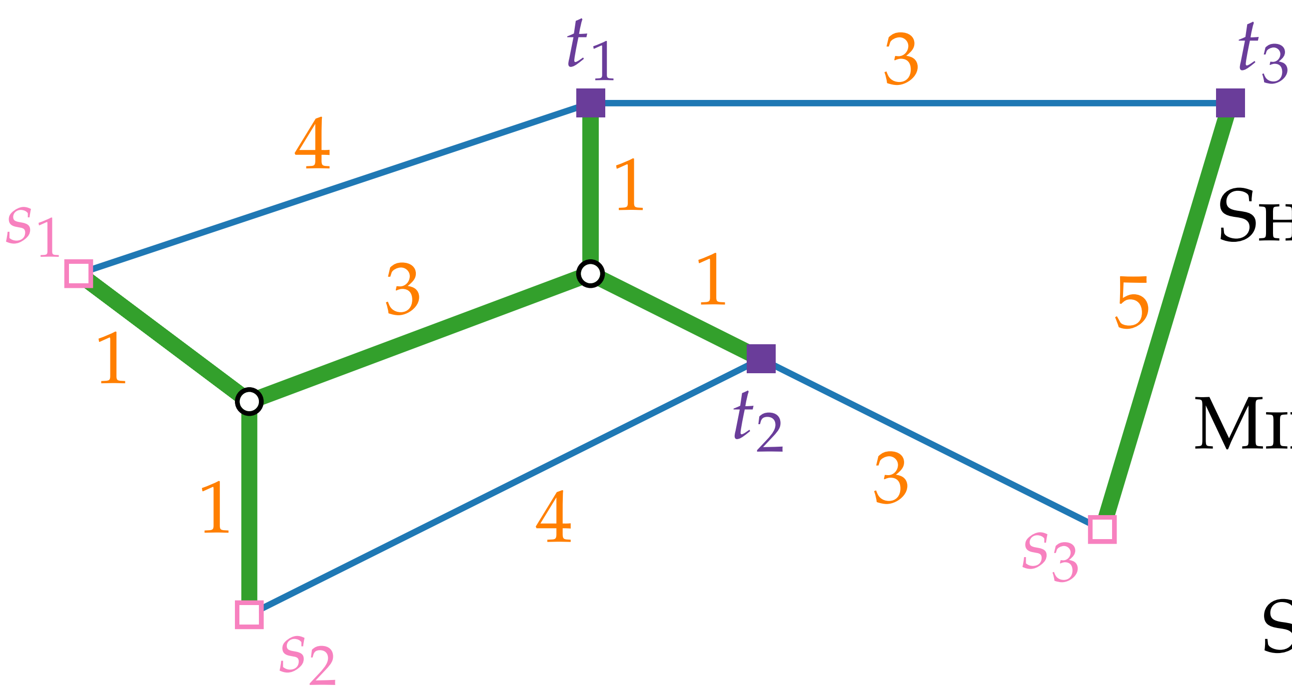


Special cases?

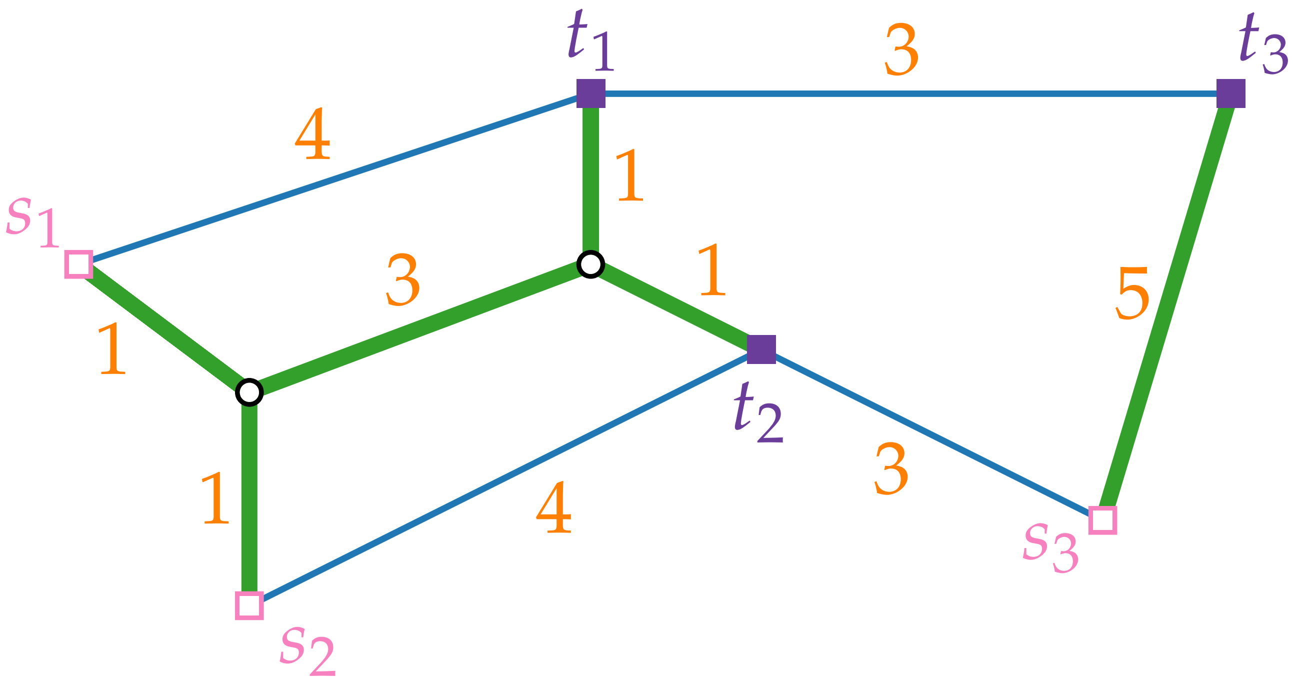
ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

SteinerTree ($R = T \times T$)

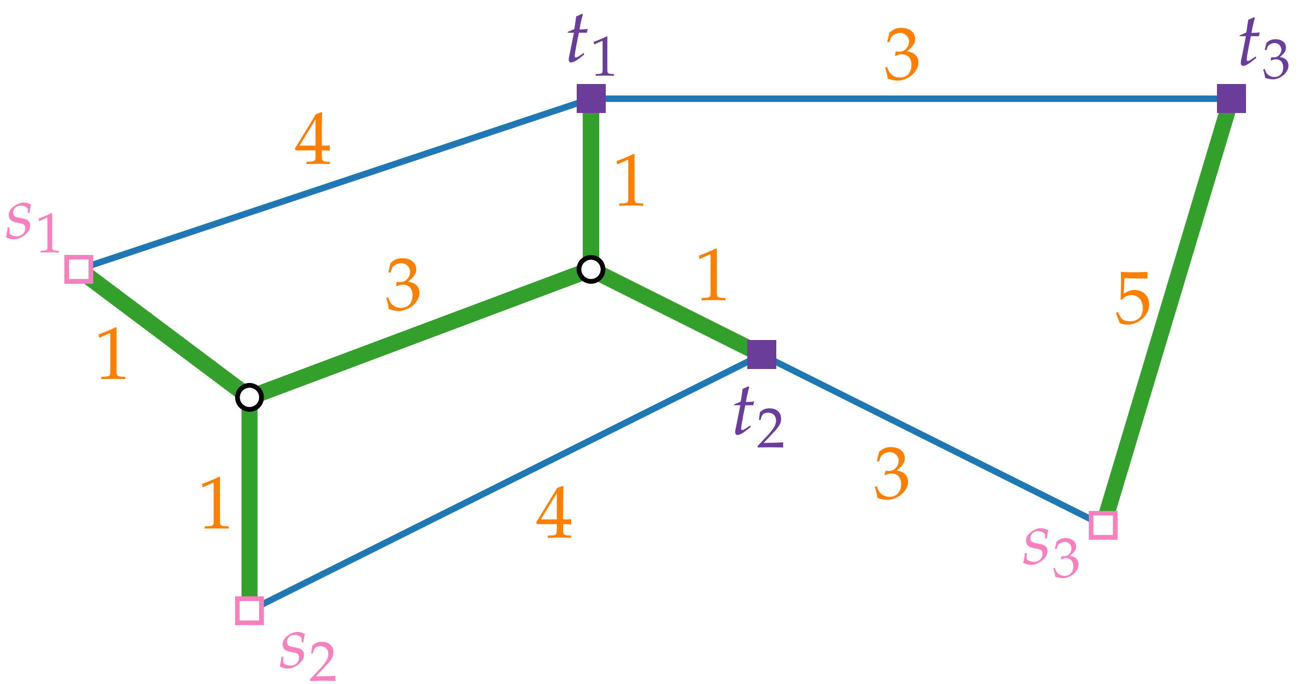
Approaches?

- Merge k shortest s_i-t_i-paths



Approaches?

- Merge k shortest s_i-t_i-paths
- SteinerTree on the set of terminals



Approaches?

- Merge k shortest s_i-t_i-paths
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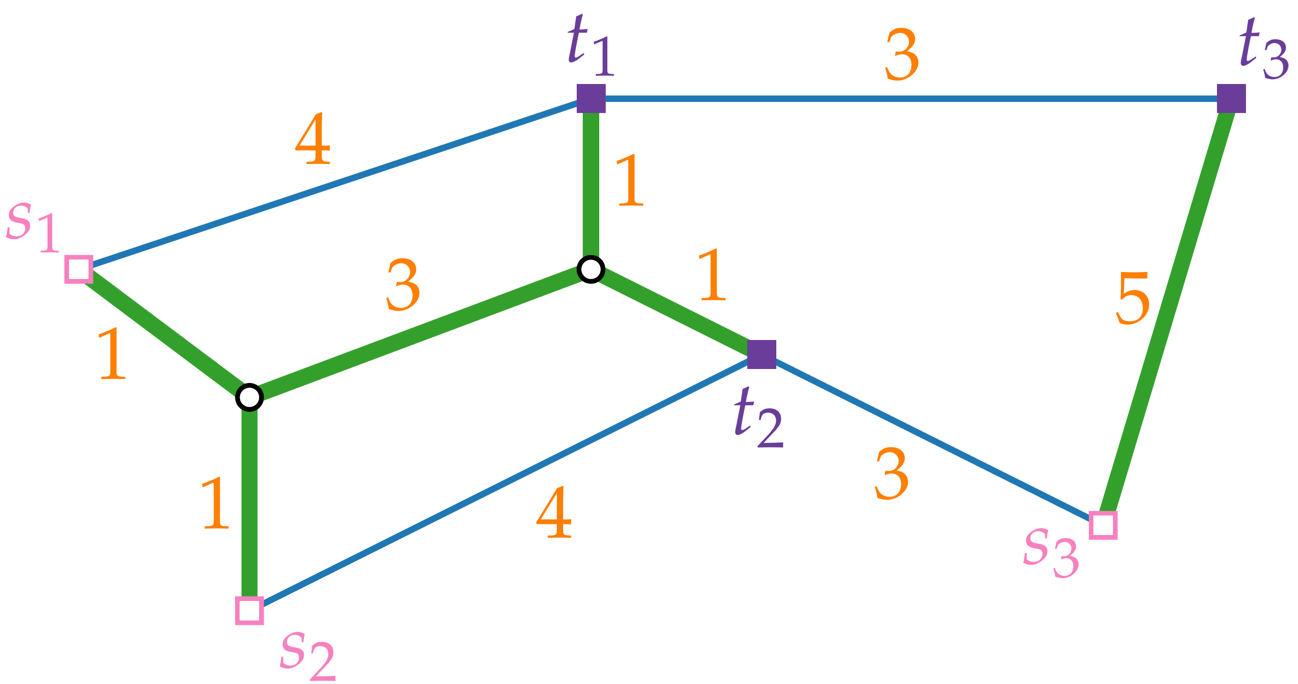
Above approaches perform poorly :-(



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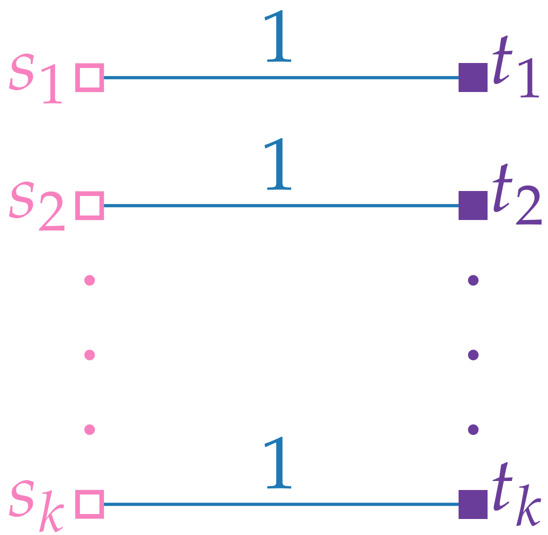

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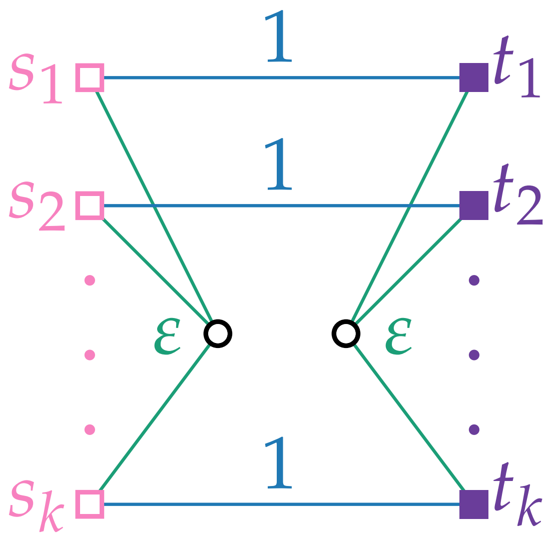

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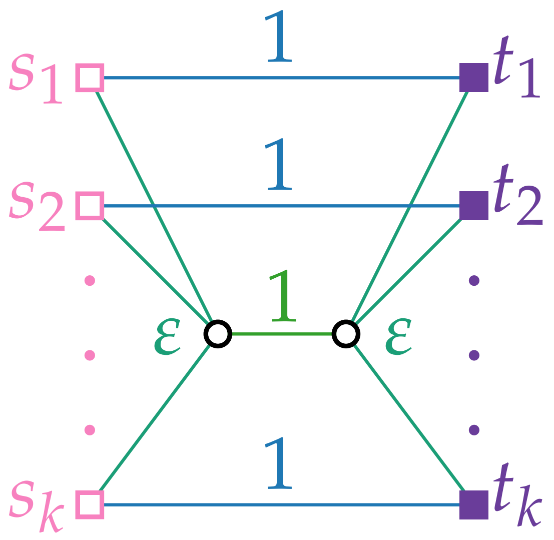

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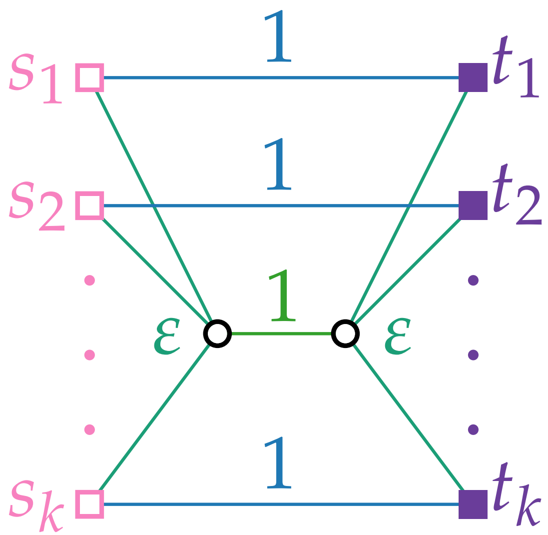

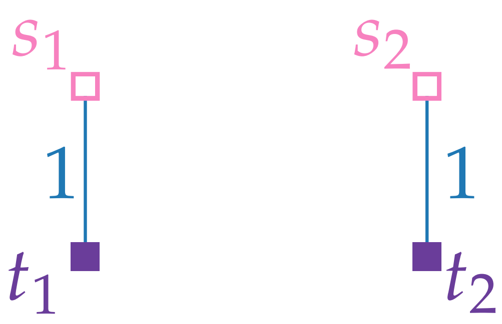

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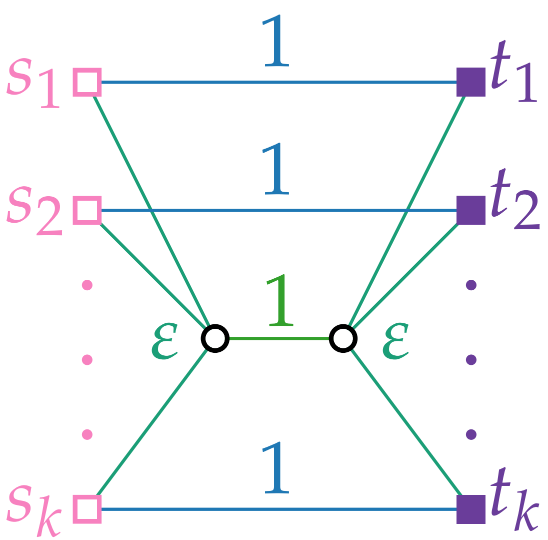

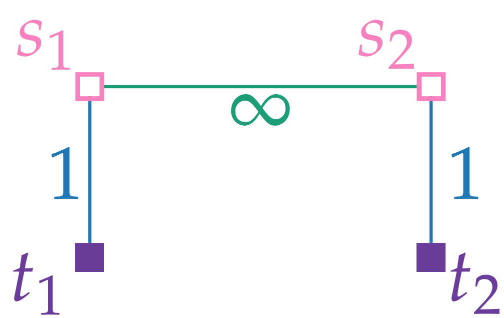

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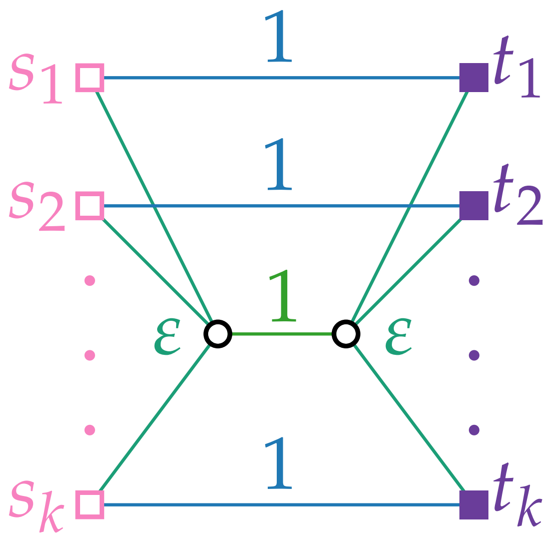
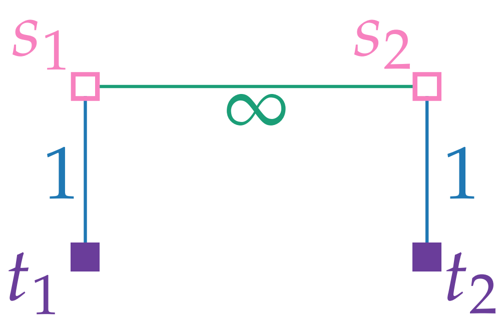
Above approaches perform poorly :-(

Difficulty: which terminals belong to the same tree of the forest?





Approximation Algorithms

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Part II:
Primal and Dual LP

Joachim Spoerhase

Winter 2020/21

An ILP

minimize

subject to

An ILP

minimize

subject to

$$x_e \in \{0, 1\} \qquad e \in E$$

An ILP

minimize $\sum_{e \in E} c_e x_e$

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t_i

s_i

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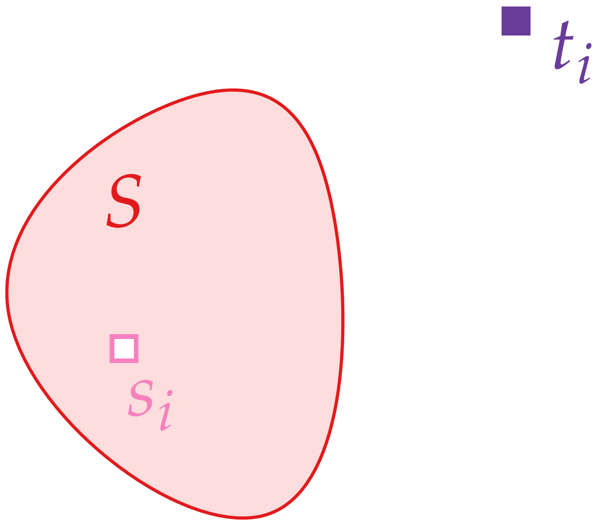

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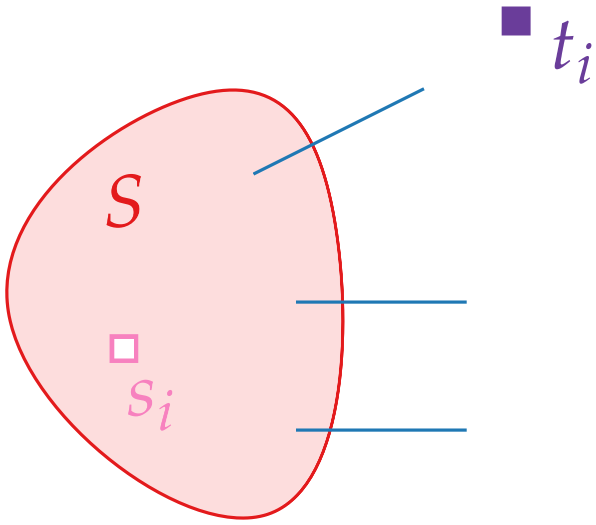

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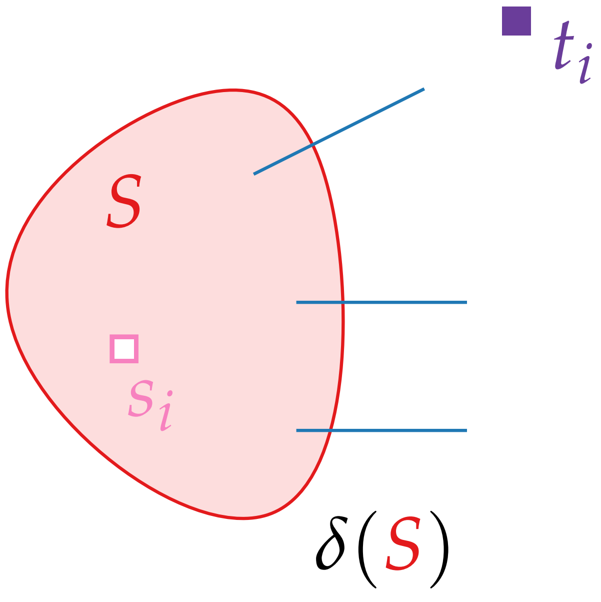

