Computational Geometry

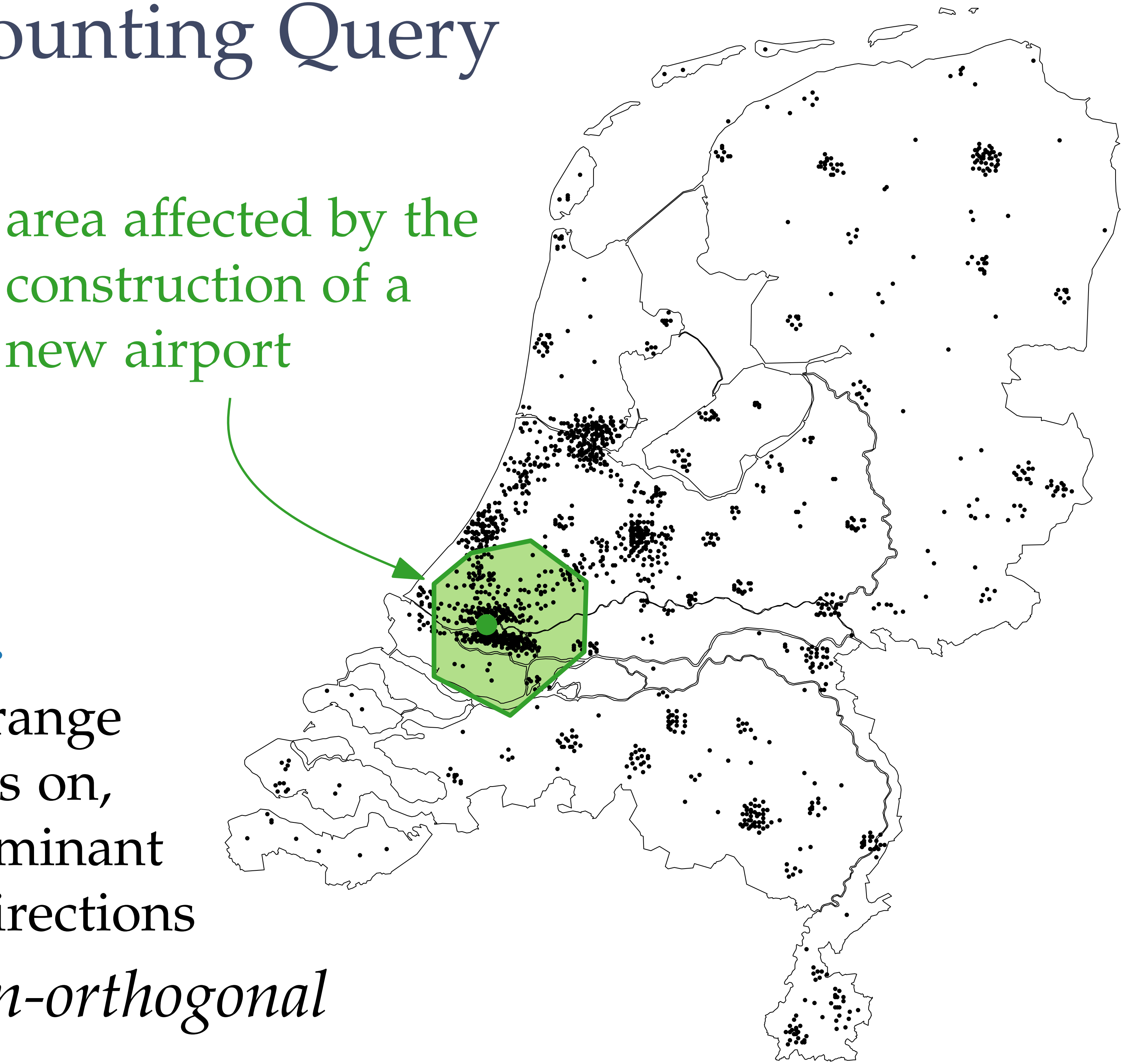
Lecture 11:
Simple Range Searching

Part I:
The 1-Dimensional Case

Philipp Kindermann

Winter Semester 2020

Range-Counting Query



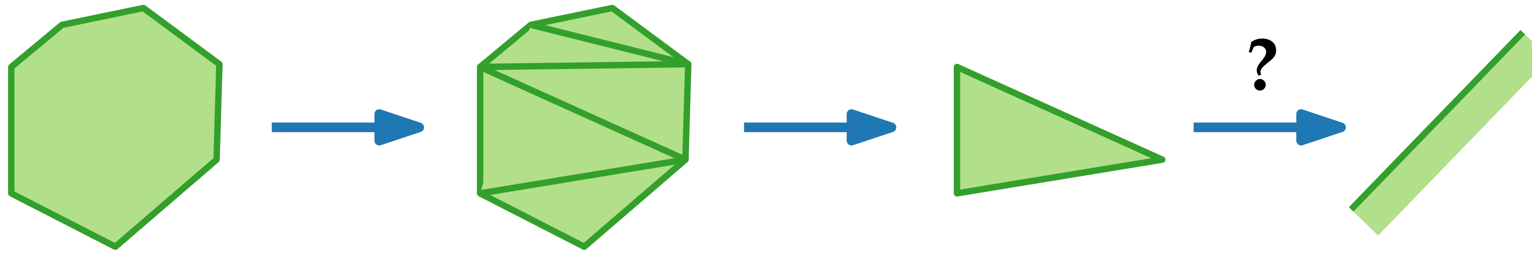
Observation.

Query range depends on, e.g., dominant wind directions

$\Rightarrow$ *non-orthogonal*

