Computational Geometry

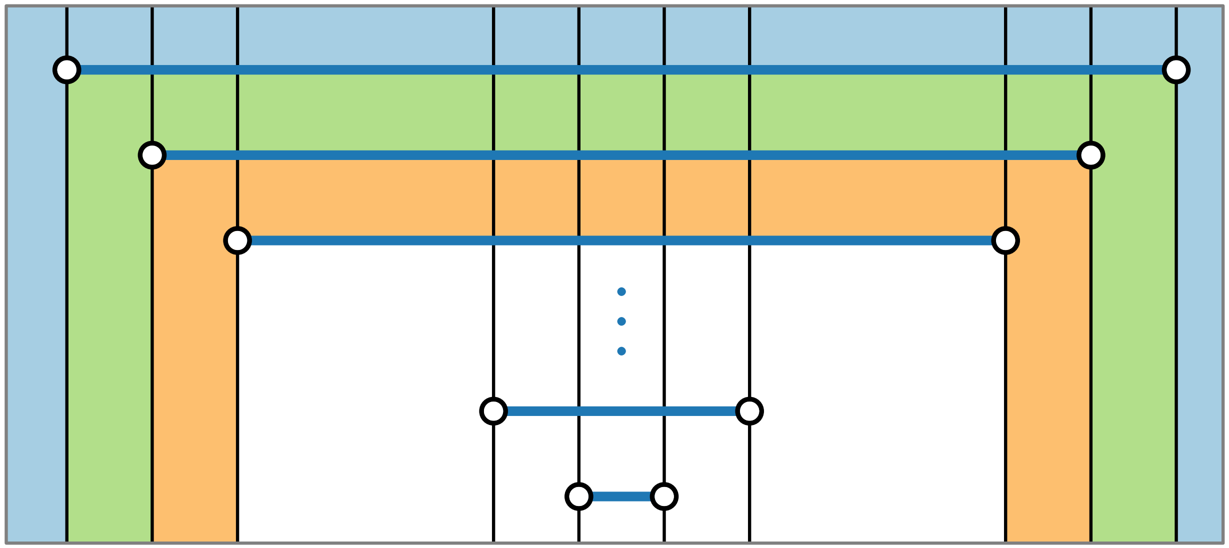
Lecture 6:
Point Localization
or
Where am I?

Part I:
Definition & First Approach

Philipp Kindermann

Winter Semester 2020

What's the Problem?



$n+1$
$+\ n-1$
$+\ n-3$
$+\ \ldots$
$=n^2/4$

Task: Given a planar subdivision $\mathcal{S}$ with n segments, preprocess $\mathcal{S}$ to allow for fast pt. location queries!

Solution: Preproc.: Partition $\mathcal{S}$ into slabs induced by vertices.

Query: – find correct slab
– search in slab

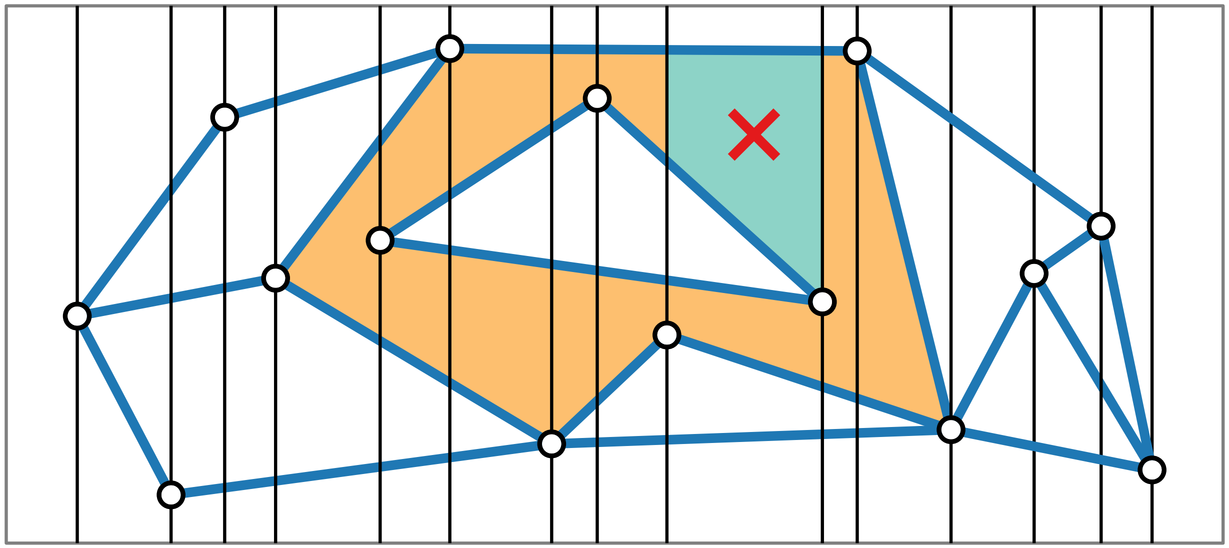
} 2 bin. searches!

$O(\log n)$ time!

But: Space? $\Theta(n^2)$

Task: Give lower-bound example!

What's the Problem?



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Solution: Preproc.: Partition $\mathcal{S}$ into slabs induced by vertices.

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But: Space? $\Theta(n^2)$ Preproc? $O(n^2 \log n)$

$O(\log n)$ time!

