Computational Geometry

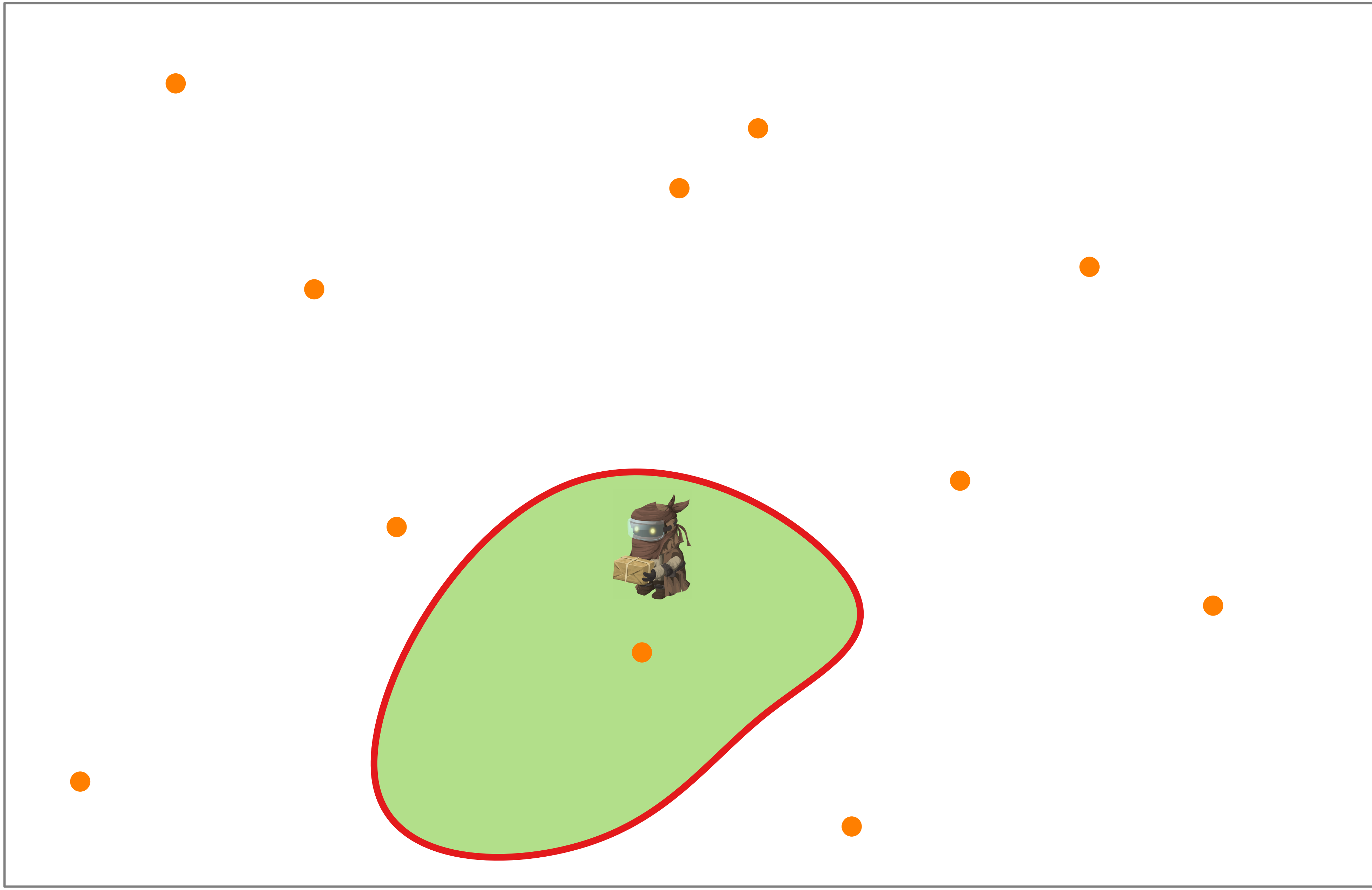
Lecture 7:
Voronoi Diagrams
or
The Post-Office Problem

