Computational Geometry

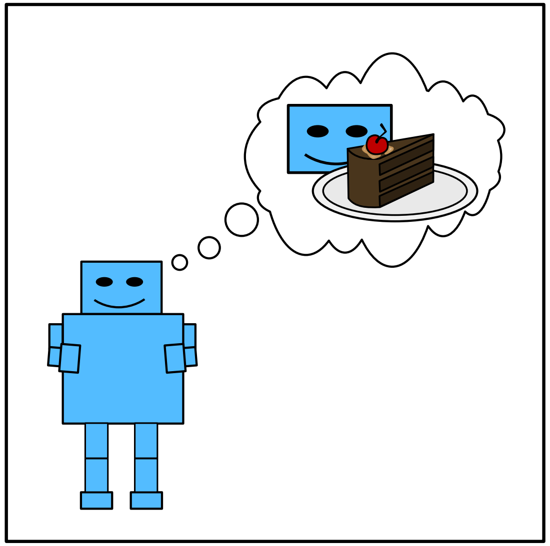
Lecture 10:
Motion Planning

Part I:
Point-Shaped Robots

Philipp Kindermann

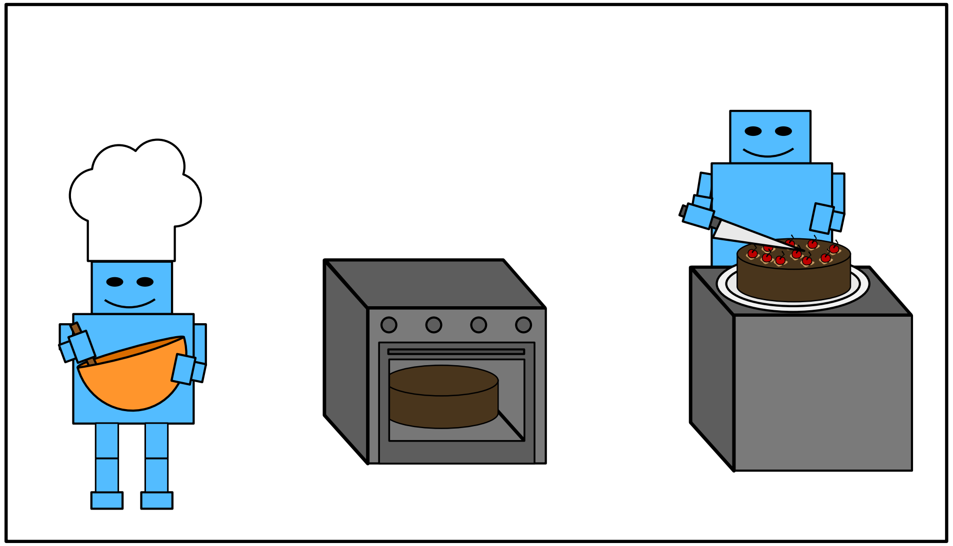
Winter Semester 2020

Planning



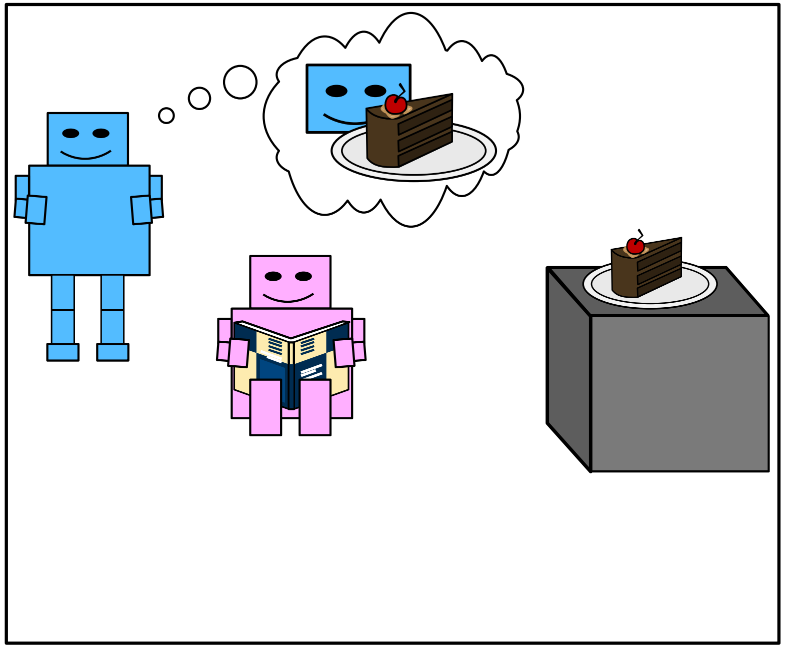
current situation,
desired situation

$\Rightarrow$



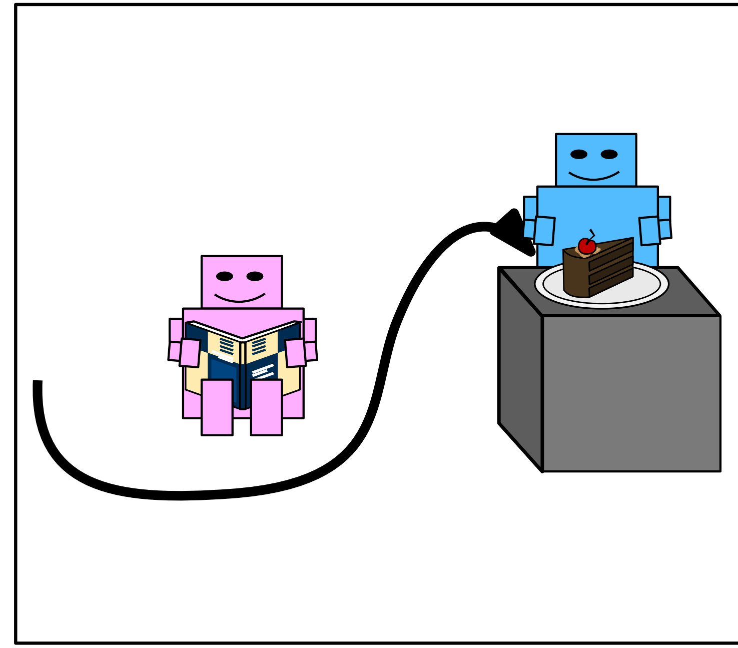
sequence of steps to reach
the one from the other