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Part II:
Decreasing the Complexity

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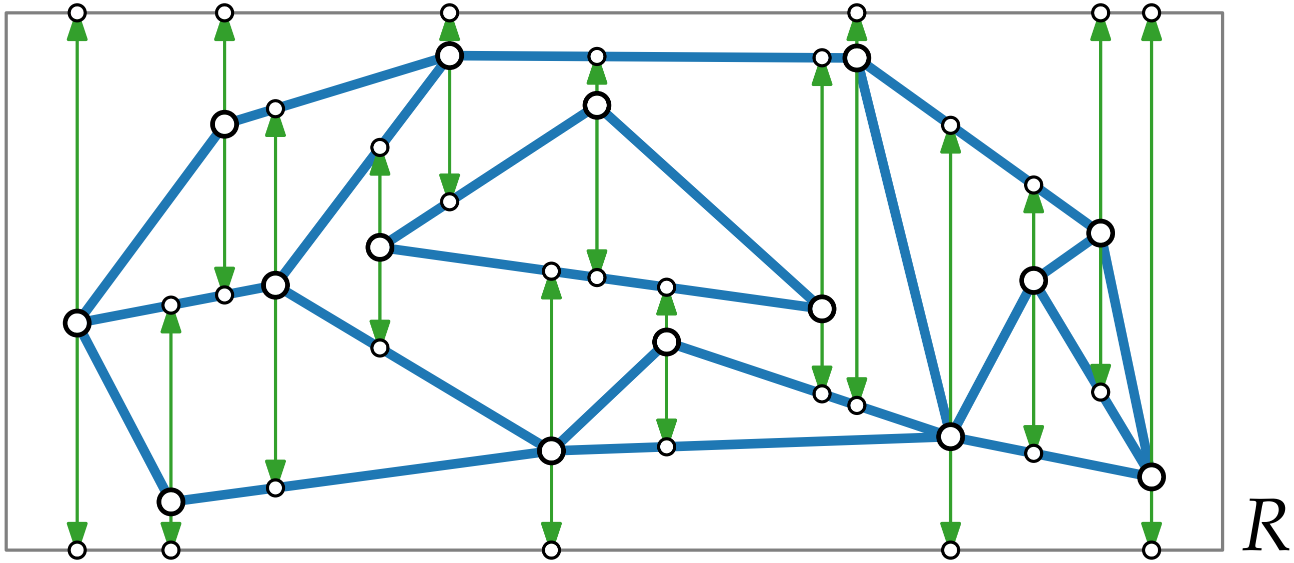
Decreasing the Complexity

Observation: The slab partition of $\mathcal{S}$ is a *refinement* $\mathcal{S}'$ of $\mathcal{S}$ that consists of (possibly degenerate) trapezoids.

Task: Find "good" refinement of $\mathcal{S}$ of low complexity!

Solution: *Trapezoidal map* $\mathcal{T}(\mathcal{S})$



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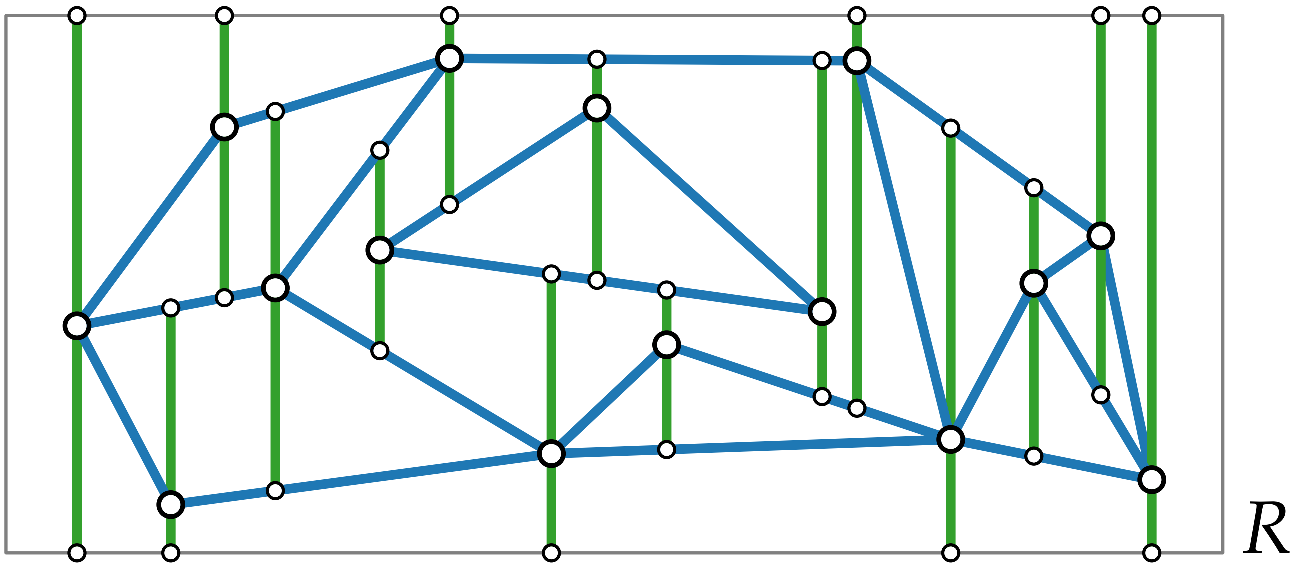
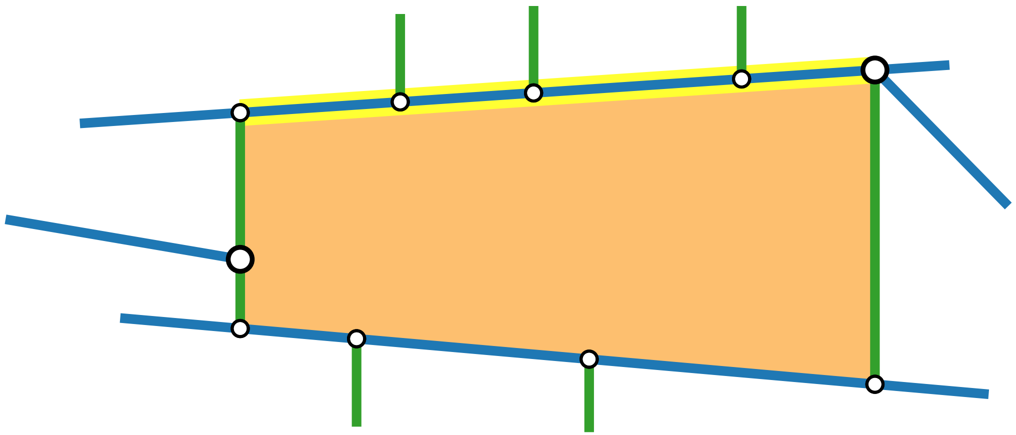