Part I:
The Post-Office Problem

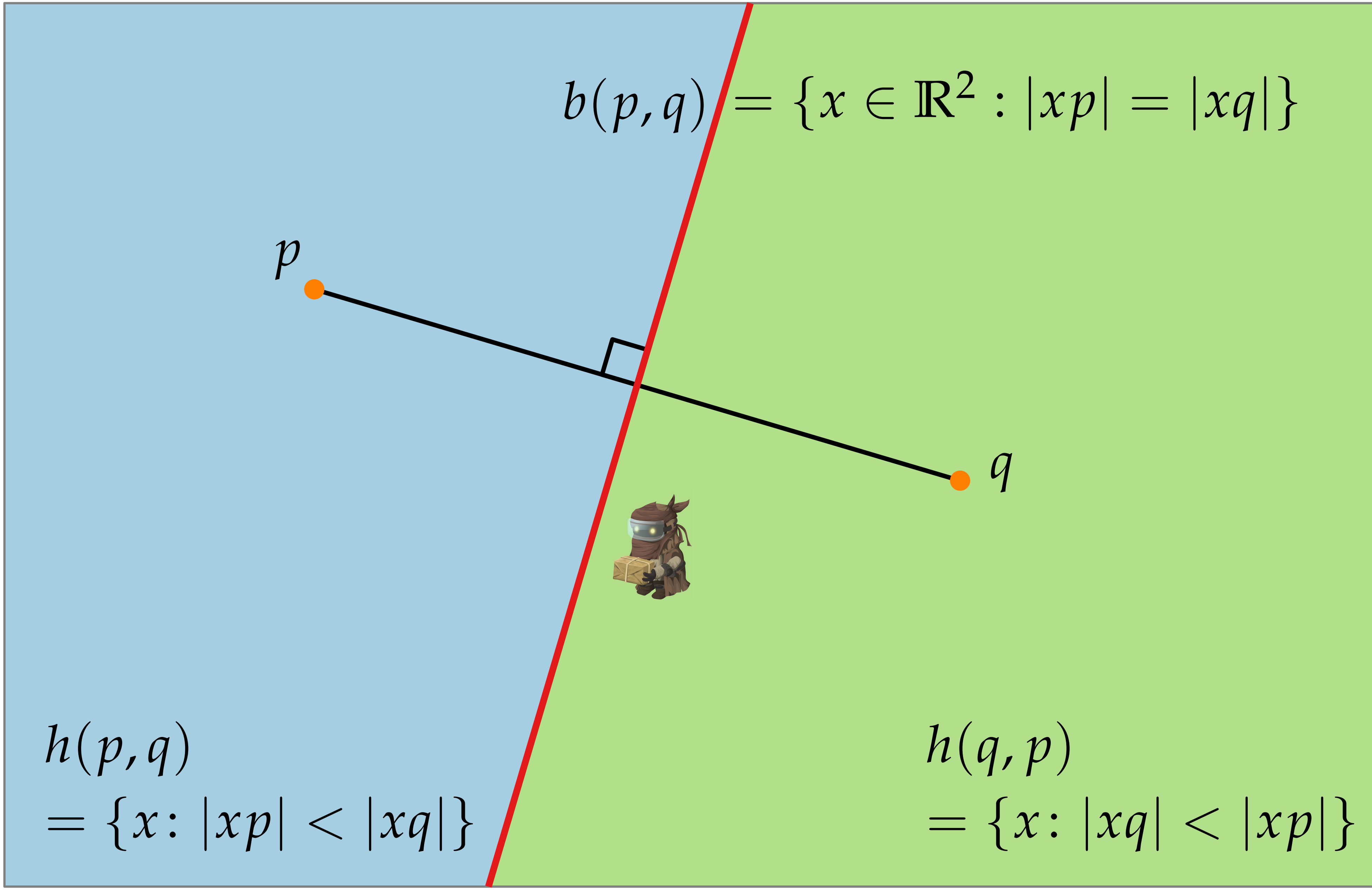
Philipp Kindermann

Winter Semester 2020

The Post-Office Problem



The Post-Office Problem



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Part II:
The Voronoi Diagram