Non-orthogonal range queries

Query range:



Problem. Given a set P of n points, preprocess P such that *half-space range-counting queries* can be answered quickly.

Task. Design a data structure for the 1-dim. case:

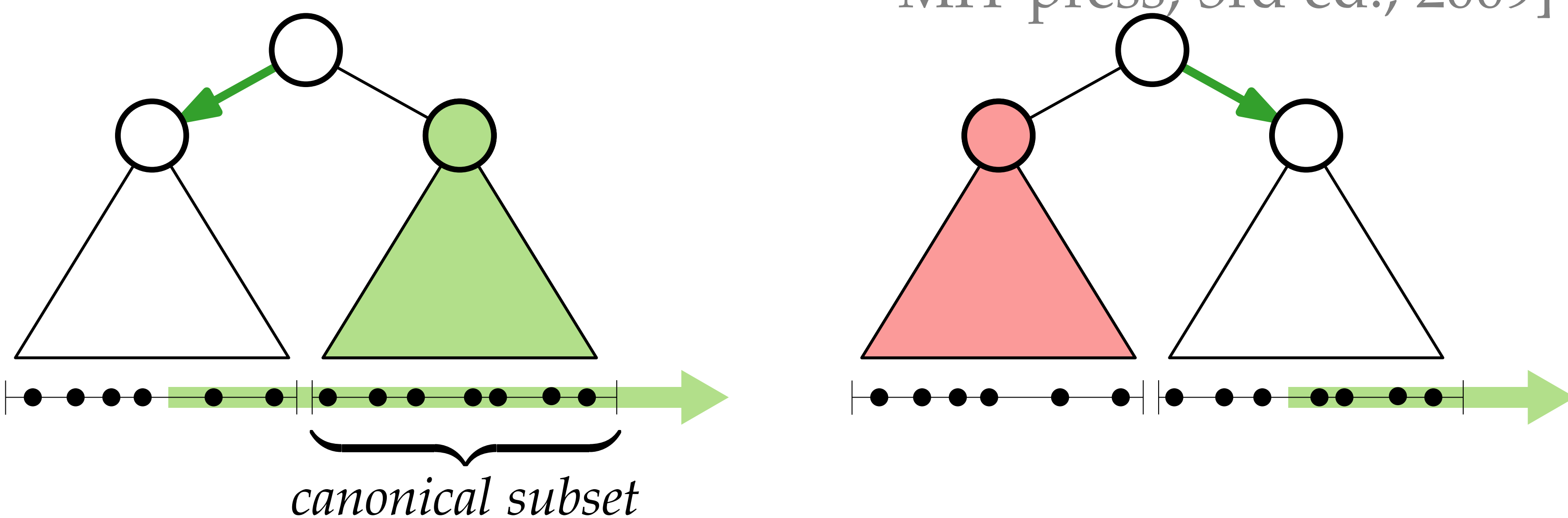
– Given a number x, return $|P \cap [x, \infty)|$.