Path Planning



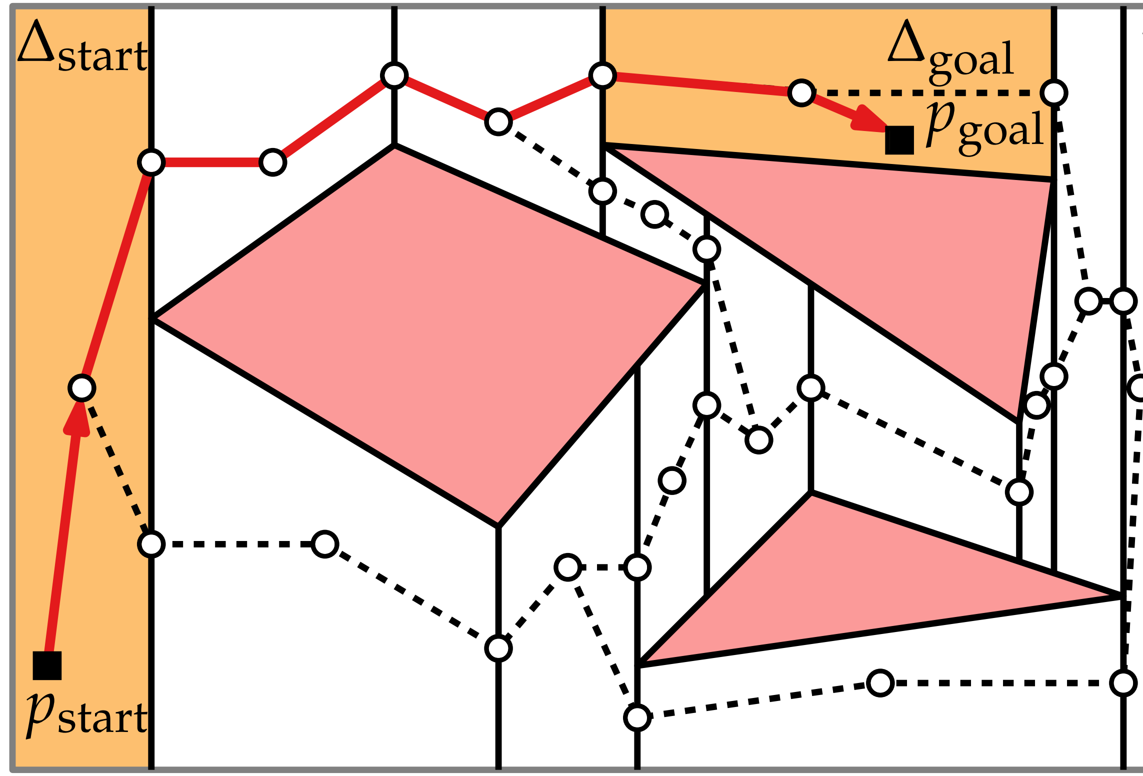
current location,
desired location

$\Rightarrow$



path to reach the
one from the other

Point-Shaped Robots



preprocessing	Create trapezoidal map of obstacle edges.	$O(n \log n)$
	Remove vertical extensions inside obstacles.	$O(n)$
	Vertices at centers of trapez. and vertical ext.	$O(n)$
	Connect "neighboring" vertices by line segm.	$O(n)$
querying	Locate $p_{\text{start}}, p_{\text{goal}}$ in map $\rightarrow \Delta_{\text{start}}, \Delta_{\text{goal}}$.	$O(\log n)$
	Do breadth-first search in the *roadmap* to find a path π from Δ_{start} to Δ_{goal}.	$O(n)$
	Connect $p_{\text{start}}, p_{\text{goal}}$ to π by line segments.	$O(1)$

A First Result

Theorem. We can preprocess a set of polygonal obstacles with a total of n edges in $O(n \log n)$ expected time such that, given a start and a goal position, we can find a collision-free path for a point robot in $O(n)$ time if it exists.

What about, say, *polygonal* robots?

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Part II:
Configuration Space

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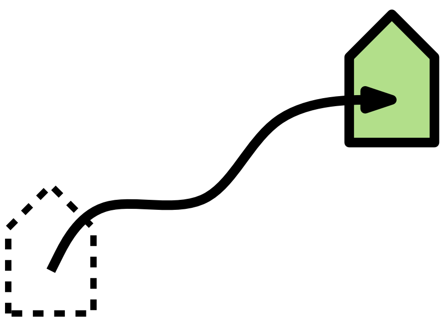
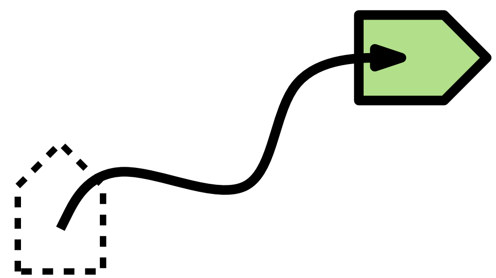
Degrees of Freedom

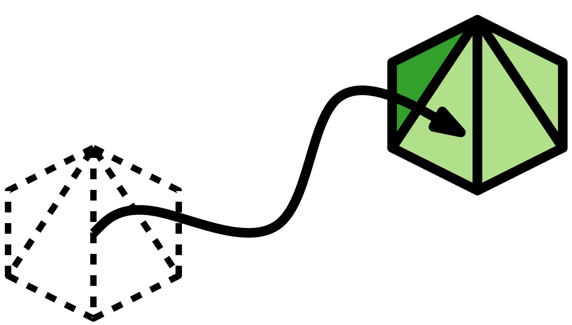
Every robot has some number *d* of *degrees of freedom*, meaning that its *configuration* with respect to the world can be specified by *d* parameters.