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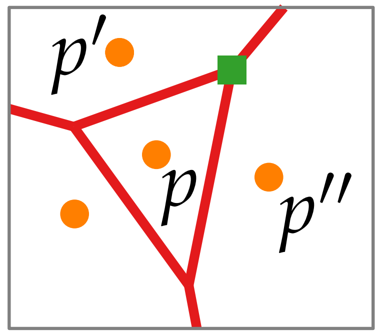
The Voronoi Diagram

Let P be a set of points in the plane and let $p, p', p'' \in P$.

[Voronoi diagram]



$\mathrm{Vor}(P)$ → subdivision of $\mathbb{R}^2$
→ geometric graph

[Voronoi cell]

$$\mathcal{V}(\{p\}) = \mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$$
$$= \textstyle\bigcap_{q \neq p} h(p, q)$$

[Voronoi edge]

$$\mathcal{V}(\{p, p'\}) = \{x \colon |xp| = |xp'| \text{ and } |xp| < |xq| \;\; \forall q \neq p, p'\}$$
$$= \text{rel-int}\big(\partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')\big) \;\; \text{(w/o the endpts)}$$

[Voronoi vertex]

$$\mathcal{V}(\{p, p', p''\}) = \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p') \cap \partial\mathcal{V}(p'')$$
$$= \{x \colon |xp| = |xp'| = |xp''| \text{ and } |xp| \leq |xq| \;\; \forall q\}$$

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Part III:
Shape and Complexity

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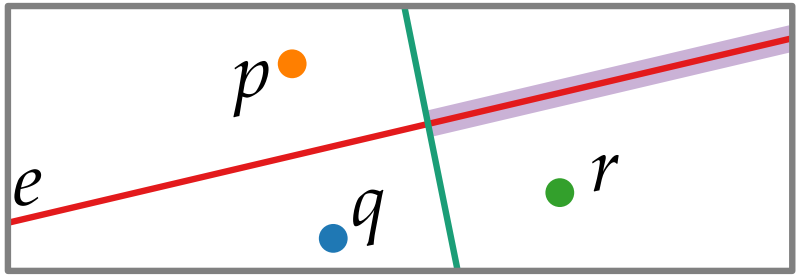
Overall Shape of Vor(P)

Theorem. Let $P \subset \mathbb{R}^2$ be a set of n pts (called *sites*). If all sites are collinear, Vor(P) consists of $n-1$ parallel lines. Otherwise, Vor(P) is connected and its edges are line segments or half-lines.

Proof. Assume that P is not collinear.

– Assume that Vor(P) contains an edge e that is a full line, say, $e = b(p, q)$.



Let $r \in P$ be not collinear with p and q.

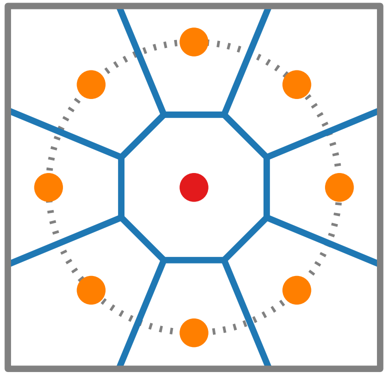
Then $e' = b(q, r)$ is not parallel to e.

$\Rightarrow$ $e \cap h(r, q)$ is closer to r than to p and q.