– Consider P static / dynamic!

The 1-Dimensional Case

Task. Design a data structure for the 1-dim. case!

Solution.
- use balanced binary search trees
- augment each node with the number of nodes in its subtree [see Cormen et al., *Introduction to Algorithms*, MIT press, 3rd ed., 2009]



Lesson. On each level, visit ≤ 1 subtree recursively!

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Part II:
Generalizing to 2 Dimensions

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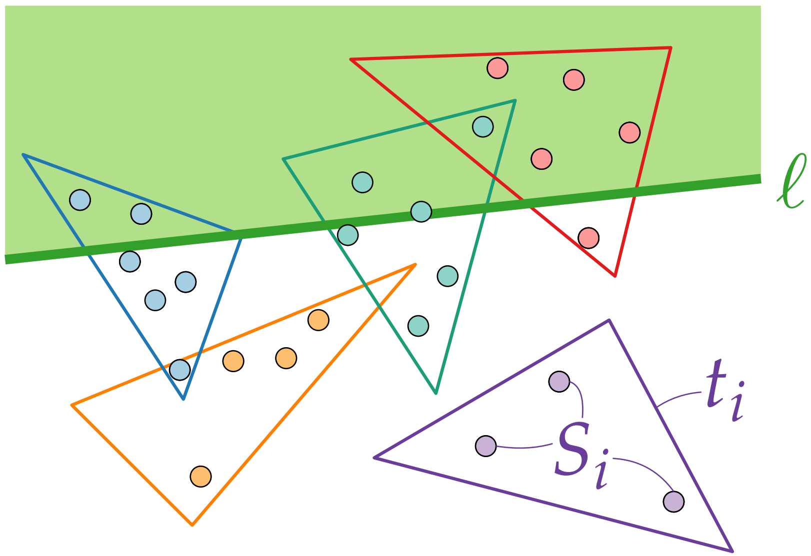
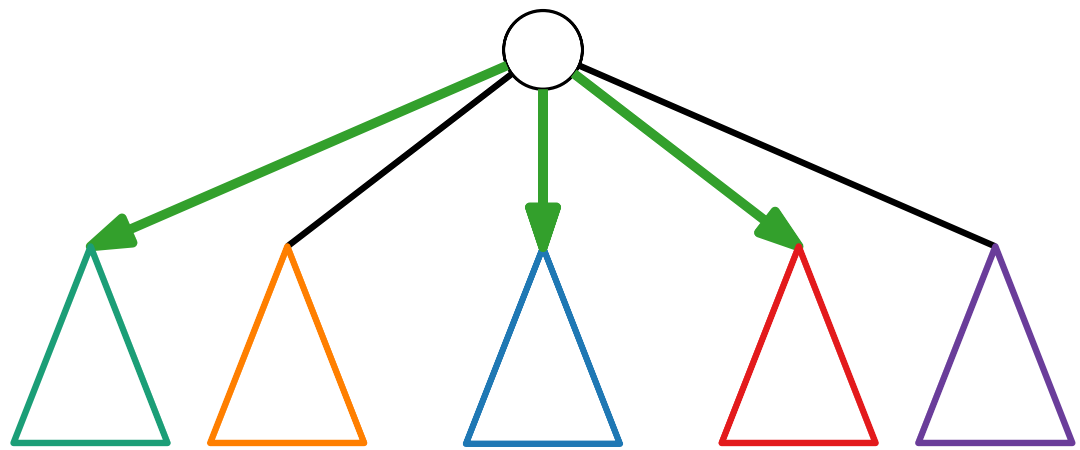
Generalizing to 2 Dimensions

Partition the input! Query… in a *partition tree* …recursively!