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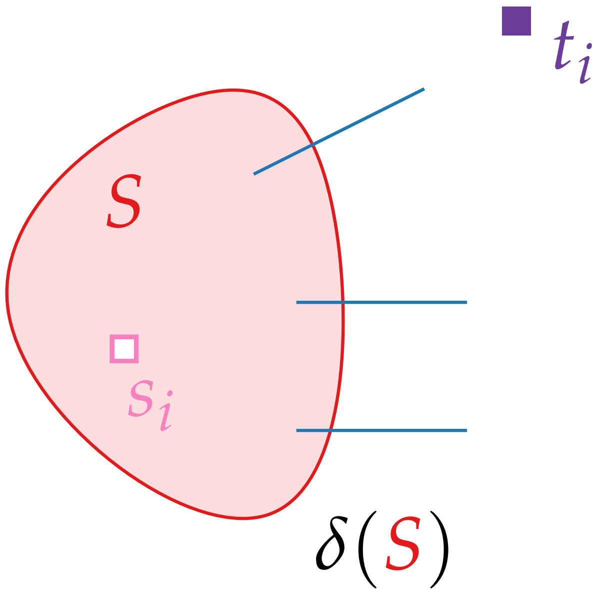
$\delta(S) := \{(u,v) \in E : u \in S \text{ and } v \notin S\}$



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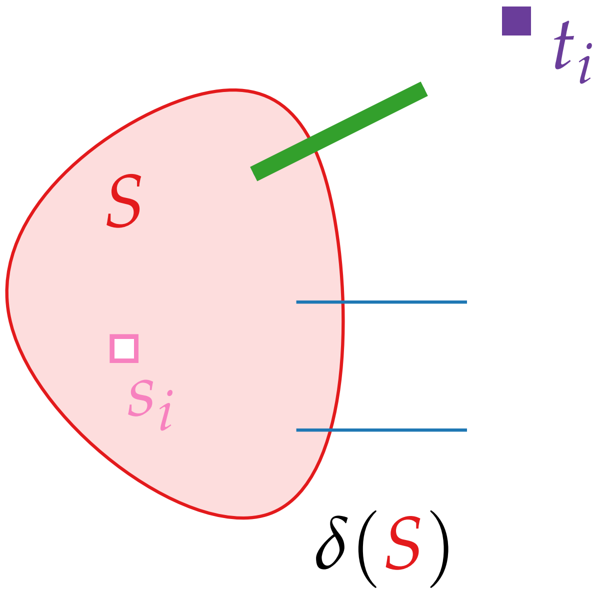

An ILP

$$\begin{array}{lll} \textbf{minimize} & \sum_{e \in E} c_e x_e & \\ \textbf{subject to} & \sum_{e \in \delta(S)} x_e \geq 1 & \\ & x_e \in \{0,1\} & e \in E \end{array}$$

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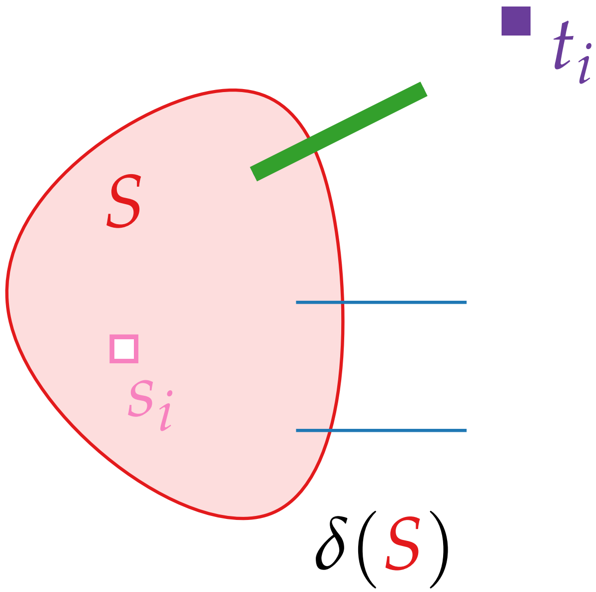

An ILP

minimize	$\sum_{e \in E} c_e x_e$	
subject to	$\sum_{e \in \delta(S)} x_e \geq 1$	$S \in \mathcal{S}_i, i = 1, \ldots, k$
	$x_e \in \{0, 1\}$	$e \in E$

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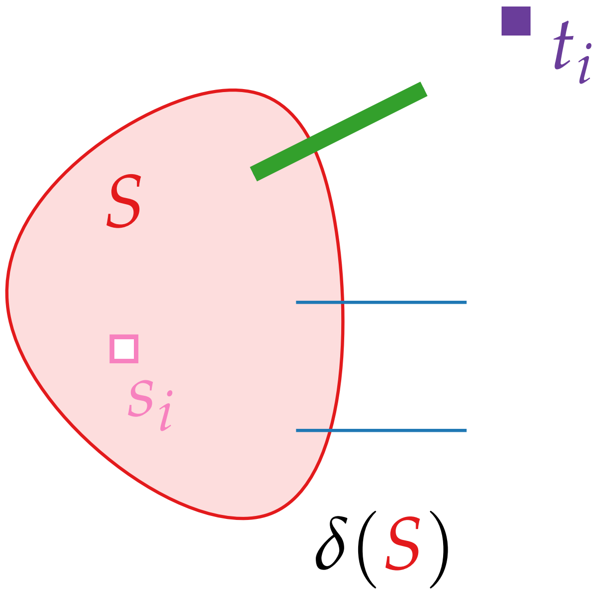

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where $\mathcal{S}_i := \{S \subseteq V \colon |S \cap \{s_i, t_i\}| = 1\}$

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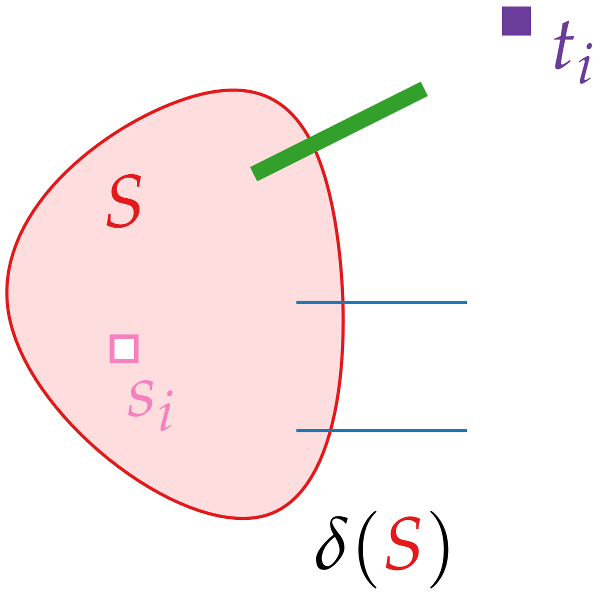

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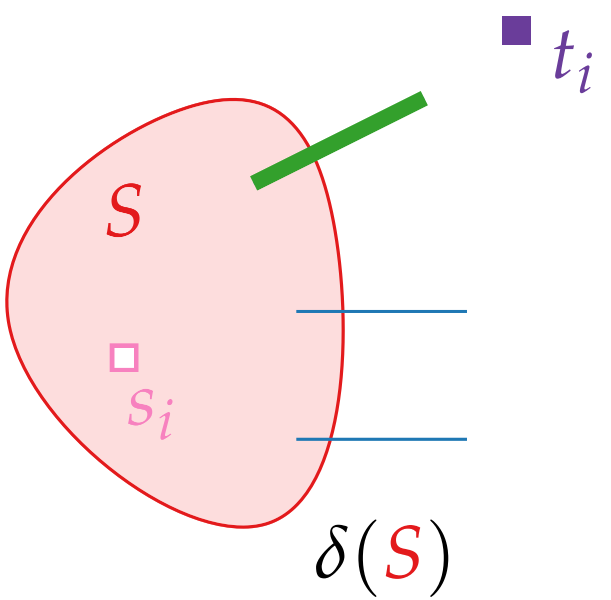
$\rightsquigarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{e \in E} c_e x_e & \\ \textbf{subject to} & \displaystyle\sum_{e \in \delta(S)} x_e \geq 1 & S \in \mathcal{S}_i, i = 1, \ldots, k \\ & x_e \geq 0 & e \in E \end{array}$$

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$$

maximize

subject to

$$
y_S \geq 0 \qquad S \in \mathcal{S}_i, i = 1, \ldots, k
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LP-Relaxation and Dual LP

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Intuition for the Dual

$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{\substack{S \in \mathcal{S}_i \\ i=1,\ldots,k}} y_S & \\ \textbf{subject to} & \displaystyle\sum_{S:\, e \in \delta(S)} y_S \le c_e & e \in E \\ & y_S \ge 0 & S \in \mathcal{S}_i, i = 1, \ldots, k \end{array}$$

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The graph is a network of **bridges**, spanning the **moats**.

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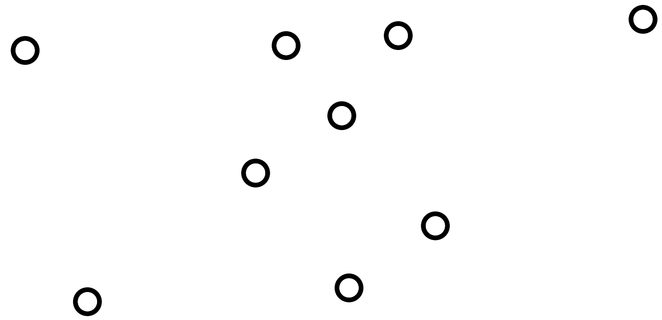

Intuition for the Dual

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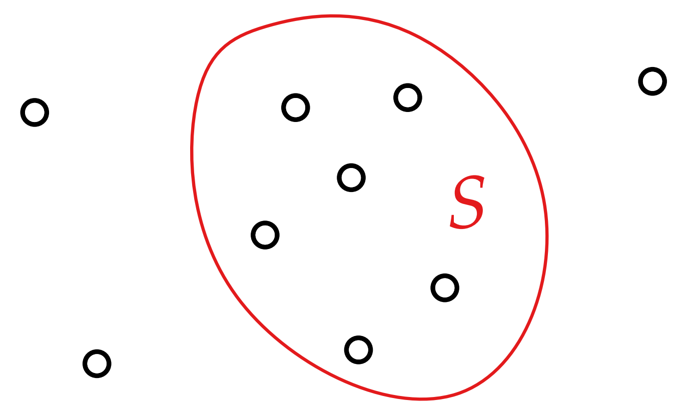
& y_S \ge 0 && S \in \mathcal{S}_i, i = 1, \ldots, k
\end{aligned}$$

The graph is a network of **bridges**, spanning the **moats**.



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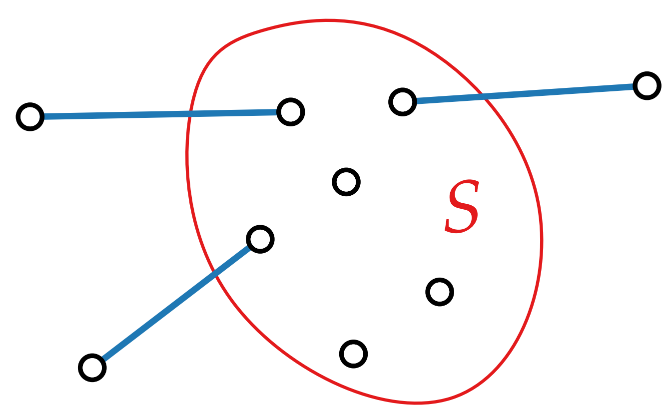

Intuition for the Dual

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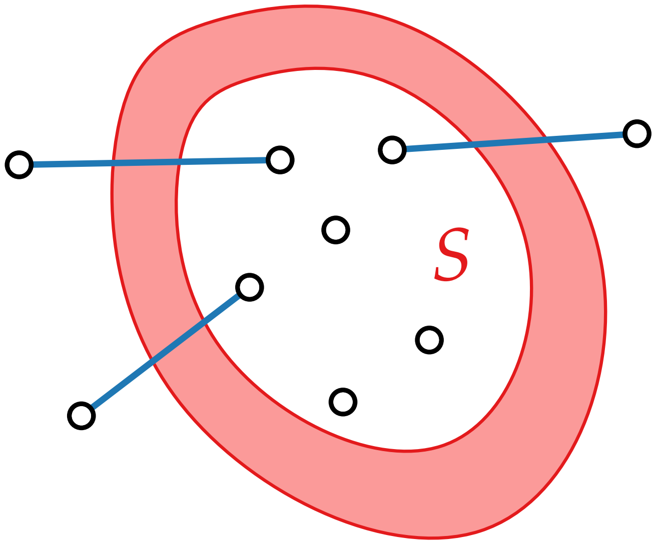

Intuition for the Dual

$$\text{maximize} \quad \sum_{\substack{S \in \mathcal{S}_i \\ i=1,\ldots,k}} y_S$$

$$\text{subject to} \quad \sum_{S:\, e \in \delta(S)} y_S \le c_e \qquad e \in E$$

$$y_S \ge 0 \qquad S \in \mathcal{S}_i, i = 1, \ldots, k$$

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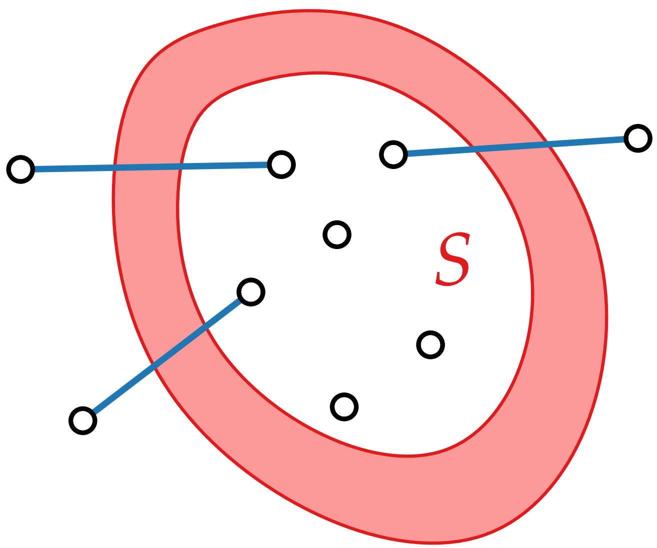


$\delta(S)$ = set of edges / bridges over the moat around S

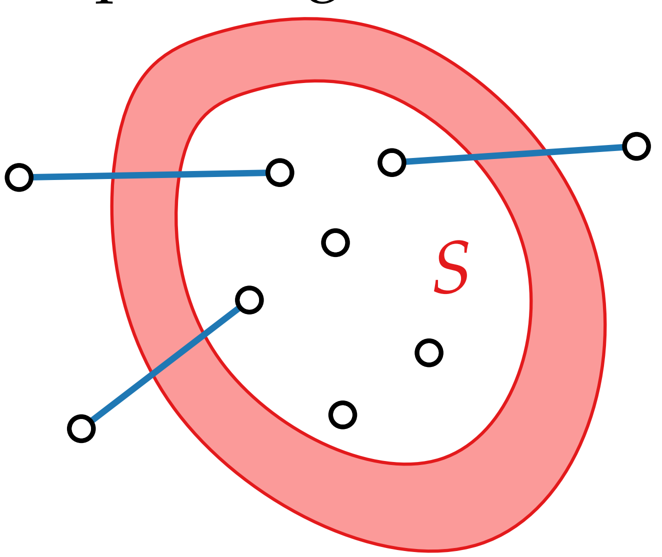
Intuition for the Dual

$$\textbf{maximize} \quad \sum_{\substack{S \in \mathcal{S}_i \\ i=1,\dots,k}} y_S$$

$$\textbf{subject to} \quad \sum_{S:\, e \in \delta(S)} y_S \le c_e \qquad e \in E$$

$$y_S \ge 0 \qquad S \in \mathcal{S}_i, i = 1, \dots, k$$

The graph is a network of **bridges**, spanning the **moats**.



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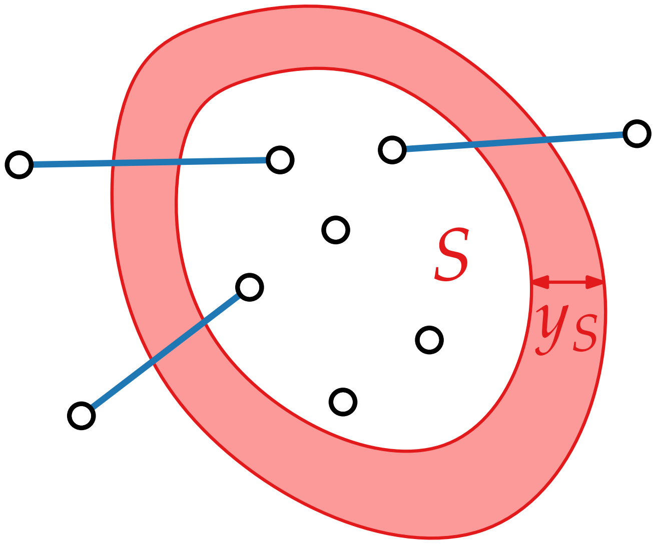

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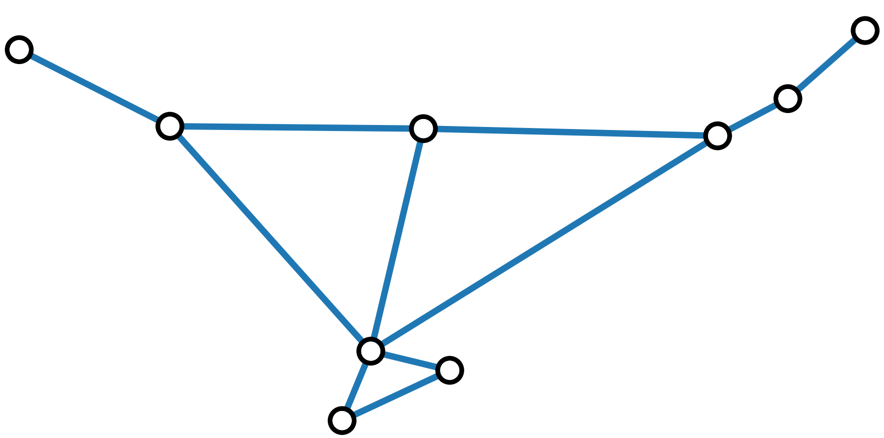

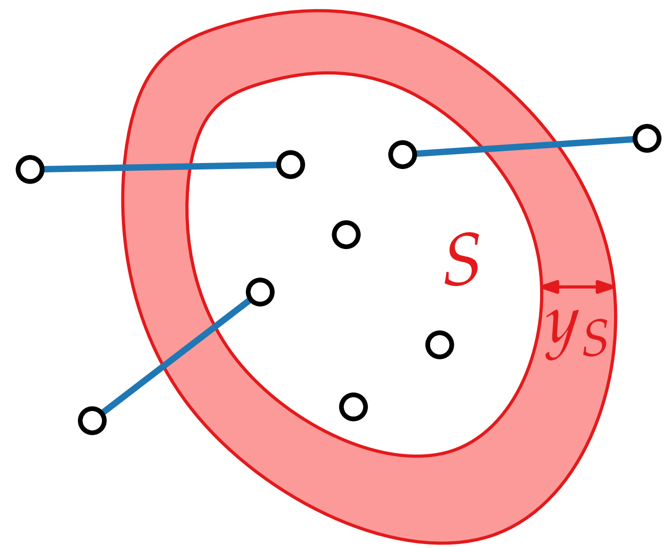

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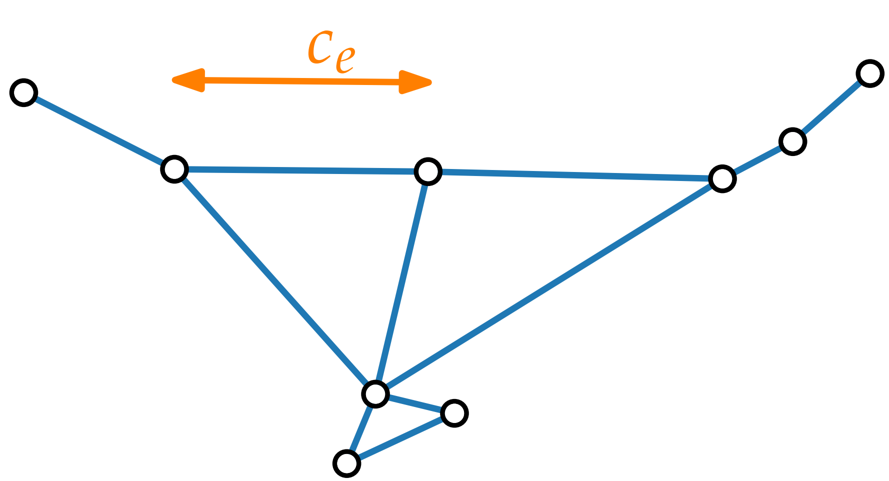

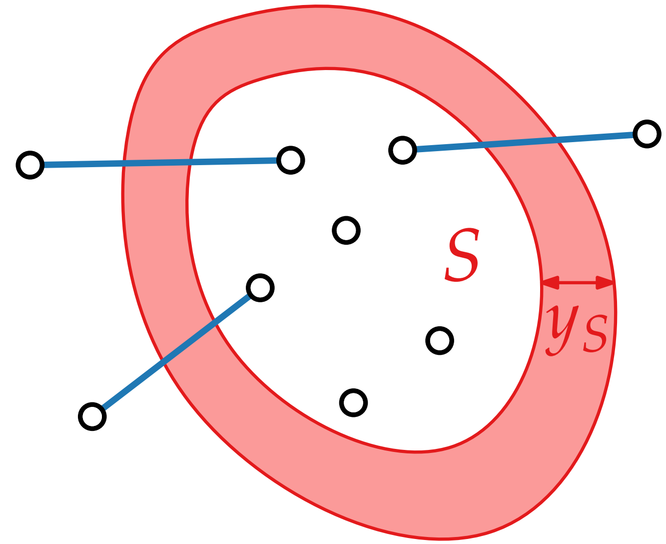

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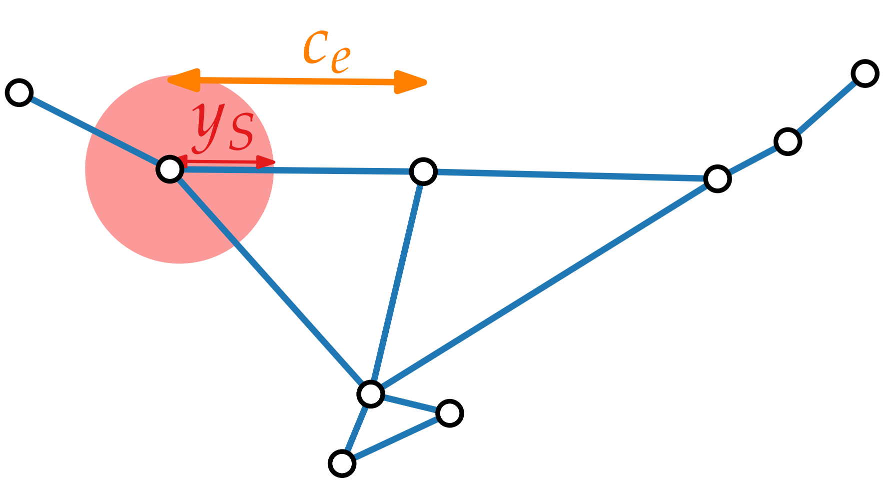

