K - Teardown

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The Problem

Problem Description

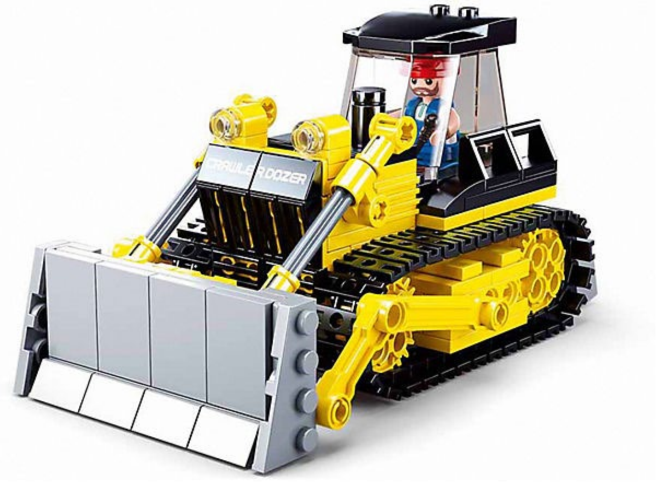
Bulldozer Time!

Given:

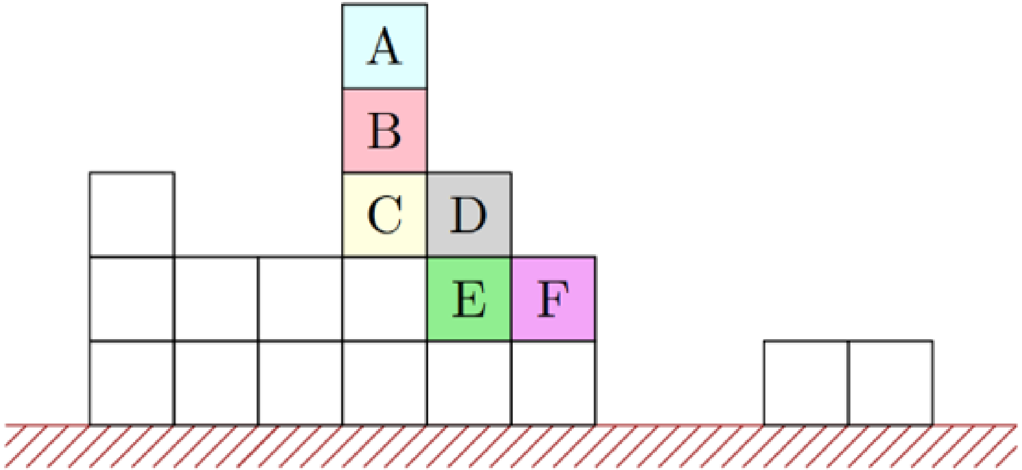
- Many buildings along a long, straight road
- modelled as individual square blocks

Objective:

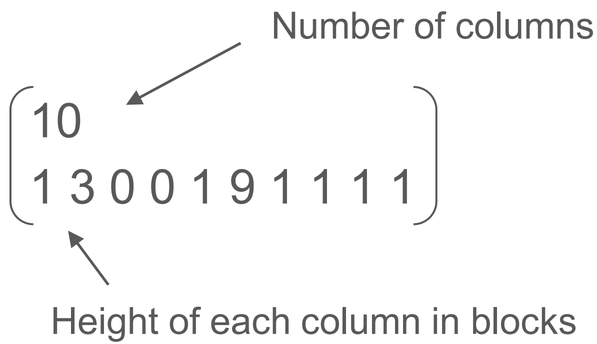
- Level all the buildings
- by getting all blocks on the ground
- by moving any block left or right
- with as few moves as possible



What is a Move?



