Computational Geometry

Lecture 9:
Convex Hulls in 3D
or
Mixing More Things

Part I:
Complexity & Visibility

Philipp Kindermann

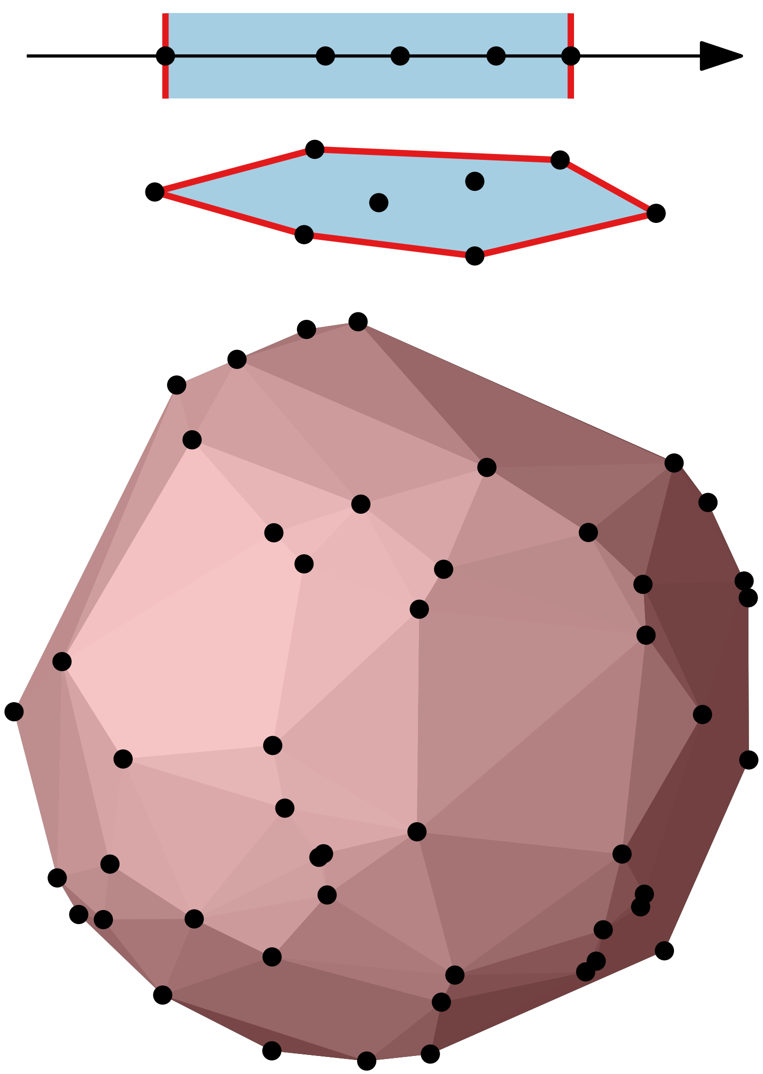
Winter Semester 2020

Complexity of the Convex Hull

Given set S of n points in $\mathbb{R}^d$, what is max. #edges on $\partial\mathrm{CH}(S)$?

dim	w-c complexity of $\mathrm{CH}(S)$
1	$2 \in \Theta(1)$
2	$n \in \Theta(n)$
3	$3n - 6 \in \Theta(n)$
d	$\Theta(n^{\lfloor d/2 \rfloor})$