Definition. $\Psi(S) = \{(S_1, t_1), (S_2, t_2), \ldots, (S_r, t_r)\}$ is a *simplicial partition* (of size r) for S if

– S is partitioned by $S_1, \ldots, S_r$ and

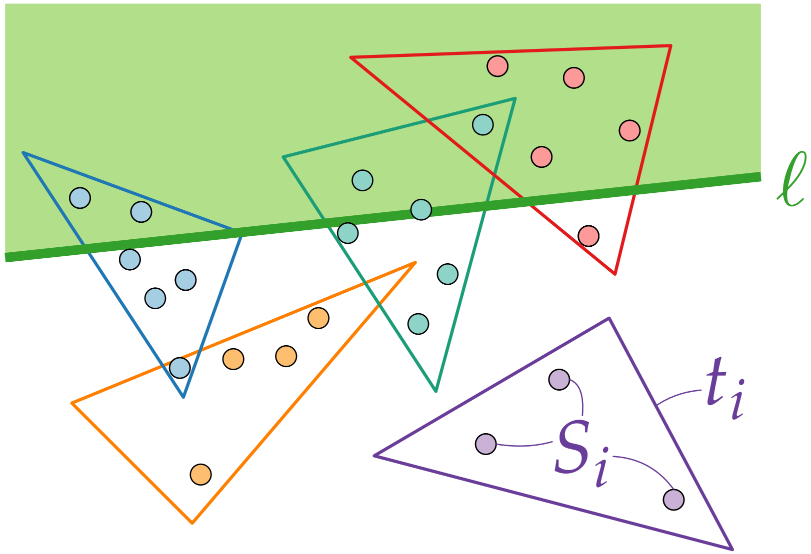
– for $1 \leq i \leq r$, t_i is a triangle and $S_i \subset t_i$.

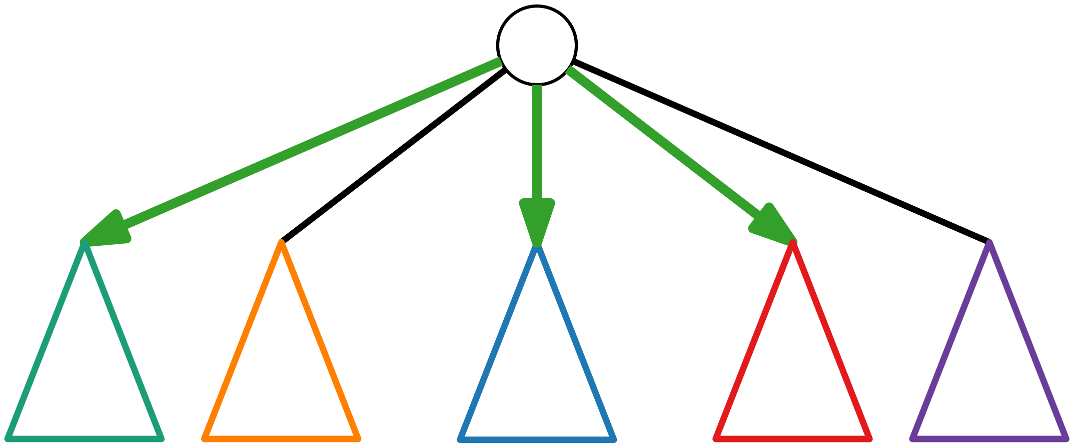
classes of S

$\Psi(S)$ is *fine* if $|S_i| \leq 2\frac{|S|}{r}$ for every $1 \leq i \leq r$.

Generalizing to 2 Dimensions

Partition the input! Query… in a *partition tree* …recursively!