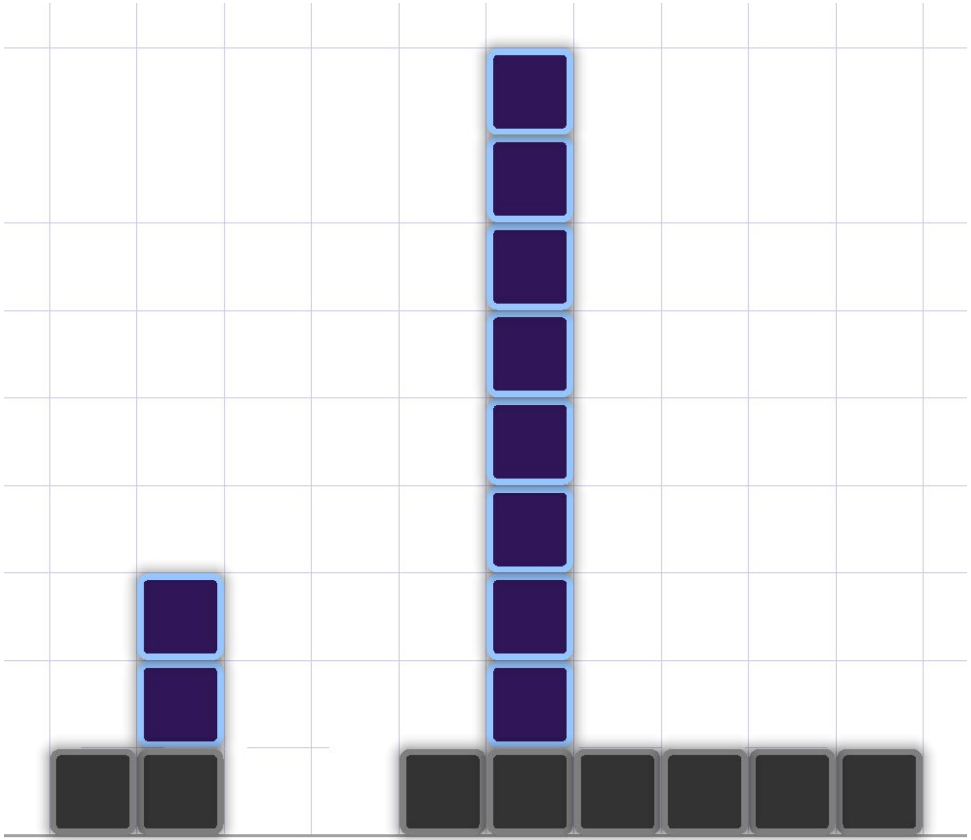
Input



Number of columns

10
1 3 0 0 1 9 1 1 1 1

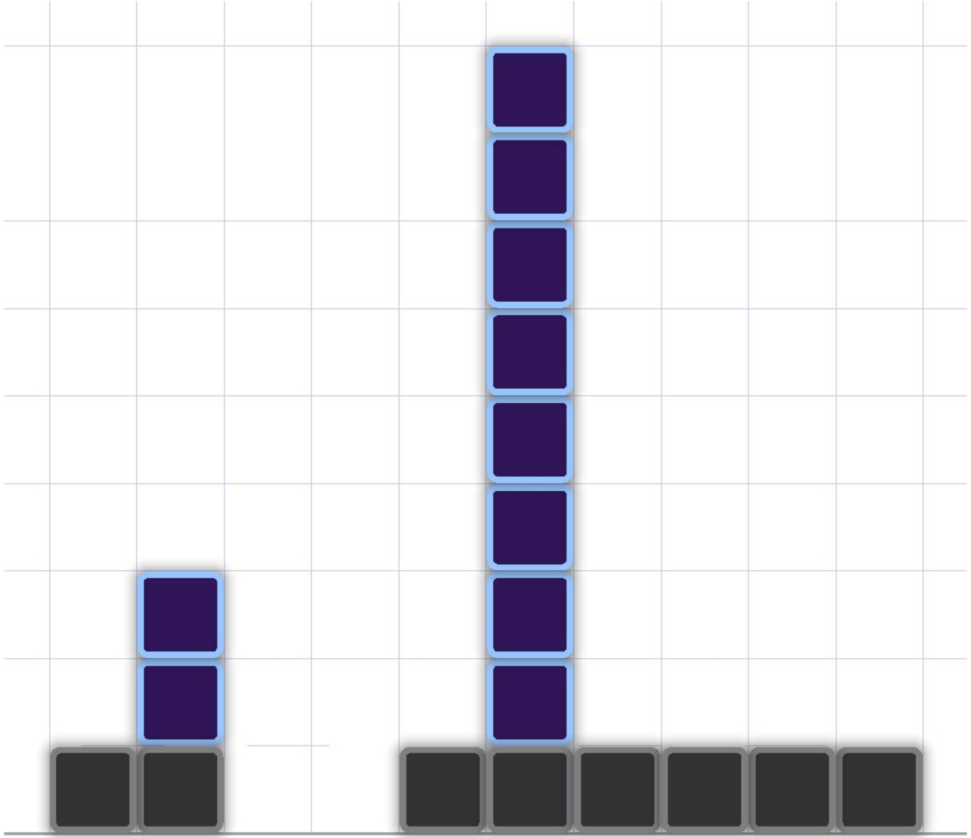
Height of each column in blocks



Output

$$\begin{bmatrix} 13 \end{bmatrix}$$

Minimum number of moves needed to get all blocks to level 0



Pause the video and play a little!

https://xdracam.itch.io/teardown

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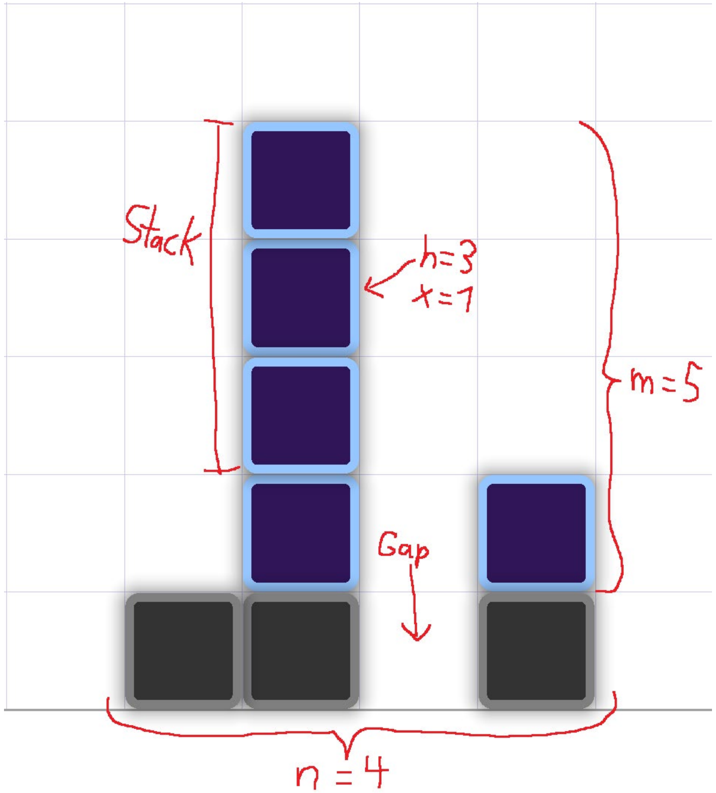
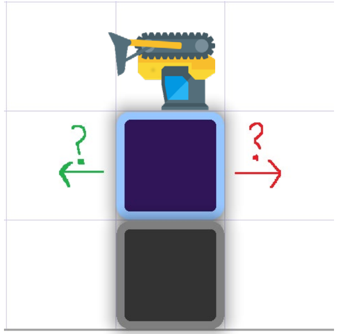
Definitions

