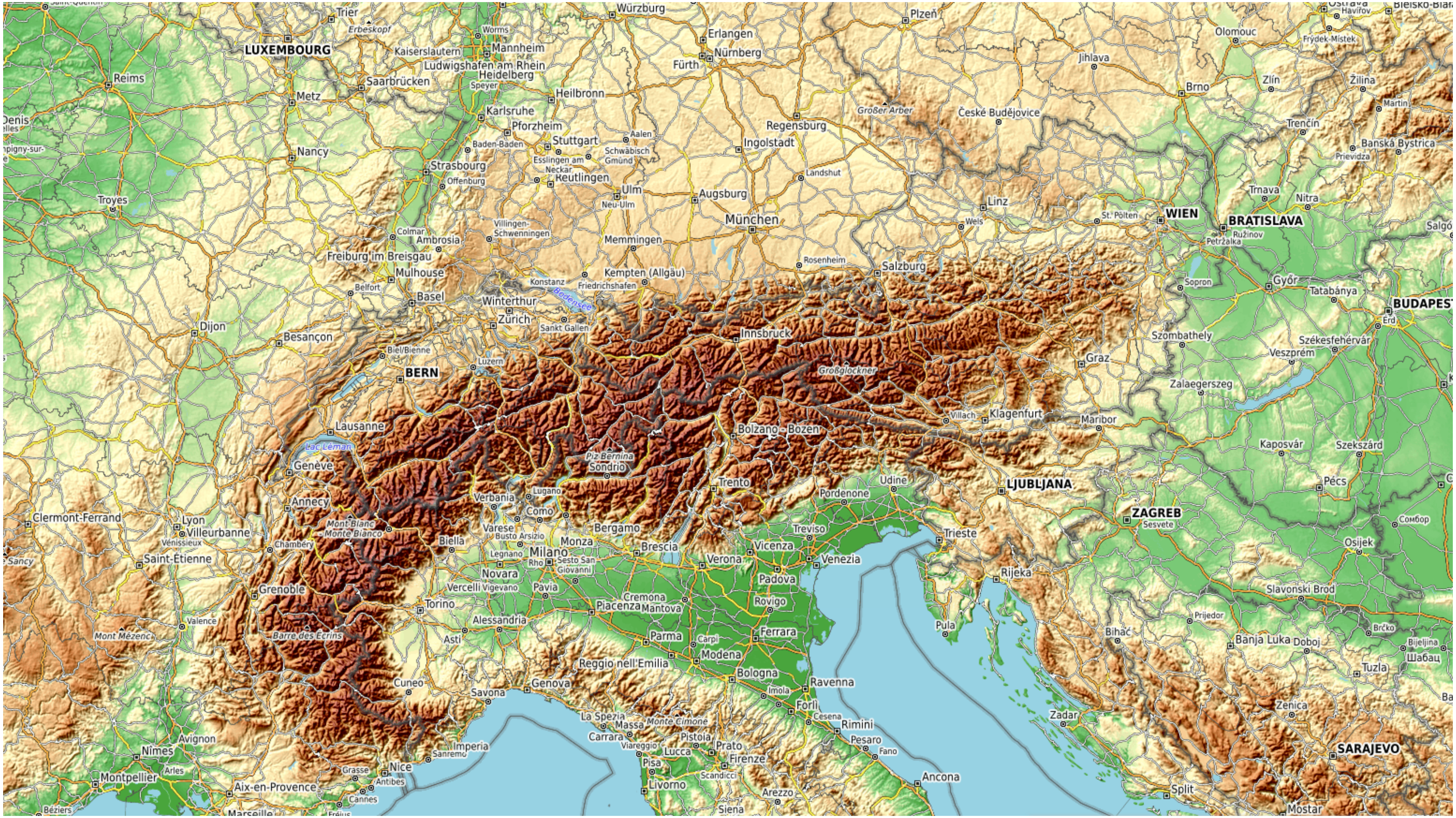
Computational Geometry

Lecture 8:
Delaunay Triangulations
or
Height Interpolation

Part I:
Height Interpolation

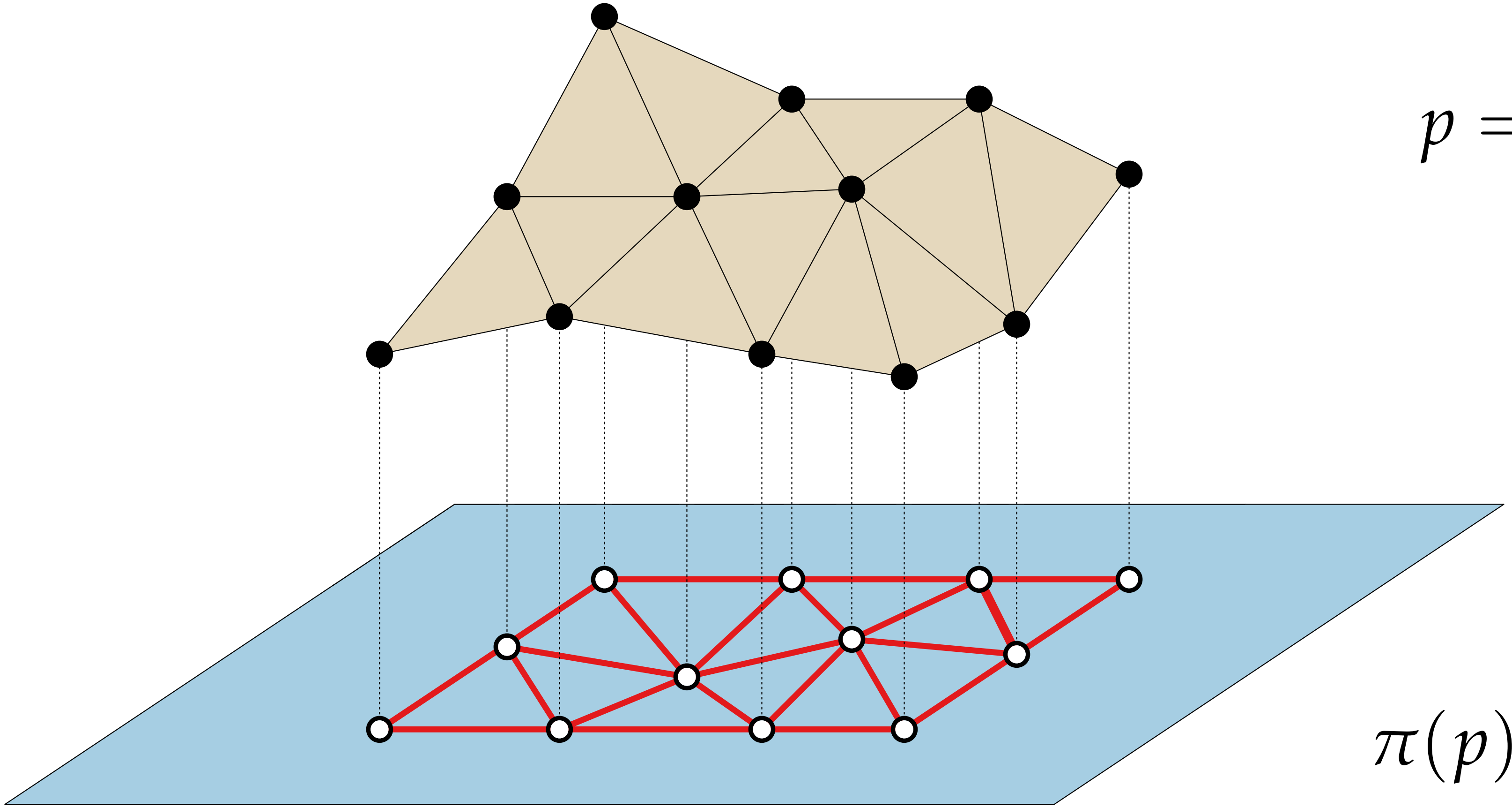
Philipp Kindermann

Summer Semester 2020



[opentopomap.org]

Height Interpolation