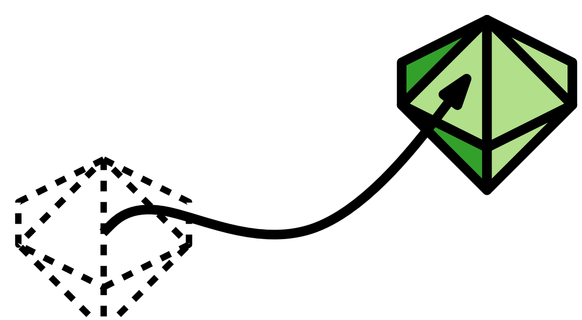
2D translating robot



2D translating, rotating robot

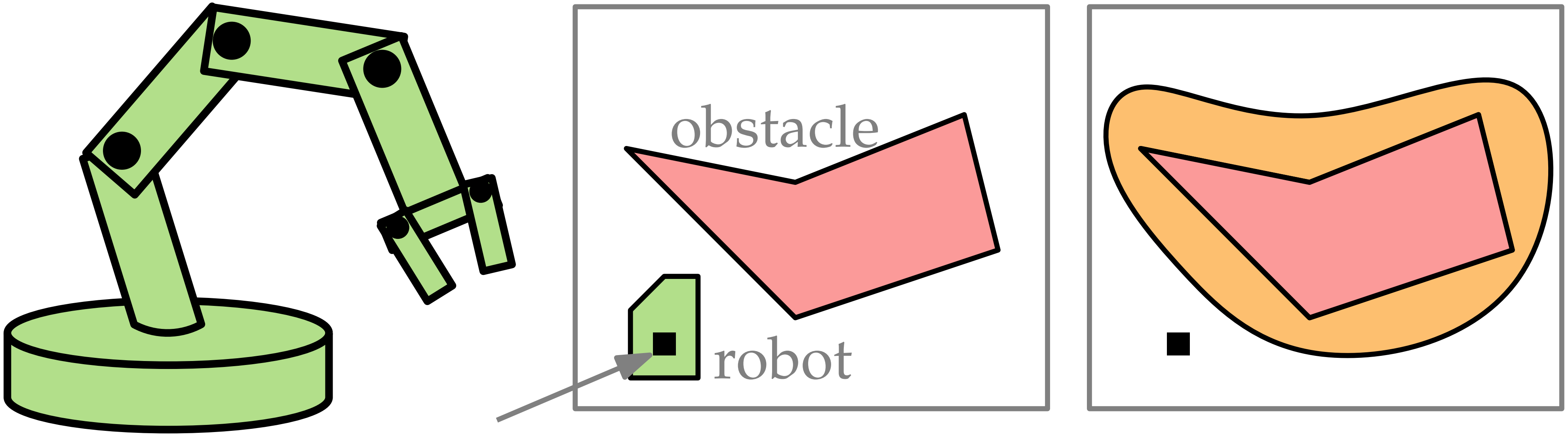


3D translating robot



3D translating, rotating robot

Configuration Space



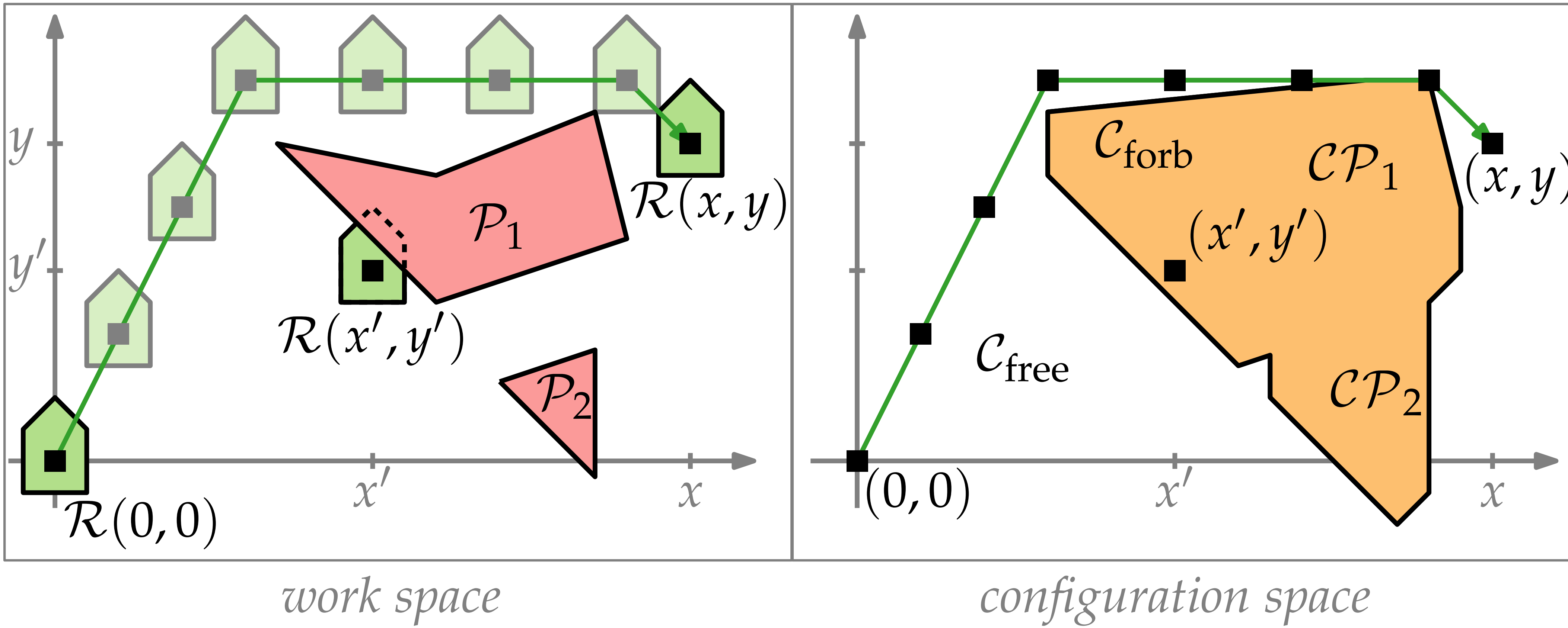
The *configuration space* is the *d*-dimensional space of all possible (i.e., obstacle avoiding) parameter value combinations.