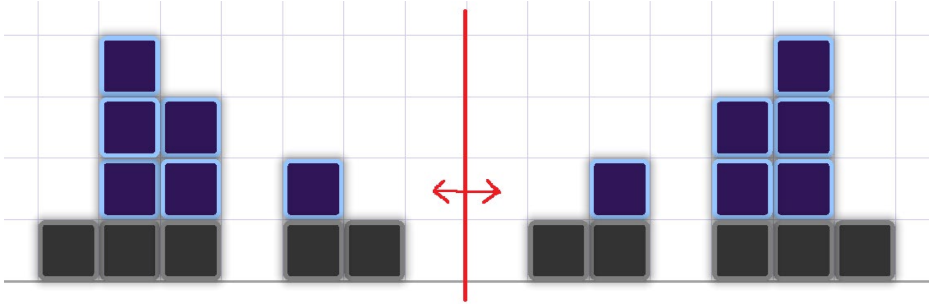
n	Number of columns
Block	Single block with clearly defined xy-coordinates
h	Height of a block = y-coordinate
m	Number of blocks with h > 0
Column	A specific x-coordinate
Gap	A column without any blocks
Stack	Multiple adjacent blocks in the same column
Split	Separation of a column into three parts that either go left, right or are leveled in place



Obvious Problem Characteristics



- always possible to find a solution
  - infinite gaps to the left and right of the instance
- solution is not unique
  - many different moves and orders can lead to the same or equivalent outcomes
- each problem instance has mirror version with left/right swapped
  - so the order with which we iterate the instance does not matter

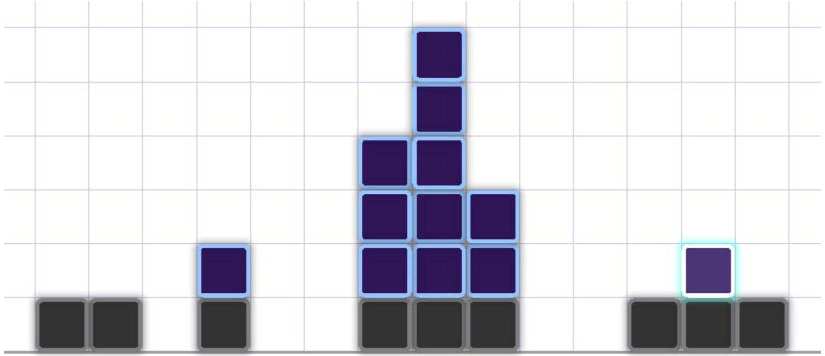


Intuitive Heuristics

- after pushing blocks into a direction, it makes no sense to push them back
- good idea to move many blocks at once
- moving towards closer/enough gaps is better

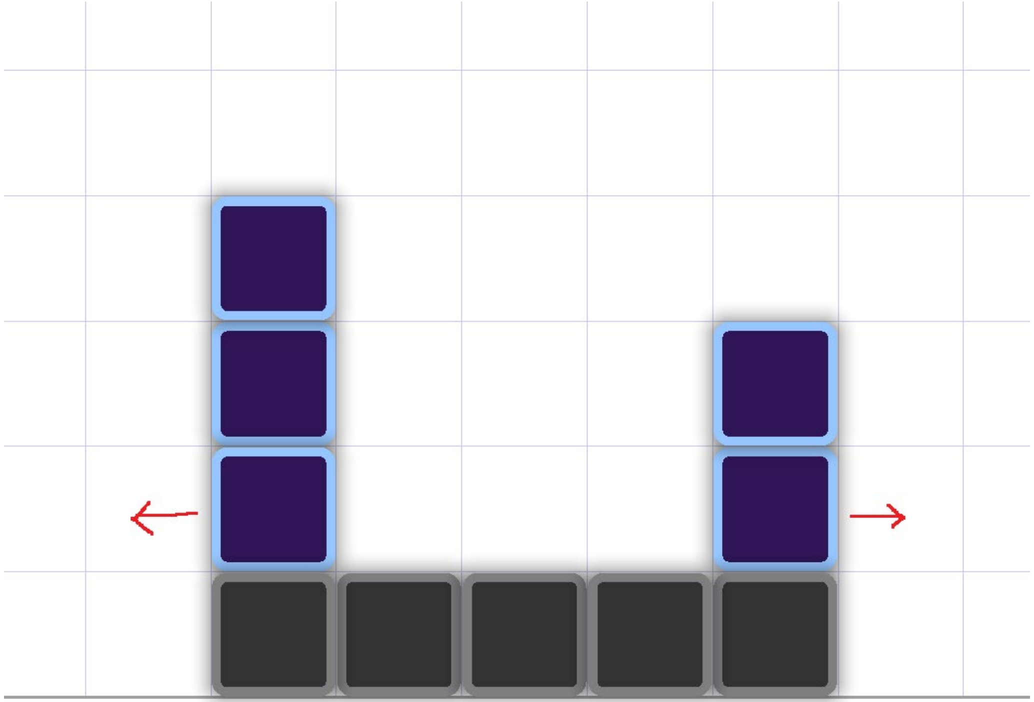
- moving blocks at h=0 is useless

Or is it?