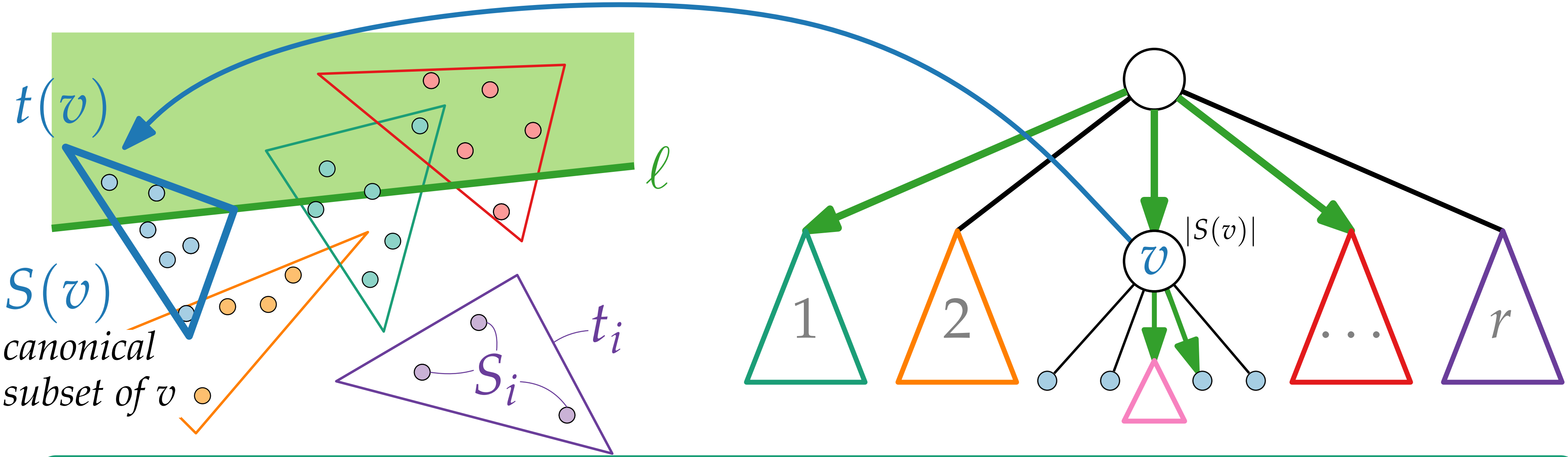


Definition. The *crossing number* of ℓ (w.r.t. $\Psi(S)$) is the number of triangles $t_1, \ldots, t_r$ *crossed* by ℓ.

The *crossing number* of $\Psi(S)$ is the maximum crossing number over all possible lines.

Generalizing to 2 Dimensions

Partition the input! Query… in a *partition tree* …recursively!



Theorem. [Matoušek, DCG 1992] For any set S of n pts and any $1 \leq r \leq n$, a fine simplicial partition of size r and crossing number $O(\sqrt{r})$ exists. For any $\varepsilon > 0$, such a partition can be built in $O(n^{1+\varepsilon})$ time.

Lemma. A partition tree for S can be constructed in $O(n^{1+\varepsilon})$ time. The tree uses $O(n)$ storage.