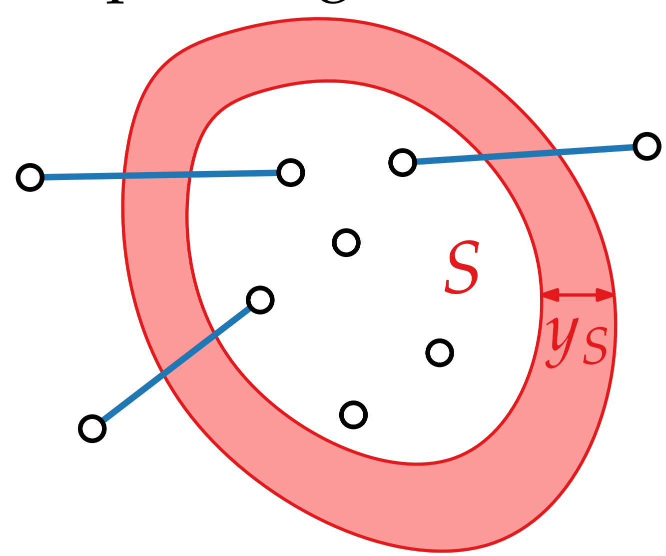

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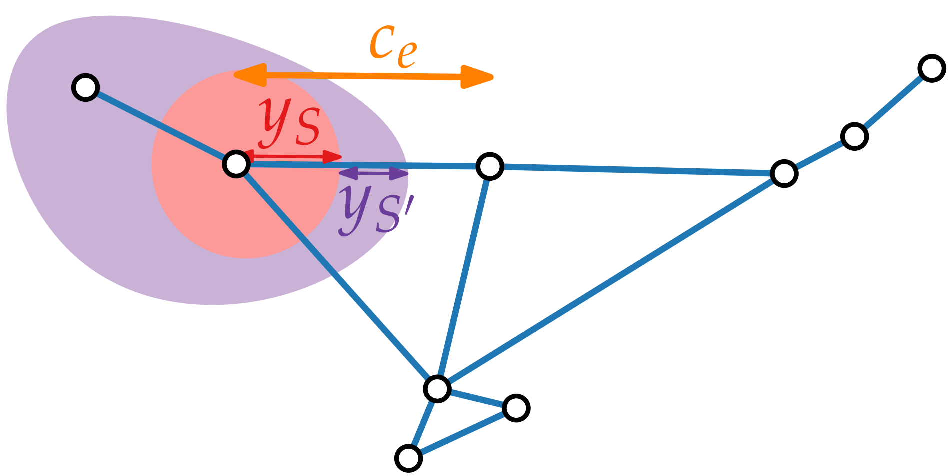

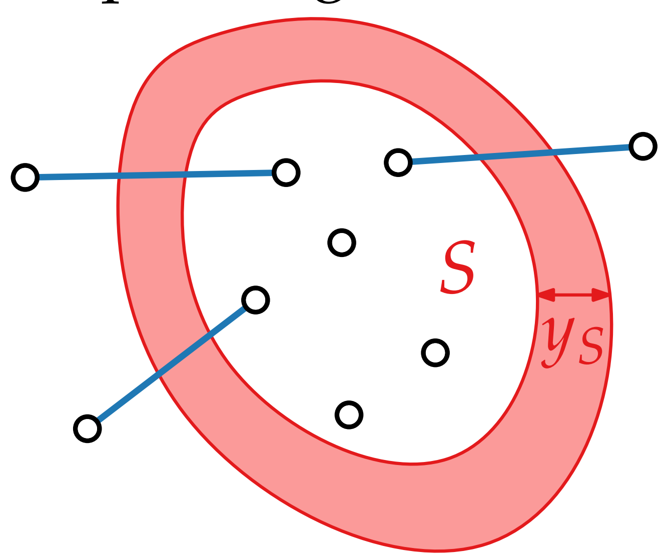

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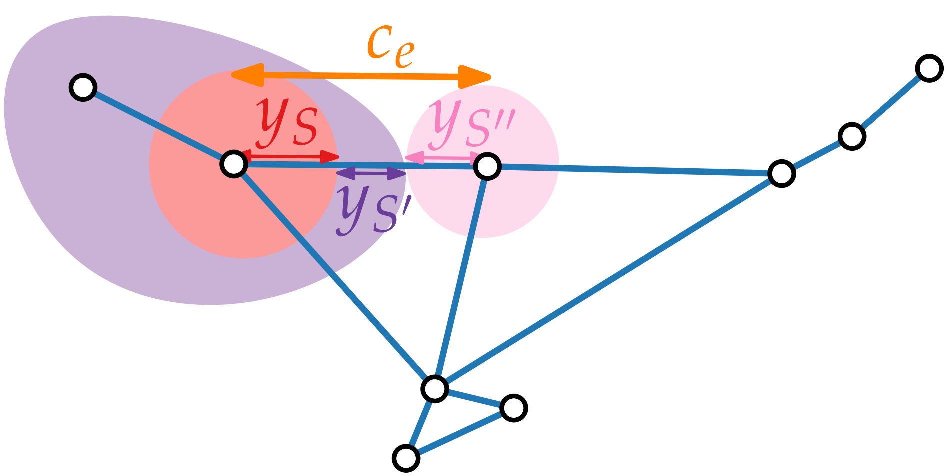

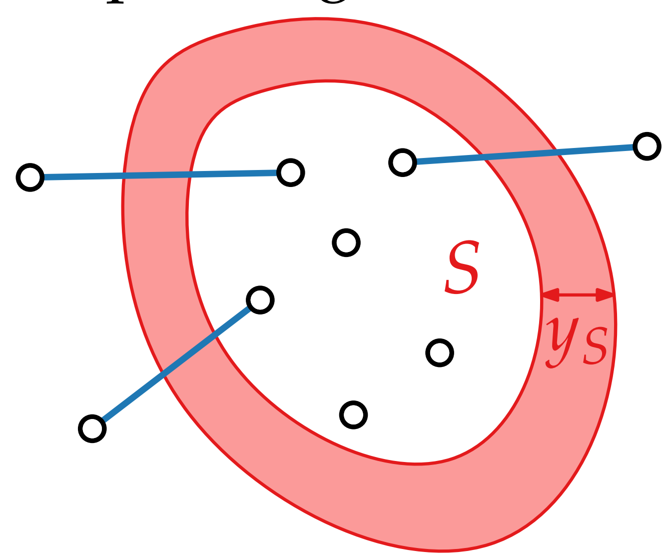

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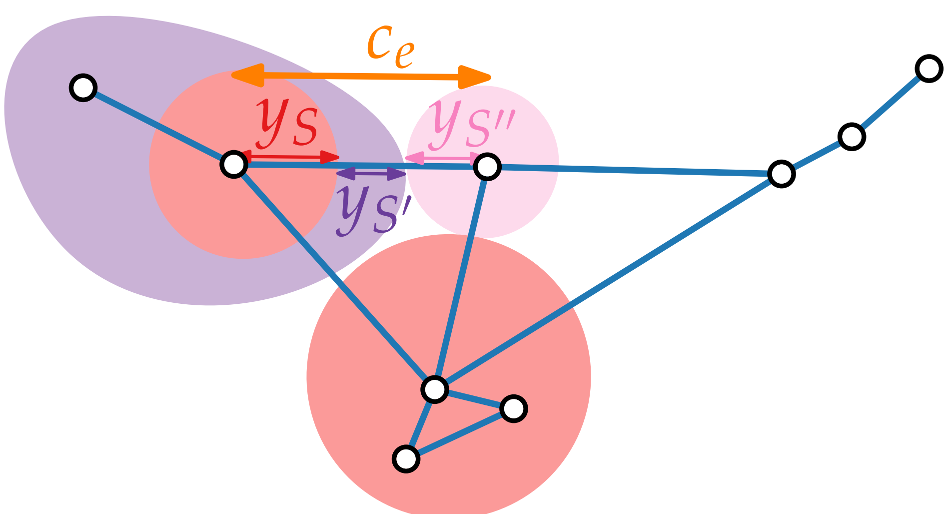

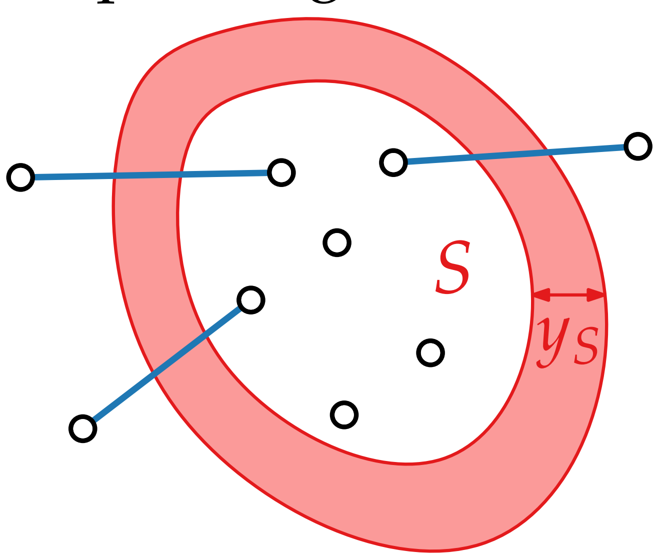

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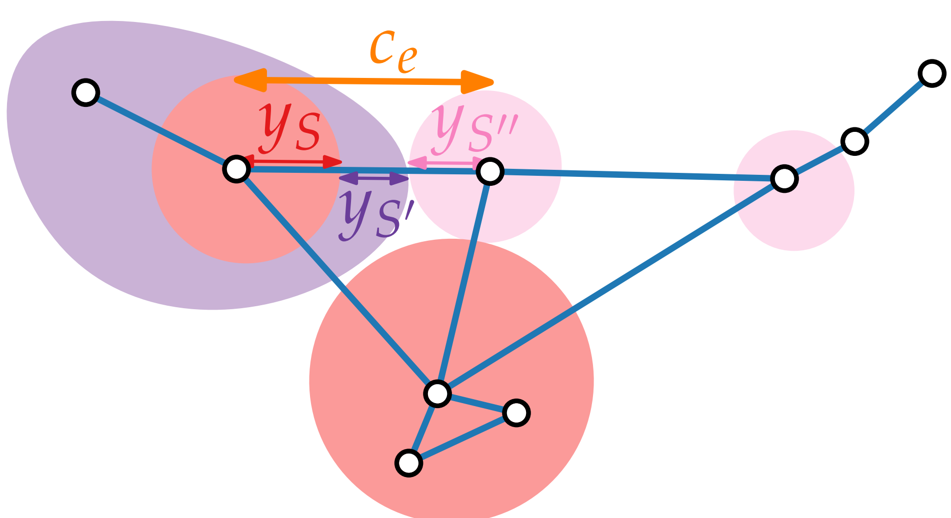

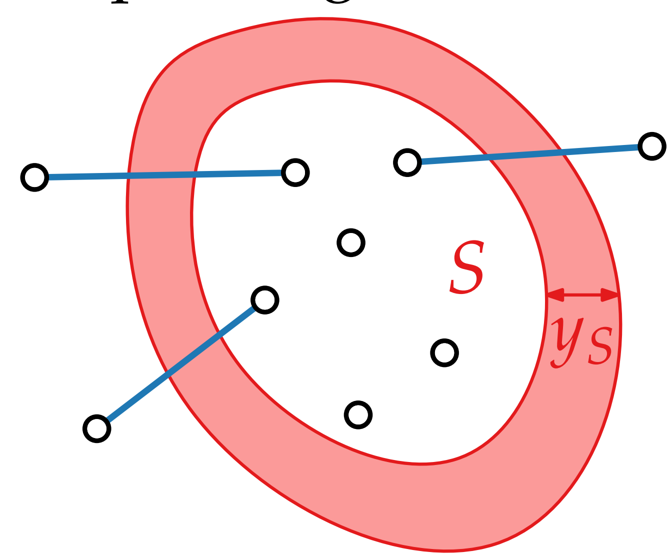

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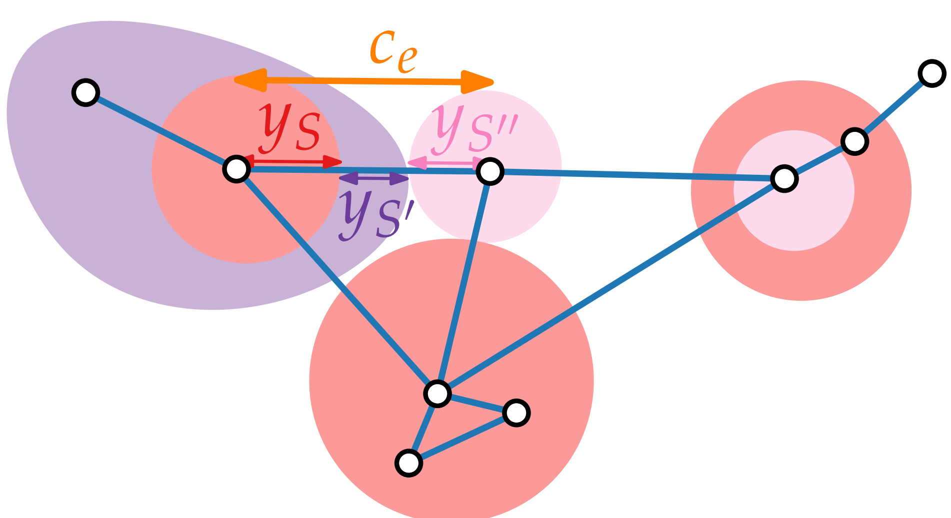

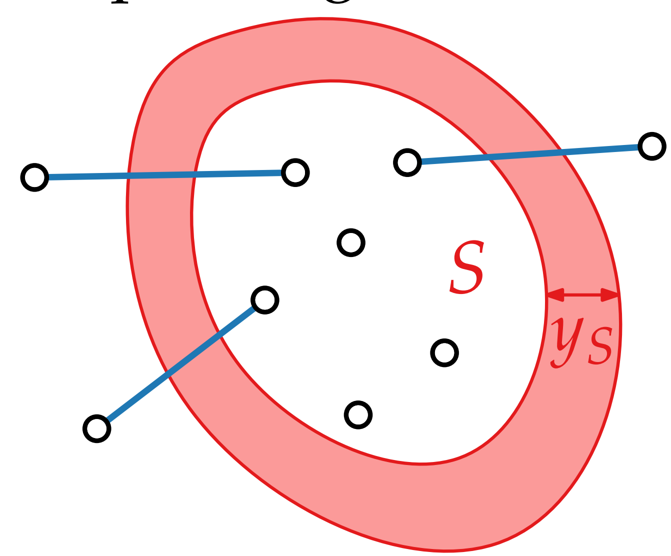

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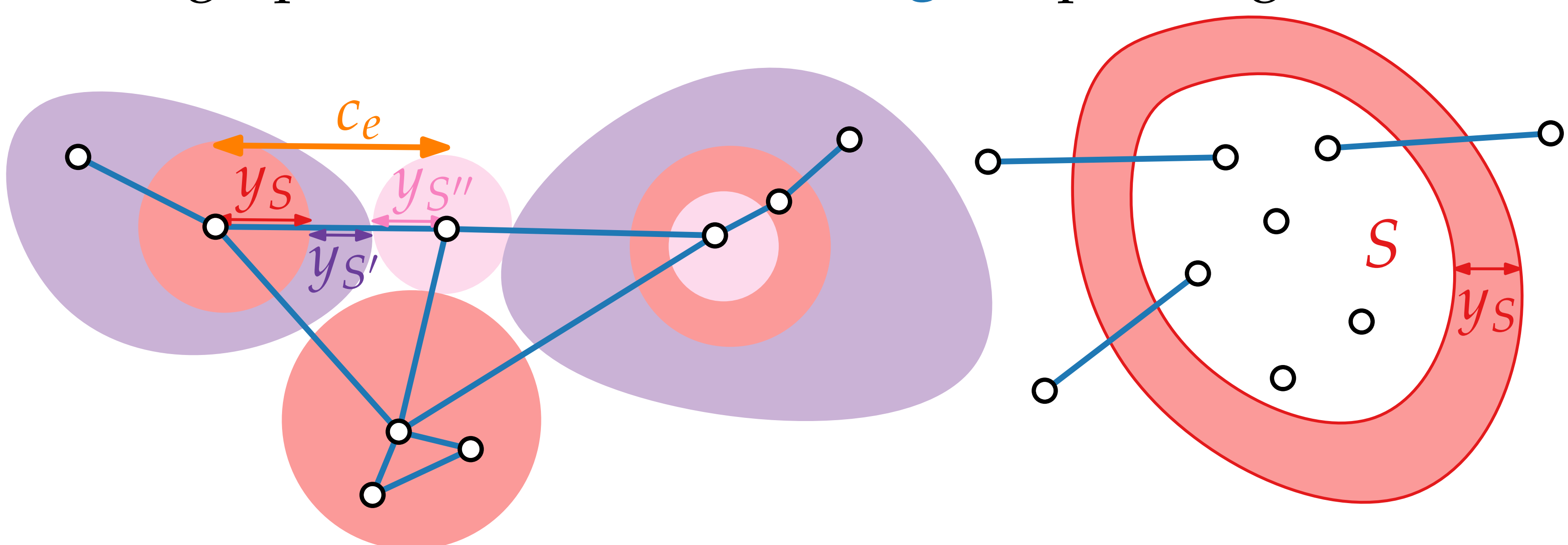

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part III:
A First Primal-Dual Approach

Joachim Spoerhase

Winter 2021/22

Complementary Slackness (Rep.)

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

A First Primal-Dual Approach

Complementary slackness: $x_e > 0 \quad \Rightarrow$

A First Primal-Dual Approach

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$\Rightarrow$ pick "critical" edges (and only those)

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Idea: iteratively build a feasible integral Primal-Solution.

A First Primal-Dual Approach

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Idea: iteratively build a feasible integral Primal-Solution.

How to find a violated primal constraint? $\left(\sum_{e \in \delta(S)} x_e < 1\right)$

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How do we iteratively improve the Dual-Solution?

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Idea: iteratively build a feasible integral Primal-Solution.

How to find a violated primal constraint? $(\sum_{e \in \delta(S)} x_e < 1)$

$\leadsto$ Consider related connected component C!

How do we iteratively improve the Dual-Solution?

$\leadsto$ increase y_C! (until some edge in $\delta(C)$ becomes critical)

A First Primal-Dual Approach

PrimalDualSteinerForestNaive(G, c, R)

A First Primal-Dual Approach

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$y \leftarrow 0, F \leftarrow \emptyset$

return F

A First Primal-Dual Approach

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while some $(s_i, t_i) \in R$ not connected in (V, F) **do**

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Running Time?

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Running Time?

Trick: Handle all y_S with $y_S = 0$ implicitly

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$$\sum_{e \in F} c_e \overset{\text{CS}}{=} \sum_{e \in F} \sum_{S\,:\, e \in \delta(S)} y_S = \sum_{S} |\delta(S) \cap F| \cdot y_S.$$

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Compare to the value of the dual objective function $\sum_S y_S$

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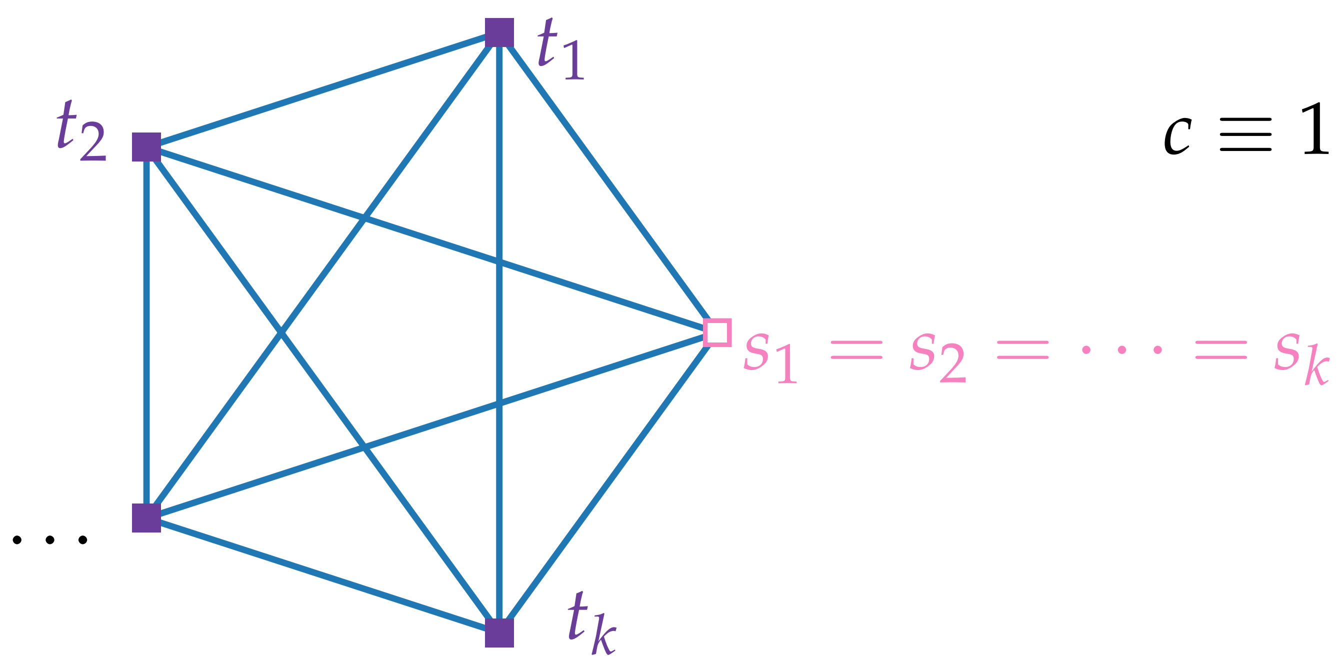