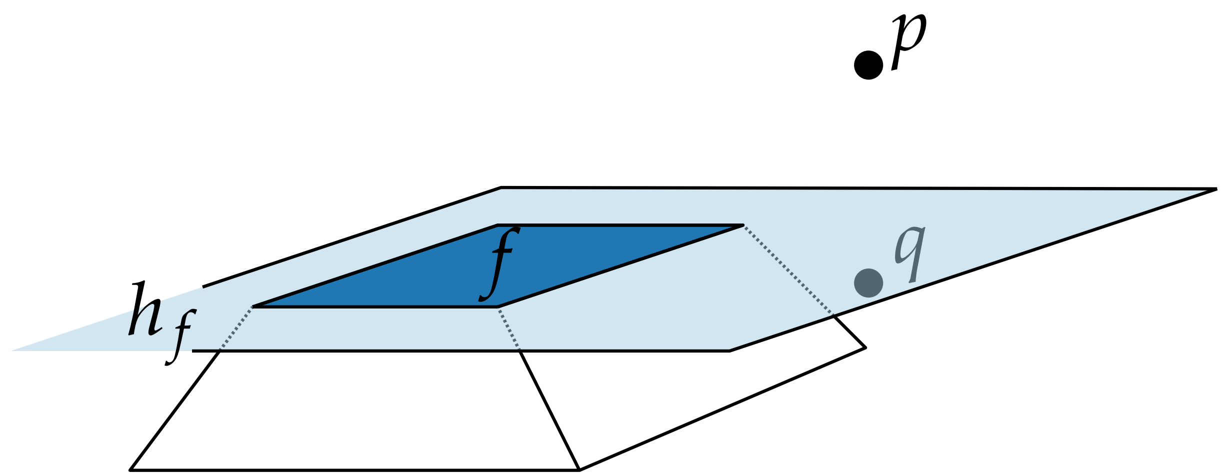
Upper Bound Theorem



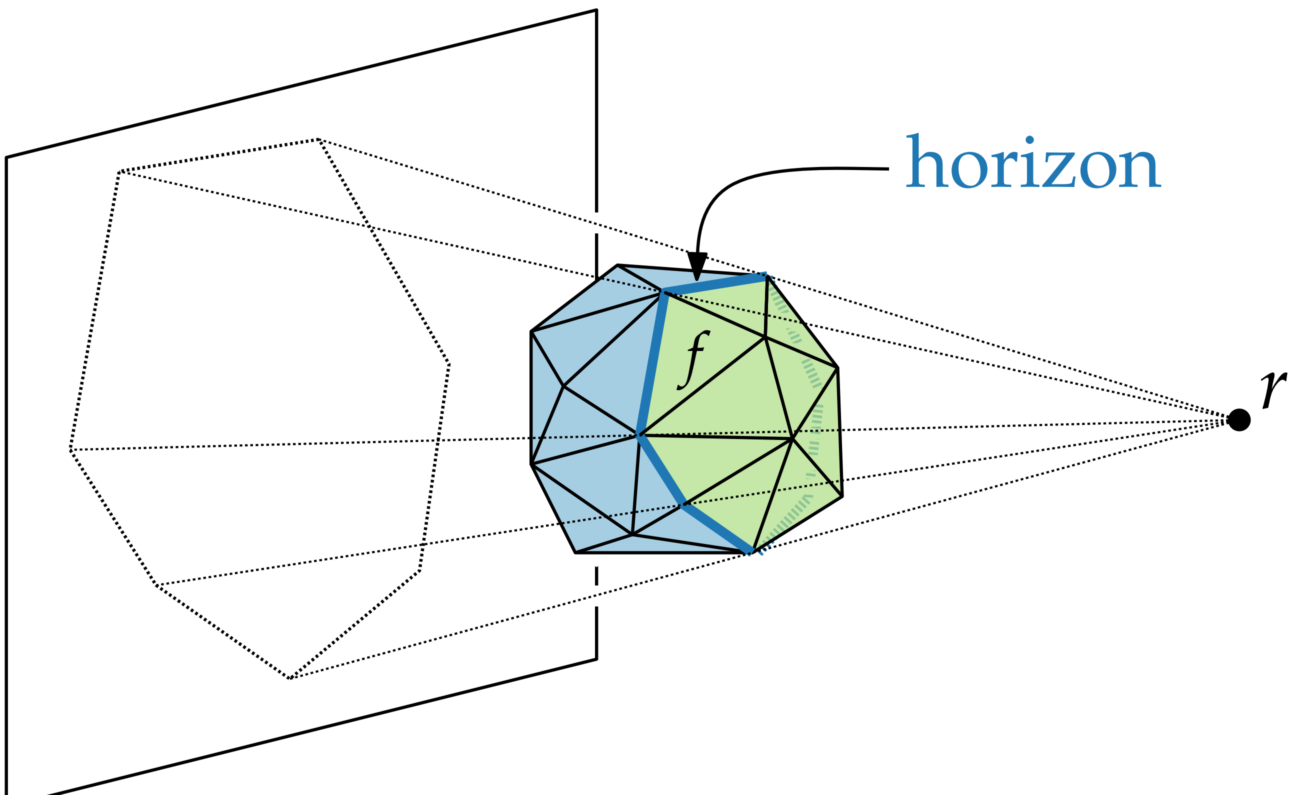
Construction

randomized-incremental!