search tree with n leaves

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Part III:
Query Algorithm

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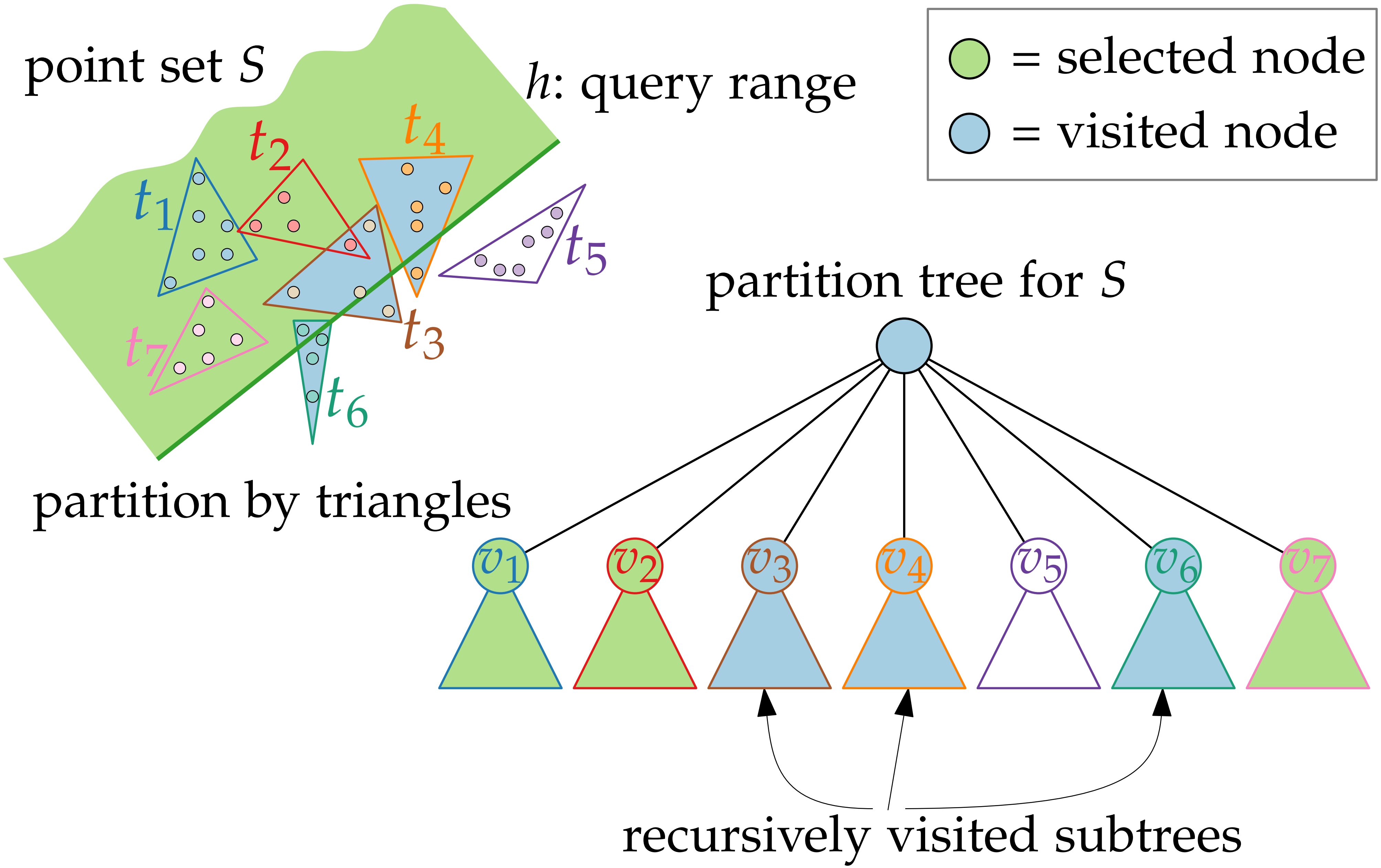
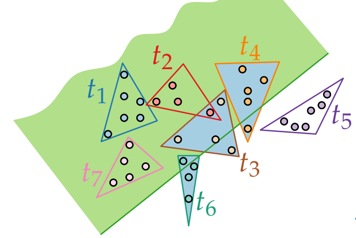
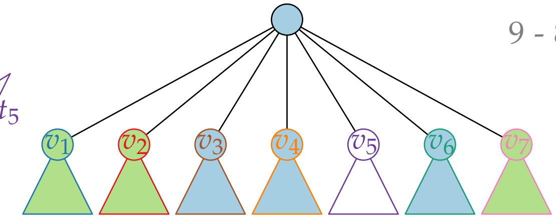
Example for a Query



Query Algorithm





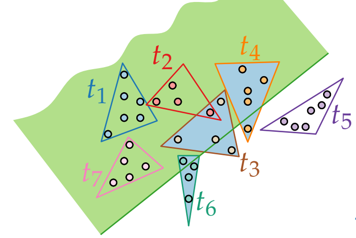
SELECTINHALFPLANE(half-plane h, partit. tree $\mathcal{T}$ for pt set S)

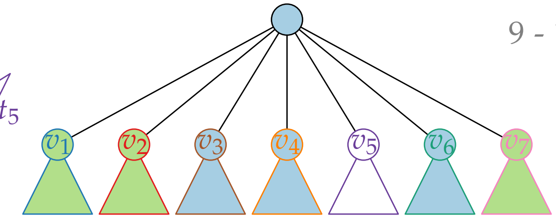
$N \leftarrow \emptyset$ // set of *selected* nodes

if $\mathcal{T} = \{\mu\}$ **then**
  if point stored at μ lies in h **then**
    $N \leftarrow \{\mu\}$
else
  foreach child ν of the root of $\mathcal{T}$ **do**
    if $t(\nu) \subset h$ **then**
      $N \leftarrow N \cup \{\nu\}$
    else
      if $t(\nu) \cap h \neq \emptyset$ **then**
        $N \leftarrow N \cup$ SELECTINHALFPLANE$(h, \mathcal{T}_\nu)$

return N // with $S \cap h = \bigcup_{\nu \in N} S(\nu)$

Query Algorithm





~~SELECT~~ COUNT InHalfplane(half-plane h, partit. tree $\mathcal{T}$ for pt set S)