$\Rightarrow$ e is bounded on at least one side. $\square$

Complexity

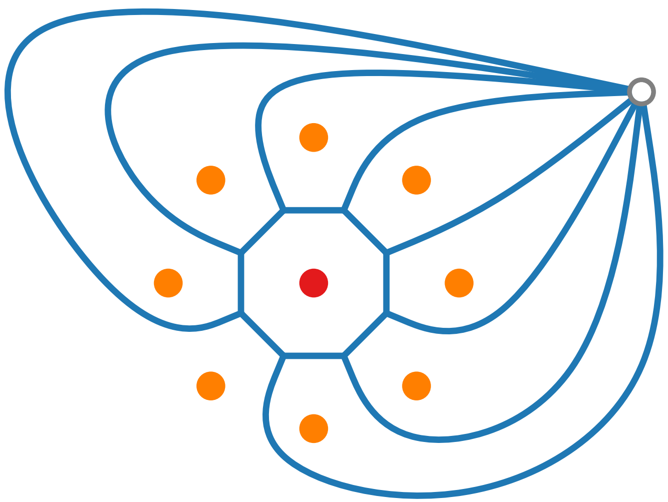
Task: Construct a set P of sites such that $\mathrm{Vor}(P)$ has a cell of linear complexity!



Theorem. Given a set $P \subset \mathbb{R}^2$ of n sites, $\mathrm{Vor}(P)$ consists of at most $2n-5$ vertices and $3n-6$ edges.

Proof. *Problem:* unbounded edges!

$\Rightarrow$ can't apply Euler directly, but…



$|F| = n \Rightarrow (|V|+1) - |E| + n = 2$

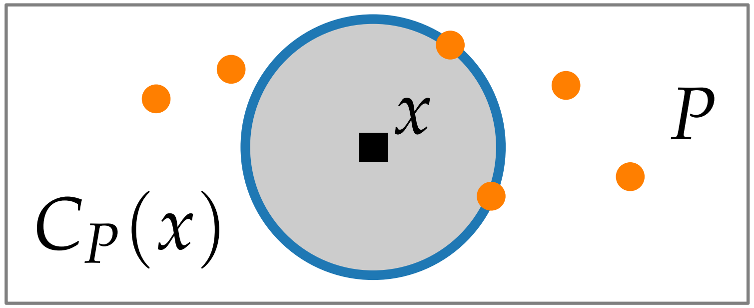
min. degree 3 $\Rightarrow 2|E| \geq 3(|V|+1)$

$\Rightarrow (|V|+1) - \frac{3}{2}(|V|+1) + n \leq 2$

$\Rightarrow \frac{1}{2}(|V|+1) \leq n-2$ $\square$

Characterization of Voronoi vtc and edges

$C_P(x) :=$ largest circle centered at x w/o sites in its interior



Theorem: (i) x Voronoi vtx $\Leftrightarrow |C_P(x) \cap P| \geq 3$

(ii) $b(p, p')$ contains a Voronoi edge $\Leftrightarrow$
$\exists x \in b(p, p') \colon C_P(x) \cap P = \{p, p'\}$