Debunking Approaches

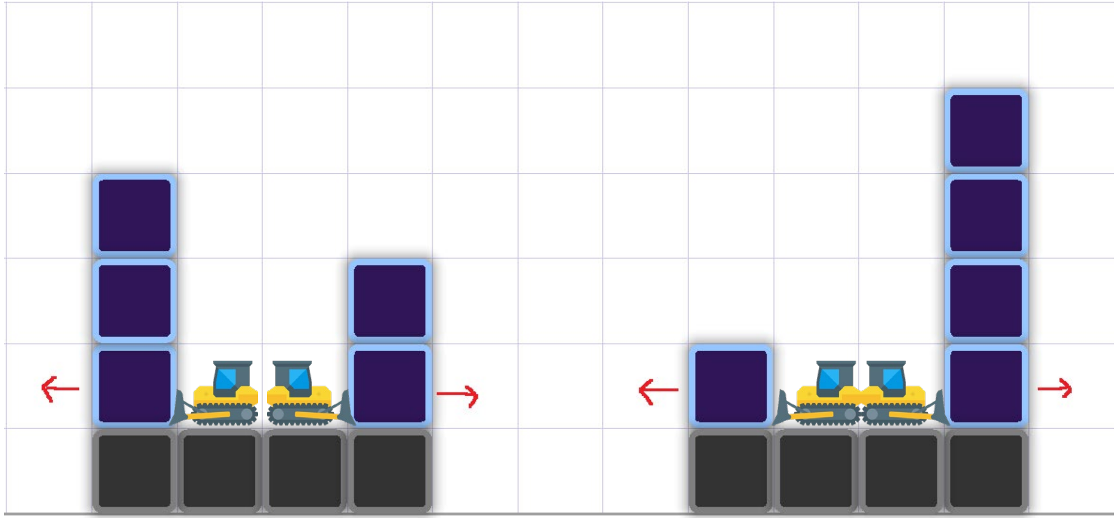
Move all blocks into same direction?



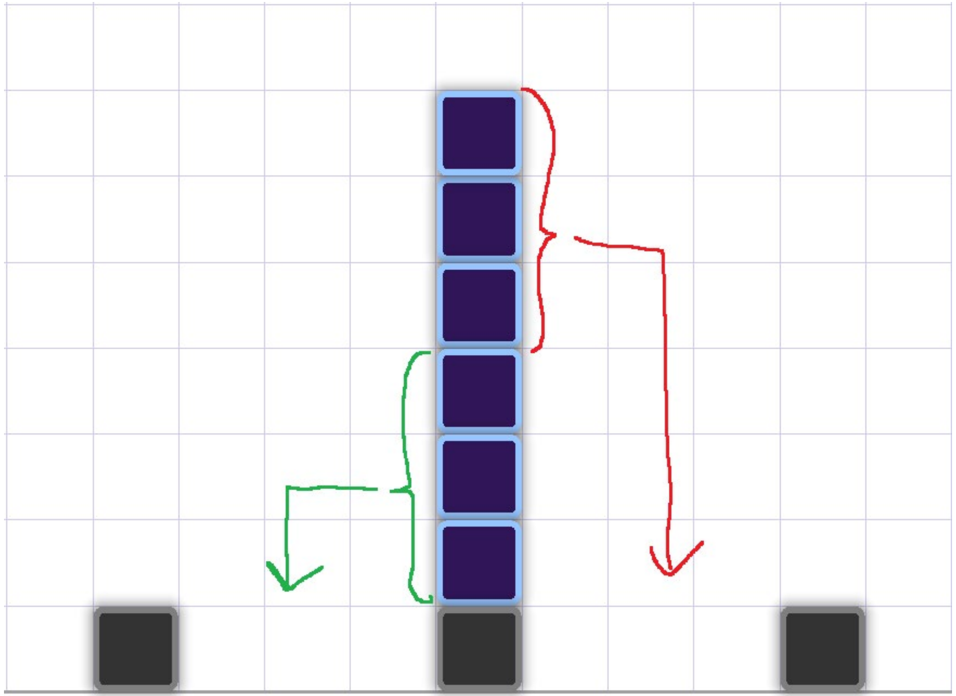
Find a column c.

Move all blocks with x ≤ c to the left

& all blocks with x > c to the right



For each column, move left or right individually?



Problem Structure

Problem Complexity

- depends on number of blocks above ground level = ***m***
- hard to solve in linear time
    - we need to split a stack in the middle sometimes
    - we can't know where to split in advance
    - so we need to consider all splits

- up to 10^9 columns with 10^5 blocks each ($m < 10^{14}$)

10^{14} bytes = 100 terabytes

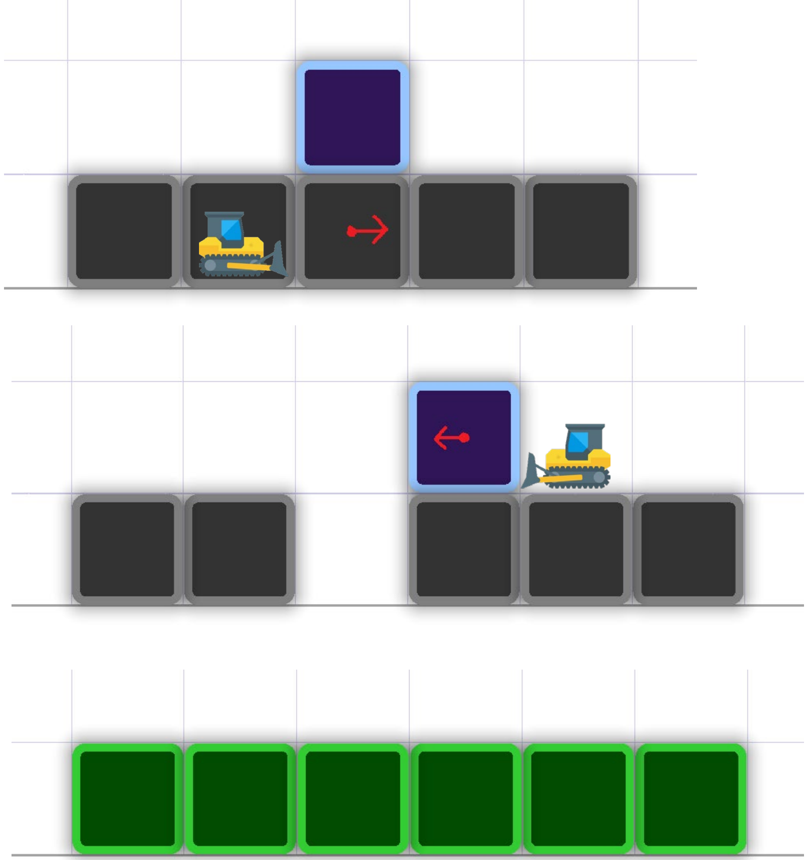
As much data as the LHC generates in one second!