$N \leftarrow$ ~~$\emptyset$~~ 0 // ~~set~~ number of *selected* nodes

if $\mathcal{T} = \{\mu\}$ **then**

- **if** point stored at μ lies in h **then**
  - $N \leftarrow$ ~~$\{\mu\}$~~ $N + 1$

else

- **foreach** child v of the root of $\mathcal{T}$ **do**
  - **if** $t(v) \subset h$ **then**
    - $N \leftarrow N$ ~~$\cup \{v\}$~~ $+|S(v)|$
  - **else**
    - **if** $t(v) \cap h \neq \emptyset$ **then**
      - $N \leftarrow N$ ~~$\cup$ SELECT~~ $+$ COUNT InHalfplane$(h, \mathcal{T}_v)$

return N // with $|S \cap h| = |\bigcup_{v \in N} S(v)|$

Task.

Turn this into a range *counting* query algorithm!

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Part IV:
Analysis of the Partition Tree

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Analysis of the Partition Tree

Lemma. For any $\varepsilon > 0$, there is a partition tree $\mathcal{T}$ for S s.t.:
for a query half-plane h,
SelectInHalfplane selects in $O(n^{1/2+\varepsilon})$ time
a set N of $O(n^{1/2+\varepsilon})$ nodes of $\mathcal{T}$
with the property that $h \cap S = \bigcup_{v \in N} S(v)$.

Proof. Let $\varepsilon > 0$. Let $r = 2(\sqrt{2}c)^{1/\varepsilon}$.

$$\Rightarrow Q(n) \leq \begin{cases} 1 & \text{if } n = 1, \\ r + \sum_{v \in C(h)} Q(|S(v)|) & \text{if } n > 1. \end{cases}$$

$C(h)$: all children v of the root s.t. h crosses $t(v)$

Theorem. [Matoušek, DCG 1992] For any set S of n pts and any $1 \leq r \leq n$, a fine simplicial partition of size r and crossing number $c\sqrt{r}$ exists. For any $\varepsilon > 0$, such a partition can be built in $O(n^{1+\varepsilon})$ time.

Analysis of the Partition Tree

Lemma. For any $\varepsilon > 0$, there is a partition tree $\mathcal{T}$ for S s.t.:
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with the property that $h \cap S = \bigcup_{\nu \in N} S(\nu)$.

Lemma. A partition tree for S can be constructed in $O(n^{1+\varepsilon})$ time. The tree uses $O(n)$ storage.

Corollary. Half-plane range counting queries can be answered in $O(n^{1/2+\varepsilon})$ time
using $O(n)$ space and $O(n^{1+\varepsilon})$ prep.

Back to *Triangular* Range Queries

Any ideas? Just use SELECTINHALFPLANE!

Theorem. Given a set S of n pts in the plane, for any $\varepsilon > 0$, a triangular range-counting query can be answered in $O(n^{1/2+\varepsilon})$ time using a partition tree.

The tree can be built in $O(n^{1+\varepsilon})$ time and uses $O(n)$ space.

The points inside the query range can be reported in $O(k)$ additional time, where k is the number of reported pts.

Can we do better?

Use cutting trees! (Chapter 16.3 [dBCvKO])

Query time $O(\log^3 n)$, prep. & storage $O(n^{2+\varepsilon})$.

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Part V:
Multi-Level Partition Trees