Path for a *point* through configuration space

$$\Downarrow$$

path for the *robot* in the original space.

Example: Translating 2D Polygonal Robots



- Compute $\mathcal{CP}_i = \{(x,y) : \mathcal{R}(x,y) \cap \mathcal{P}_i \neq \emptyset\}$ for each $\mathcal{P}_i$.
- Compute their union $\mathcal{C}_{\text{forb}} = \bigcup_i \mathcal{CP}_i$.
- Find a path for a point in the complement $\mathcal{C}_{\text{free}}$ of $\mathcal{C}_{\text{forb}}$.
  $\Rightarrow$ collision-free path for the robot in work space

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Part III:
Characterizing Configuration Spaces

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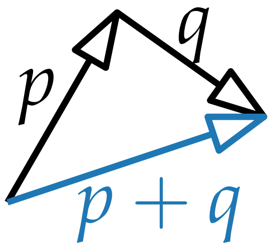
Some Linear Algebra

Vector sums

Algebra: $(p_x, p_y) + (q_x, q_y) = (p_x + q_x, p_y + q_y)$

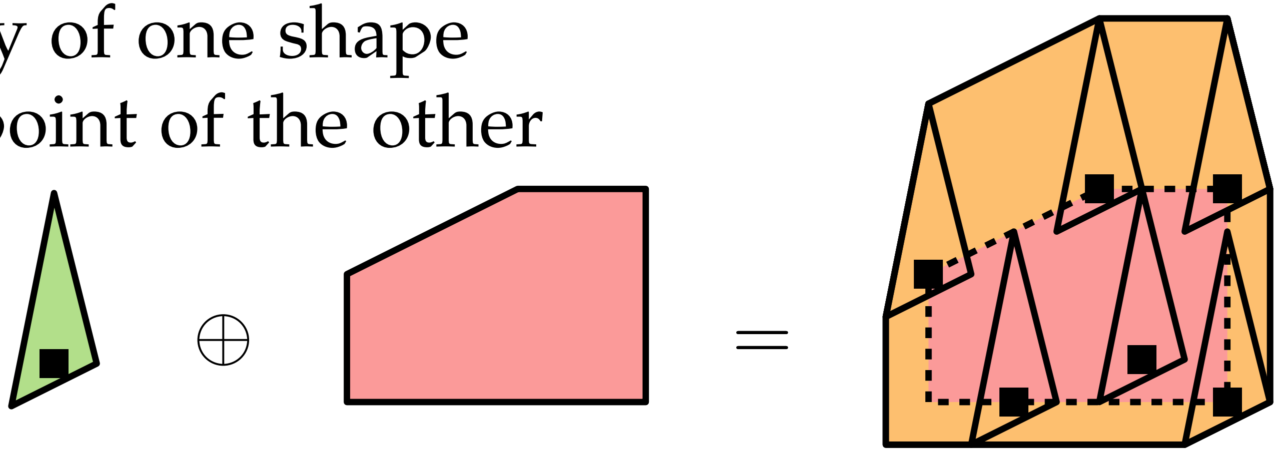
Geometry: place vectors head to tail



Minkowski sums

Algebra: $S_1 \oplus S_2 = \{p + q \mid p \in S_1, q \in S_2\}$

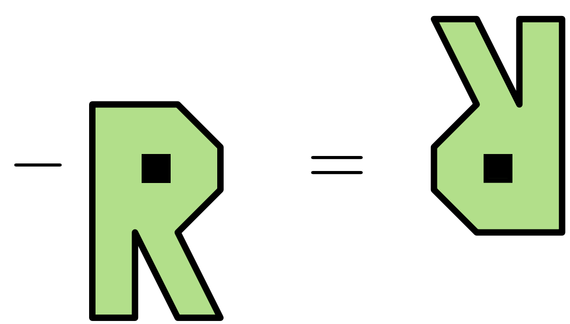
Geometry: place copy of one shape at every point of the other



Inversion

Algebra: $-S = \{-p \mid p \in S\}$

Geometry: rotate 180° (point-mirror) around reference point



Characterizing $\mathcal{CP}$

Recall that $\mathcal{CP} = \{(x,y) \colon \mathcal{R}(x,y) \cap \mathcal{P} \neq \emptyset\}$ for an obstacle $\mathcal{P}$.