Assumption: $\mathcal{S}$ is in *general position*, that is, no two vertices have the same x-coordinates.

Notation

Definition: A *side* of a face of $\mathcal{T}(\mathcal{S})$ is a segment of max. length contained in the boundary of the face.



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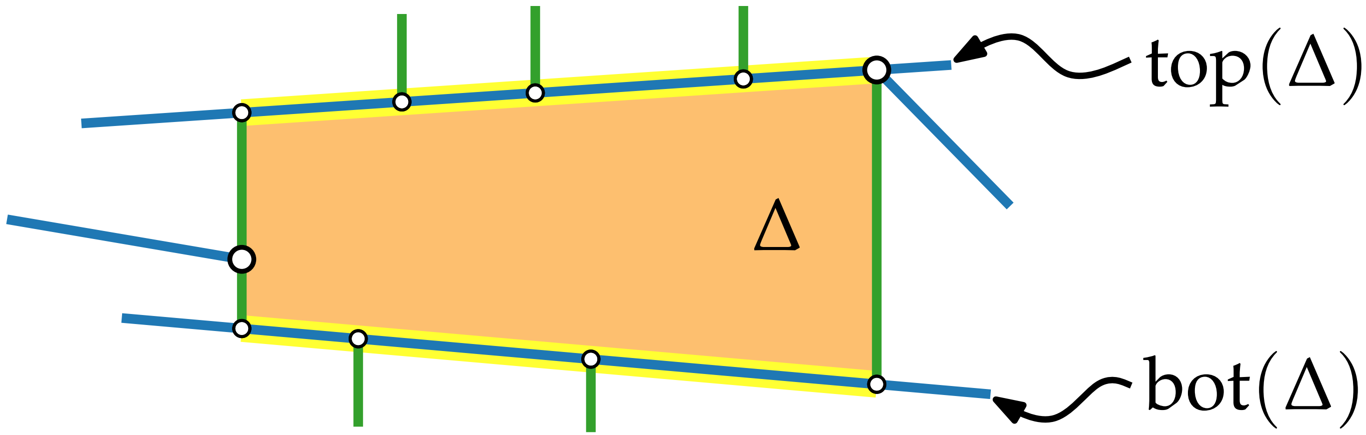
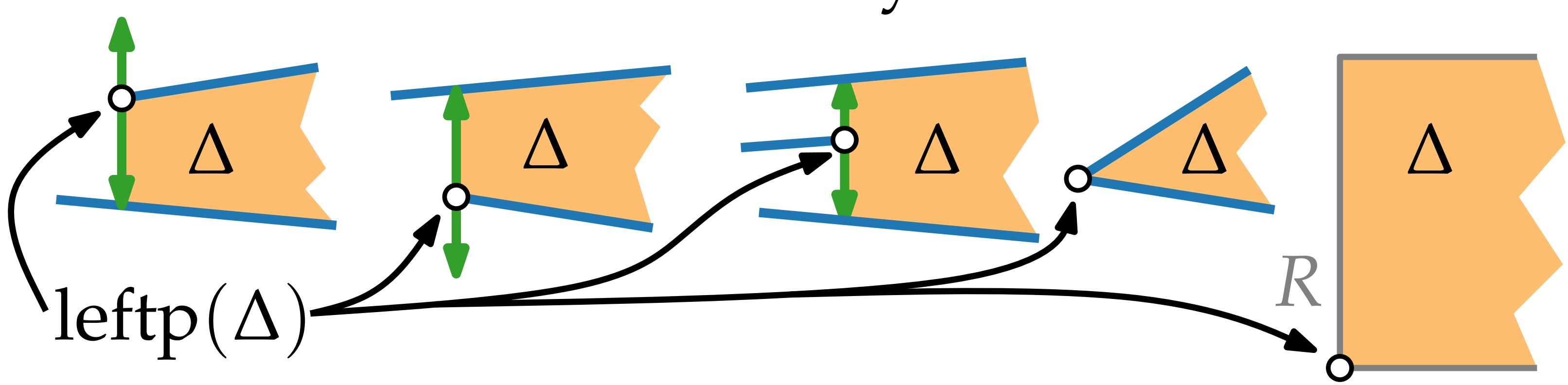