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Part IV:
The Beachline

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Computation

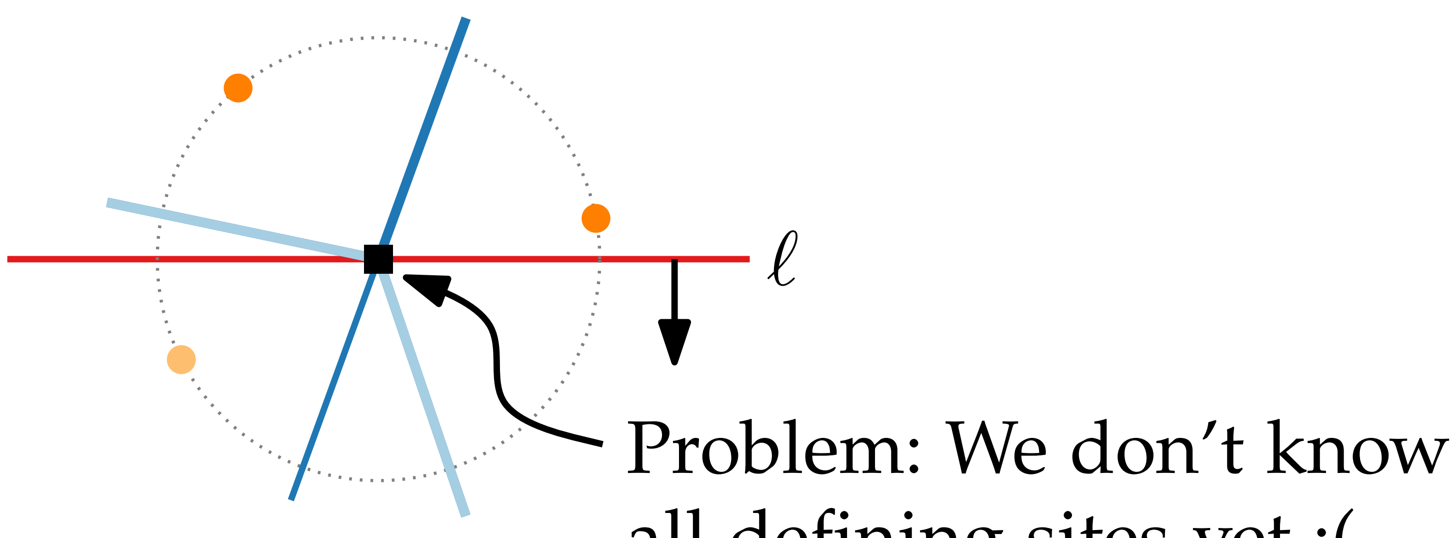
Brute force: For each $p \in P$, compute $\mathcal{V}(p) = \underbrace{\bigcap_{p' \neq p} h(p, p')}$.

$$\underbrace{\begin{array}{rl} \text{[Lect. 2, map-overlay / line-segment alg]} & O(n \log^2 n) \text{ time} \\ \text{[Lect. 4, half-plane intersection]} & O(n \log n) \text{ time} \end{array}}$$

in total: $O(n^2 \log n)$ time

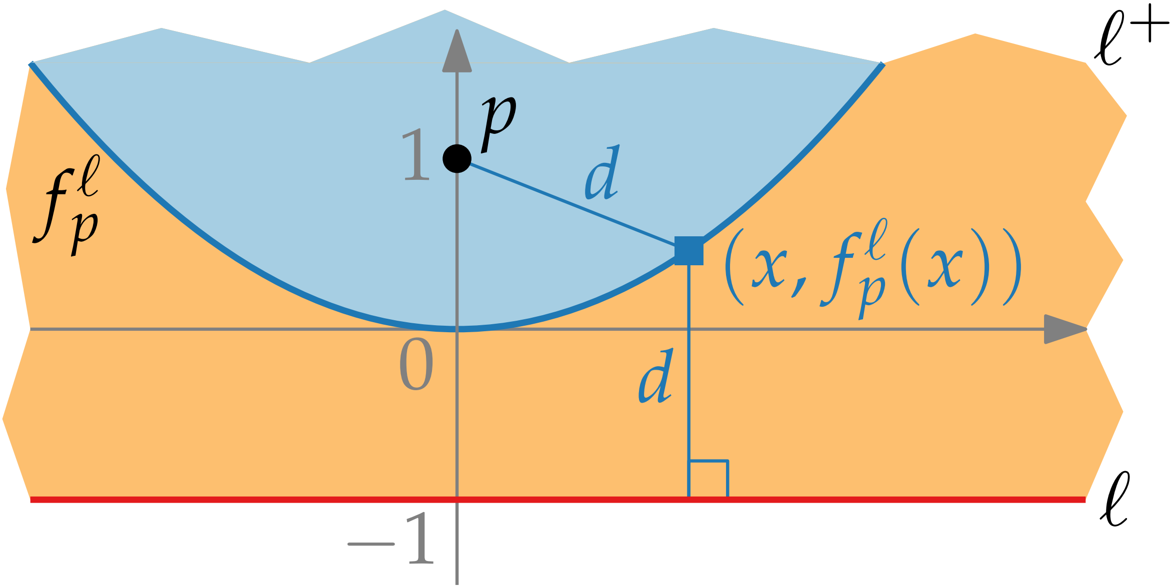
– but the complexity of $\mathrm{Vor}(P)$ is *linear!*

Sweep?