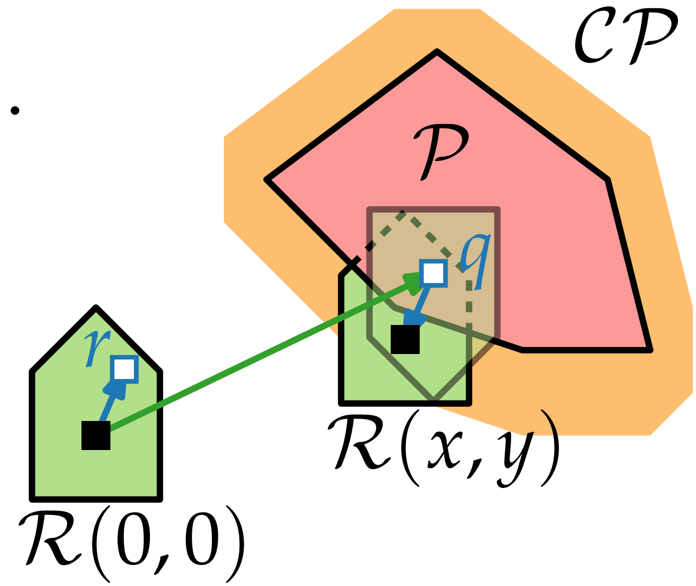
In other words: $\mathcal{R}(x,y)$ intersects $\mathcal{P} \quad \Leftrightarrow \quad (x,y) \in \mathcal{CP}$.

Theorem. $\mathcal{CP} = \mathcal{P} \oplus (-\mathcal{R}(0,0))$

Proof. Show: $\mathcal{R}(x,y)$ intersects $\mathcal{P} \Leftrightarrow (x,y) \in \mathcal{P} \oplus (-\mathcal{R}(0,0))$.

"$\Rightarrow$" Suppose $\mathcal{R}(x,y)$ intersects $\mathcal{P}$.
Let $q \in \mathcal{R}(x,y) \cap \mathcal{P}$. Then...

"$\Leftarrow$" Let $(x,y) \in \mathcal{P} \oplus (-\mathcal{R}(0,0))$.
Then there are points
$q \in \mathcal{P}$ and $r \in \mathcal{R}(0,0)$
such that ...



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Motion Planning

Part IV:
Minkowski Sum: Complexity & Computation

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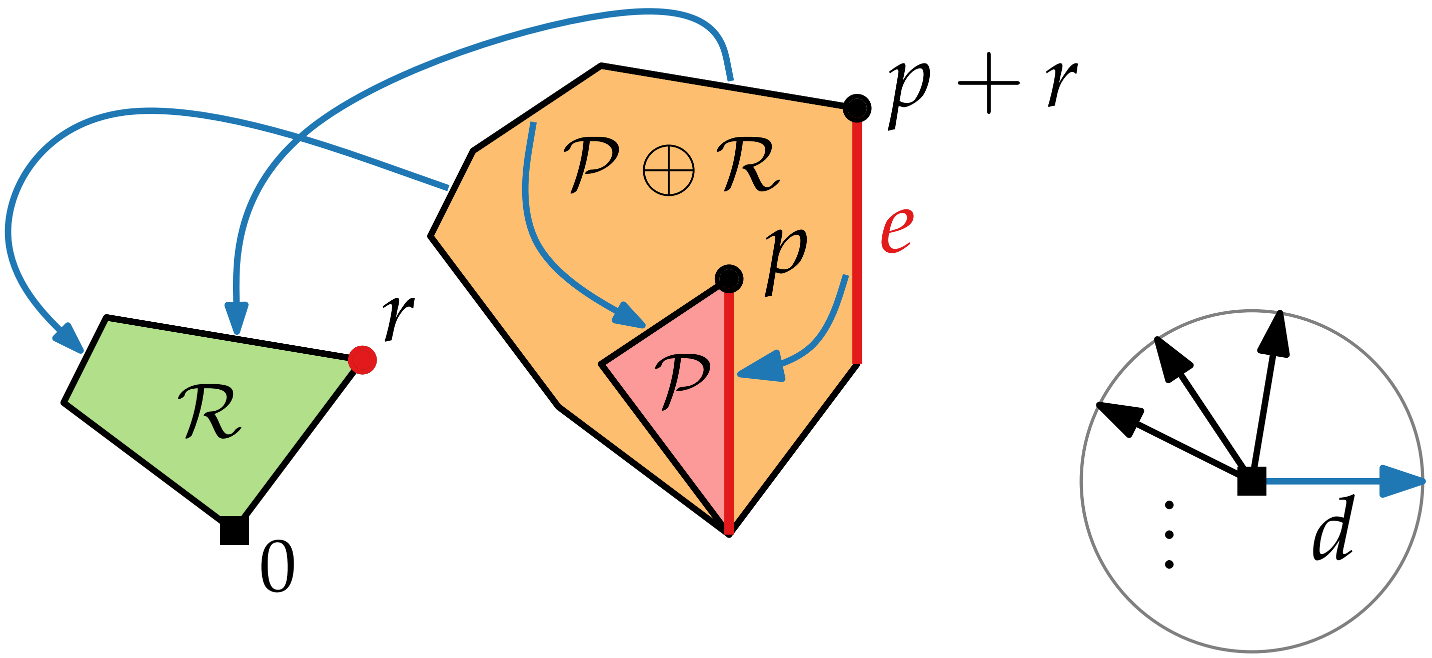
Winter Semester 2020

Minkowski Sums: Complexity

Theorem. If $\mathcal{P}$ and $\mathcal{R}$ are convex polygons with n and m edges, respectively, then $\mathcal{P} \oplus \mathcal{R}$ is a convex polygon with at most $n + m$ edges.