Observation: $\mathcal{S}$ in gen. pos. $\Rightarrow$ each face Δ of $\mathcal{T}(\mathcal{S})$ has:

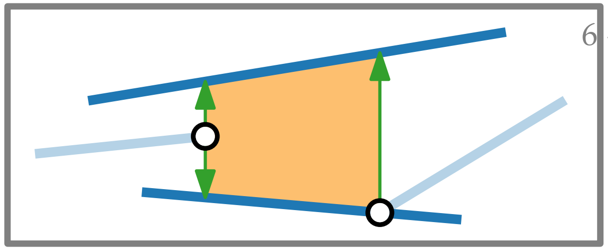
– one or two vertical sides

– exactly 2 non-vertical sides

Left side:



Complexity of $\mathcal{T}(\mathcal{S})$



Observe: A face Δ of $\mathcal{T}(\mathcal{S})$ is uniquely defined by top(Δ), bot(Δ), leftp(Δ), and rightp(Δ).

Lemma. $\mathcal{S}$ planar subdivision in gen. pos. with n segments $\Rightarrow \mathcal{T}(\mathcal{S})$ has $\leq 6n+4$ vtc and $\leq 3n+1$ trapezoids.

Proof. The vertices of $\mathcal{T}(S)$ are

$$\left.\begin{array}{ll} \text{– endpts of segments in } \mathcal{S} & \leq 2n \\ \text{– endpts of vertical extensions} & \leq 2\cdot 2n \\ \text{– vertices of } R & 4 \end{array}\right\} \leq 6n+4$$

Bound #trapezoids via Euler or directly (segments/leftp).

Approach: Construct trapezoidal map $\mathcal{T}(\mathcal{S})$ and point-location data structure $\mathcal{D}(\mathcal{S})$ for $\mathcal{T}(\mathcal{S})$ *incrementally*!

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Part III:
The 1D Problem

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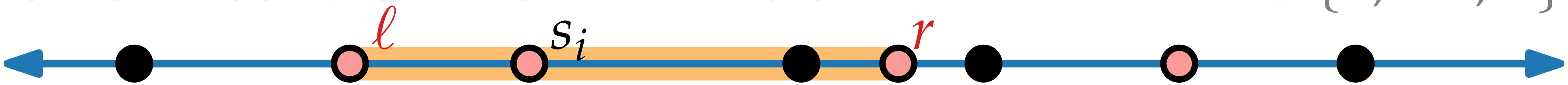
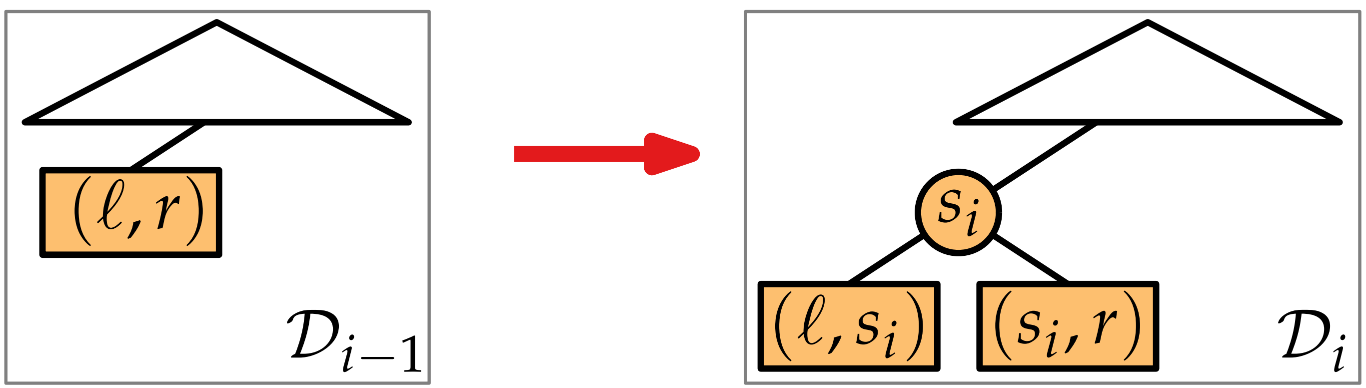
The 1D Problem

Given a set S of n real numbers... $i \in \{1, \ldots, n\}$



$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, I_{i-1} := set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

- pick an arbitrary point s_i from $S \setminus S_{i-1}$
- locate s_i in the search structure $\mathcal{D}_{i-1}$ of S_{i-1}
- split interval (ℓ, r) of I_{i-1} containing s_i
- build $\mathcal{D}_i$:



Problem: *looong* search paths!