⇒ we cannot possibly use O(m) memory

Upper and Lower Bounds

- m = number of blocks above ground level
- need a minimum of m moves
  - every move can only level at most one block
  - blocks on floor are already leveled
- need a maximum of 2m moves

⇒ **2-approximation strategy**



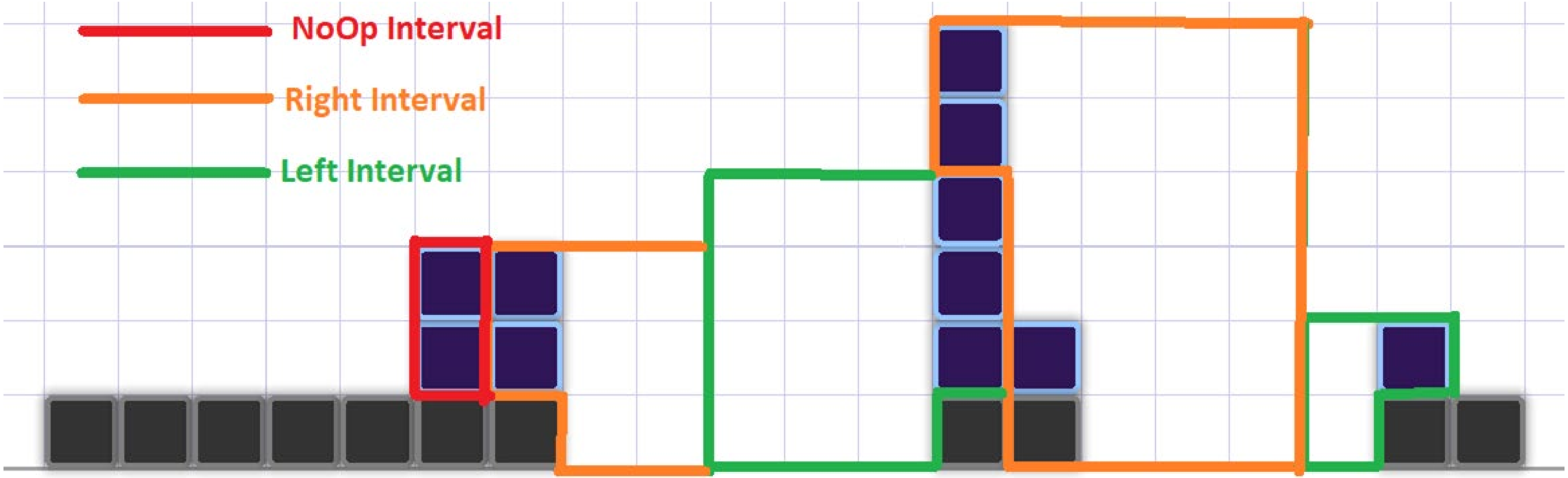
Basic Solution Idea

Partition all blocks with h > 0 into *non-overlapping intervals*

- Every block in an interval is leveled with the same strategy
- In *Left/Right intervals*, all blocks are moved in the same direction until leveled
- In a *NoOp interval*, all blocks are leveled with the 2-approximation strategy
    - every block in a NoOp interval requires exactly 2 moves to be leveled

⇒ Partitioning of **all** blocks with h > 0 into **non-overlapping** intervals so that the *sum of required moves* is minimal

Visualization: Interval Partitioning



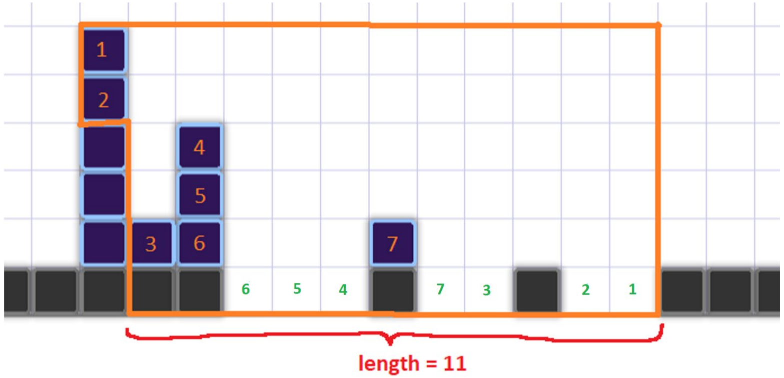
required moves: 4+2+3+4+1 = 14

Definition: Left / Right Intervals

- every block with h > 0 is moved in the same direction until leveled
- contain a start stack and a continuous sequence of complete columns
  - can include gap columns outside the problem instance!

Clearly defined by:

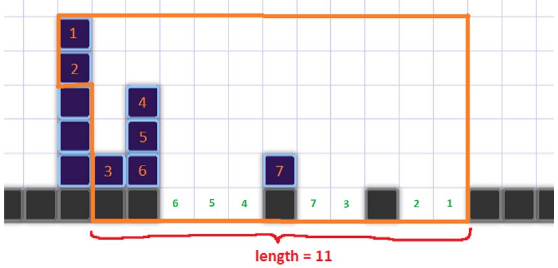
- start column index
- end column index
- number of blocks moved
  in start column *(= start stack size)*



length of an interval = number of included columns - 1

= end column index - start column index

Left / Right Intervals: Observations



- start stack always has at least 1 block with h > 0
  - otherwise there would be nothing to move, so why include?
- for every block with h > 0, includes at least one matching gap
  - otherwise we could not have leveled that block in the interval
- end column is always a gap
  - interval ends as soon as we have found a gap for each non-leveled block
- a single column can include start stacks of both a left and a right interval

