Lehrstuhl für Informatik I
Algorithmen und Komplexität
Universität Würzburg

Wüzburg, den 28. April 2021
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Vorlesung Algorithmen für Geographische Informationssysteme SS 2021
Markov-Ketten

Abgabe: in 2er Gruppen bis Mittwoch, den 5.5.2021, 10:15 Uhr, in analoger Form oder über die E-Learning-Seite der Lehrveranstaltung. Bitte geben Sie auf Ihrer Ausarbeitung die Namen beider Gruppenteilnehmer an.

1 Maximum-likelihood estimation

Consider a (possibly biased) coin with sides H and T. The probability distribution for a single flip is $\Pr[\texttt{H}] = p$ and $\Pr[\texttt{T}] = 1 - p$, where $p \in [0, 1]$ is a parameter. We flip the coin n times, independently, and k of these flips result in H. Derive the maximum-likelihood estimate for p; prove your result. *(4 Punkte)*

2 Markov students

Consider a student who is, at every time step, in one of the states in $Z = \{\texttt{Study}, \texttt{Party}, \texttt{Sleep}\}$. The transition probabilities are as follows.

$$\Pr[X_{n+1} = \texttt{Study} \mid X_n = \texttt{Study}] = 0.3$$
$$\Pr[X_{n+1} = \texttt{Party} \mid X_n = \texttt{Study}] = 0.5$$
$$\Pr[X_{n+1} = \texttt{Sleep} \mid X_n = \texttt{Study}] = 0.2$$

$$\Pr[X_{n+1} = \texttt{Study} \mid X_n = \texttt{Party}] = 0.0$$
$$\Pr[X_{n+1} = \texttt{Party} \mid X_n = \texttt{Party}] = 0.5$$
$$\Pr[X_{n+1} = \texttt{Sleep} \mid X_n = \texttt{Party}] = 0.5$$

$$\Pr[X_{n+1} = \texttt{Study} \mid X_n = \texttt{Sleep}] = 0.3$$
$$\Pr[X_{n+1} = \texttt{Party} \mid X_n = \texttt{Sleep}] = 0.3$$
$$\Pr[X_{n+1} = \texttt{Sleep} \mid X_n = \texttt{Sleep}] = 0.4$$

(a) Draw the state transition diagram, with a node for every state and arrows indicating the transition probabilities. *(1 Punkte)*

(b) You may notice that, independent of the initial distribution $\Pr[X_1]$, the probability distribution for later time steps always seems to converge to the same distribution as time goes on. (Try some.)

This doesn't hold for all possible Markov chains, but it is in fact true for this particular Markov chain: independently of $\Pr[X_1]$, the probability distribution $\Pr[X_n]$ converges as n goes to infinity. This limit distribution is called a *stationary distribution*, defined as a probability distribution such that $\Pr[X_{n+1}] = \Pr[X_n]$ in a particular Markov chain. (It is called "stationary" because it "doesn't move.") Calculate the stationary distribution for the given Markov chain; give exact values. *(4 Punkte)*

3   Im Gefängnis (nur zu Besuch)

Consider the well-known boardgame Monopoly, where the players travel along a loop of 40 game tiles (or possibly are 'in jail'). See Figure 3 (a) for a version of the game board. During the course of the game, the player can buy and trade tiles in order to later make money from 'rent' paid by other players that land on the tile. This means that any tiles that players are more likely to land on are more valuable.

We want to model the movement of a player along the board as a Markov chain, so that we can easily calculate a stationary distribution: as the game goes longer and (infinitely) longer, what is the probability of a player landing on each tile? This information is useful when making buying decisions in the game. See Figure 3 (b) for the stationary distribution of Dutch-rules Monopoly, which has been calculated in this way.

Model, as Markov chain, a basic version of how a player goes around the board during the game. What are the states and transition probabilities? Then find out about more advanced rules and incorporate some of them in your model. The exact rules vary from edition to edition, but certainly discuss: going to jail, getting out of jail, and chance cards. *(5 Punkte)*

(a) A version of the game board.
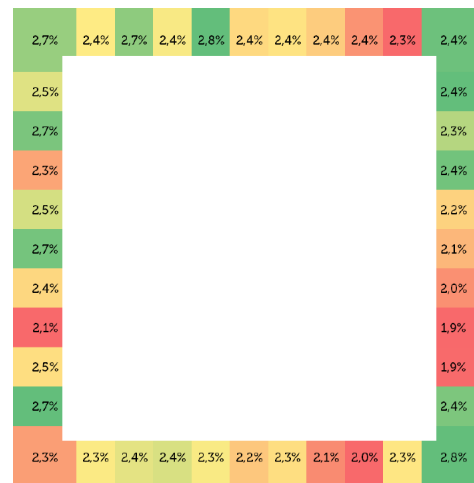
(b) Stationary distribution using the Dutch chance cards, colour-coded from red (low probability) to green (high probability). There is an additional 5.5% probability of being in jail.

Abbildung 1: Monopoly