The 1D Problem

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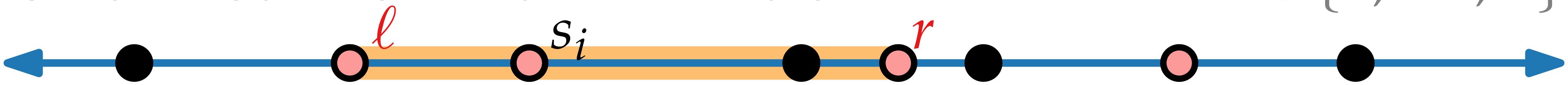

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, $I_{i-1} :=$ set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

Solution: *random!*

- pick ~~an arbitrary~~ point s_i from $S \setminus S_{i-1}$
- locate s_i in the search structure $\mathcal{D}_{i-1}$ of S_{i-1}
- split interval (ℓ, r) of I_{i-1} containing s_i
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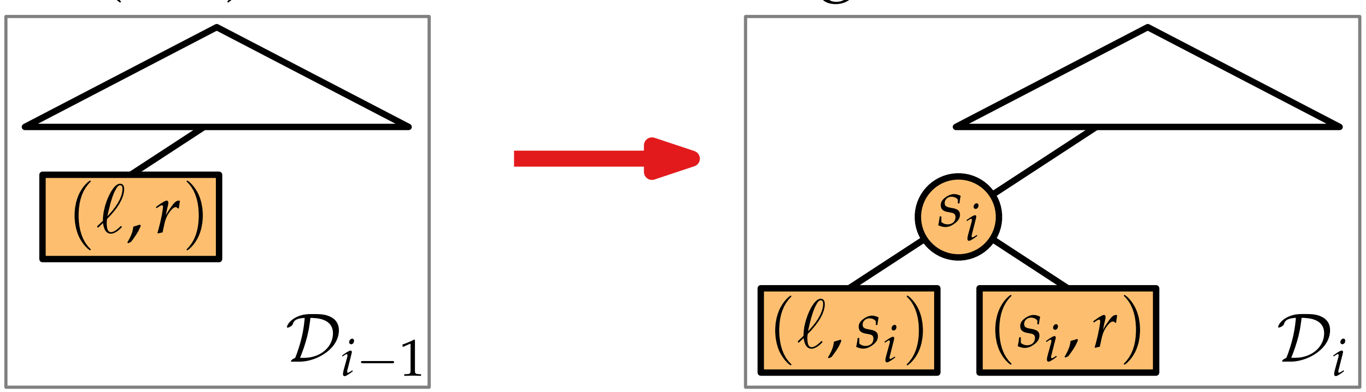
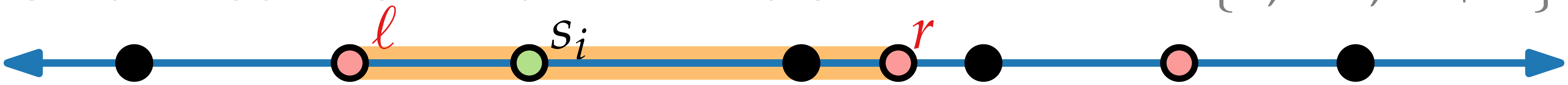


~~**Problem:** *looong* search paths!~~

The 1D Result

Given a set S of n real numbers... $i \in \{1, \ldots, n+1\}$



$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, $I_{i-1} :=$ set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

Theorem. The randomized-incremental alg. preproc. a set S of n reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

Proof. Let $q \in \mathbb{R}$ (wlog. $q \notin S$) and $I_i(q) = \arg\{I \in I_i : q \in I\}$.

Define random variable $X_i = \begin{cases} 1 & \text{if } I_i(q) \neq I_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$

$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = ?$

Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] = 2/i$$
$$= \text{probability that } I_i(q) \neq I_{i-1}(q)\text{, i.e., } s_i \in I_{i-1}(q).$$

Backwards analysis:
(see Lecture 04)

Consider S_i fixed.
If we *remove* a randomly chosen pt from S_i,
what's the probability that the interval
containing q changes?
– we have i choices, identically distributed
– at most two of these change the interval

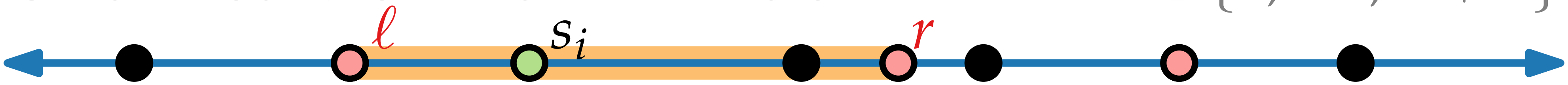
Define random variable $X_i = \begin{cases} 1 & \text{if } I_i(q) \neq I_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$
$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \text{?}$