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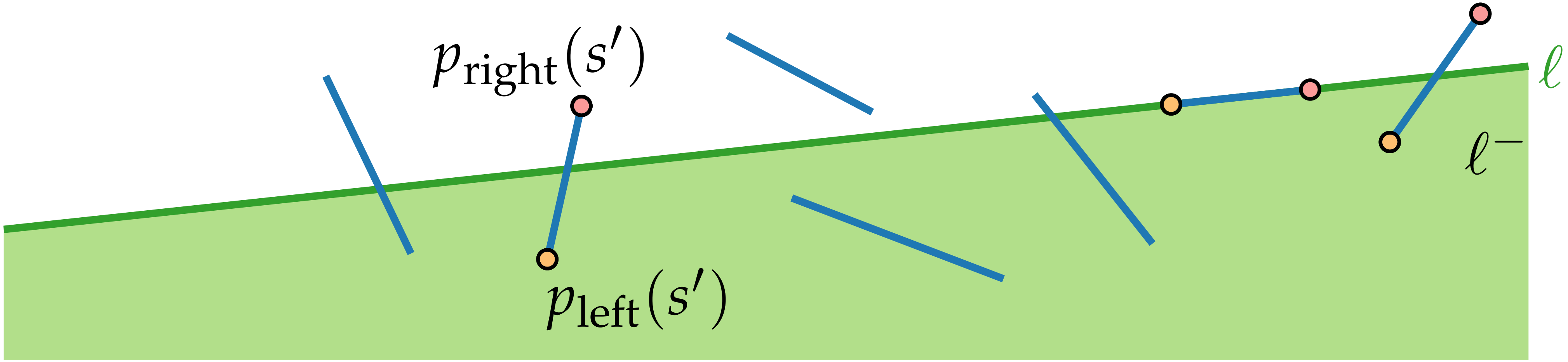
Multi-Level Partition Trees

$|S(v)|$

Idea. Store with each internal node not just a number, but another data structure!

Task. Design a fast data structure for line segments that counts all segments intersecting a query line ℓ.



Query Algorithm

For $S' \subseteq S$, let
$P^{\text{left}}_{\text{right}}(S') = \{p^{\text{left}}_{\text{right}}(s) \mid s \in S'\}$

SelectIntSegments(line ℓ, two-level partition tree $\mathcal{T}$ for S)

– first-level tree stores $P_{\text{right}}(S)$
– second-level trees store subsets of $P_{\text{left}}(S)$

$N \leftarrow \emptyset$
if $\mathcal{T} = \{\mu\}$ **then**
  if segment stored in μ intersects ℓ **then** $N \leftarrow \{\mu\}$
else
  foreach child ν of $\mathcal{T}$'s root **do**
    if $t(\nu) \subset \ell^{+}$ **then**
      $N \leftarrow N \cup$ SelectInHalfplane$(\ell^{-}, \mathcal{T}_\nu^{\text{assoc}})$
    else
      if $t(\nu) \cap \ell \neq \emptyset$ **then**
        $N \leftarrow N \cup$ SelectIntSegments$(\ell, \mathcal{T}_\nu)$
return N

stores $P_{\text{left}}(S_{\text{seg}}(\nu))$, where
$S_{\text{seg}}(\nu) = \{s \mid p_{\text{right}}(s) \in S(\nu)\}$

below above ?

!!! $\bigcup_{\nu \in N} S(\nu) = \{s \in S \mid p_{\text{right}}(s)$ above ℓ and $p_{\text{left}}(s)$ below $\ell\}$.

Results

Lemma. A 2-level partition tree for line-intersection queries among a set of n segments uses $O(n \log n)$ storage.

Lemma. Let S be a set of n segments in the plane. For any $\varepsilon > 0$, there is a 2-level partition tree $\mathcal{T}$ for S s.t.

- given a query line ℓ, we can select $O(n^{1/2+\varepsilon})$ nodes from $\mathcal{T}$ whose canonical subsets represent the segments intersected by ℓ.
- The selection takes $O(n^{1/2+\varepsilon})$ time.

Corollary. Let S be a set of n segments in the plane. We can count the number of segments in S intersected by a query line in $O(n^{1/2+\varepsilon})$ time using $O(n \log n)$ space and $O(n^{1+\varepsilon})$ prep.

Results

Lemma. A 2-level partition tree for line-intersection queries among a set of n segments uses $O(n \log n)$ storage.

Lemma. Let S be a set of n segments in the plane. For any $\varepsilon > 0$, there is a 2-level partition tree $\mathcal{T}$ for S s.t.

- given a query line ℓ, we can select $O(n^{1/2+\varepsilon})$ nodes from $\mathcal{T}$ whose canonical subsets represent the segments intersected by ℓ.
- The selection takes $O(n^{1/2+\varepsilon})$ time.

δ-level objects

Corollary. Let S be a set of n ~~segments~~ in the plane. We can count the number of ~~segments~~ in S in a δ-level query ~~intersected by a query line~~ in $O(n^{1/2+\delta\varepsilon})$ time using $O(n \log^{\delta-1} n)$ space and $O(n^{1+\delta\varepsilon})$ prep.