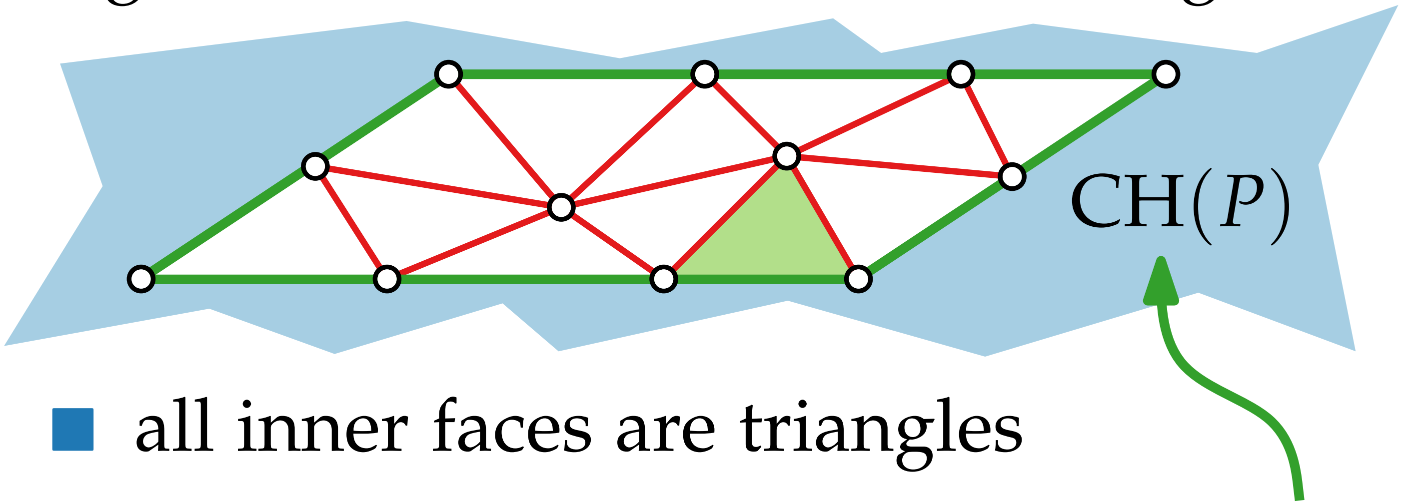


$$p = (x_p, y_p, z_p)$$

$$\pi(p) = (x_p, y_p, 0)$$

Triangulation of Planar Point Sets

Definition. Given $P \subset \mathbb{R}^2$, a *triangulation* of P is a maximal planar subdivision with vtx set P, that is, no edge can be added without crossing.