$O(\log n)$

The 1D Result

Given a set S of n real numbers... $i \in \{1, \ldots, n+1\}$



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Part IV:
The 2D Problem

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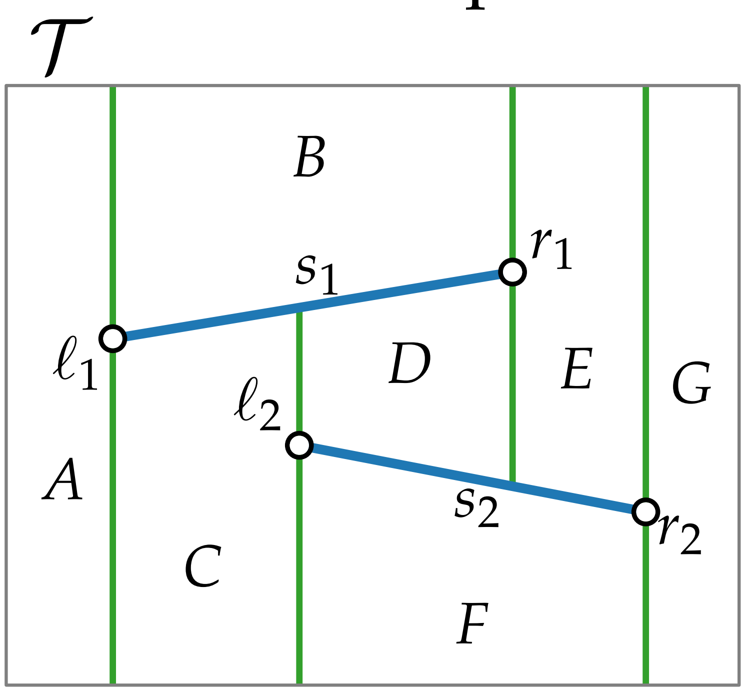
The 2D Problem

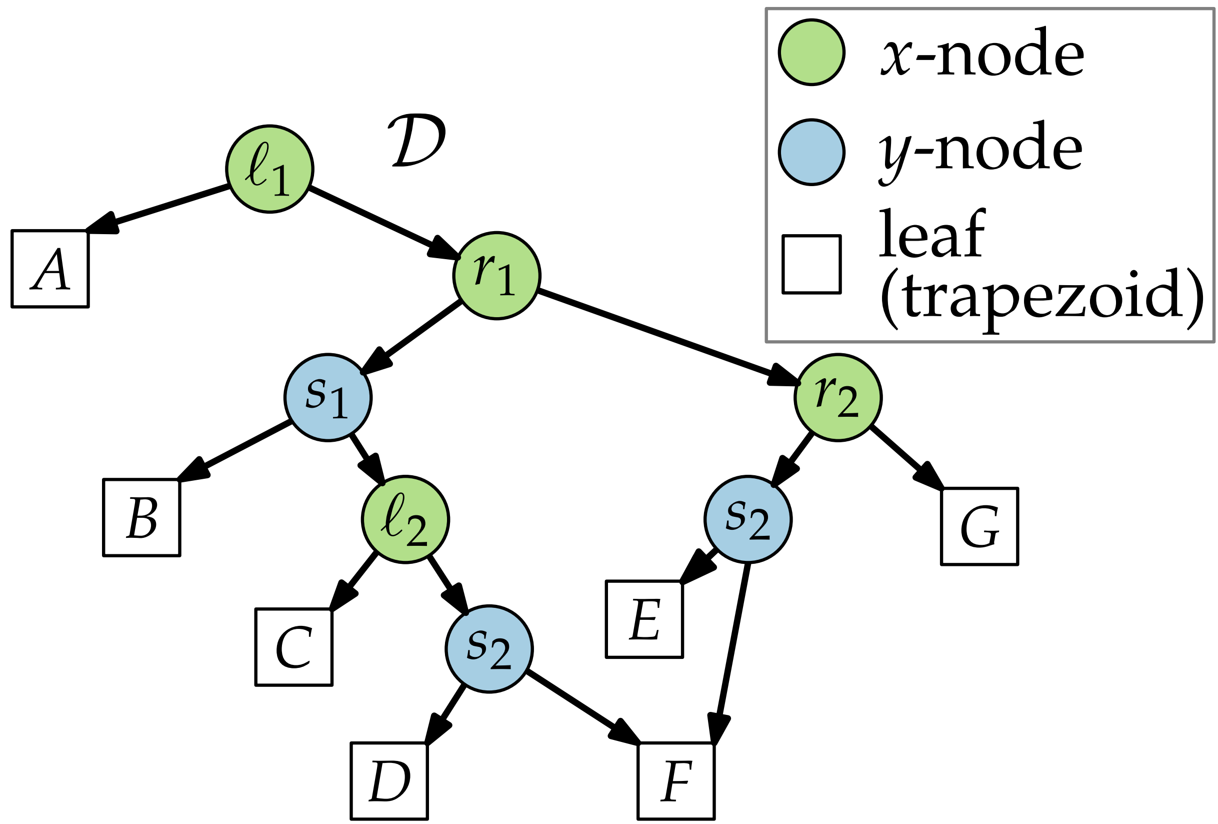
point-location data structure (DAG)

trapezoidal map

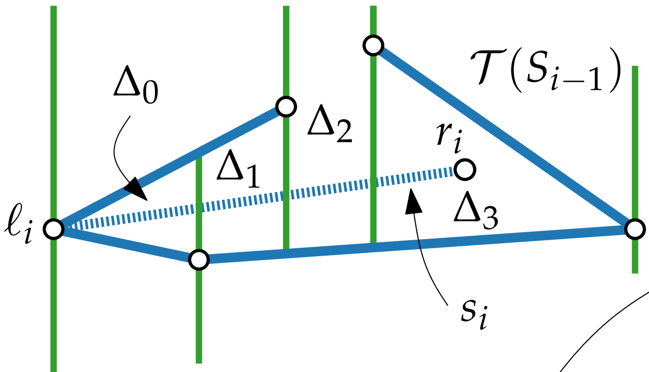
Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