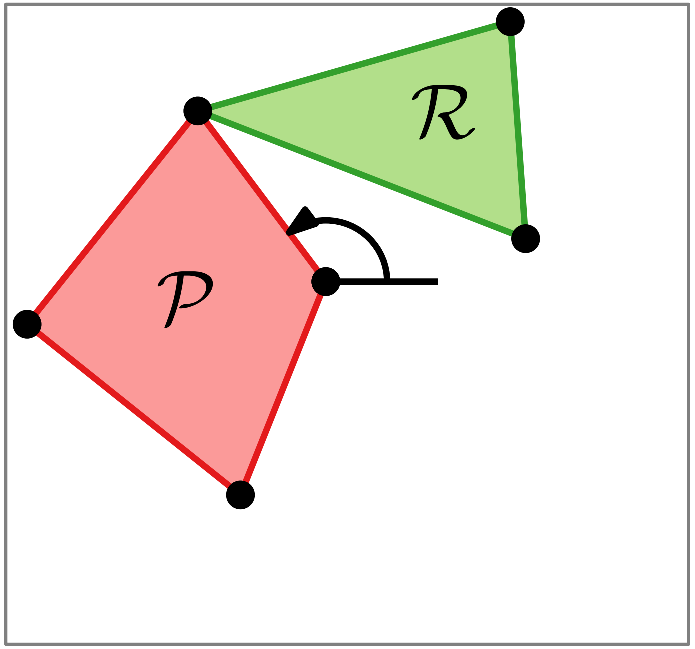
Minkowski Sums: Computation

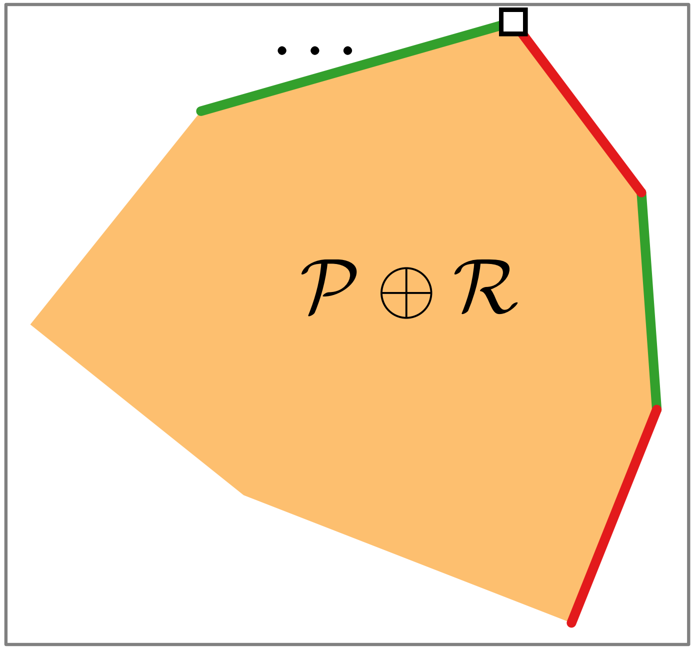
Task. How would you compute $\mathcal{P} \oplus \mathcal{R}$ given $\mathcal{P}$ and $\mathcal{R}$?

Idea. $\mathcal{P} \oplus \mathcal{R} = \mathrm{CH}\big(\underbrace{\{p + r \mid p \in \mathcal{P}, r \in \mathcal{R}\}}\big)$ *(Proof?)*

Problem. complexity $\in \Theta(|\mathcal{P}| \cdot |\mathcal{R}|)$:-(

Theorem. The Minkowski sum of two convex polygons $\mathcal{P}$ and $\mathcal{R}$ can be computed in $O(|\mathcal{P}| + |\mathcal{R}|)$ time.





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Part V: Pseudodisks

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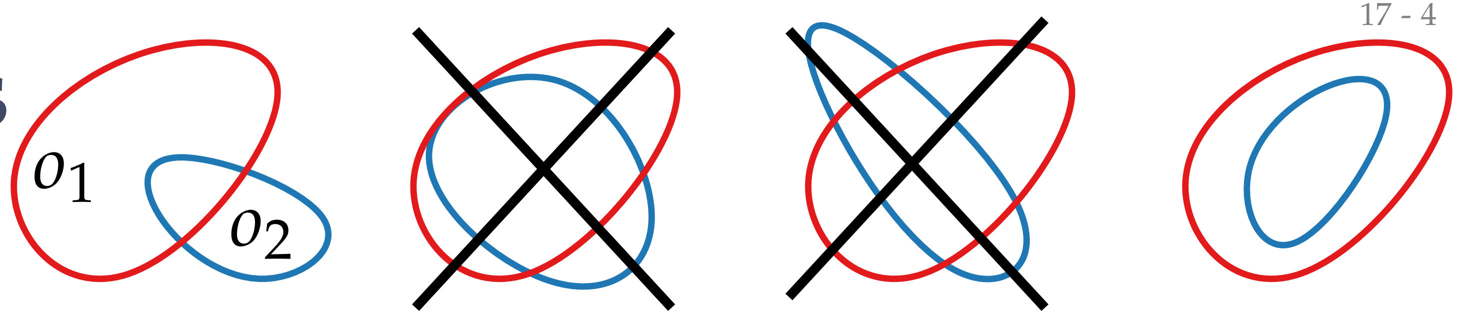
Pseudodisks



Definition. A pair of planar objects o_1 and o_2 is a pair of pseudodisks if:

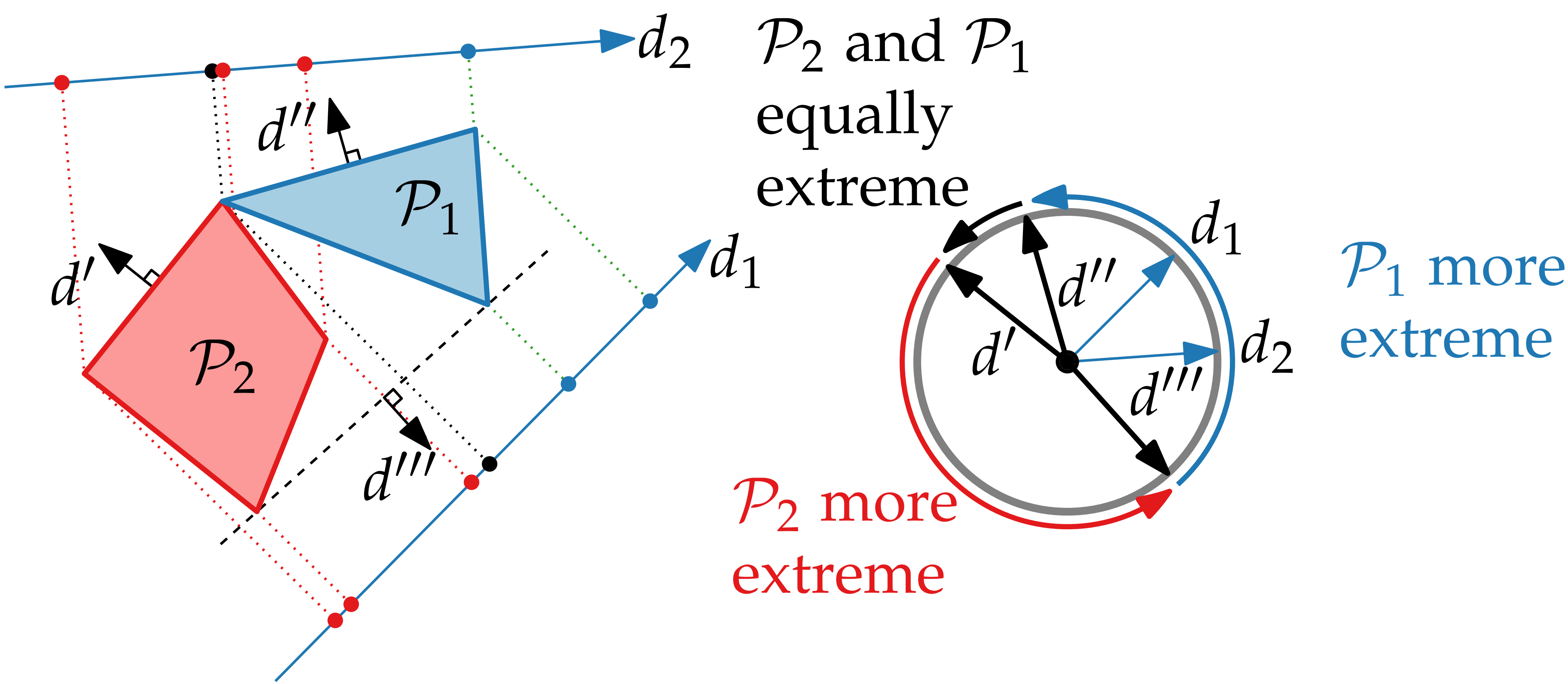
- $\partial o_1 \cap \text{int}(o_2)$ is connected, and
- $\partial o_2 \cap \text{int}(o_1)$ is connected.

$p \in \partial o_1 \cap \partial o_2$ is a *boundary crossing* if ∂o_1 crosses at p from the interior to the exterior of o_2.

Observation. A pair of polygonal pseudodisks defines at most two boundary crossings.

Extreme Directions

Observation. Let $\mathcal{P}_1$, $\mathcal{P}_2$ be interior-disjoint convex polygons. Let d_1 and d_2 be directions in which $\mathcal{P}_1$ is more extreme than $\mathcal{P}_2$. Then $\mathcal{P}_1$ is more extreme than $\mathcal{P}_2$ either in $[d_1, d_2]$ or in $[d_2, d_1]$.

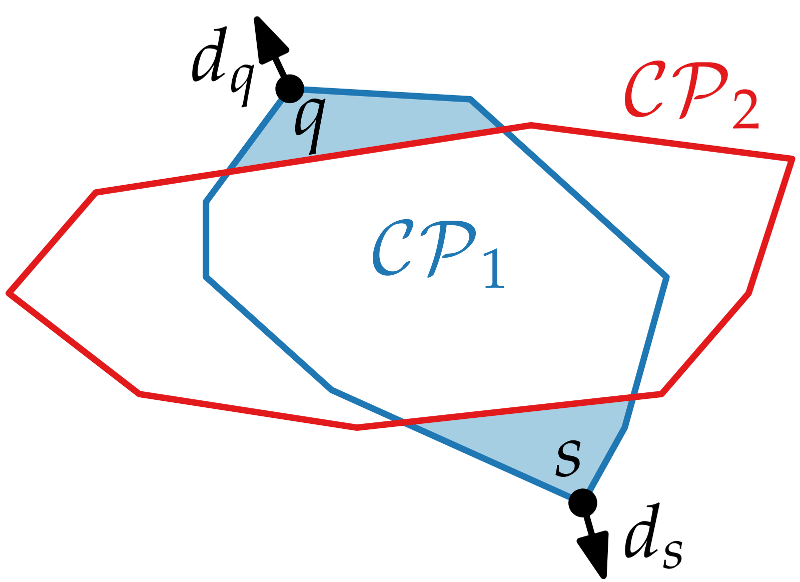


Polygonal Pseudodisks

Theorem. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex polygons with disjoint interiors, and $\mathcal{R}$ is another convex polygon, then $\underbrace{\mathcal{P}_1 \oplus \mathcal{R}}_{\mathcal{CP}_1}$ and $\underbrace{\mathcal{P}_2 \oplus \mathcal{R}}_{\mathcal{CP}_2}$ is a pair of pseudodisks.