Observe.

- all inner faces are triangles
- outer face is complement of a convex polygon

Theorem. Let $P \subset \mathbb{R}^2$ be a set of n sites, not all collinear, and let h be the number of sites on $\partial\mathrm{CH}(P)$.

Then *any* triangulation of P has $2n - 2 - h$ triangles and $3n - 3 - h$ edges.

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Part II:
Angle-Optimal Triangulation

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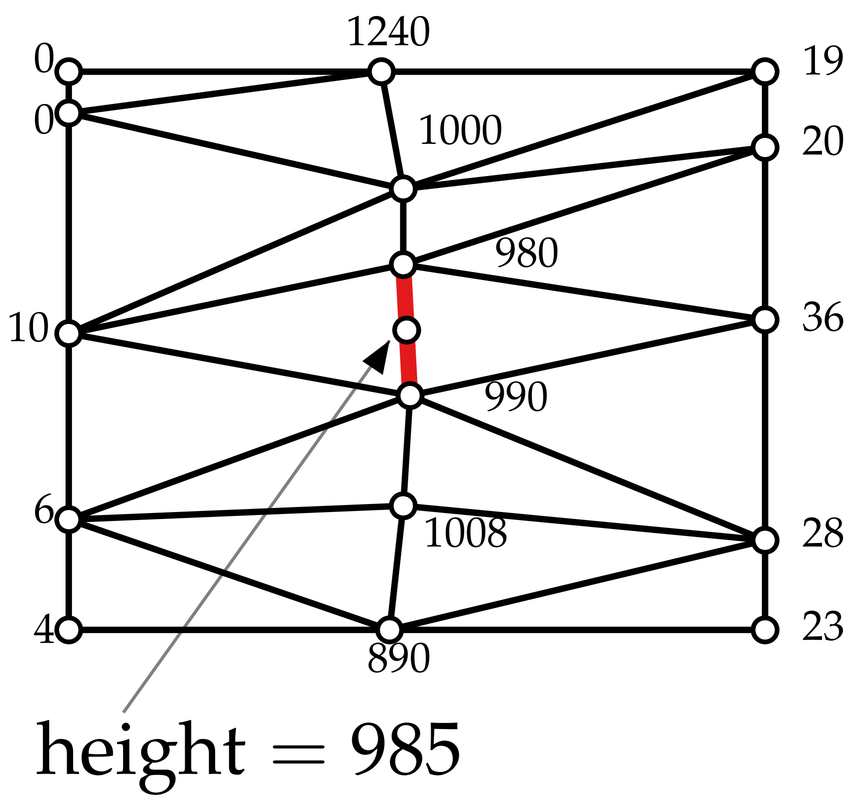
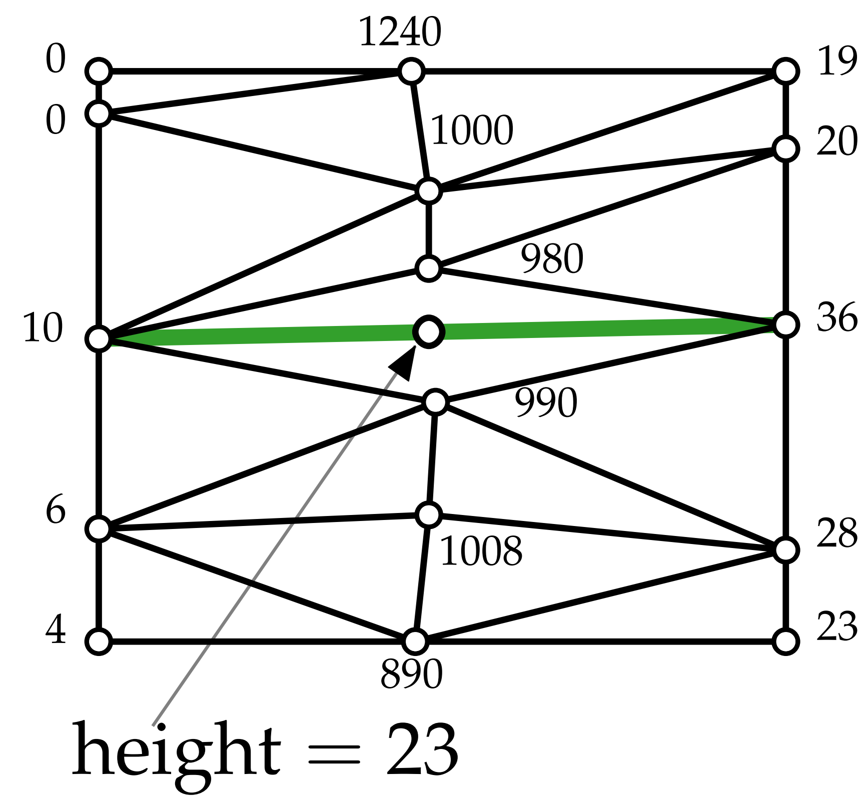
Back to Height Interpolation





Intuition. Avoid “skinny” triangles!