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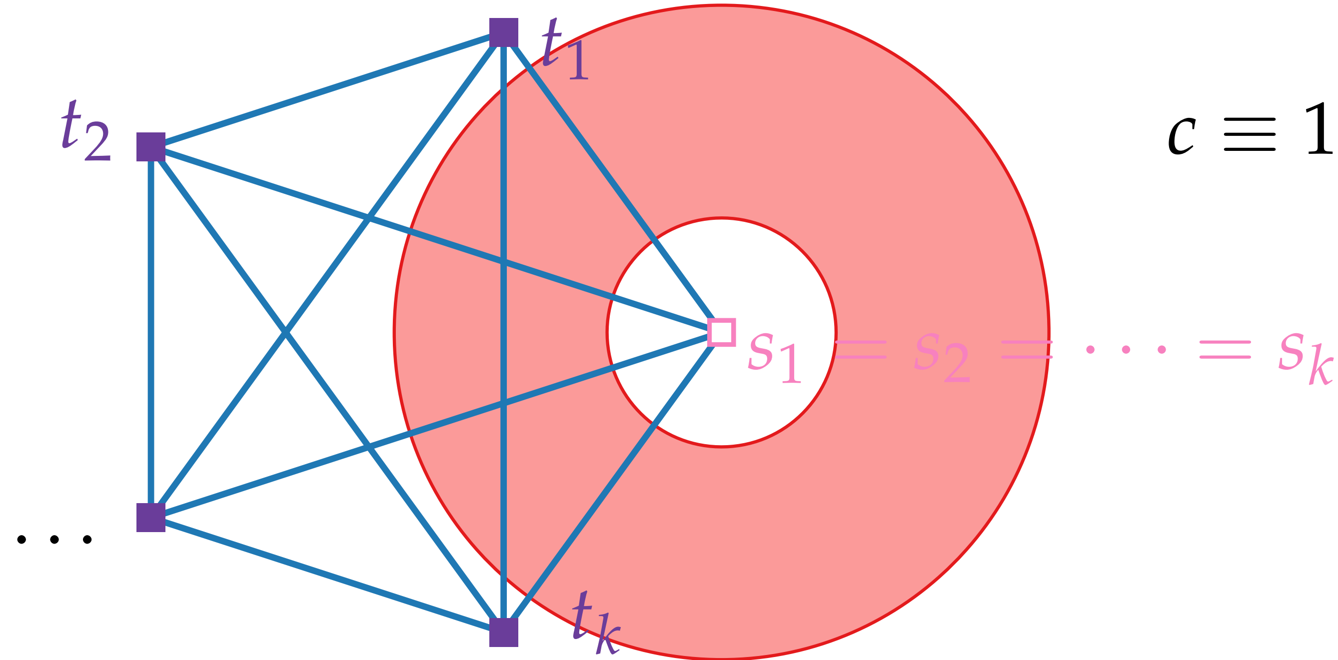

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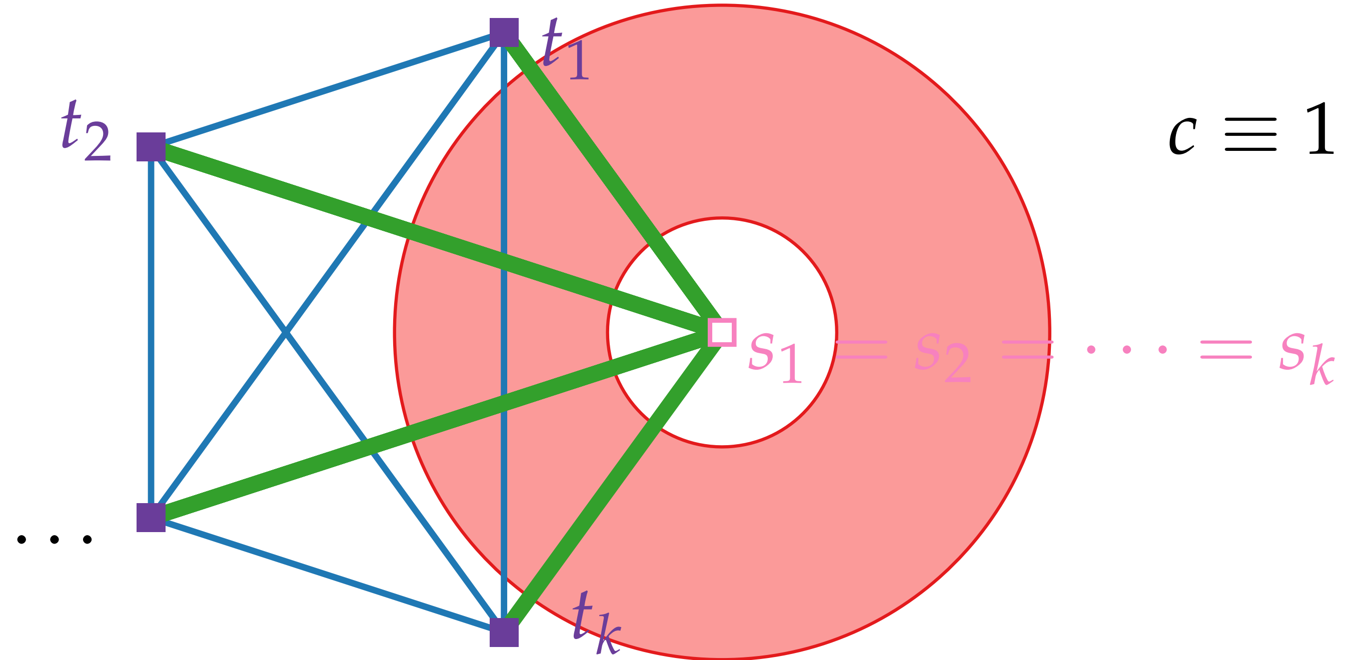

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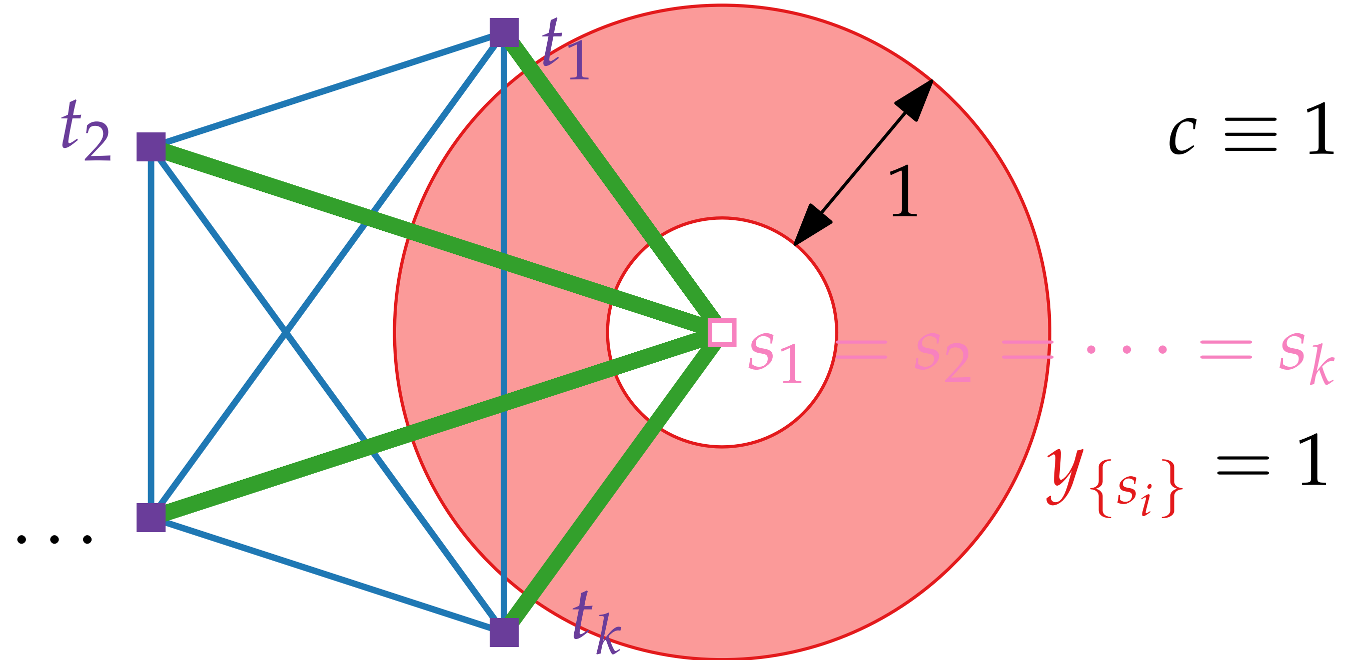

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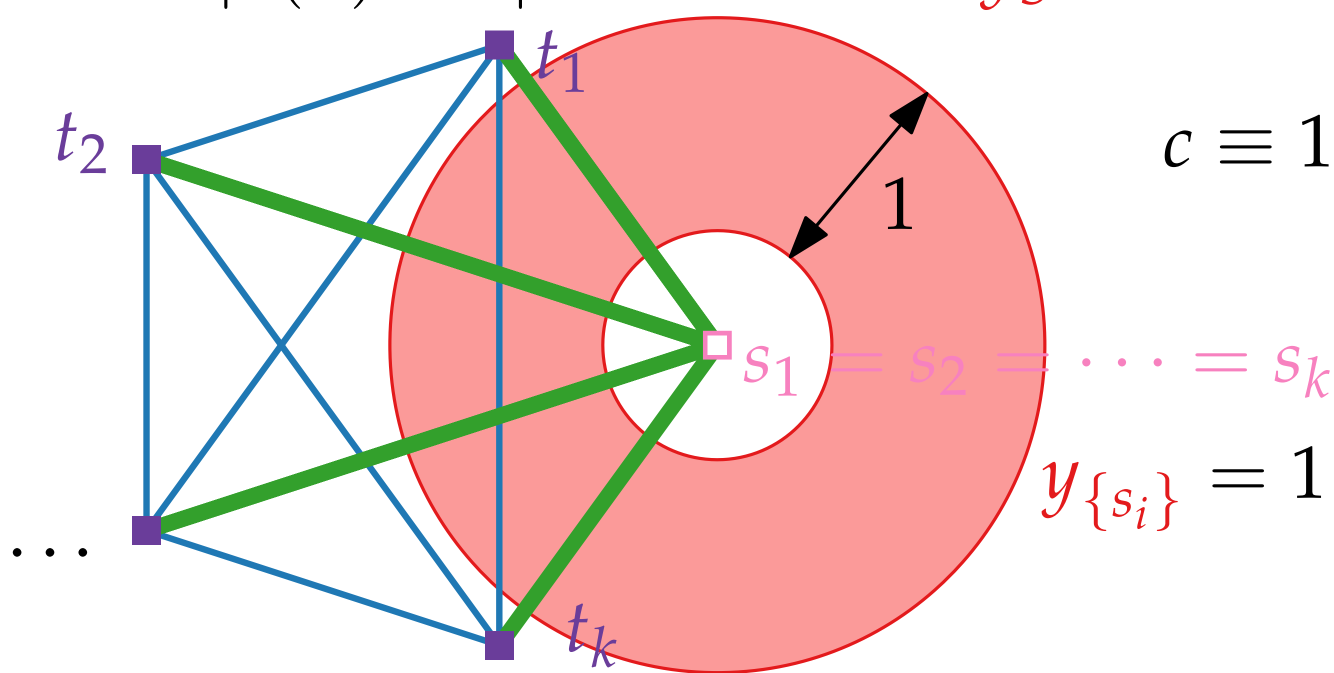

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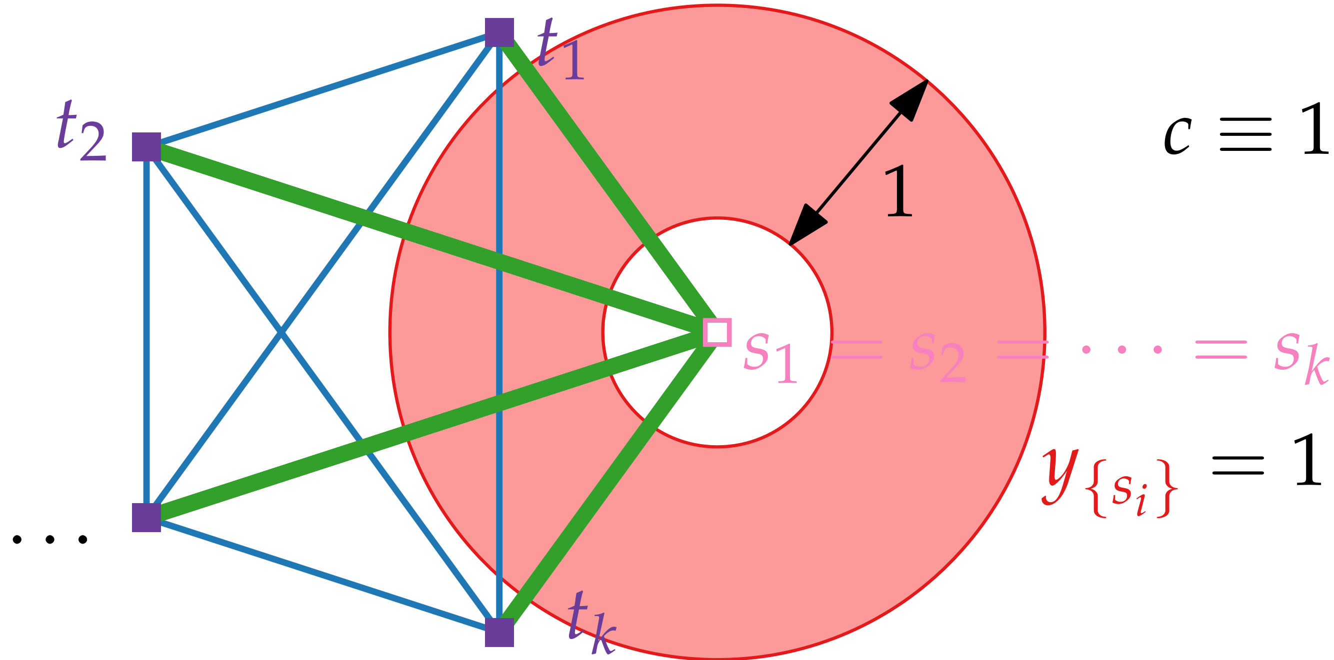
There are examples with $|\delta(S) \cap F| = k$ for each $y_S > 0$:

But: Average degree of component is 2!

$\Rightarrow$ Increase y_C for all components C simultaneously!



Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part IV:
Primal-Dual with Synchronized Increases

Joachim Spoerhase

Winter 2021/22

Primal-Dual with Synchronized Increases

PrimalDualSteinerForest(G, c, R)

$y \leftarrow 0, F \leftarrow \emptyset, \ell \leftarrow 0$

while some $(s_i, t_i) \in R$ not connected in (V, F) **do**

  $\ell \leftarrow \ell + 1$

  $F \leftarrow F \cup \{e_\ell\}$

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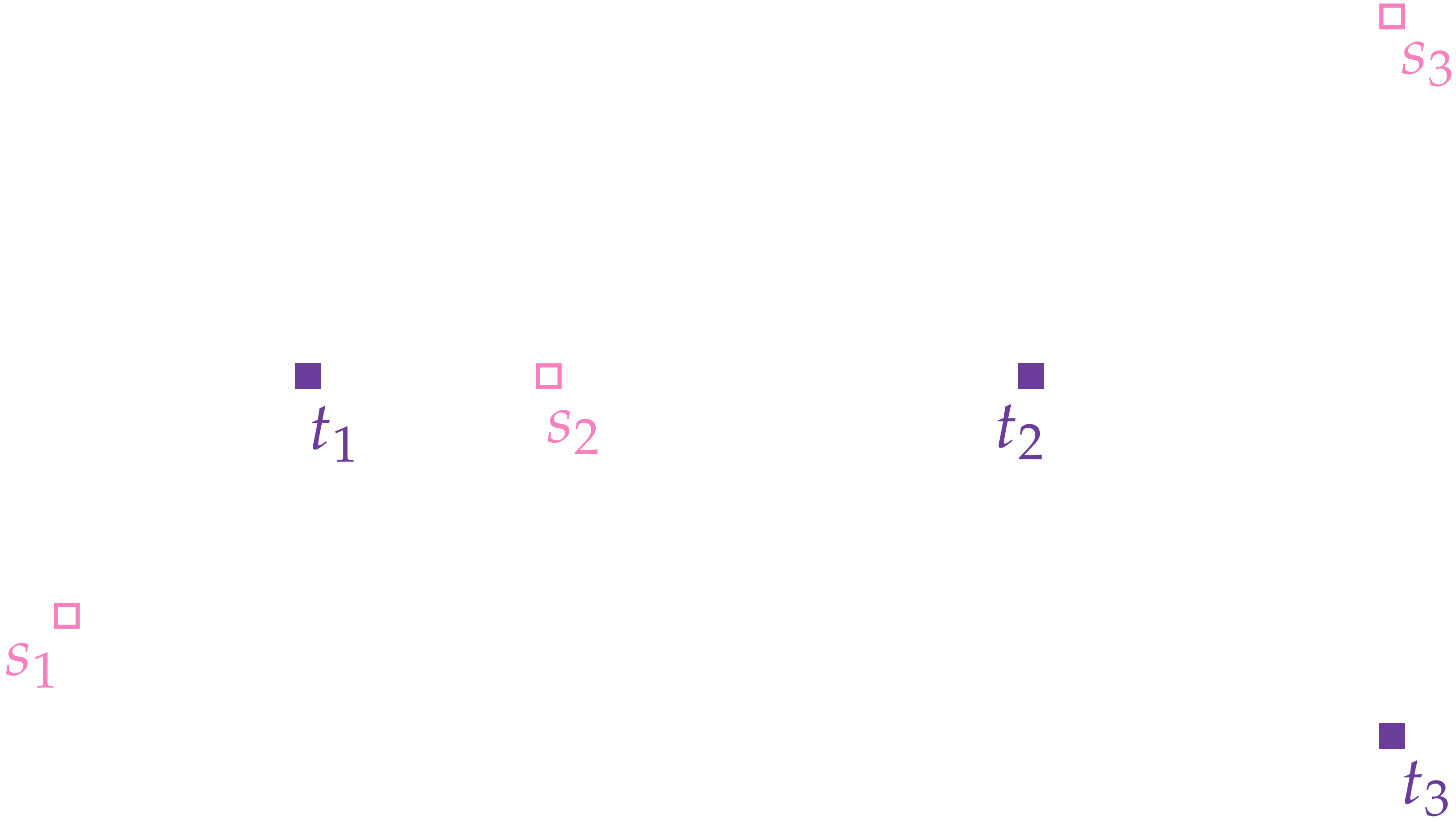
Illustration

$G = K_6$ with Euclidean edge costs



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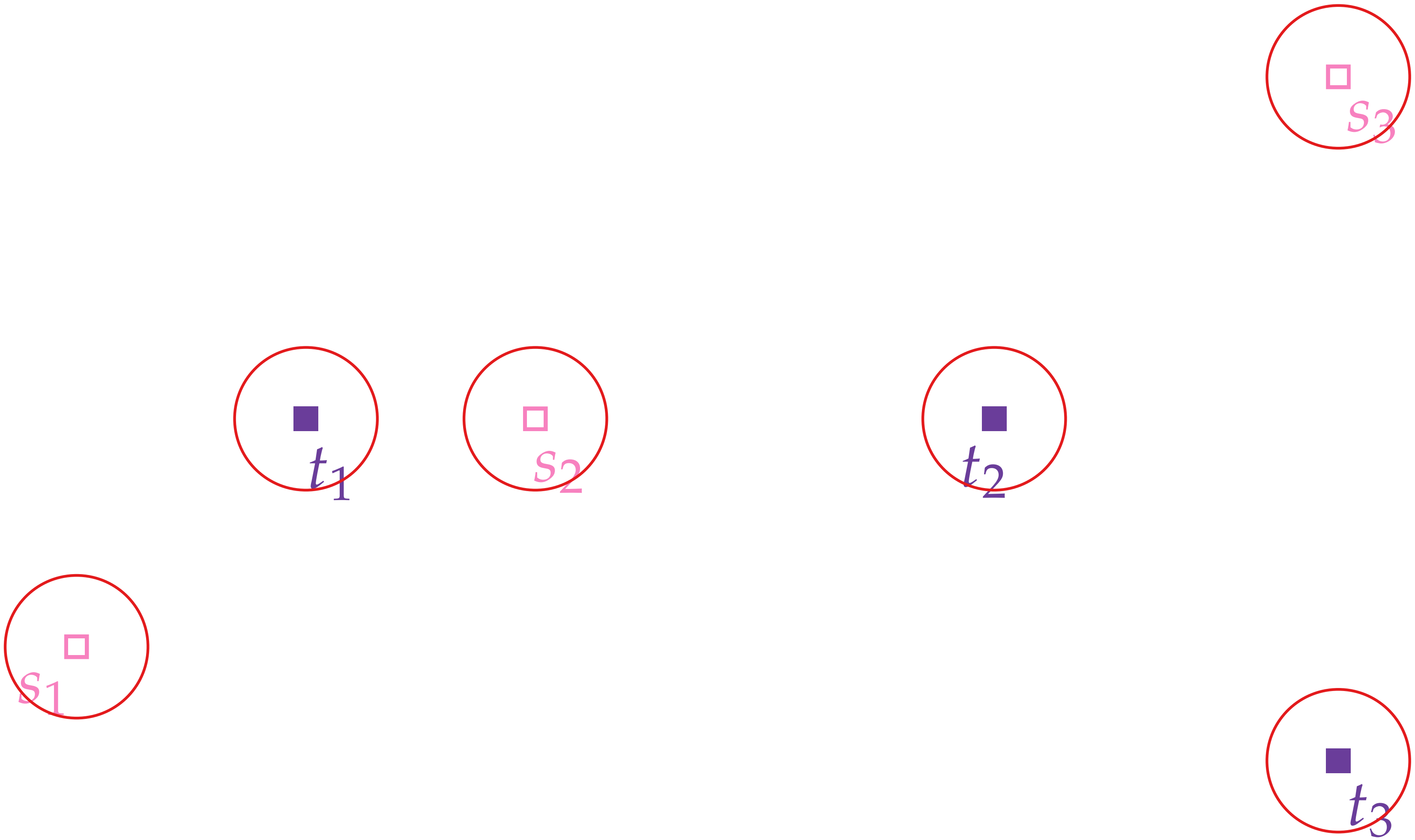