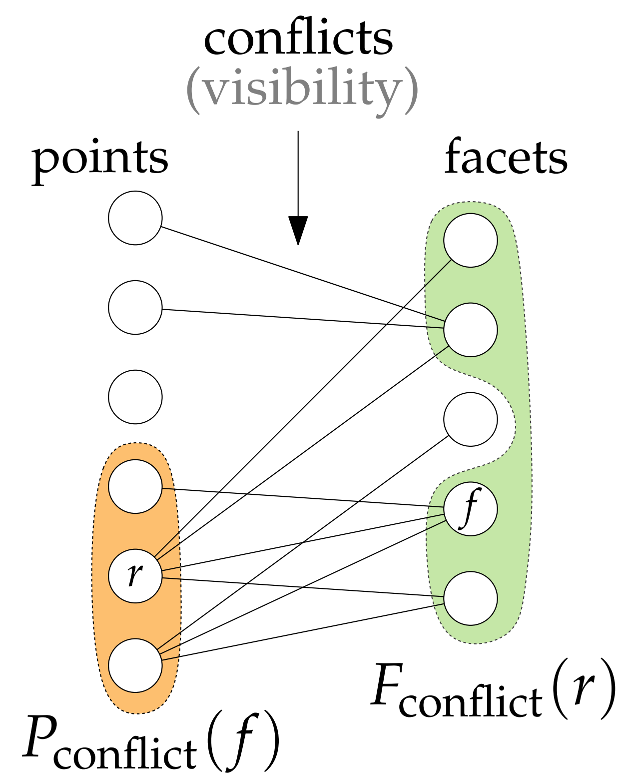
Visibility



Face f is *visible* from p but not from q.



Define conflict graph G:



Computational Geometry

Lecture 9:
Convex Hulls in 3D
or
Mixing More Things

Part II:
Randomized Incremental Algorithm

Philipp Kindermann

Winter Semester 2020

Rand3DConvexHull($P \subset \mathbb{R}^3$)

pick non-coplanar set $P' = \{p_1, \ldots, p_4\} \subseteq P$

$C \leftarrow \mathrm{CH}(P')$

compute rand. perm. $(p_5, \ldots, p_n)$ of $P \setminus P'$

initialize conflict graph G

for $r = 5$ **to** n **do**

  if $F_{\text{conflict}}(p_r) \neq \emptyset$ **then** $\{\ p_r \notin C\ \}$

    delete all facets in $F_{\text{conflict}}(p_r)$ from C

    $\mathcal{L} \leftarrow$ list of horizon edges visible from p_r

    foreach $e \in \mathcal{L}$ **do**

      $f \leftarrow C$.create_facet(e, p_r); create vtx for f in G

      $(f_1, f_2) \leftarrow$ previously_incident$_C(e)$

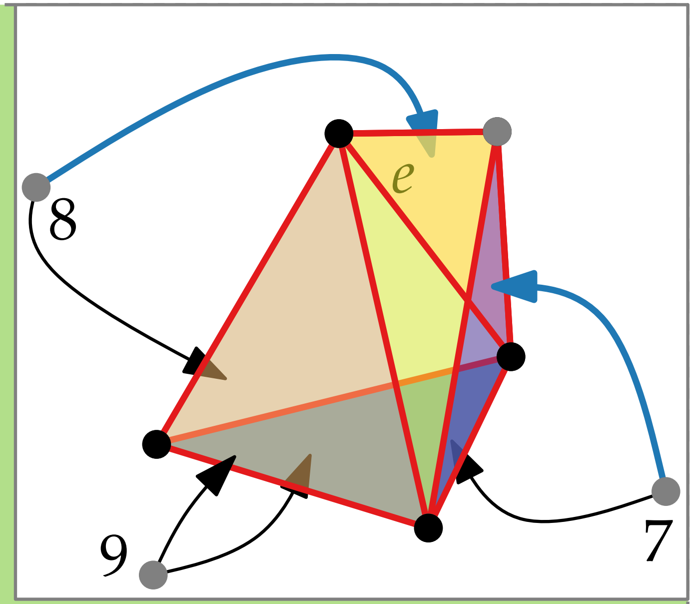
      $P(e) \leftarrow P_{\text{conflict}}(f_1) \cup P_{\text{conflict}}(f_2)$

      foreach $p \in P(e)$ **do**

        if f is visible from p **then** add edge (p, f) to G

    delete vtc $\{p_r\} \cup F_{\text{conflict}}(p_r)$ from G

return C



Worst-case running time: $O(n^3)$

Computational Geometry

Lecture 9:
Convex Hulls in 3D
or
Mixing More Things

Part III:
Analysis

Philipp Kindermann

Winter Semester 2020

Analysis

Idea. Bound expected *structural change*, that is, the total #facets created by the algorithm.