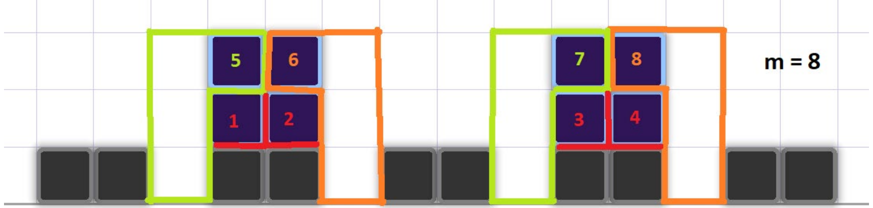
- **required moves to level = length** of the interval
  - in a right-interval, we need to move the leftmost block into the rightmost gap
  - the leftmost block is in the start stack, the rightmost gap is the end column
  - all other blocks on the way will be leveled before the leftmost block reaches the end gap

How Many Intervals?

- each block with $h > 0$ can be in either a left, right or noop interval

  ⇒ up to 3m possible intervals in a problem instance

- up to m non-overlapping intervals at the same time



⇒ $O(2^m)$ interval partitionings to consider!

but $m < 10^{14}$ ⇒ impossible to calculate all partitionings

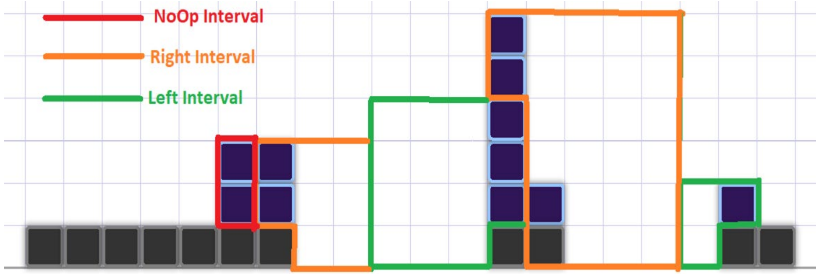
The Algorithm

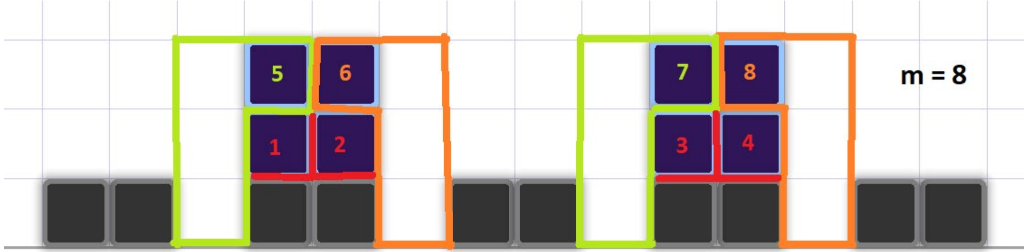
Incremental Calculation

Too many possible interval partitions

Cannot calculate them all

➔ **Dynamic Programming**

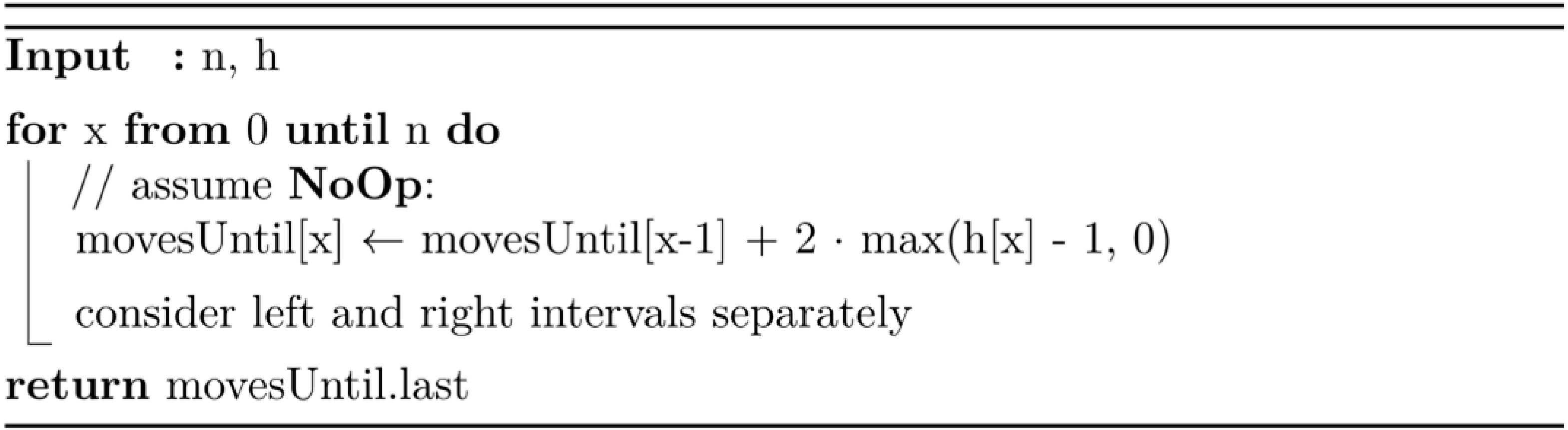




Basic Approach

Iterate over columns from left to right

For each column **x**, remember min number of moves required to level everything to the left (including **x**) in **movesUntil[x]**



```
Input : n, h
for x from 0 until n do
    // assume NoOp:
    movesUntil[x] ← movesUntil[x-1] + 2 · max(h[x] - 1, 0)
    consider left and right intervals separately
return movesUntil.last
```