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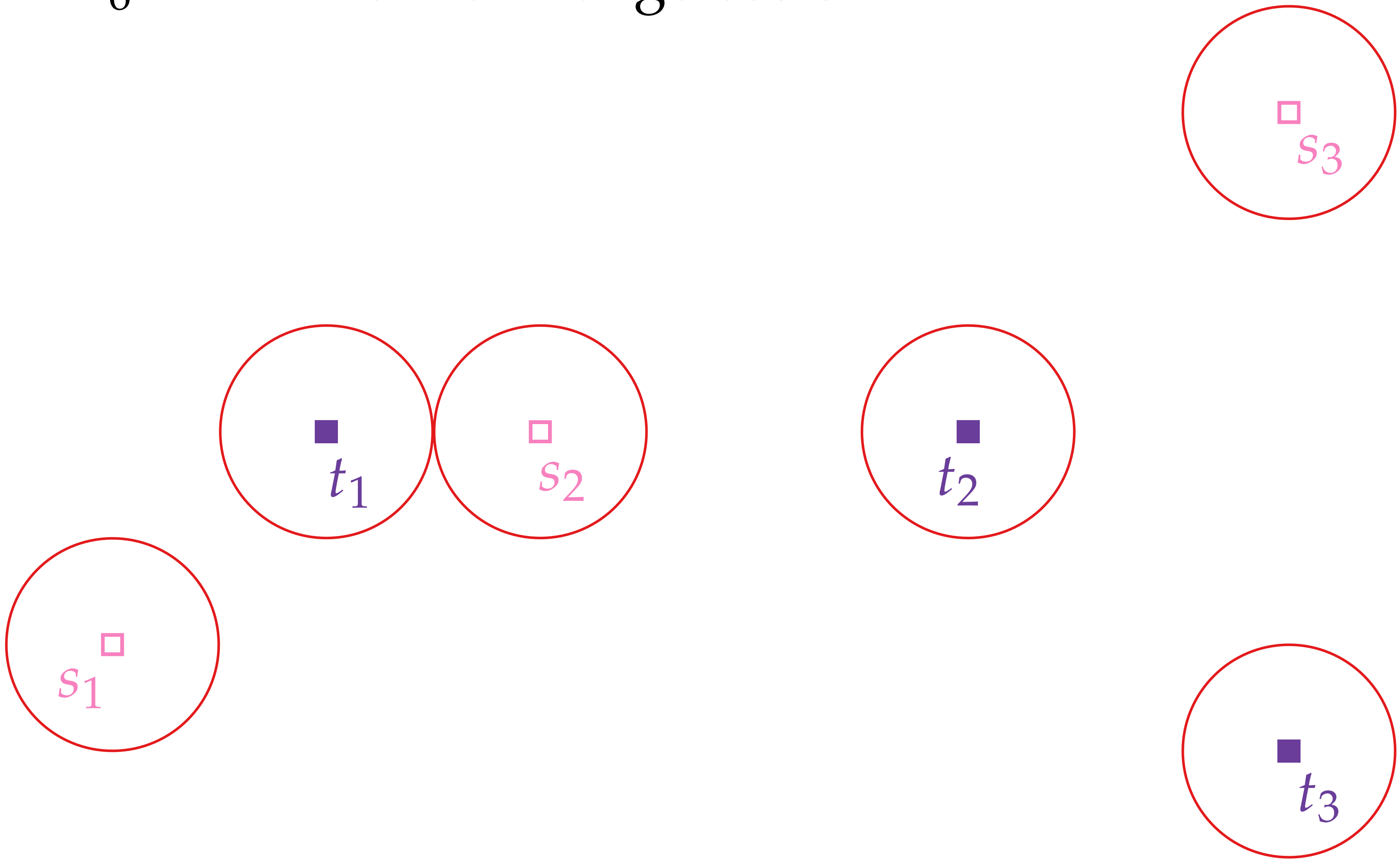

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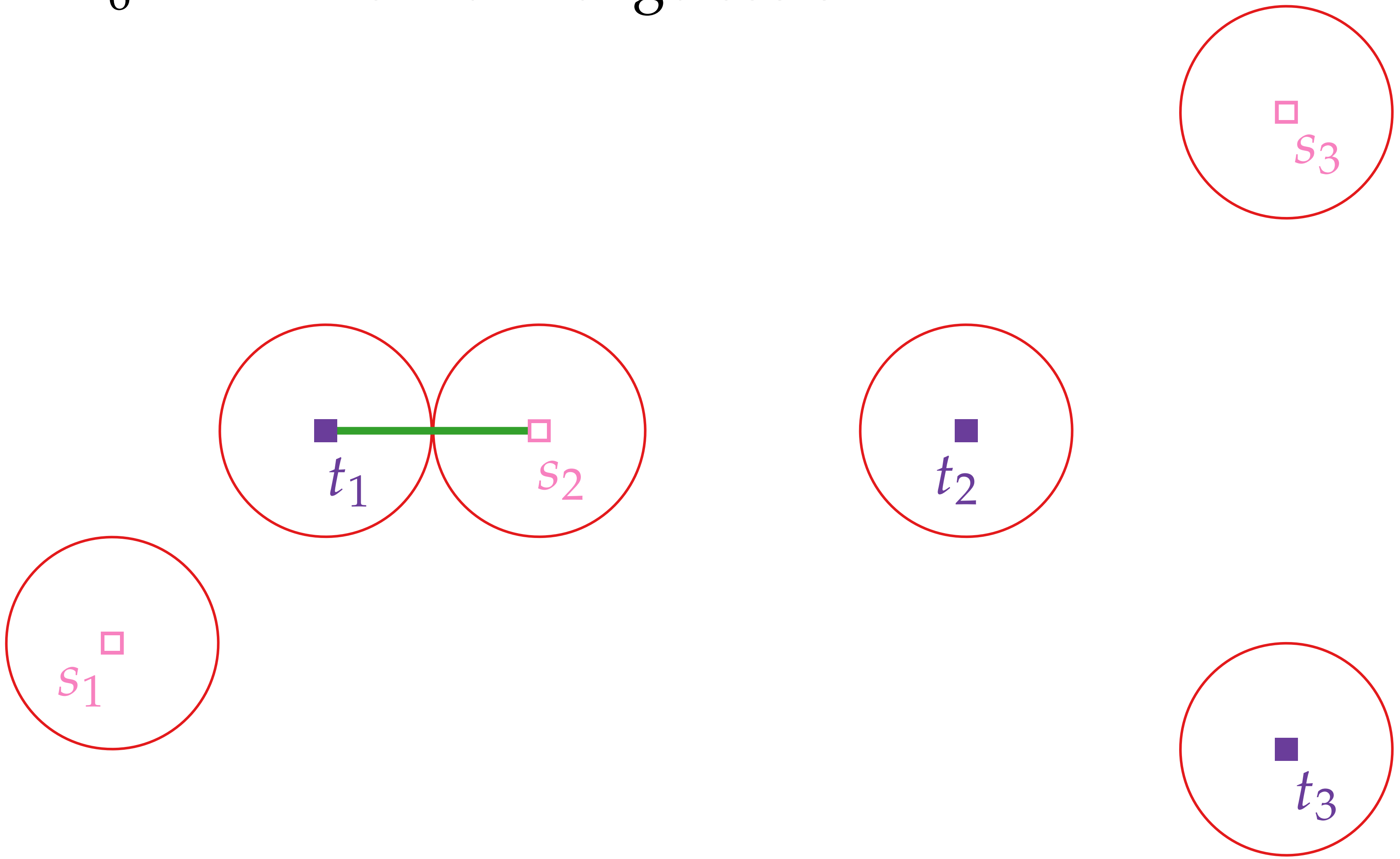

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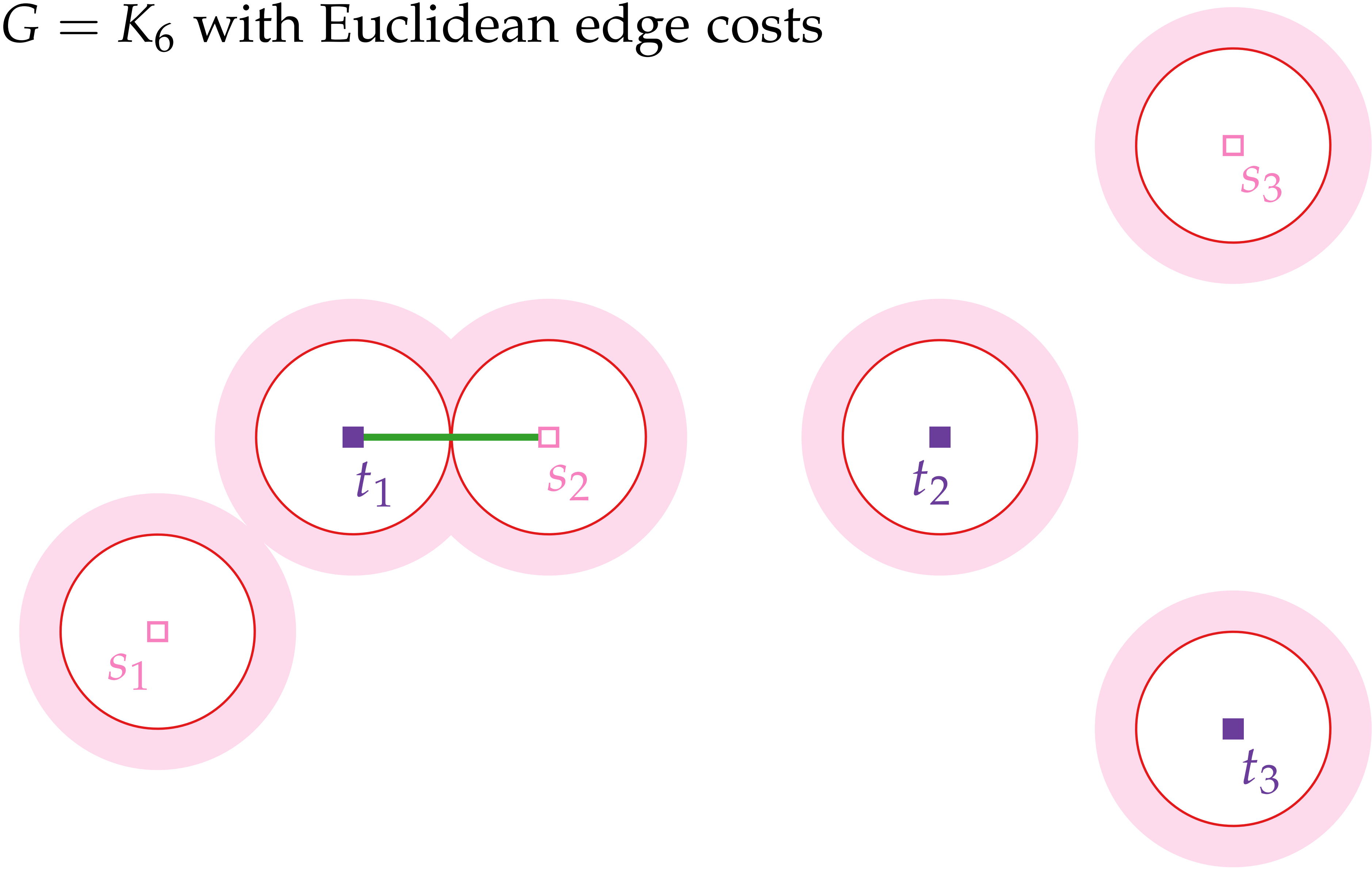

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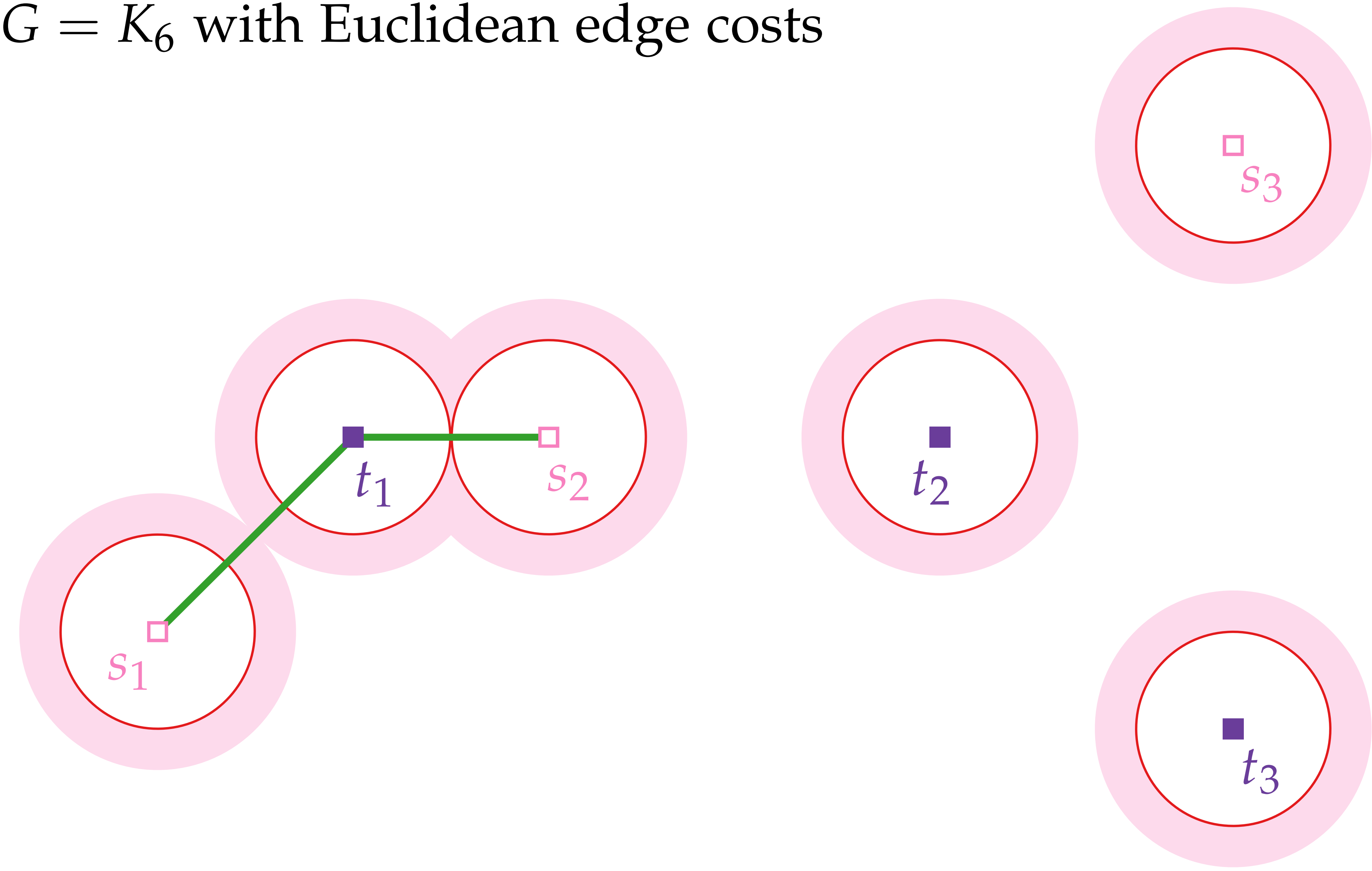

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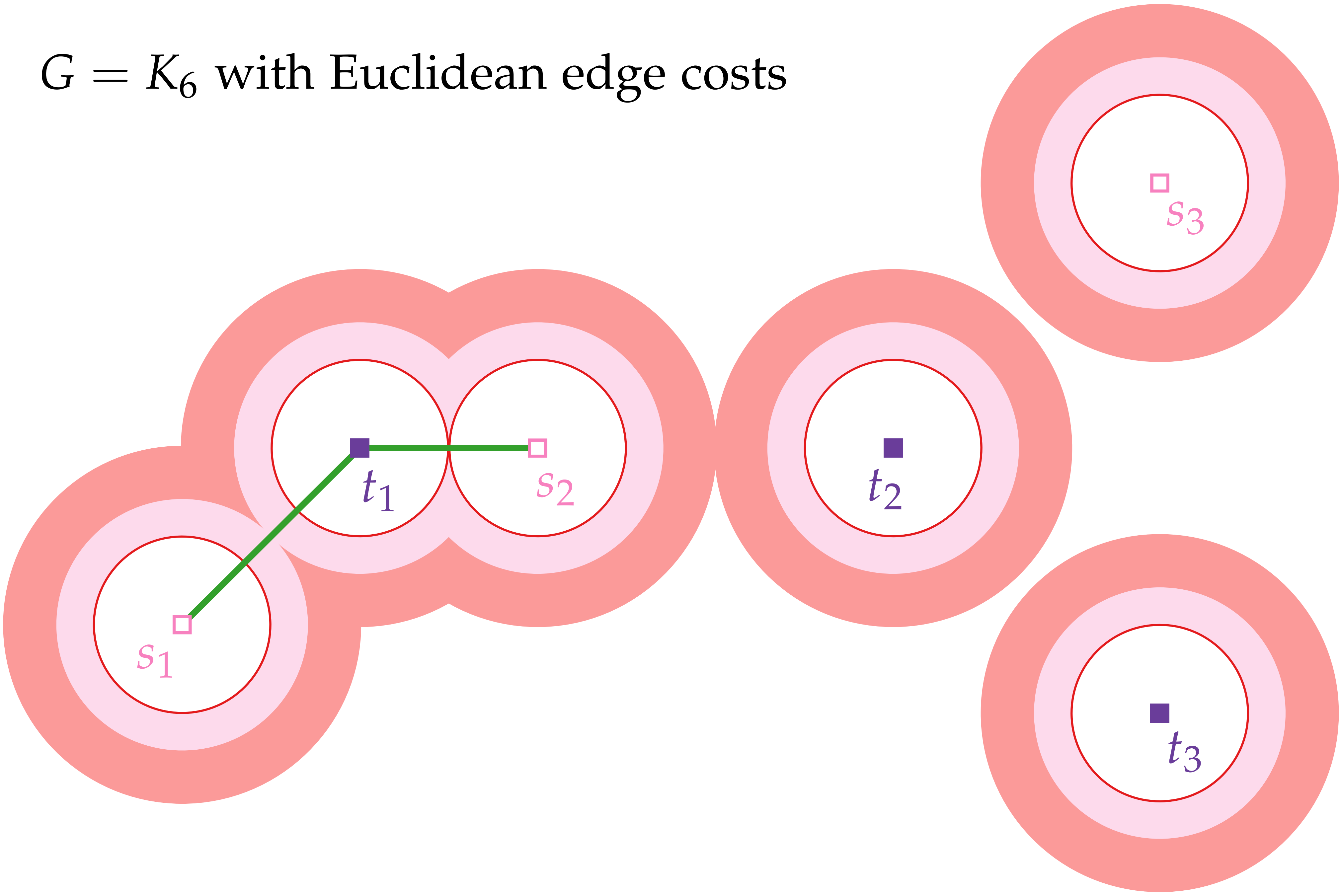

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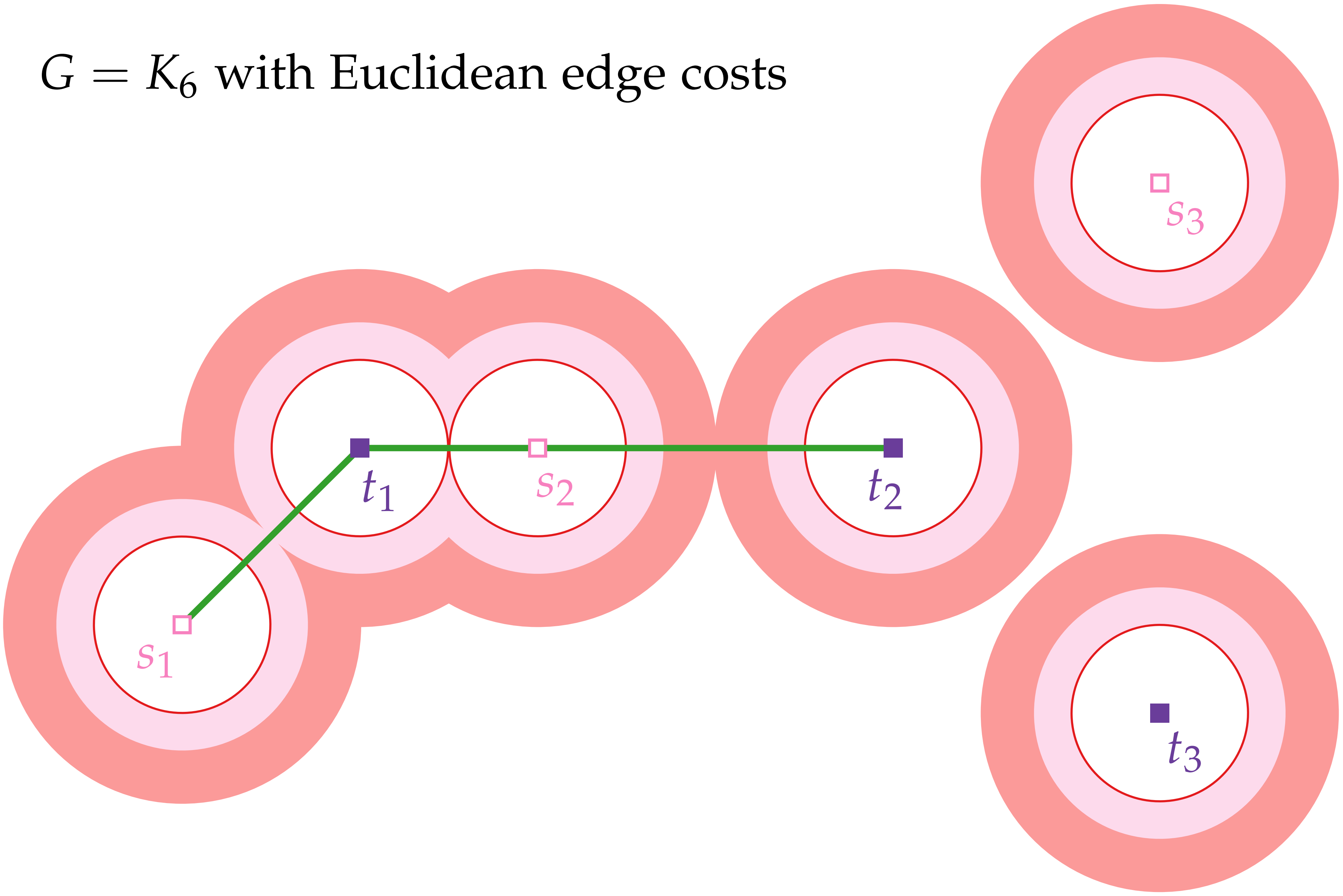