Lemma. The expected #facets created is at most $6n - 20$.

Proof. $E[\#\text{facets created}] =$

$$= 4 + \sum_{r=5}^{n} E\underbrace{[\#\text{facets incident to } p_r \text{ in } \mathrm{CH}(P_r)]}_{\deg(p_r, \mathrm{CH}(P_r))} \leq \begin{matrix} 6n \\ -20 \end{matrix}$$

#edges

For $r > 4$:

$$E[\deg(p_r, \mathrm{CH}(P_r))] = \frac{1}{r-4} \sum_{i=5}^{r} \deg(p_i, \mathrm{CH}(P_r))$$

$$\leq \frac{1}{r-4} \Big[\underbrace{\Big(\sum_{i=1}^{r} \deg(p_i)\Big)}_{2 \,\cdot\, \#\text{ edges of } \mathrm{CH}(P_r)} - 12\Big]$$

$$\leq \frac{1}{r-4} \big[2 \cdot (3r - 6) - 12\big] \leq 6$$

Running Time

$\mathbb{R}^d, d > 3$

Theorem. The convex hull of a set of n pts in $\mathbb{R}^3$ can be computed in $O(n \log n)$ expected time.

$O(n^{\lfloor d/2 \rfloor})$

$O(n)$ time

```
Rand3DConvexHull(P ⊂ ℝ³)
  pick non-coplanar set P' = {p_1, ..., p_4} ⊆ P
  C ← CH(P')
  compute rand. perm. (p_5, ..., p_n) of P \ P'
  initialize conflict graph G
  for r = 5 to n do
      if F_conflict(p_r) ≠ ∅ then
          delete all facets in F_conflict(p_r) from C
          ℒ ← list of horizon edges visible from p_r
          foreach e ∈ ℒ do
              f ← C.create_facet(e, p_r); create vtx for f in G
              (f_1, f_2) ← previously_incident_C(e)
              P(e) ← P_conflict(f_1) ∪ P_conflict(f_2)
              foreach p ∈ P(e) do
                  if f visible from p then add edge (p, f) to G
          delete vtc {p_r} ∪ F_conflict(p_r) from G
  return C
```

using *configuration spaces*, Section 9.5 [Comp. Geom A&A]

Stage r of for-loop (w/o foreach loop) takes time $O(|F_{\text{conflict}}(p_r)|) =$ $O(\#\text{facets del. when adding } p_r)$

This part of for-loop in total: $E[\#\text{facets deleted}] =$
$\leq E[\#\text{facets created}] = O(n)$.

Lemma

Outer foreach-loop:
– in stage r: $O(\sum_{e \in \mathcal{L}} |P(e)|)$
– in total:

$$O\left(\sum_{e \text{ on horizon at some time}} |P(e)|\right)$$

$$= \cancel{O(n^2)}\ O(n \log n)$$

Computational Geometry

Lecture 9:
Convex Hulls in 3D
or
Mixing More Things

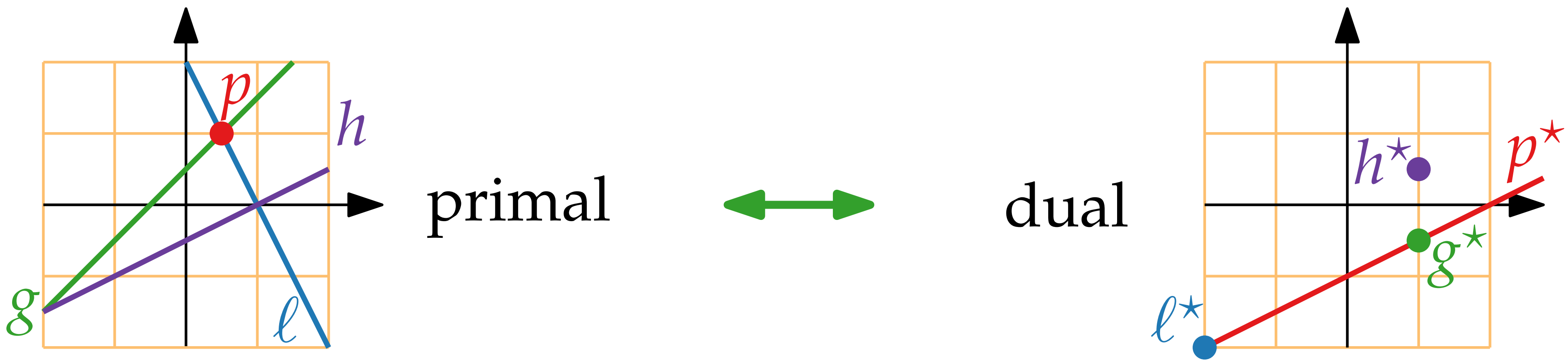
Part IV:
Half-Space Intersections

Philipp Kindermann Winter Semester 2020

Convex Hulls and Half-~~Space~~ Plane Intersections

Define dualtity $\star$ between pts and (non-vertical) lines:

For $p = (p_x, p_y)$, define the line $p^\star \colon y = p_x x - p_y$.



For $\ell \colon y = mx + b$, define $\ell^\star$ to be the pt q with $q^\star = \ell$,

that is, $\ell^\star = (m, -b)$.

Observe. Let $p \in \mathbb{R}^2$ and let ℓ be a non-vertical line.

$\star$ is incidence-preserving: $p \in \ell \;\Leftrightarrow\; \ell^\star \in p^\star$

$\star$ is order-preserving: p above $\ell \;\Leftrightarrow\; \ell^\star$ above $p^\star$