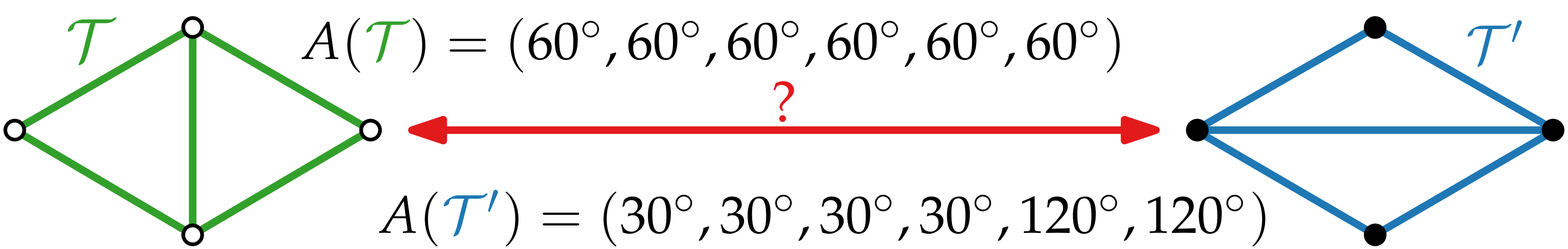
In other words: avoid small angles!

Angle-Optimal Triangulations

Definition. Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of P, let m be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the *angle vector* of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$
if $\exists i \in \{1, ..., 3m\} : \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.

$\mathcal{T}$ is *angle-optimal* if
$A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.



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Part III:
Edge Flips & Legal Triangulations

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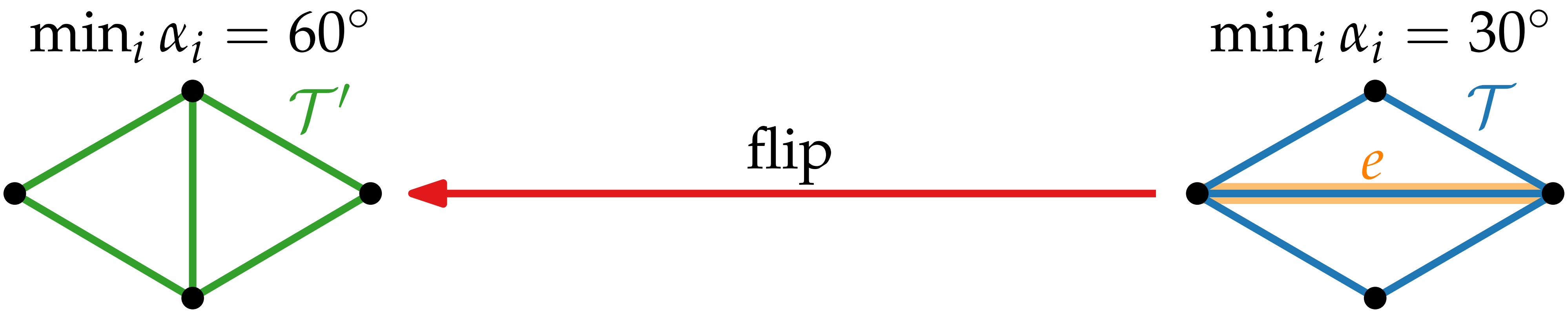
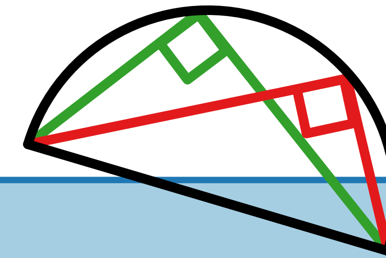
Edge Flips

Definition. Let $\mathcal{T}$ be a triangulation. An edge e of $\mathcal{T}$ is *illegal* if the minimum angle in the two triangles adjacent to e increases when flipping.

Observe. Let e be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.



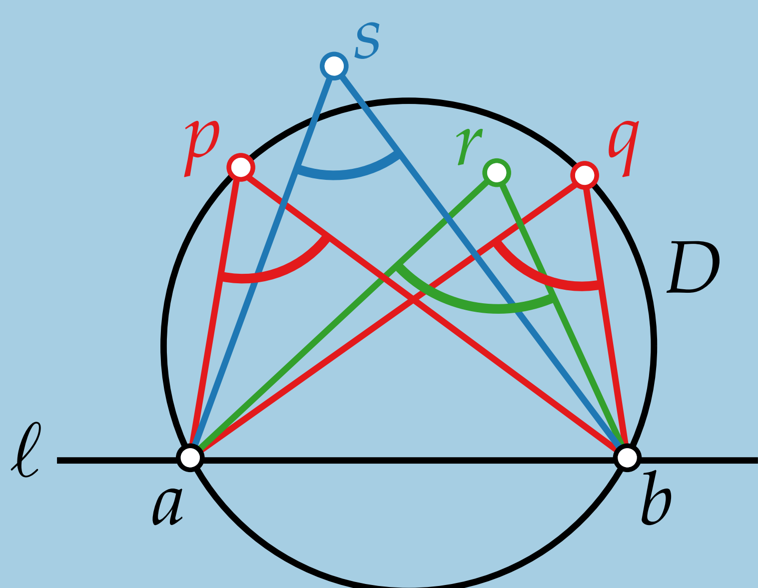
This is all Greek to me...