Sweep?

Which part of the plane above ℓ is fixed by what we've seen?

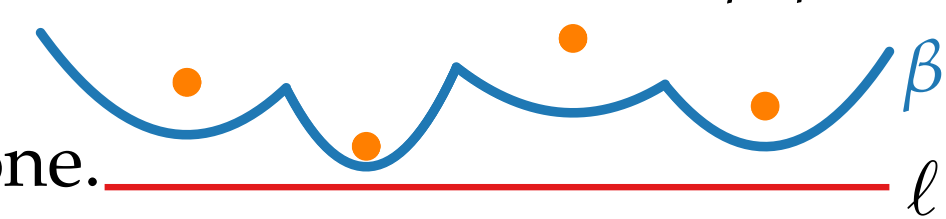


Solution:

f_p^ℓ is the parabola with focus p and directrix ℓ.

Task: Compute f_p^ℓ for $p = (0,1)$ and $\ell \colon y = -1$!

Definition. *beachline* $\beta \equiv$ lower envelope of $(f_p^\ell)_{p \in P \cap \ell^+}$



Observation. β is x-monotone.

The Beachline β

Question: What does β have to do with $\mathrm{Vor}(P)$?

Answer: "Breakpoints" of β trace out the Voronoi edges!

Lemma. New arcs on β only appear through *site events,* that is, whenever ℓ hits a new site.

Corollary. β consists of at most $2n - 1$ arcs.

Definition. *Circle event:* ℓ reaches lowest pt of a circle through three sites above ℓ whose arcs are consecutive on β.

Lemma. Arcs disappear from β only at circle events.

Lemma. The Voronoi vtc correspond 1:1 to circle events.

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Part V:
Fortune’s Sweep

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Winter Semester 2020

Fortune's Sweep

VoronoiDiagram($P \subset \mathbb{R}^2$)

$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted by y-coord.

$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)

$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)

while not $\mathcal{Q}$.empty() **do**

  $p \leftarrow \mathcal{Q}$.ExtractMax()

  if p site event **then**

    HandleSiteEvent(p)

  else

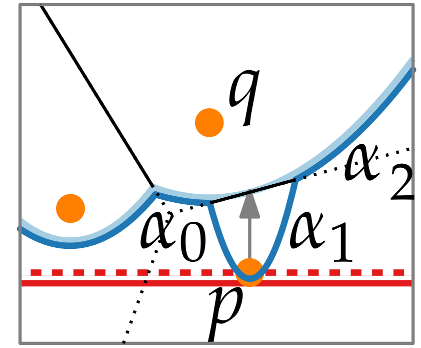
    $\alpha \leftarrow$ arc on β that will disappear

    HandleCircleEvent(α)

treat remaining int. nodes of $\mathcal{T}$ ($\equiv$ unbnd. edges of Vor(P))