Proof. It suffices to show: $\mathcal{CP}_1 \setminus \mathcal{CP}_2$ is connected.

Suppose $\mathcal{CP}_1 \setminus \mathcal{CP}_2$ is not connected...



↯ to previous observation!

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Part VI:
Union Complexity

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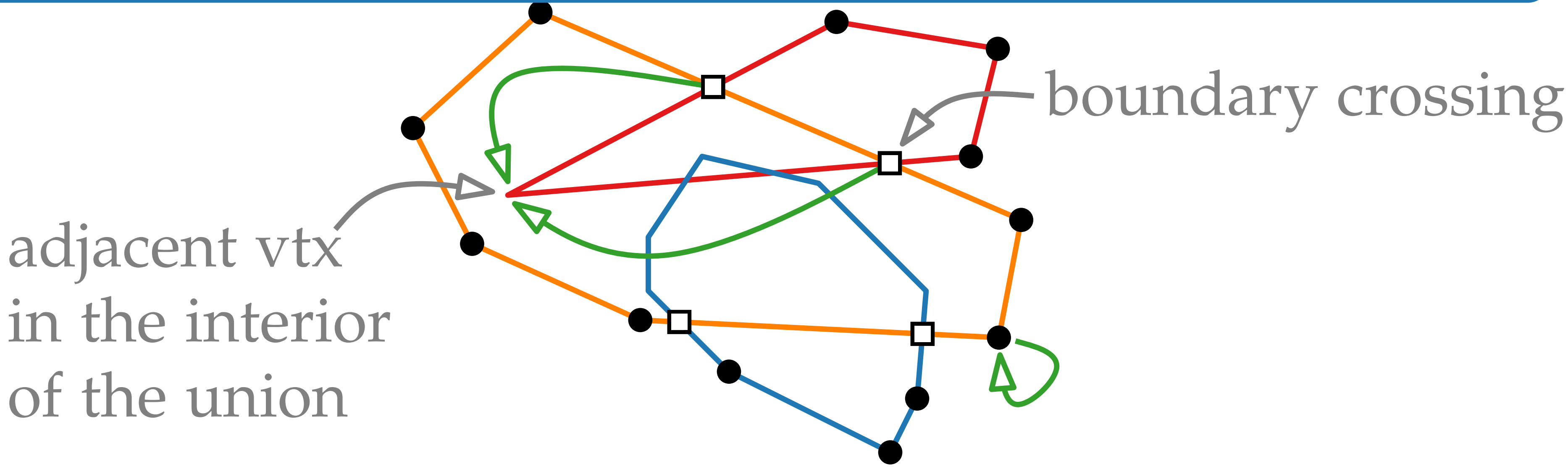
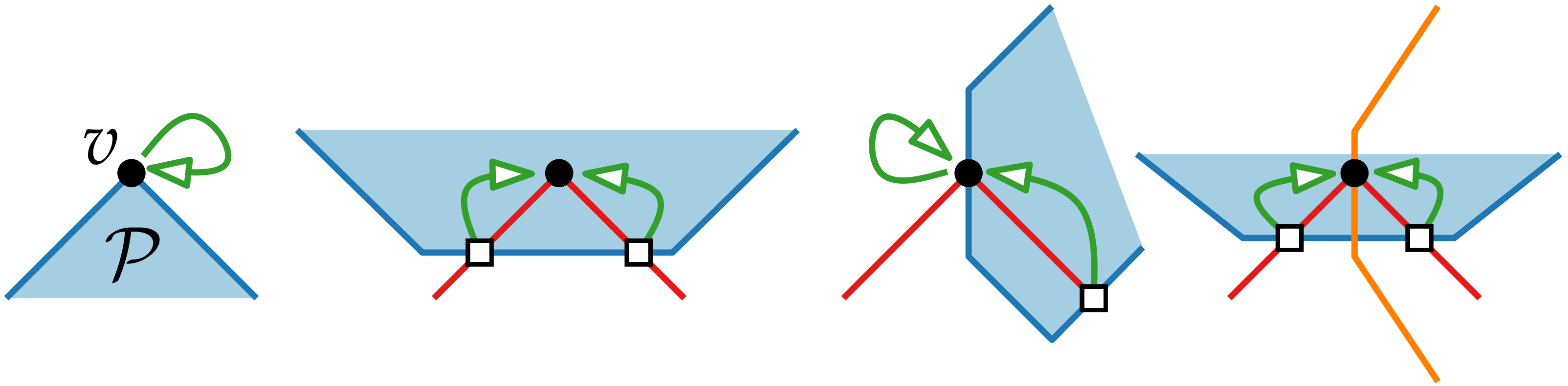
Union Complexity

Theorem. A collection S of convex polygonal pseudodisks with n vtc in total has a union with $\leq 2n$ vtc.



Proof. Charge every vtx of the union to a polygon vtx s.t. every polygon vtx is charged at most twice.



Summary and Main Result

Theorem. Let $\mathcal{R}$ be a constant-complexity convex robot, translating among a set S of disjoint polygonal obstacles with n edges in total.
We can preprocess S in $O(n \log^2 n)$ time such that, given any start and goal position, we can compute in $O(n)$ time a collision-free path for $\mathcal{R}$ if it exists.

Proof.

- $O(n \log n)$ Triangulate the obstacles if not convex. Lec. 3
- $O(n)$ Compute $\mathcal{CP}_i$ for every convex obstacle $\mathcal{P}_i$.
- $O(n \log^2 n)$ Compute their union $\mathcal{C}_{\text{forb}} = \bigcup_i \mathcal{CP}_i$ using div. and conq. (merge by sweeping – Lec. 2)

  [Argue carefully about the number of intersection pts!]
- $O(n)$ Find a path for a point in the complement $\mathcal{C}_{\text{free}}$.