Theorem. [Thales]

The diameter of a circle always subtends a right angle to any point on the circle.

Theorem: [Thales++]



$$\{a, b\} := \ell \cap \partial D \ \ (a \neq b)$$

$$p, q \in \partial D$$

$$r \in \operatorname{int}(D)$$

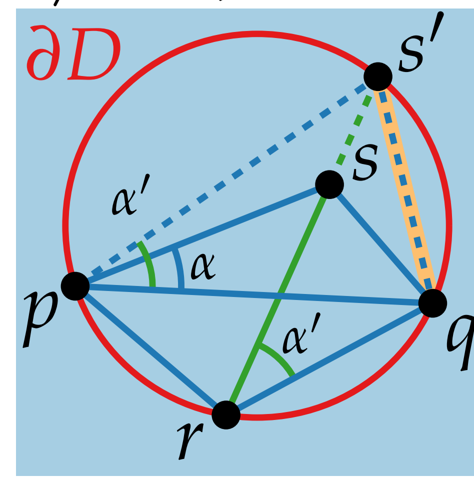
$$s \notin D$$

$$\angle asb < \angle apb = \angle aqb < \angle arb$$

Legal Triangulations

Lemma. Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$.
Then edge pq is illegal iff $s \in \mathrm{int}(D)$.

If p, q, r, s in convex position and $s \notin \partial D$, then either pq or rs is illegal.



Proof. Show: $\forall \alpha'$ in $\mathcal{T}'$ $\exists \alpha$ in $\mathcal{T}$ s.t. $\alpha < \alpha'$.
(“$\Rightarrow$”) Use Thales++ w.r.t. qs'. □

Note. If $s \in \partial D$, both pq *and* rs legal.

Definition. A triangulation is *legal* if it has no illegal edge.

Existence? Algorithm: Let $\mathcal{T}$ be any triangulation of P.
While $\mathcal{T}$ has an illegal edge e, flip e. Return $\mathcal{T}$.

algorithm terminates

$A(\mathcal{T})$ goes up! & #(triangulations of P) $< \infty$

Legal vs. Angle-Optimal

Clearly… Every angle-optimal triangulation is legal.

But is every legal triangulation angle-optimal??

Let's see.

To clarify things, we'll introduce yet another type of triangulation…

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Part IV:
Delaunay Triangulation

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Voronoi & Delaunay

Recall: Given a set P of n points in the plane...

$\mathrm{Vor}(P)$ = subdivision of the plane into Voronoi cells, edges, and vertices

$\mathcal{V}(p) = \{x \in \mathbb{R}^2 \colon |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$

Voronoi cell of $p \in P$

Definition: The graph $\mathcal{G} = (P, E)$ with

$\{p, q\} \in E \;\Leftrightarrow\; \mathcal{V}(p)$ and $\mathcal{V}(q)$ share an edge