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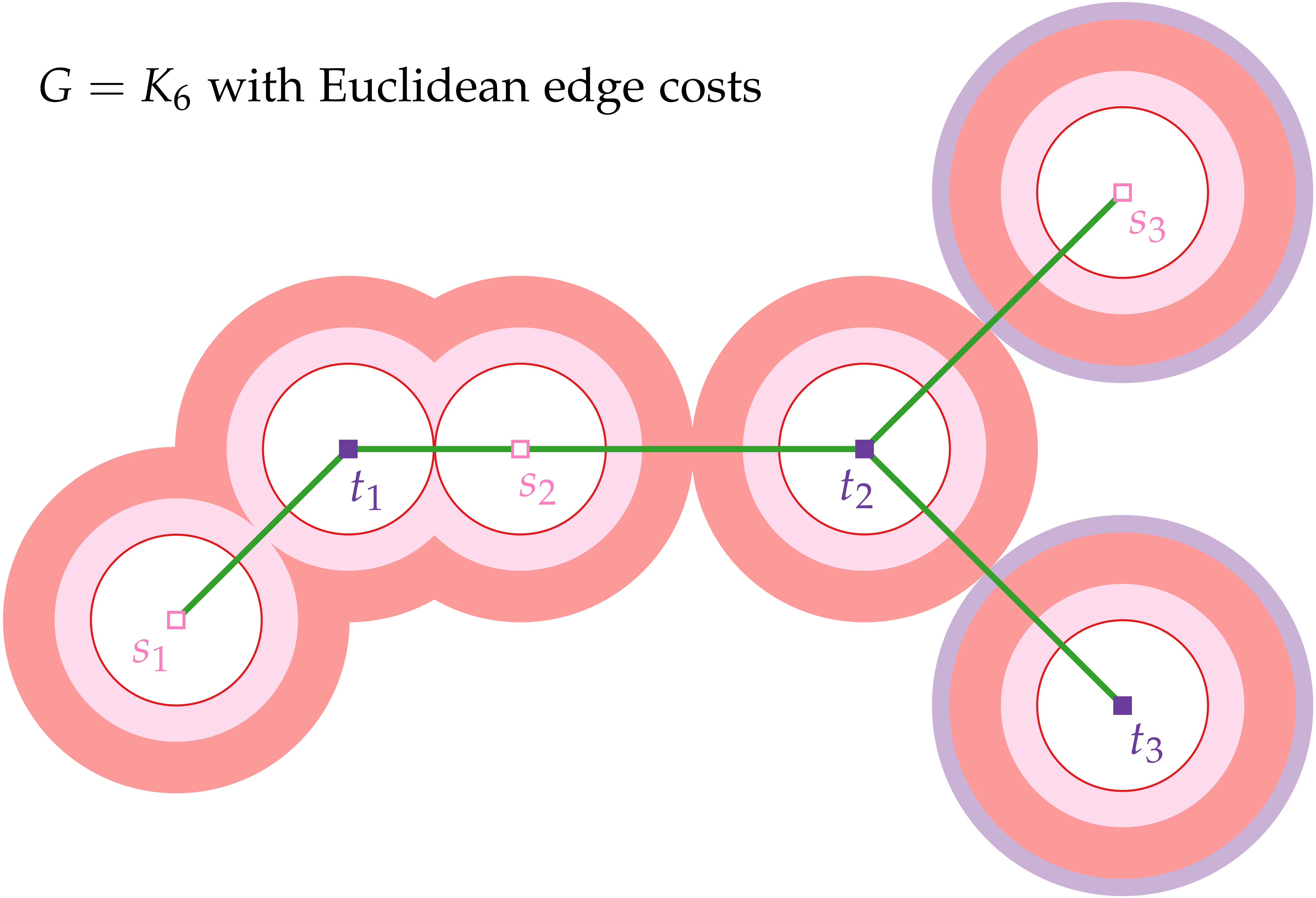

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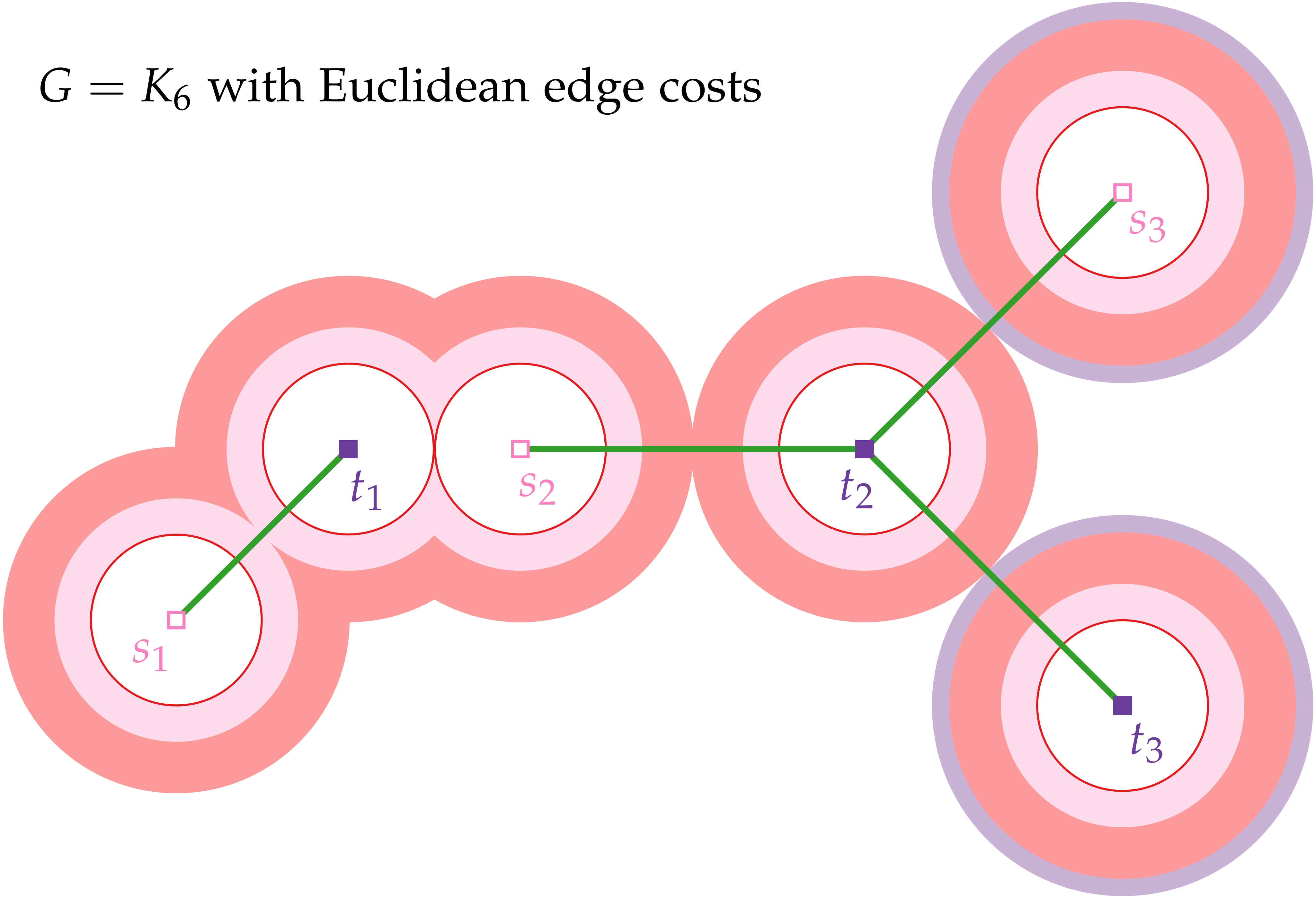

Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part V:
Structure Lemma

Joachim Spoerhase

Winter 2021/22

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Lemma. For each $\mathcal{C}$ of an iteration of the algorithm:

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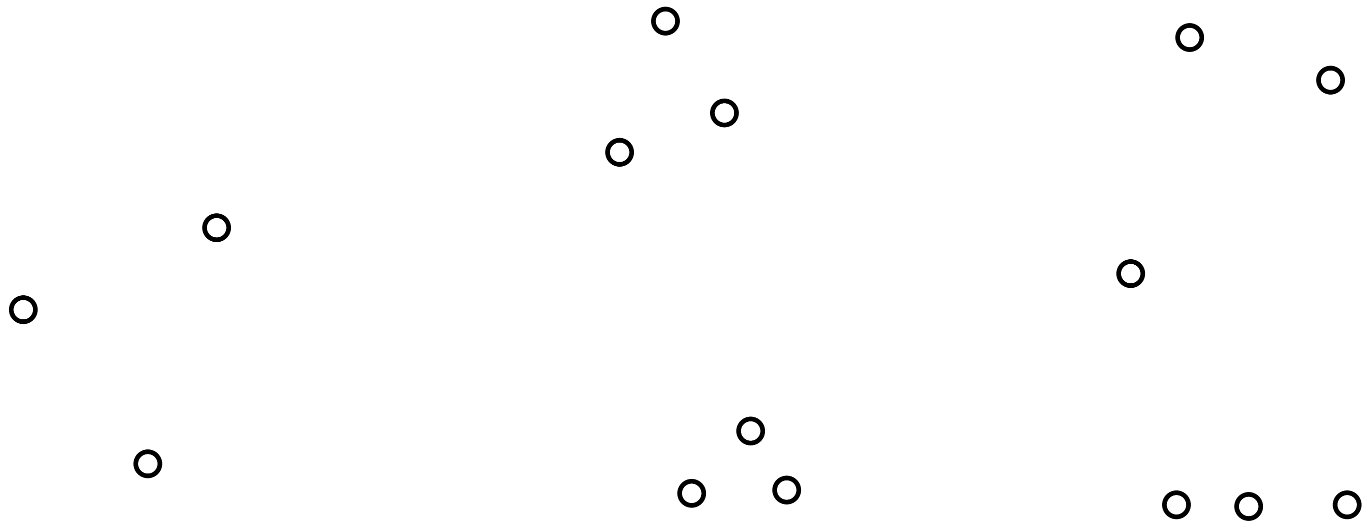

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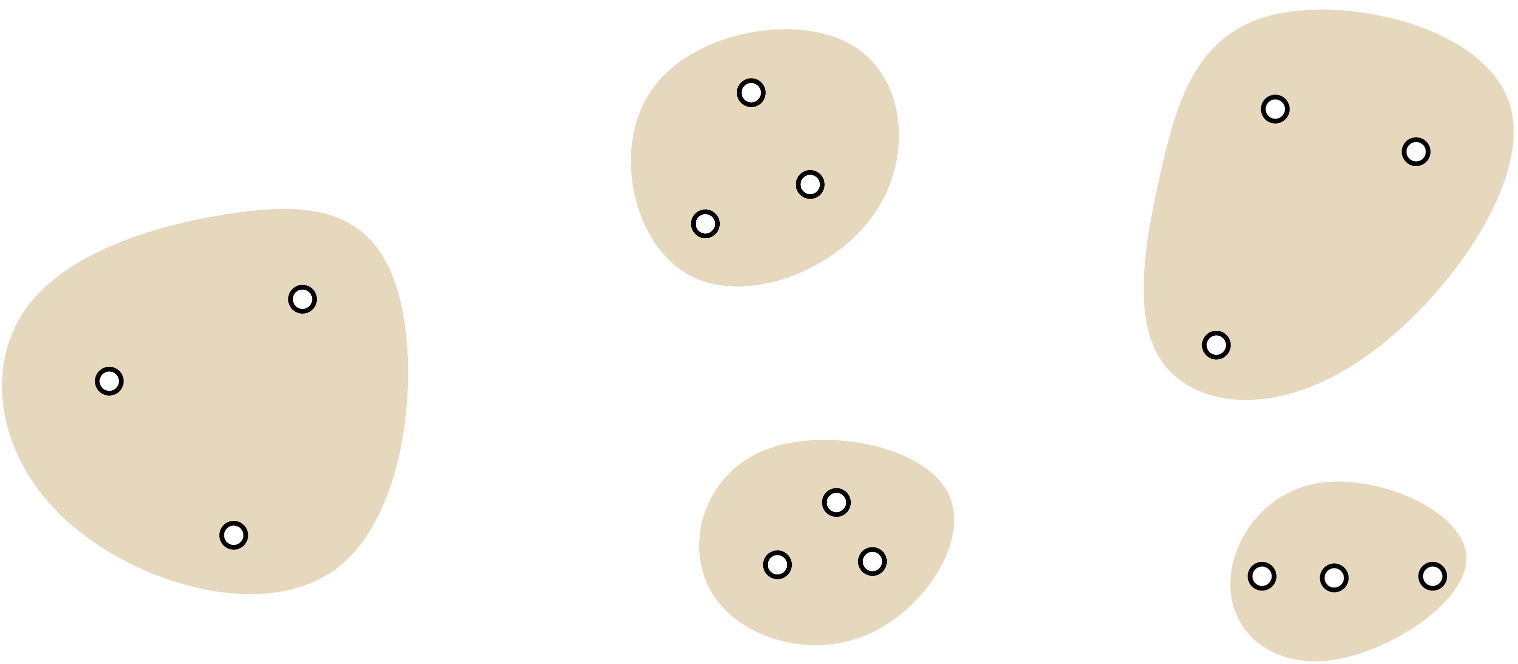

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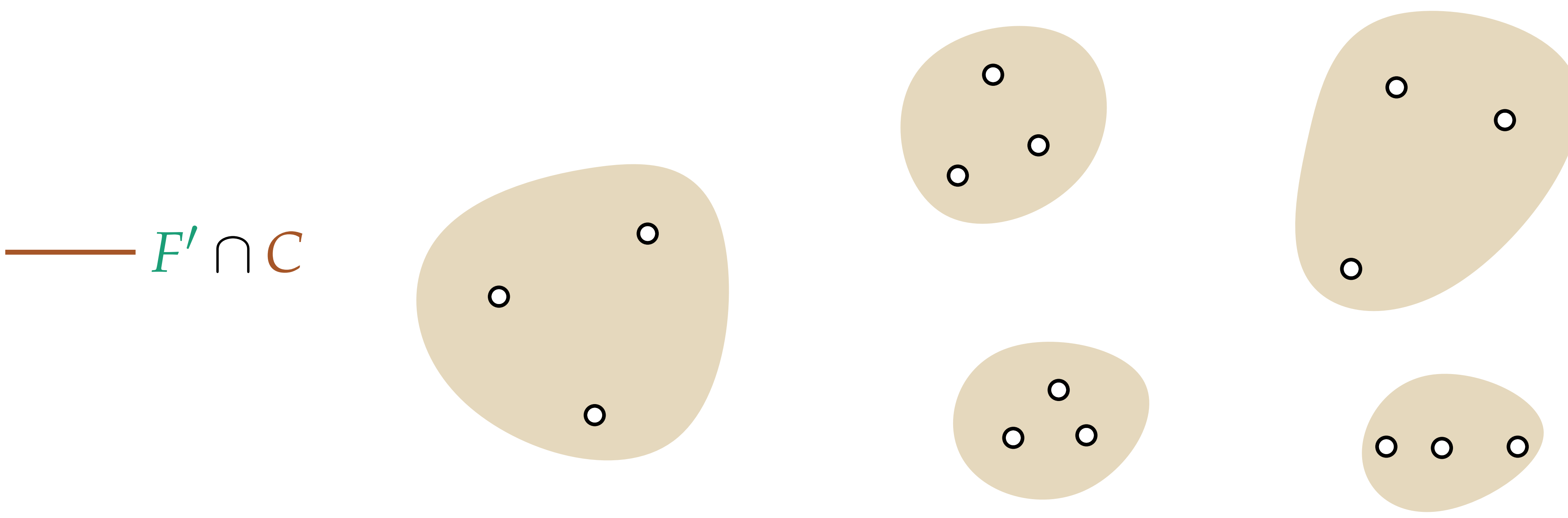

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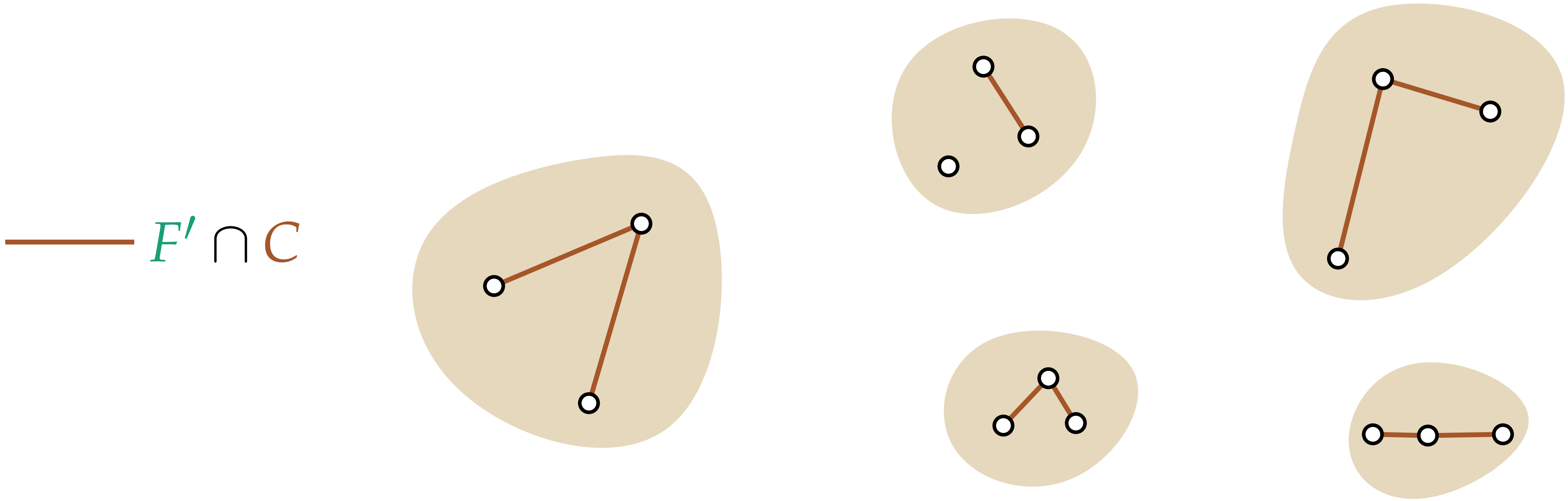

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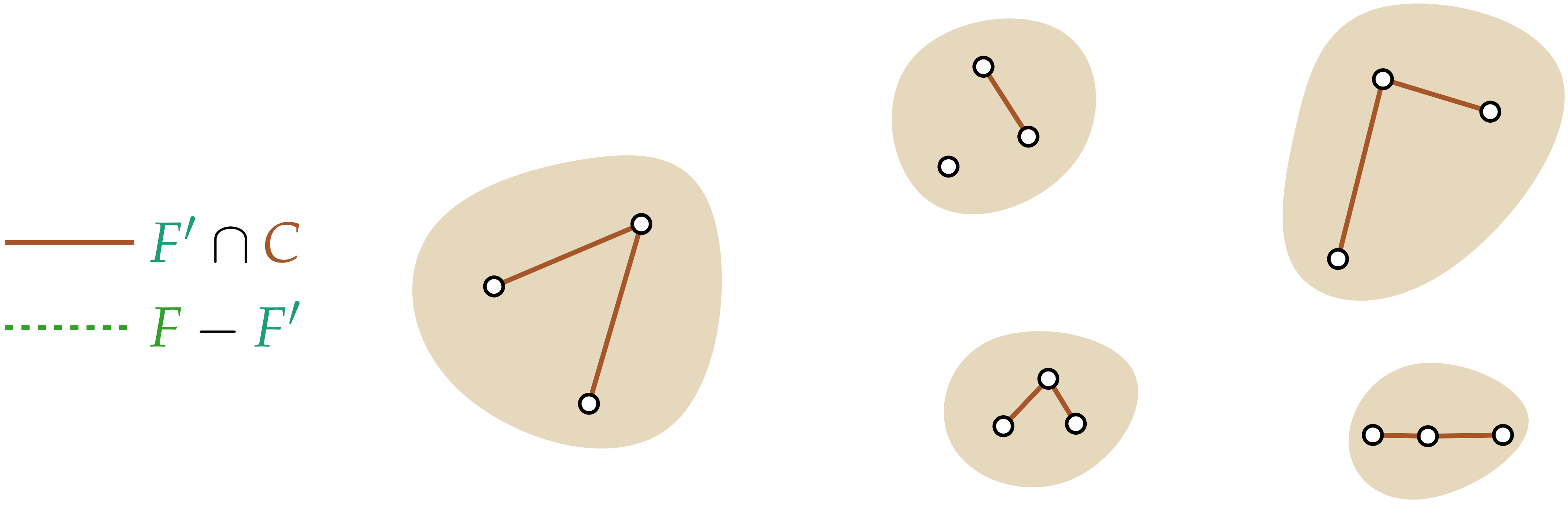

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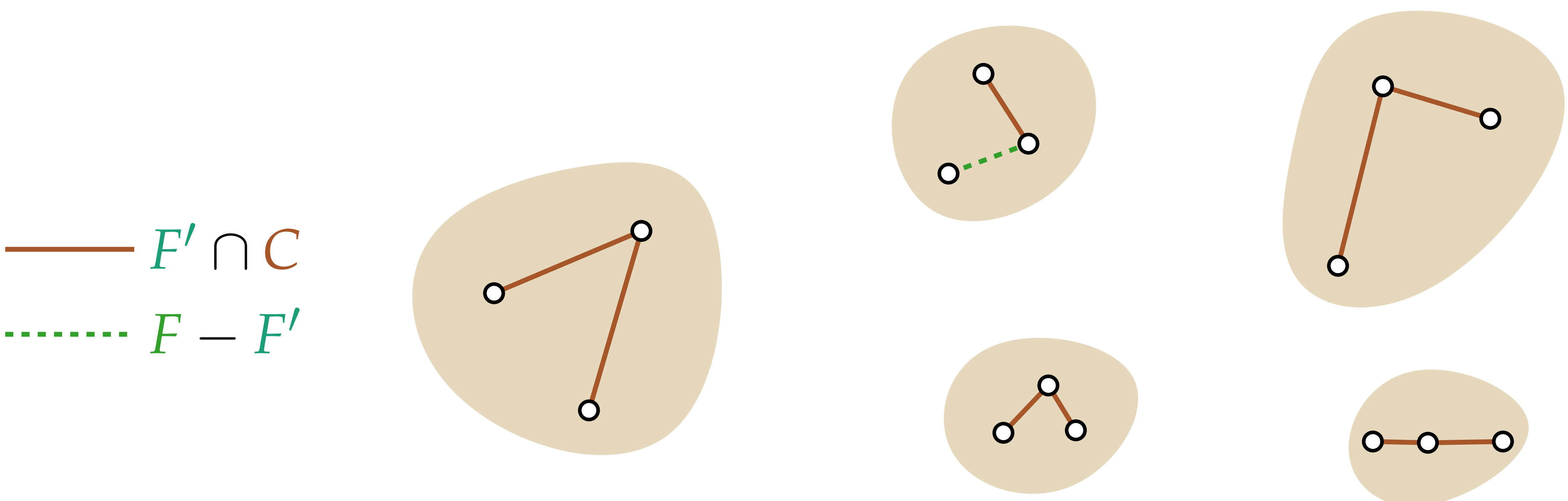