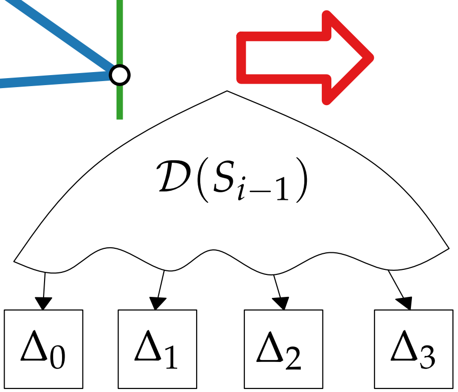
- use $\mathcal{D}$ to locate left endpoint of next segment s
- "walk" along s through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting s
- construct new trapezoids of $\mathcal{T}$ (adjacent to s)
- update $\mathcal{D}$

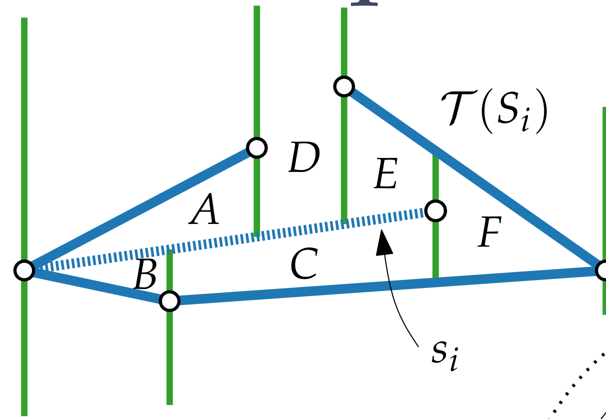


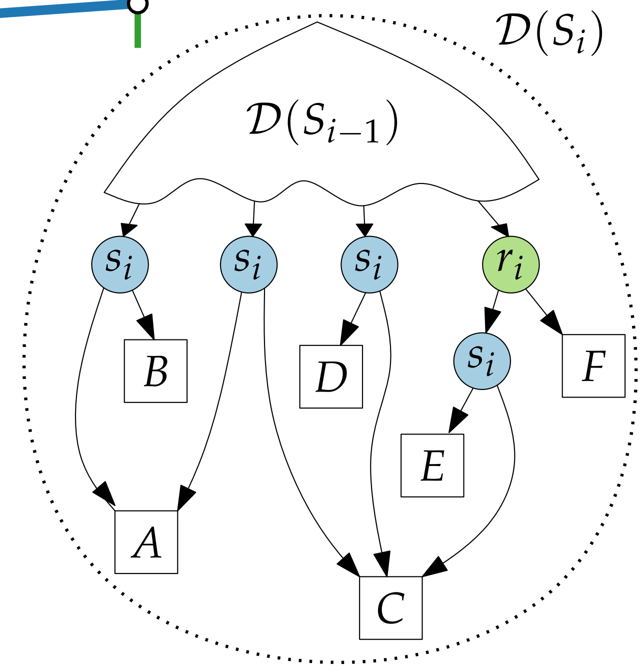


Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$









TrapezoidalMap(set S of n non-crossing seg.)
  $R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{T}$.add(new trapezoids incident to s_i)
    $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{D}$.add_leaves(new trapez. incident to s_i)
    $\mathcal{D}$.add_new_inner_nodes()

The 2D Result

Theorem. TrapezoidalMap(S) computes $\mathcal{T}(S)$ for a set of n line segments in general position and a search structure $\mathcal{D}$ for $\mathcal{T}(S)$ in $O(n \log n)$ expected time. The expected size of $\mathcal{D}$ is $O(n)$ and the expected query time is $O(\log n)$.

Invariant: Before step i, $\mathcal{T}$ is a trapezoidal map for S_{i-1} and $\mathcal{D}$ is a valid search structure for $\mathcal{T}$.

Proof.
– Correctness by loop invariant.
– Query time similar to 1D analysis.
$\Rightarrow$ construction time

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Part V:
Query Time and Size

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Query Time

Let $T(q)$ be the query time for a fixed query pt q.

$\Rightarrow T(q) = O(\text{length of the path from } \mathcal{D}.\text{root to } q)$.

height($\mathcal{D}$) increases by at most 3 in each step. $\Rightarrow T(q) \leq 3n$.

We are interested in the *expected* behaviour of $\mathcal{D}$:

$\Rightarrow$ average of $T(q)$ over all $n!$ insertion orders (permut. of S)

$X_i :=$ # nodes that are added to the query path in iteration i.

S and q are fixed.

$\Rightarrow X_i$ random var. that depends only on insertion order of S.

$\Rightarrow$ expected path length from $\mathcal{D}$.root to q is

$\mathbf{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbf{E}[X_i] = ?$

Query Time (cont'd)

$p_i =$ prob. that the search path Π_q of q in $\mathcal{D}$ contains a node that was created in iteration i.