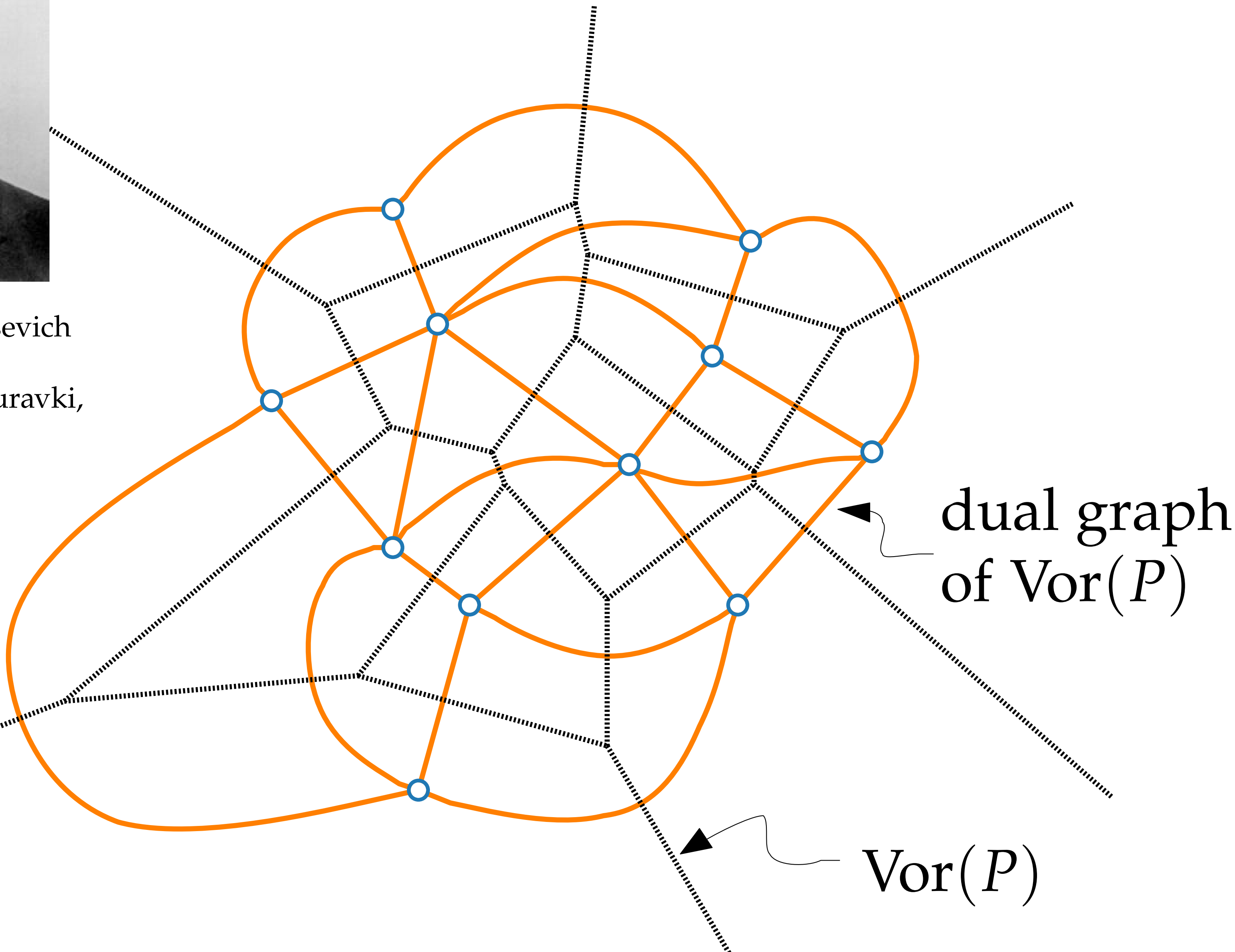
is the *dual graph* of $\mathrm{Vor}(P)$

Definition: The *Delaunay graph* $\mathcal{DG}(P)$ is the straight-line drawing of $\mathcal{G}$.

From Voronoi to Delaunay

$P \subset \mathbb{R}^2$

Georgy Feodosevich Voronoy (1868–1908 Zhuravki, now Ukraine)



From Voronoi to Delaunay

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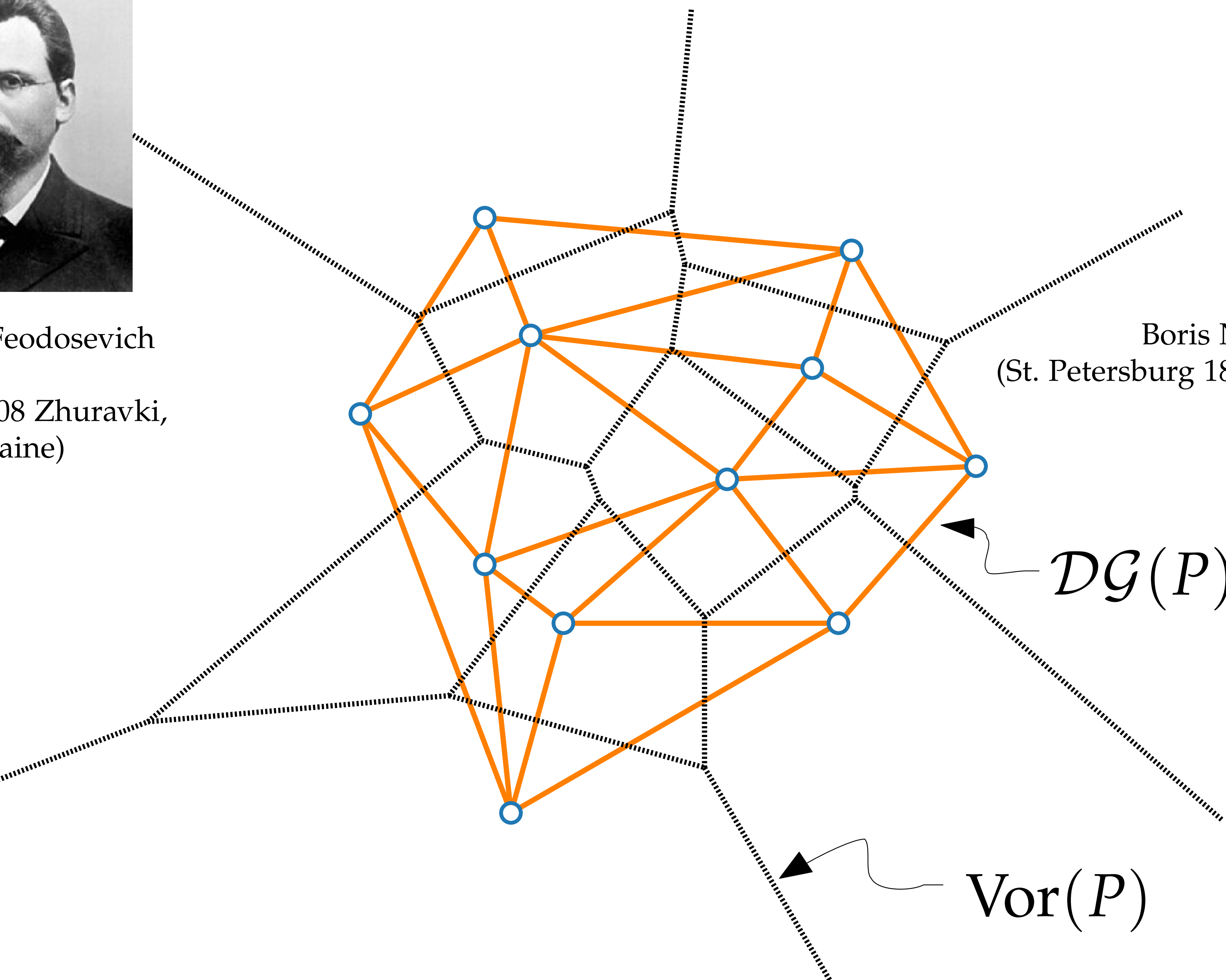
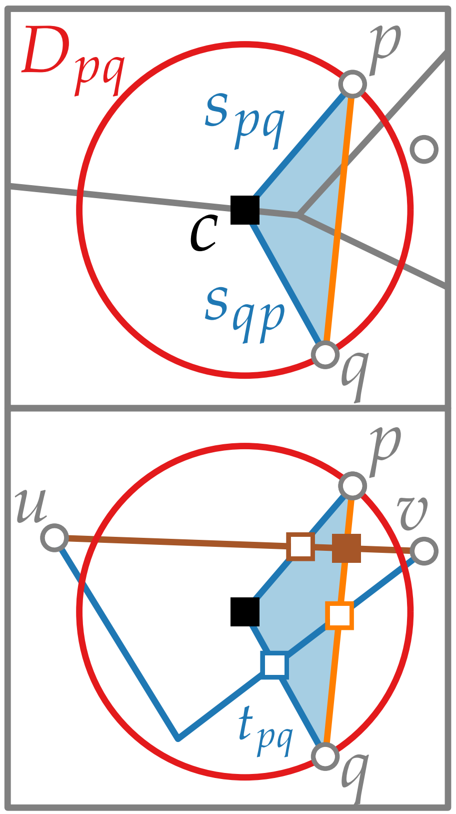