Calculating a Left Interval

Naive approach: go left until we have gaps for all found blocks

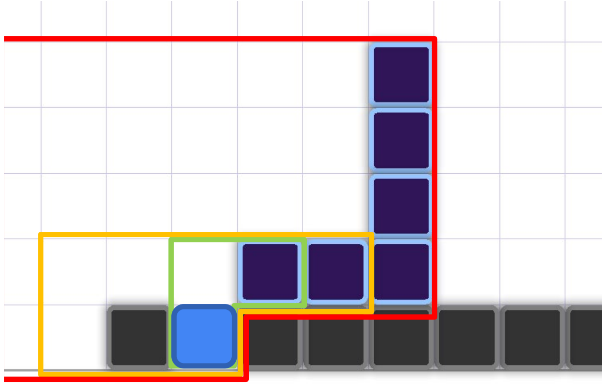
➔ Inefficient, will result in **O(m²)** runtime for left moves alone

Idea: Keep

- a stack of open gaps we found
- a counter how many gaps to the left of 0 have been filled

Calculating a Left Interval

```
Input : n, h
let gaps ← empty stack
let gapsFilledBeyondLeftBorder ← 0
let leftSplits ← 2-dim array
for x from 0 until n do
    leftSplits[x][0] ← movesUntil[x-1]
    if h[x] = 0 then
        push x to gaps
    else
        for y from 1 until h[x] do
            if gaps is not empty then
                leftBound ← gaps.pop()
            else
                gapsFilledBeyondLeftBorder +=1
                leftBound ← -gapsFilledBeyondLeftBorder
            leftSplits[x][y] ← leftSplits[x][y-1] + 2
            let leftMoves ← movesUntil[leftBound] + x - leftBound
            if leftSplits[x][y] > leftMoves then
                leftSplits[x][y] ← leftMoves
        movesUntil[x] ← leftSplits[x][h[x]-1]
```



Calculating a Right Interval

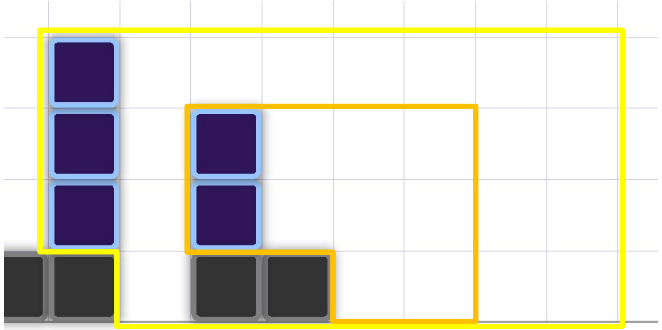
Handle right intervals at their end column➔ **x** must be a gap

A search for each gap would be inefficient

Idea: New columns have to be leveled completely before a right interval can end

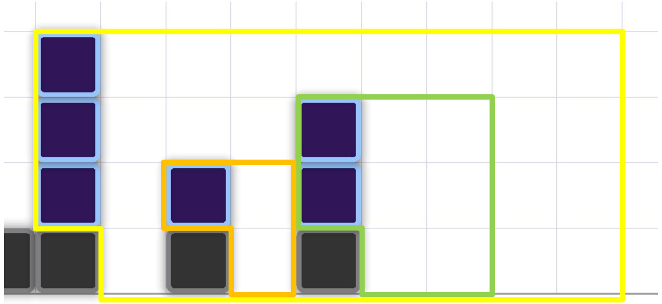
➔ Keep stack of possible right intervals

Each with **{ leftCol, remainingBlocks }**



Calculating a Right Interval

```
Input : n, h
let openRightIntervals ← empty stack of { leftCol, remainingBlocks }
for x from 0 until n do
    if h[x] > 1 then
        push { x, h[x] - 1 } to openRightIntervals
        (left interval handling)
    else if h[x] = 0 then
        movesUntil[x] ← movesUntil[x - 1]
        if openRightIntervals is not empty then
            let ri ← openRightIntervals.top
            let $x_\ell$ ← ri.leftCol
            let blocksTaken ← h[$x_\ell$ ] - ri.remainingBlocks
            let totalMoves ← leftSplits[$x_\ell$][blocksTaken] + x - $x_\ell$
            if totalMoves < movesUntil[x] then
                movesUntil[x] ← totalMoves
            ri.remainingBlocks -= 1
            if ri.remainingBlocks = 0 then
                openRightIntervals.pop()
Handle all remaining intervals in stack
Right column is always lastRightBound + .remainingBlocks (lastRightBound is n - 1 for the first one)
```



Necessary Optimizations

Current Performance

Need to iterate over every block with h > 1

- Once for left-intervals, once for right-intervals

➔ **O(m) runtime**

Need to save min move value for each possible split

➔ **O(m) memory**

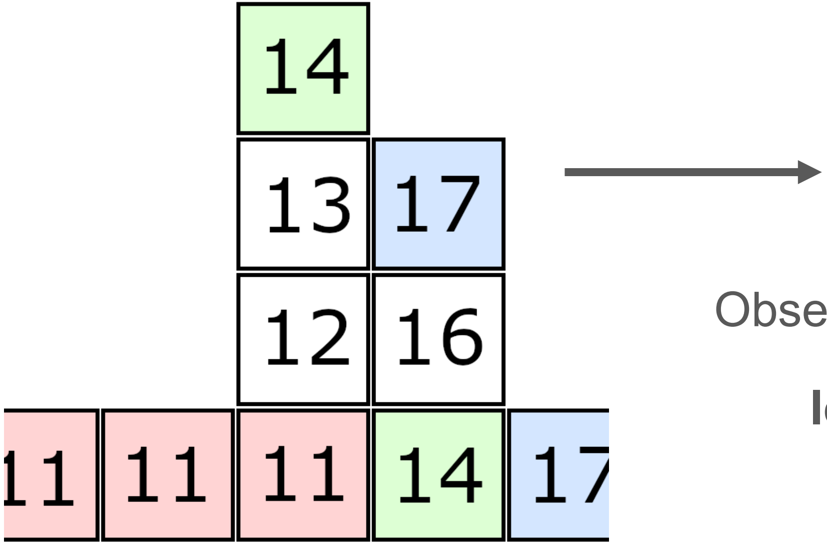
- Worst case: No Gaps
  - ➔ All columns on right-interval stack
  - ➔ Need to handle all splits at the end
  - ➔ Actually need all the values until the end