$\Rightarrow \mathbf{E}[X_i] = \sum_{j=0}^{3} j \cdot \mathbf{P}[X_i = j] \leq\ 3 \cdot \mathbf{P}[X_i \geq 1] =\ 3p_i$

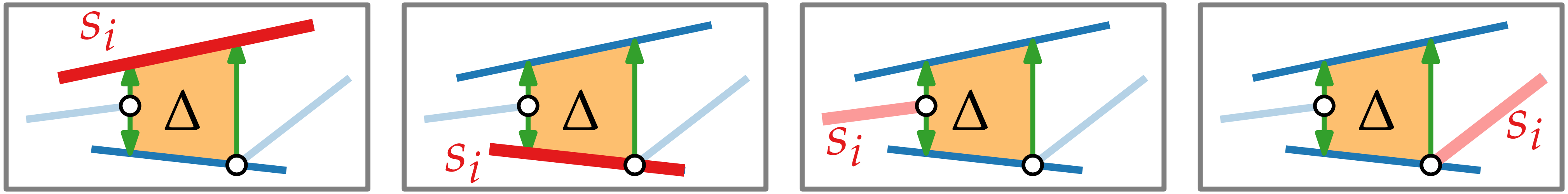
$\Delta_q(S_i) :=$ trapezoid in $\mathcal{T}(S_i)$ that contains q.

Key idea: Iteration i contributes a node to Π_q iff $\Delta_q(S_{i-1}) \neq \Delta_q(S_i)$.

In this case $\Delta_q(S_i)$ must have been created in iteration i.

$\Rightarrow \Delta :=\ \Delta_q(S_i)$ is adjacent to the new segment s_i.

$\Rightarrow$ top$(\Delta) = s_i$, bot$(\Delta) = s_i$, leftp$(\Delta) \in s_i$, or rightp$(\Delta) \in s_i$.



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$\Rightarrow \Delta := \Delta_q(S_i)$ is adjacent to the new segment s_i.

$\Rightarrow \text{top}(\Delta) = s_i$, $\text{bot}(\Delta) = s_i$, $\text{leftp}(\Delta) \in s_i$, or $\text{rightp}(\Delta) \in s_i$.

Trick: $\mathcal{T}(S_i)$ (and thus Δ) is uniquely determined by S_i.
Consider $S_i \subseteq S$ fixed.
$\Rightarrow \Delta$ does *not* depend on insertion order.

Query Time (cont'd)

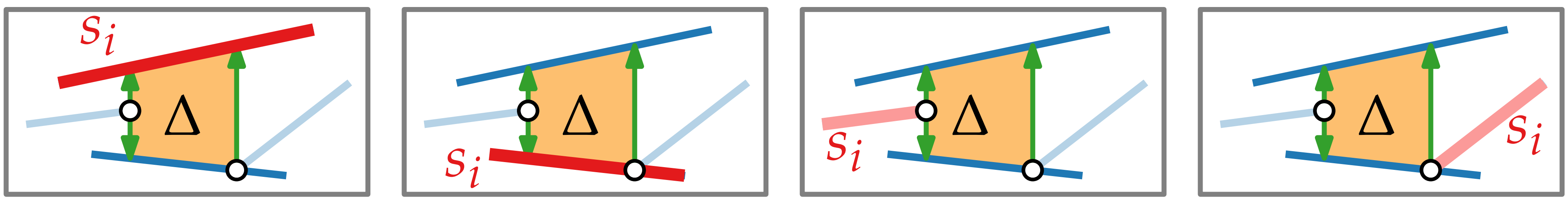
$p_i =$ prob. that the search path Π_q of q in $\mathcal{D}$ contains a node that was created in iteration i, i.e., prob. that Δ changes when inserting s_i.

Aim: bound p_i.

Tool: *Backwards analysis!*

$p_i =$ prob. that Δ changes when s_i is removed

Four cases:



$\mathbf{P}(\text{top}(\Delta) = s_i) = 1/i$ (since exactly 1 of i segments is top(Δ)).

$\Rightarrow p_i \leq 4/i$

$$\Rightarrow \mathbf{E}[\textstyle\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbf{E}[X_i] \leq \sum_{i=1}^n 3 \cdot p_i$$

$$= 12 \textstyle\sum_{i=1}^n 1/i \in O(\log n) \qquad \square$$

Size

Lemma. $\mathcal{S}$ planar subdivision in gen. pos. with n segments
$\Rightarrow \mathcal{T}(\mathcal{S})$ has $\leq 6n+4$ vtc and $\leq 3n+1$ trapezoids.

$X_i :=$ # int. nodes that are added to $\mathcal{D}$ in iteration i.
Size: $|\mathcal{T}(S)| + \sum_{i=1}^{n} \mathbf{E}[X_i] = O(n) + \sum_{i=1}^{n} \mathbf{E}[X_i]$

$$Y_i(\Delta) = \begin{cases} 1 & \text{if } \Delta \text{ disapp. from } \mathcal{T}(S_i) \text{ when } s_i \text{ is removed} \\ 0 & \text{otherwise} \end{cases}$$

(p_i = prob. that Δ changes when s_i is removed)
$\Rightarrow \mathbf{E}[Y_i(\Delta)] = p_i \leq 4/i$

$\Rightarrow \mathbf{E}[X_i] = \sum_{\Delta \in \mathcal{T}(S)} \mathbf{E}[Y_i(\Delta)] \leq (3i+1) \cdot 4/i = 12 + 4/i \leq 13$

$\Rightarrow$ Size: $O(n) + \sum_{i=1}^{n} \mathbf{E}[X_i] \leq O(n) + 13n \in O(n)$ $\square$

The 2D Result

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