return $\mathcal{D}$

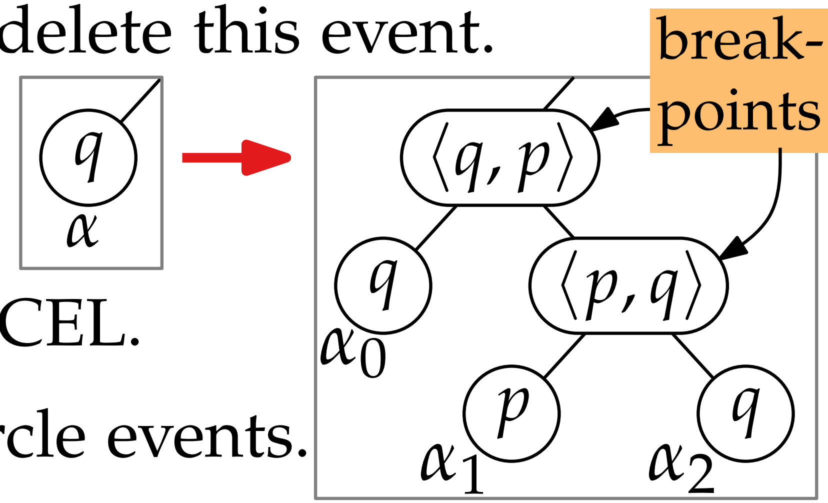
Handling Events



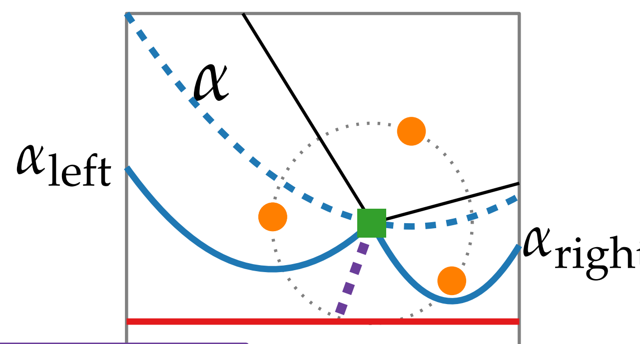
HandleSiteEvent(point p)

- Search in $\mathcal{T}$ for the arc α vertically above p.
  If α has pointer to circle event in $\mathcal{Q}$, delete this event.
- Split α into α_0 and α_2.
  Let α_1 be the new arc of p.
- Add Vor-edges $\langle q, p\rangle$ and $\langle p, q\rangle$ to DCEL.
- Check $\langle \cdot, \alpha_0, \alpha_1\rangle$ and $\langle \alpha_1, \alpha_2, \cdot\rangle$ for circle events.

In $\mathcal{T}$:



HandleCircleEvent(arc α)



- $\mathcal{T}$.delete(α); update breakpts
- Delete all circle events involving α from $\mathcal{Q}$.
- Add Vor-vtx $\boxed{\alpha_{\text{left}} \cap \alpha_{\text{right}}}$ and Vor-edge $\boxed{\langle \alpha_{\text{left}}, \alpha_{\text{right}}\rangle}$ to DCEL.
- Check $\langle \cdot, \alpha_{\text{left}}, \alpha_{\text{right}}\rangle$ and $\langle \alpha_{\text{left}}, \alpha_{\text{right}}, \cdot\rangle$ for circle events.

Running time? $O(\log n)$ per event. . .

Running Time?

VoronoiDiagram($P \subset \mathbb{R}^2$)
$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted by y-coord.
$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)
$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)
while not $\mathcal{Q}$.empty() **do**
    $p \leftarrow \mathcal{Q}$.ExtractMax()
    if p site event **then**
        HandleSiteEvent(p) exactly n such events
    else
        $\alpha \leftarrow$ arc on β that will disappear
        HandleCircleEvent(α) at most $2n - 5$ such events
treat remaining int. nodes of $\mathcal{T}$ ($\equiv$ unbnd. edges of Vor(P))
return $\mathcal{D}$

Summary

Theorem. Given a set P of n pts in the plane, Fortune's sweep computes $\mathrm{Vor}(P)$ in $O(n \log n)$ time and $O(n)$ space.

Steven Fortune
Bell Labs

Steven Fortune. A sweepline algorithm for Voronoi diagrams. *Proc. 2nd Annual ACM Symposium on Computational Geometry.* Yorktown Heights, NY, pp. 313–322. 1986.