Computational Geometry

Lecture 9:
Convex Hulls in 3D
or
Mixing More Things

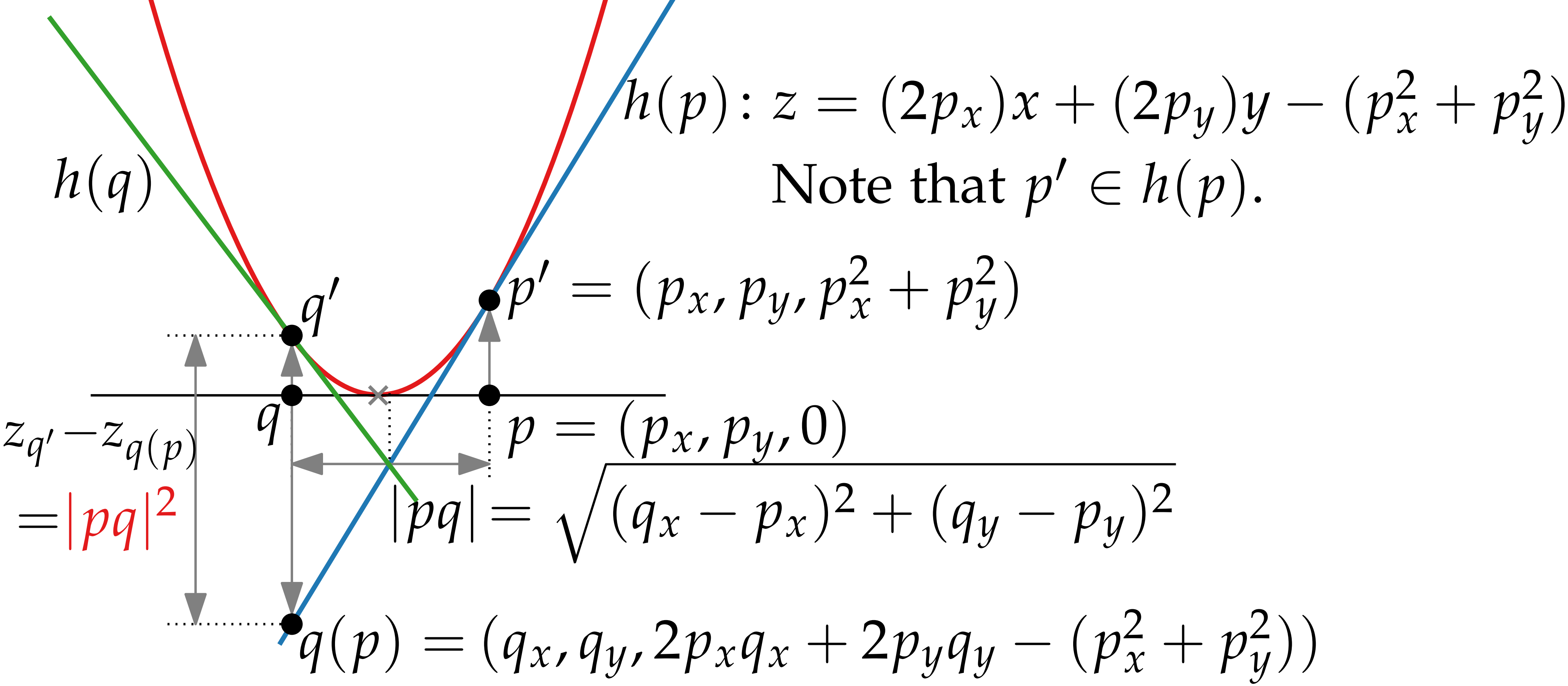
Part V:
Voronoi Diagrams Revisited

Philipp Kindermann

Winter Semester 2020

Voronoi Diagrams Revisited

Let U: $z = x^2 + y^2$ be the *unit paraboloid* in $\mathbb{R}^3$.

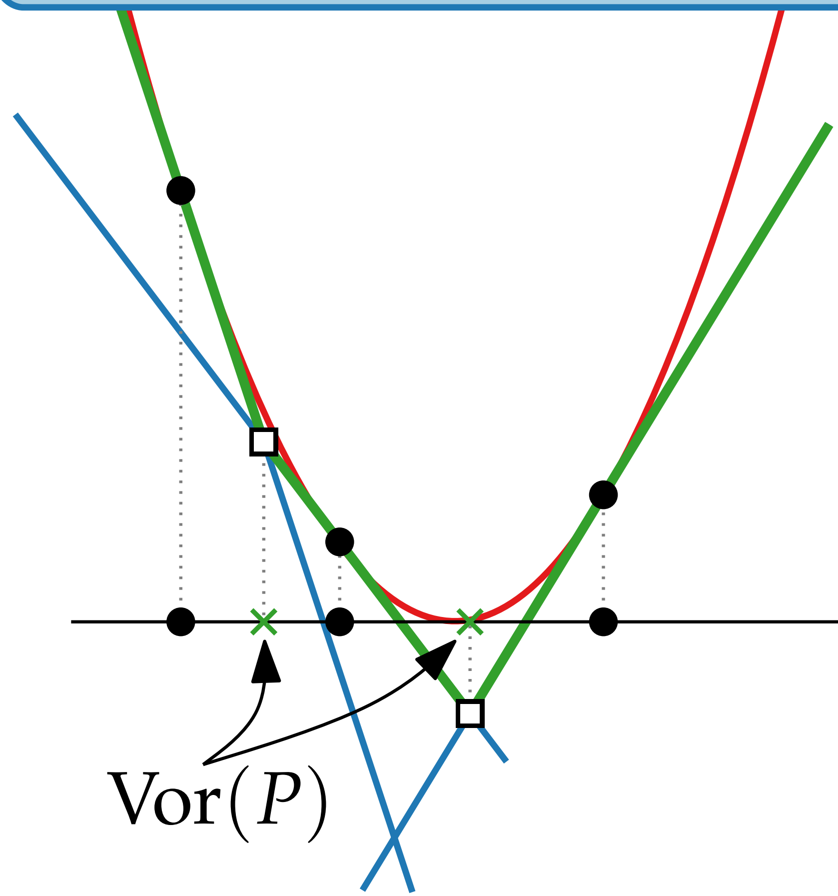


$\Rightarrow$ $h(p)$ and U encode dist. betw. p and any other pt in $z{=}0$.

$\Rightarrow$ $h(p) \cap U = \{p'\}$ $\Rightarrow$ $h(p)$ is tangent to U (in p')

The Upper Envelope Strikes Back

Theorem. Let $P \subset \mathbb{R}^2 \times \{0\}$ and $\mathcal{H} = \{h(p) \mid p \in P\}$.
Let $\mathcal{E}(\mathcal{H})$ be the upper envelope of $\mathcal{H}$.
The projection of $\mathcal{E}(\mathcal{H})$ on $z = 0$ is $\mathrm{Vor}(P)$.



can compute $\mathrm{Vor}(P)$ in $\mathbb{R}^2$
via upper envelope in $\mathbb{R}^3$

upper envelope in $\mathbb{R}^3$ is in
one-to-one correspondence to
lower convex hull of pt set $\mathcal{H}^\star$

use algorithm Rand3DConvexHull!