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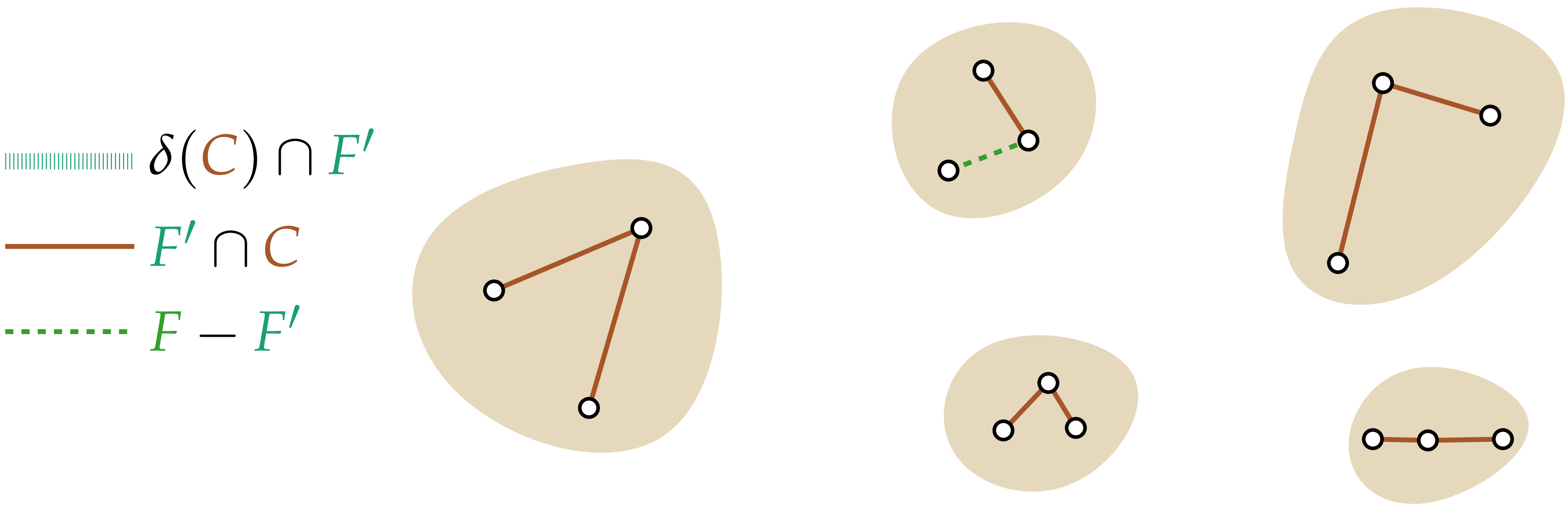

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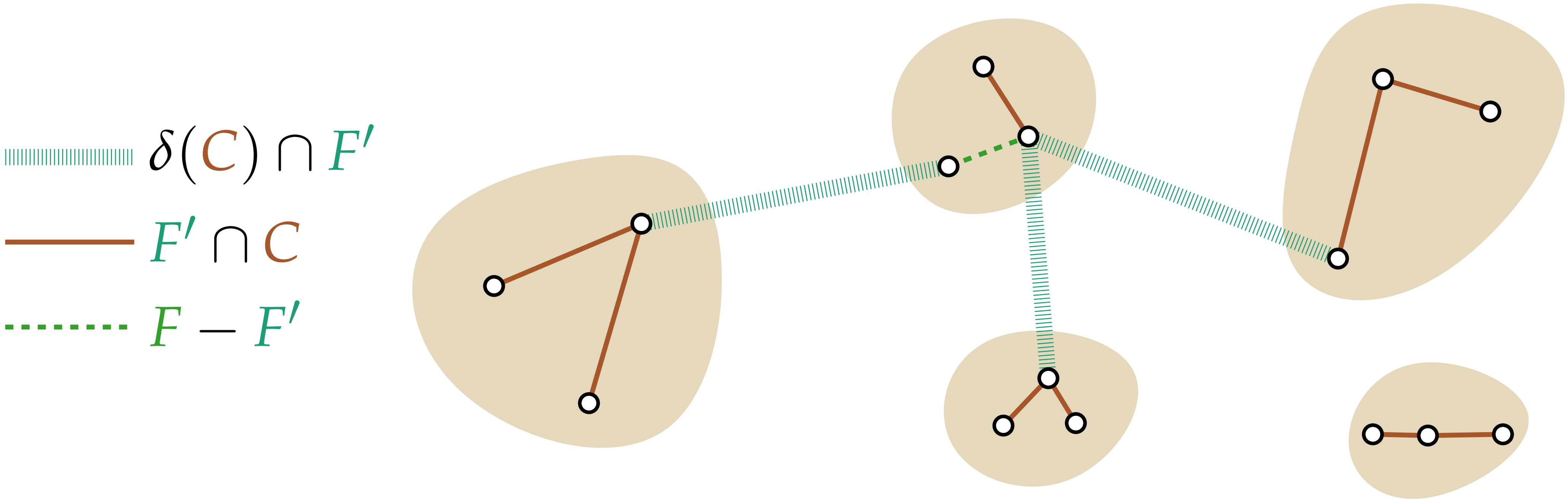

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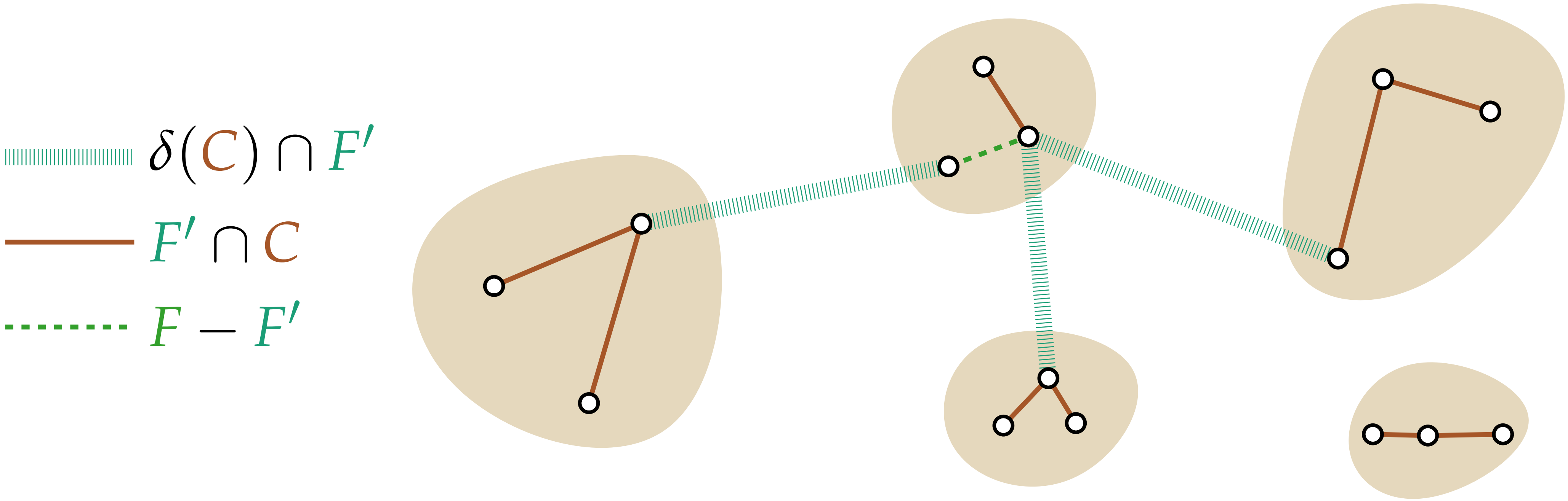

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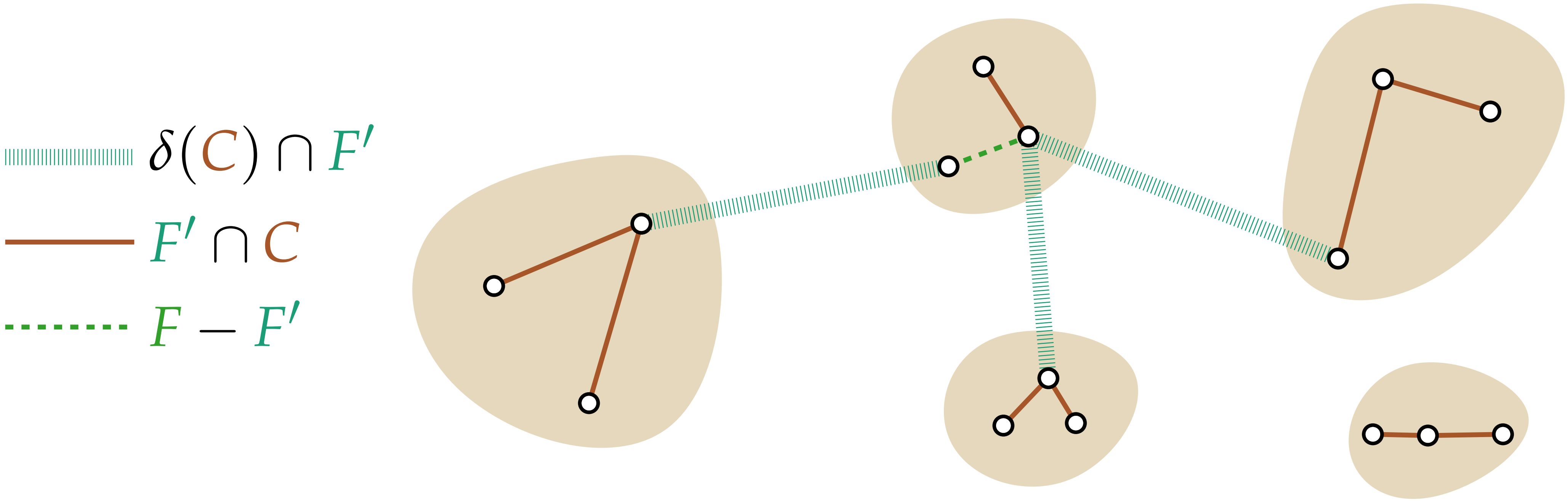

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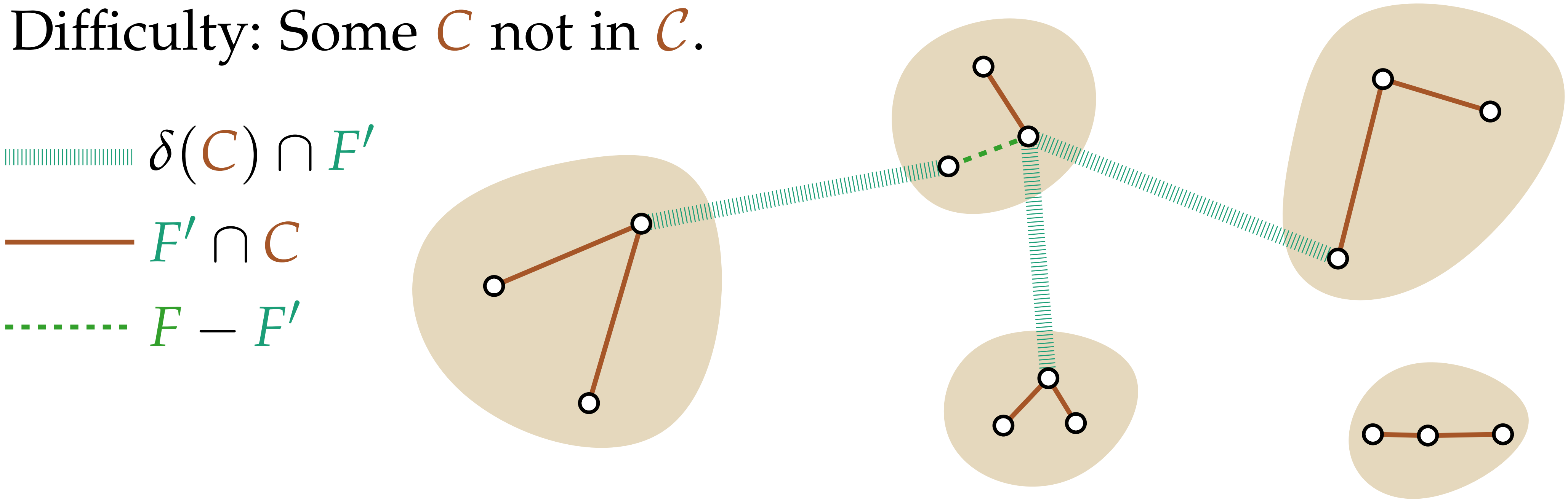
Difficulty: Some C not in $\mathcal{C}$.



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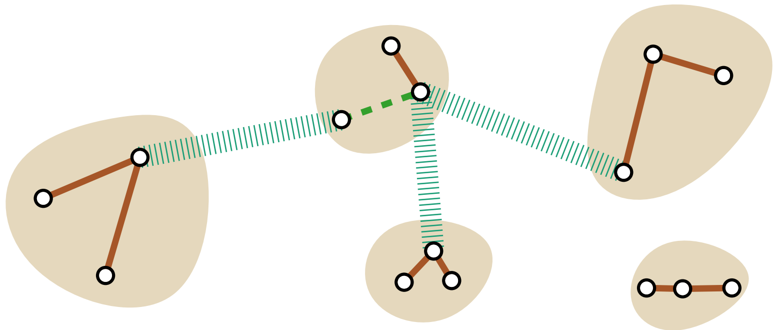

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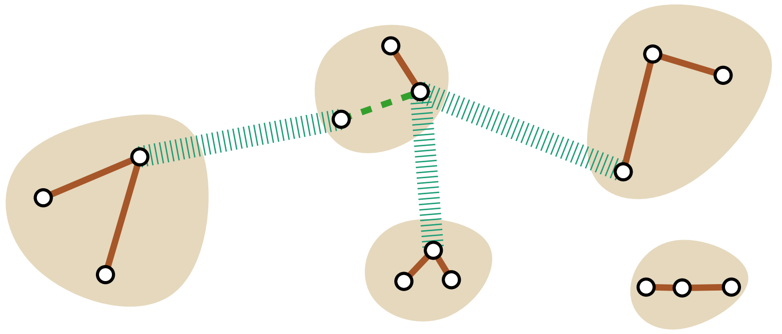

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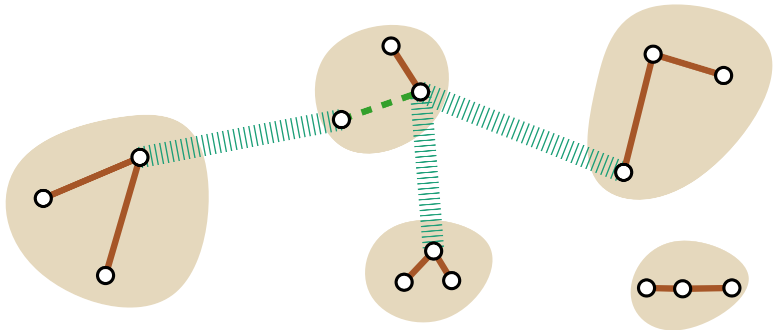

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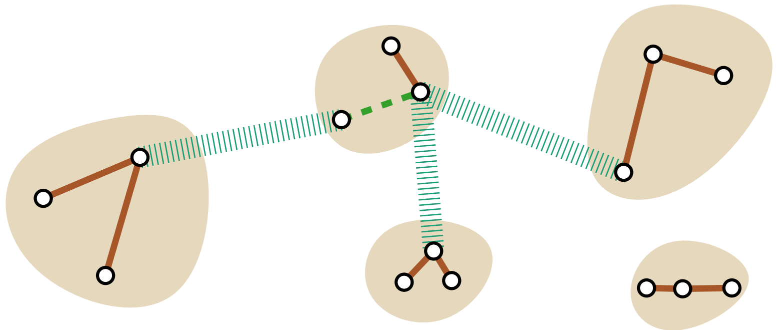

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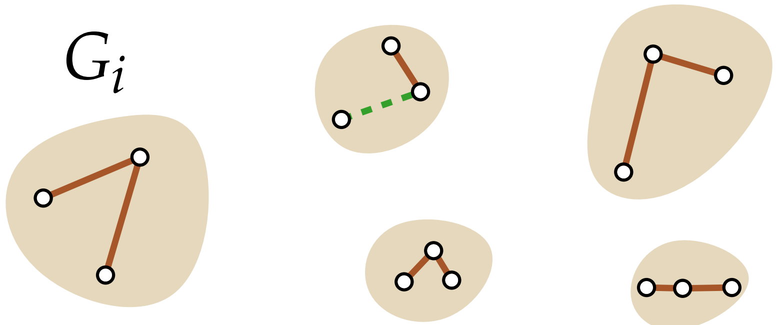

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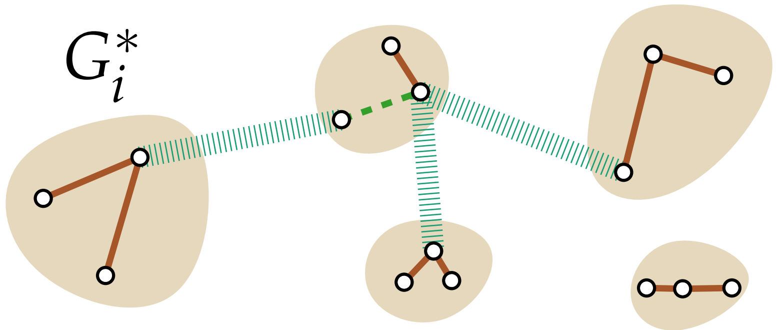

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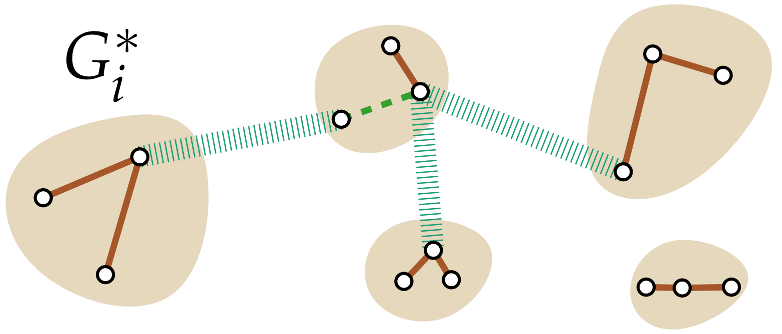

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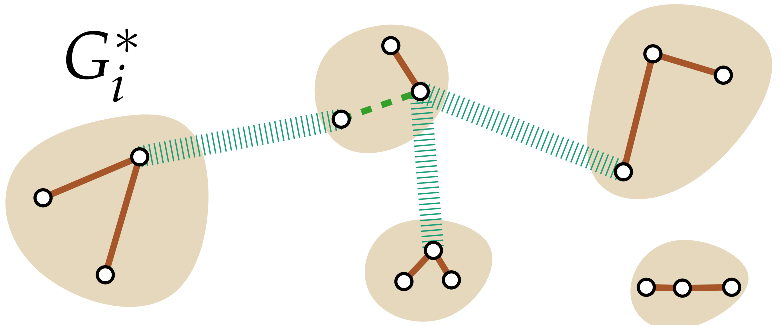

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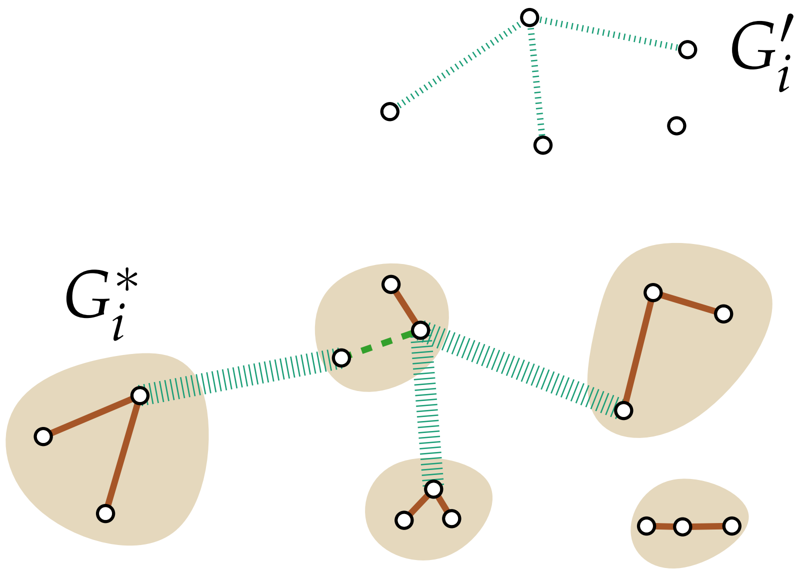

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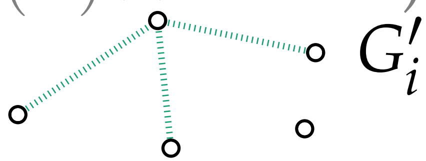

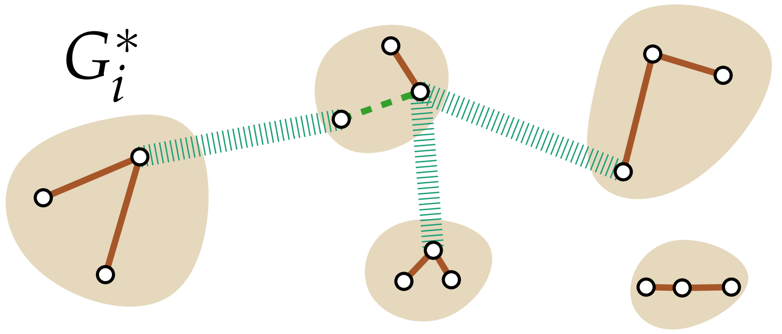

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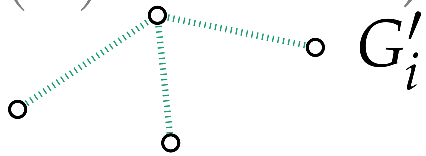

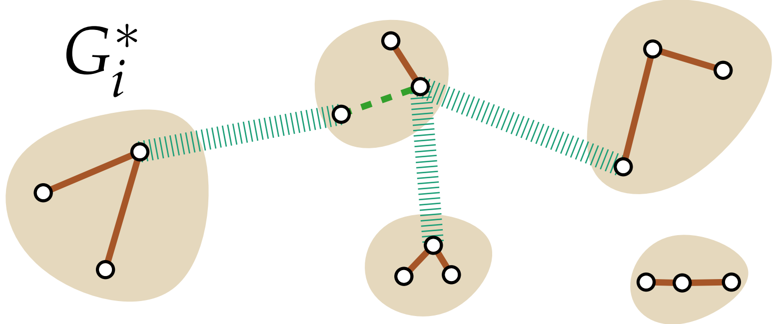

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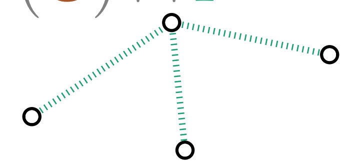

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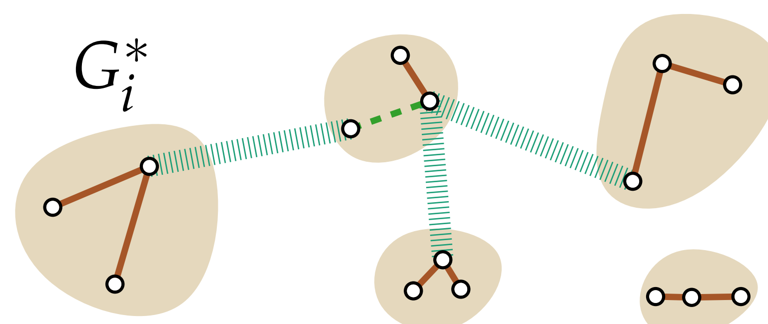