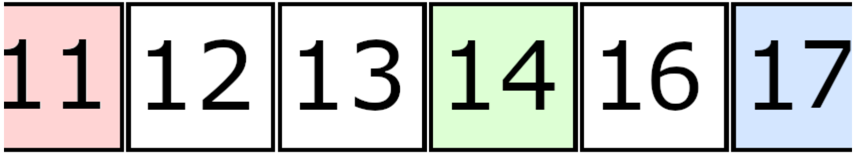
Remember: up to 10^{14} blocks
1 byte per block ➔ 100 terabyte

O(m) definitely doesn't work for extreme cases.

Idea: Implement **leftSplits** as sparse data

Step one: flatten the array





Observation if h > 1:

leftSplit[x][h[x]-1] == leftSplit[x+1][0]

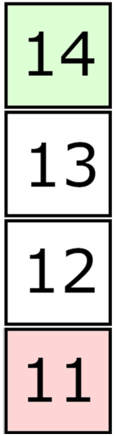
(== movesUntil[x])

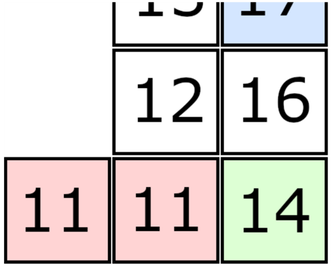
Idea: Implement **leftSplits** as sparse data

Assumption: No gaps

How do the values in the array develop?

- +1 For most blocks
- +2 For a new column





Idea: Implement **leftSplits** as sparse data

Generalizing to gaps:

- At every gap, the required moves do not increase (one non-positive change)

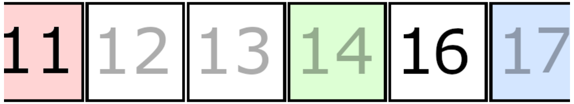
➔ Only **O(n)** non-1 differences in **leftSplits**

➔ If we only save non-1 differences, we can reduce the memory usage from **O(m)** to **O(n)**

Getting the required data

But how do we get the minimum number of moves until starting a left split when considering a right split?

- Use a search tree (C++ **std::map**, Java **TreeMap**)
- Keys: indices of old array
- When entry is present, then done
- If not, search the tree for the next smaller key
- **Result:** Value at present entry + difference between the keys



11	12	13	14	16	17

→ **O(log n)** lookup instead of **O(1)**

Logarithmic factors can often be ignored for actual runtimes ☺

Idea: Skip Left Interval calculations

When calculating left intervals, we only jump from gap to gap (at most **n**)

➜ As long as we stay in bounds (left column index ≥ 0), total left split calculation is in **O(n log n)**, as there can be at most **n** gaps

➜ **O(m log n)** only applies when leaving bounds

Idea: If we do leave the left bound, there will be infinite gaps ➜ Every additional block only adds +1 move

Since we don’t save those, we can simply break once we found a worthy (= better than 2-approx strategy) left split across the left bound ➜ **O(n log n)**

Idea: Cleanup right intervals faster

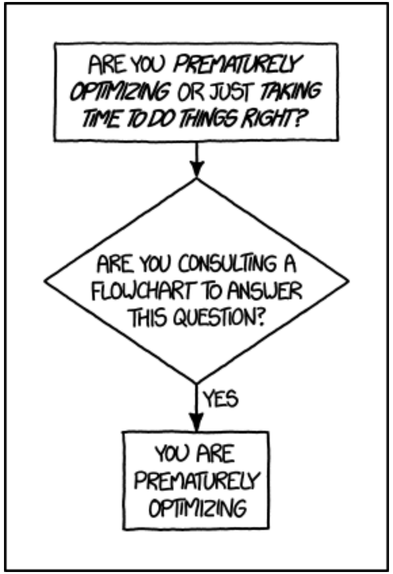
Same approach: Infinite consecutive gaps after handling all columns

- No need to find **.remainingBlocks** next gaps, can just calculate end column
- Partitionally leveling stack is not necessary, taking all blocks is optimal

Only have to handle all right splits ending before right bounds (at most **n**) and one right interval per column that exceeds bounds (at most **n**)

➔ **O(n log n)** in total

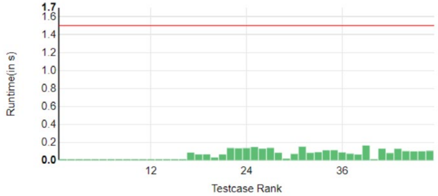
Implementation Tips



https://xkcd.com/1691/

- Use long (int64) for most numbers!
  - large instances can easily cause ints to overflow
  - performance will be fine, we promise
- Watch out for offsets!

  +/- 1 issues can easily happen
  depending on how you keep track of values



- Use expressive variable names!
  - which values are in/exclusive w.r.t. column indices?, etc
- Ignore micro-optimizations until the very end
  - can get up to factor ~3 faster
  - but algorithmic improvements can lead to ~1000 times faster code!

Summary

$\Rightarrow O(n \log n)$ runtime

$\Rightarrow O(n)$ space

Iterate through all columns and keep track of:

- min number of moves required to level everything so far
- number of blocks with $h > 0$ encountered so far (= key for **leftSplits**)

If height of column > 1:

- push to **openRightIntervals**
- calculate possible **leftSplits** by iterating through the blocks
- stop iterating early when all following blocks would only need 1 more move

If column is a gap:

- push it to the **gaps** stack
- check whether including the top of **openRightIntervals** yields a better result

After iterating, iterate backwards through **openRightIntervals** and check for a better result