Boris Nikolaevich Delone (St. Petersburg 1890–1980 Moscow)

Planarity

Lemma. $P \subset \mathbb{R}^2$ finite $\Rightarrow$ $\mathcal{DG}(P)$ plane.

Proof. Recall property of Voronoi edges:



Edge pq is in $\mathcal{DG}(P)$ $\Leftrightarrow \exists D_{pq}$ closed disk s.t.

- $p, q \in \partial D_{pq}$ and
- $\{p, q\} = D_{pq} \cap P$.

$c =$ center(D_{pq}) lies on edge betw. $\mathcal{V}(p)$ & $\mathcal{V}(q)$.

Suppose $\exists uv \neq pq$ in $\mathcal{DG}(P)$ that crosses pq.

$u, v \notin D_{pq} \Rightarrow u, v \notin t_{pq} \Rightarrow$
uv crosses another edge of t_{pq}

$p, q \notin D_{uv} \Rightarrow p, q \notin t_{uv} \Rightarrow$
pq crosses another edge of t_{uv}

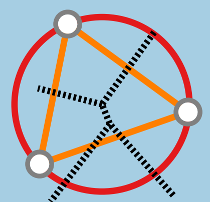
$\Rightarrow$ one of s_{pq} or s_{qp} crosses one of s_{uv} or s_{vu}

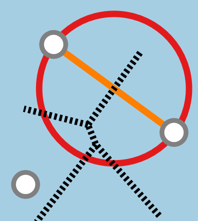
↯ $s_{pq} \subset \mathcal{V}(p)$, $s_{qp} \subset \mathcal{V}(q)$, $s_{uv} \subset \mathcal{V}(u)$, $s_{vu} \subset \mathcal{V}(v)$.

Characterization

Characterization of Voronoi vertices and Voronoi edges $\Rightarrow$

Lemma. $P \subset \mathbb{R}^2$ finite. Then

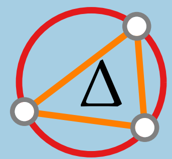




(i) Three pts $p, q, r \in P$ are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow \text{int}(C(p,q,r)) \cap P = \emptyset$

(ii) Two pts $p, q \in P$ form an edge of $\mathcal{DG}(P) \Leftrightarrow$ there is a disk D with
- $\partial D \cap P = \{p, q\}$ and
- $\text{int}(D) \cap P = \emptyset$.

Lemma. $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of P. Then

$C(\Delta)$ 

$\mathcal{T}$ Delaunay $\Leftrightarrow$ for each triangle Δ of $\mathcal{T}$: $\text{int}(C(\Delta)) \cap P = \emptyset$.