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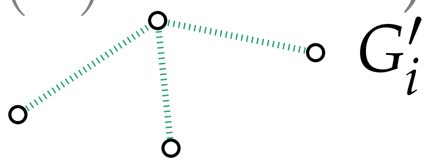

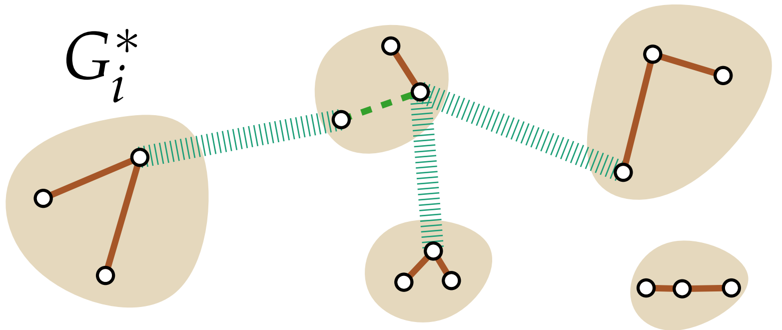

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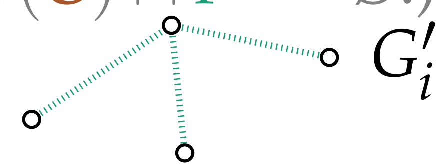

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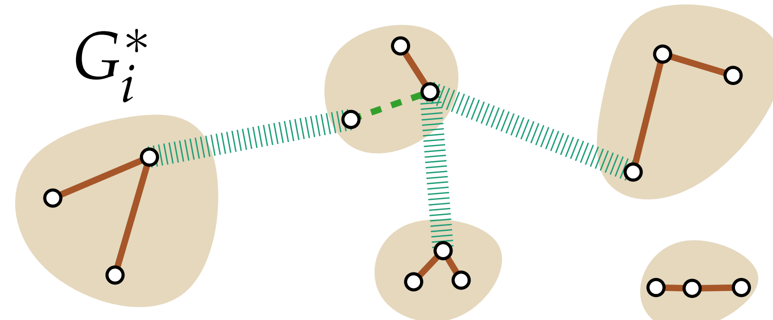

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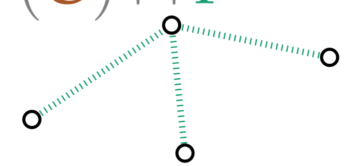

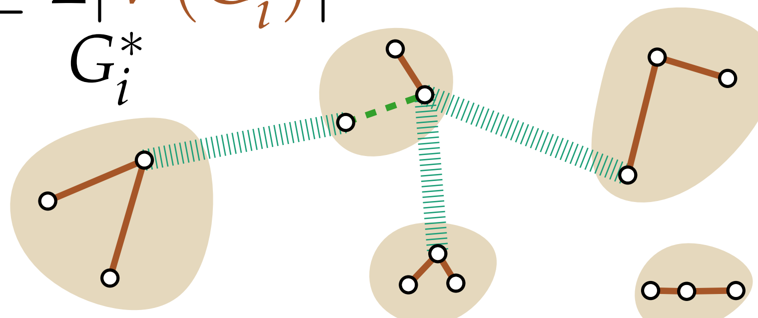

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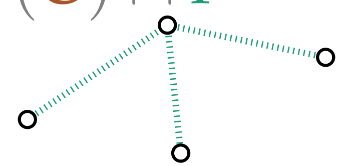

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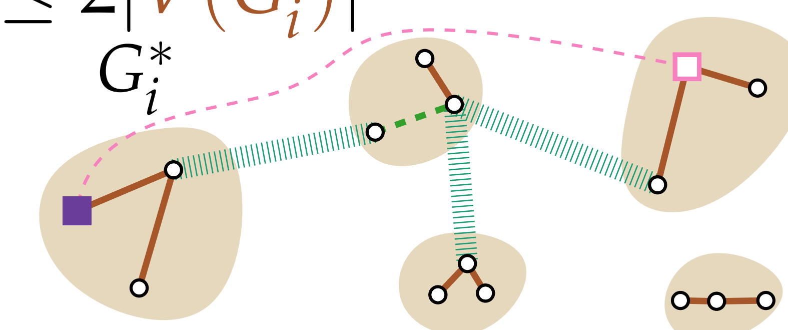

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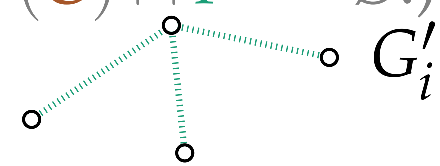

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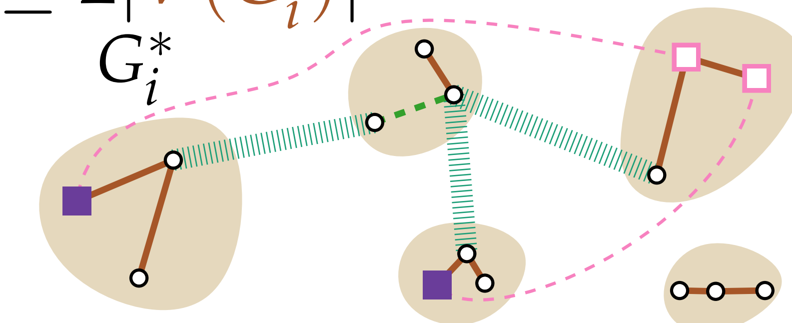

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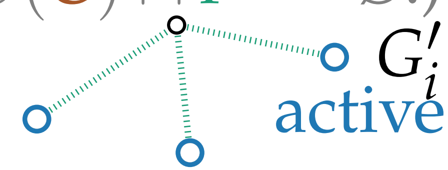

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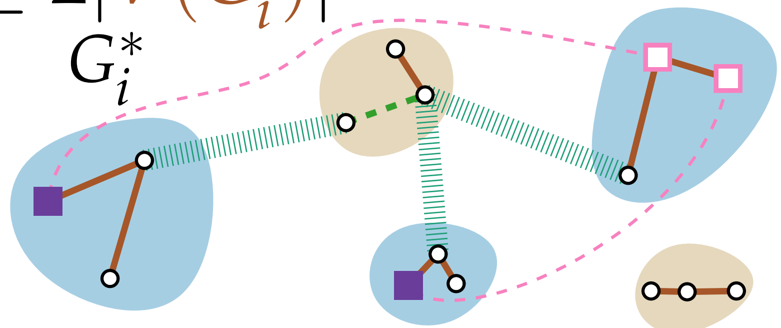

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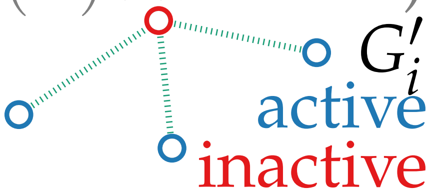

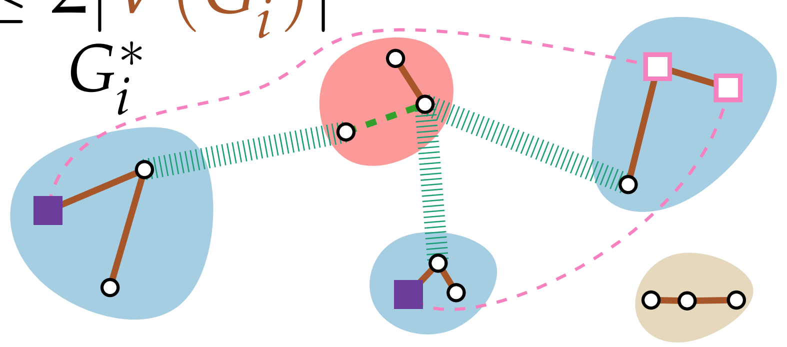

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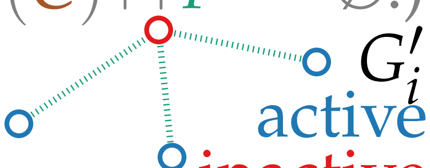

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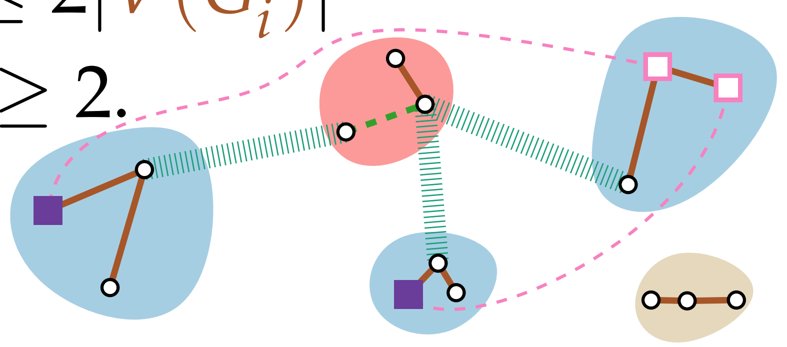

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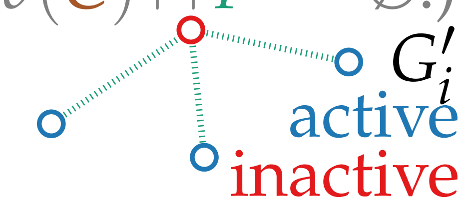

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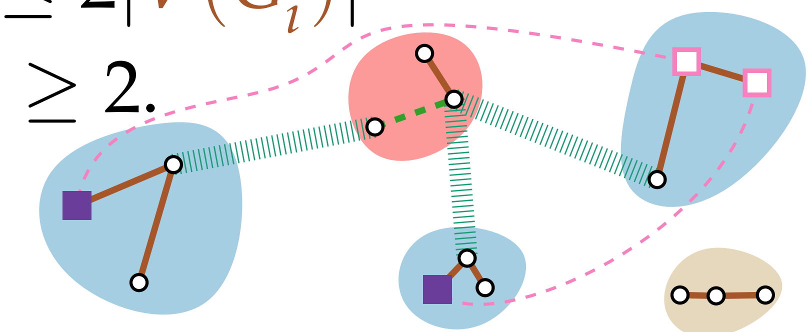

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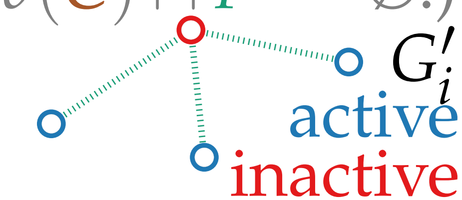

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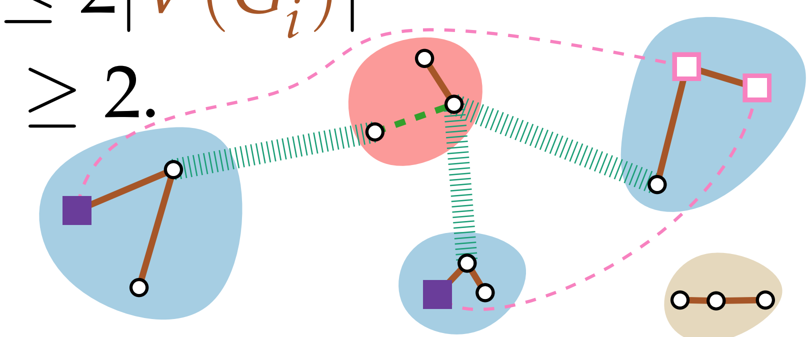

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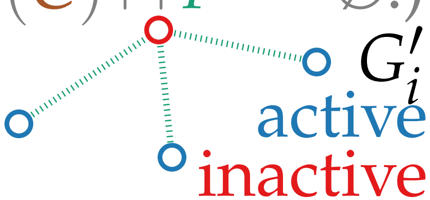

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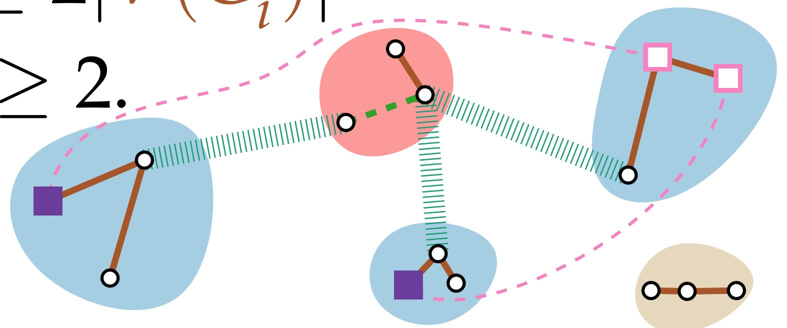

Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part VI:
Analysis

Joachim Spoerhase

Winter 2021/22

Analysis

Theorem. The Primal-Dual algorithm with synchronized increases gives a 2-approximation for SteinerForest.

Proof.

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As before

$$\sum_{e \in F'} c_e \stackrel{\text{CS}}{=} \sum_{e \in F'} \sum_{S \colon e \in \delta(S)} y_S = \sum_S |\delta(S) \cap F'| \cdot y_S.$$

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Base case trivial since we start with $y_S = 0$ for each S.

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Thus, by the Structure Lemma, $(*)$ also holds after the active iteration. □

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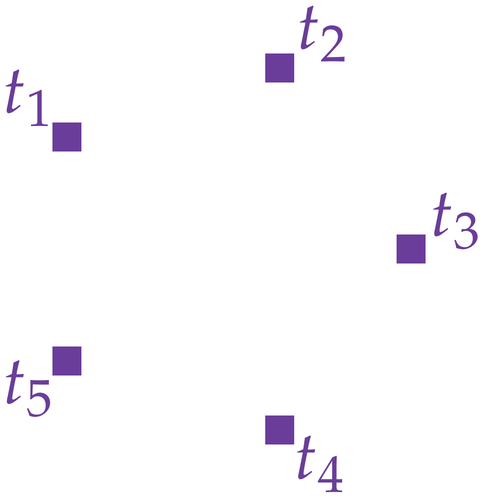

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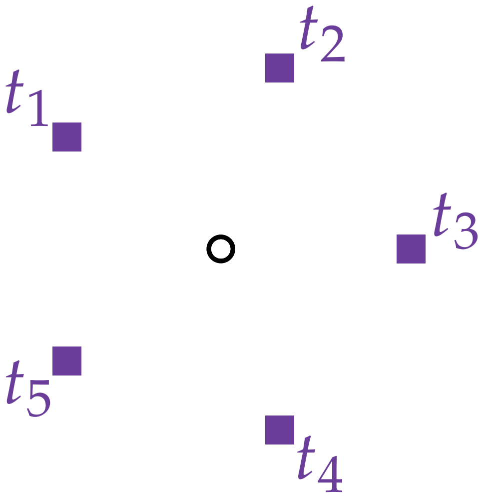

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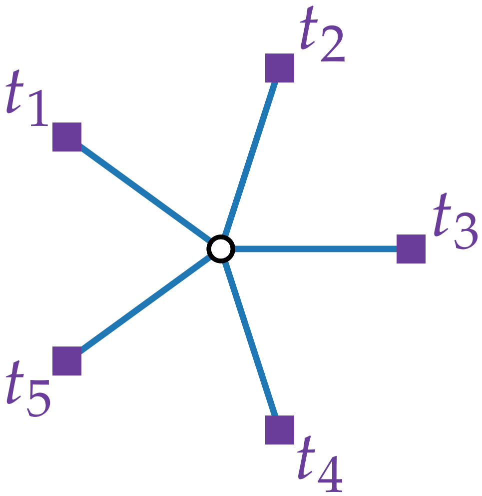

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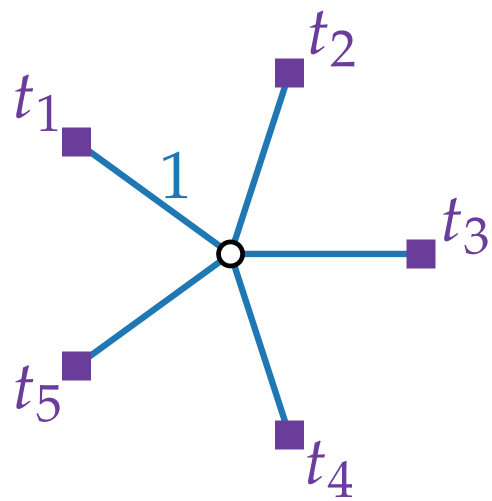

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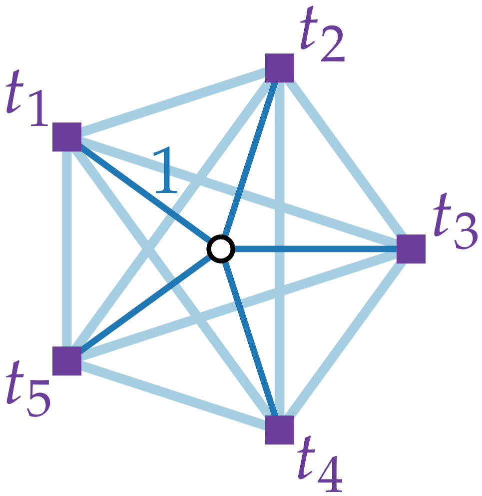

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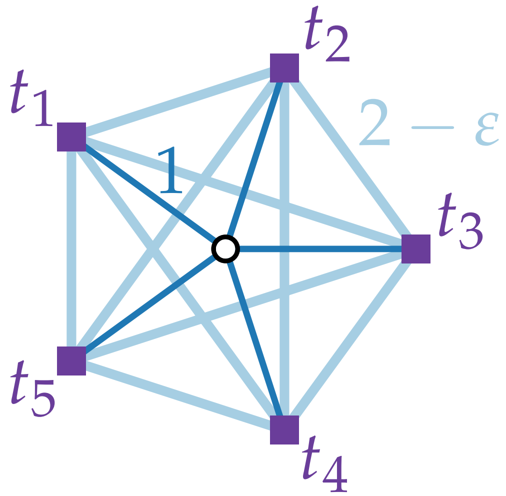

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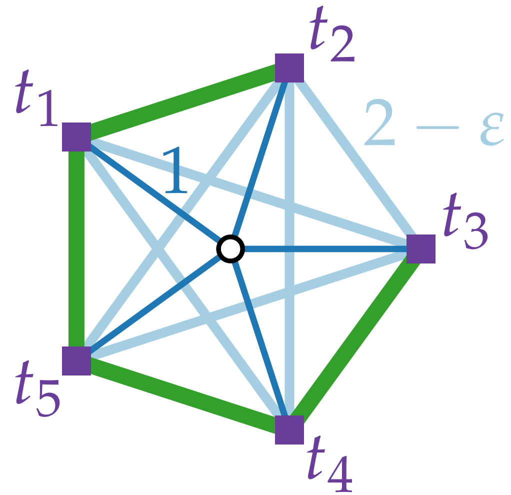

ALG $= (2 - \varepsilon)(n - 1)$

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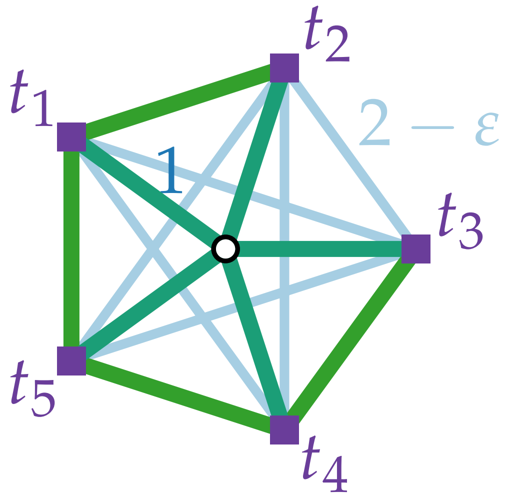

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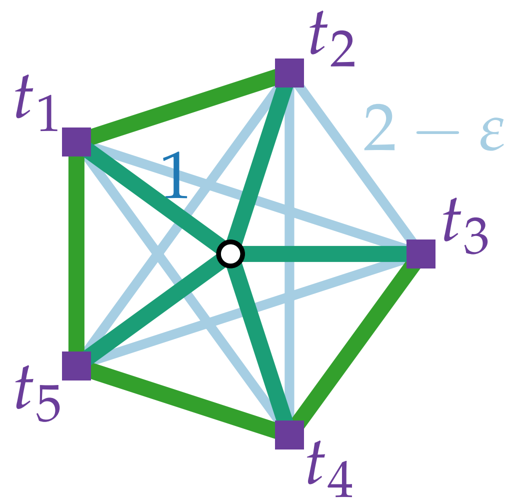

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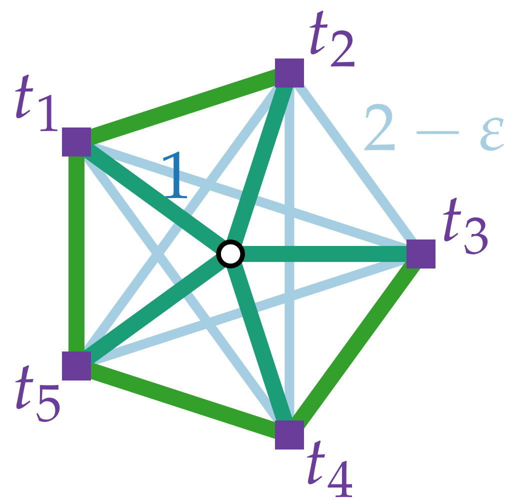

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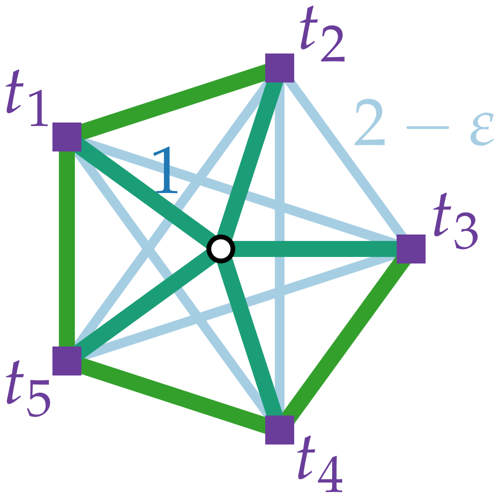

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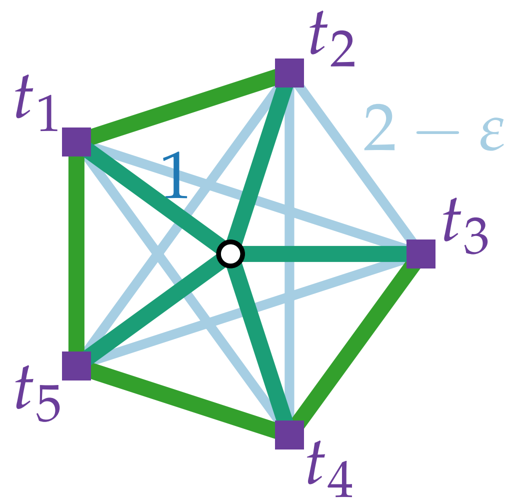

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SteinerForest (as SteinerTree) cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P=NP) [Chlebik & Chlebikova '08]