("empty-circumcircle property")

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Part V:
Correctness & Computation

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Main Result

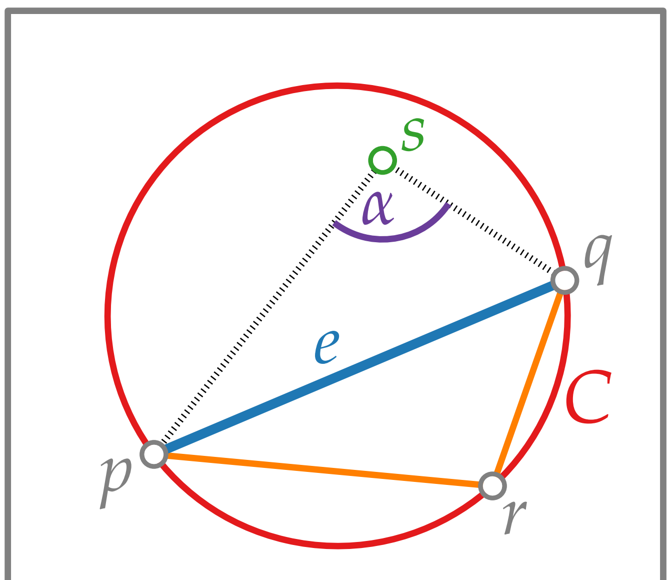
Theorem. $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of P.
Then $\mathcal{T}$ legal $\Leftrightarrow$ $\mathcal{T}$ Delaunay.

Proof. "$\Leftarrow$" implied by empty-circumcircle prop. & Thales++

"$\Rightarrow$" by contradiction:

Assume $\mathcal{T}$ is legal triang. of P, but *not* Delaunay.

$\Rightarrow \exists \Delta pqr$ such that $\mathrm{int}(C(\Delta pqr))$ contains $s \in P$.



Wlog. let $e = pq$ be the edge of Δpqr such that s "sees" pq before the other edges of Δpqr.

Among all such pairs $(\Delta pqr, s)$ in $\mathcal{T}$ choose one that maximizes $\alpha = \angle psq$.

Proof of Main Result (cont'd)

Consider the triangle Δpqt adjacent to e in $\mathcal{T}$.

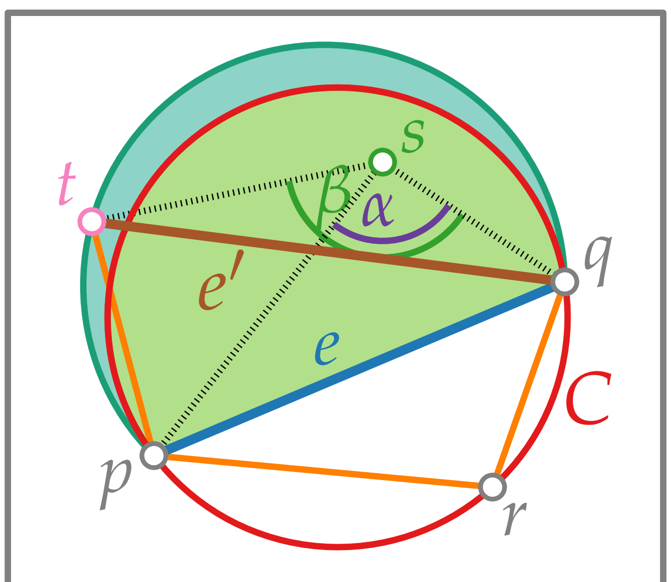
$\mathcal{T}$ legal $\Rightarrow e$ legal $\Rightarrow t \notin \text{int}(C(\Delta pqr))$

$\Rightarrow C(\Delta pqt)$ contains $C(\Delta pqr) \cap e^+$. { halfplane supported by e that contains s

$\Rightarrow s \in C(\Delta pqt)$

Wlog. let $e' = qt$ be the edge of Δpqt that s sees.

$\Rightarrow \beta = \angle tsq \quad > \quad \alpha = \angle psq$



↯ Contradiction to choice of the pair $(\Delta pqr, s)$. □

Main Result

Theorem. $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of P.
Then $\mathcal{T}$ legal $\Leftrightarrow$ $\mathcal{T}$ Delaunay.

no 4 pts on an empty circle!

Observation. Suppose P is in general position.
$\Rightarrow$ Delaunay triangulation unique [$\mathcal{DG}(P)$!]
$\Rightarrow$ legal triangulation unique
$\Downarrow$ angle-optimal $\Rightarrow$ legal [by def.]
Delaunay triangulation is angle-optimal!

Suppose P is *not* in general position. . .
$\Rightarrow$ Delaunay graph has convex "holes" bounded by co-circular pts
$\Downarrow$ Thales++ *homework exercise!*
All Delaunay triang. have same min. angle.

Computation

Theorem. A Delaunay triangulation of an arbitrary set of n pts in the plane can be computed in $O(n \log n)$ time.

[Compute dual of Vor(P), fill holes.]

Corollary. An angle-optimal triangulation of a set of n pts in general position can be computed in $O(n \log n)$ time. [$\mathcal{DG}$!]

Corollary. Given an arbitrary set of n pts, a triangulation maximizing the minimum angle can be computed in $O(n \log n)$ time. [Use Theorem.]

Corollary. An angle-optimal triangulation of an arbitrary set of n pts can be computed in $O(n^2)$ time.

[How?]