

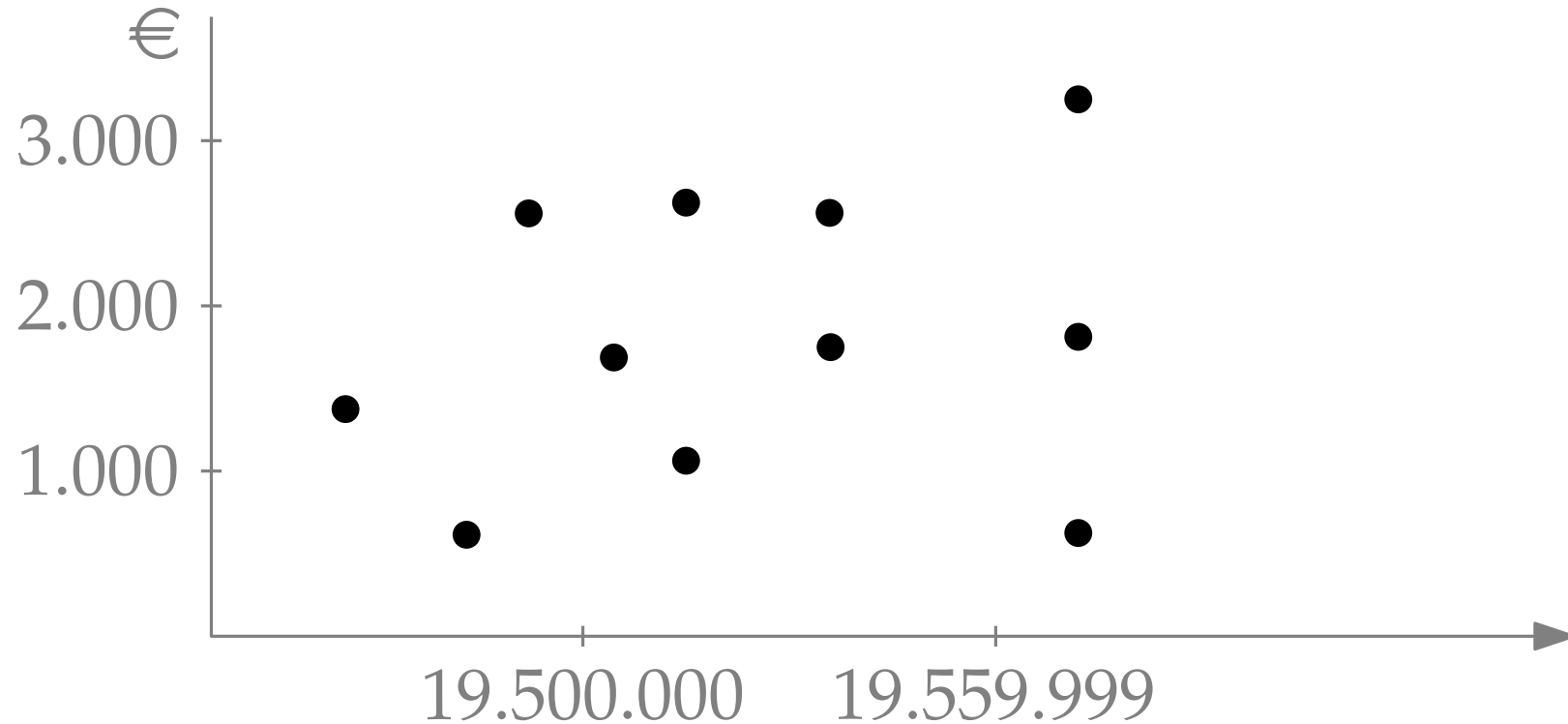
Computational Geometry

Lecture 5: Orthogonal Range Queries or Fast Access to Data Bases

Part I: 1D Range Searching

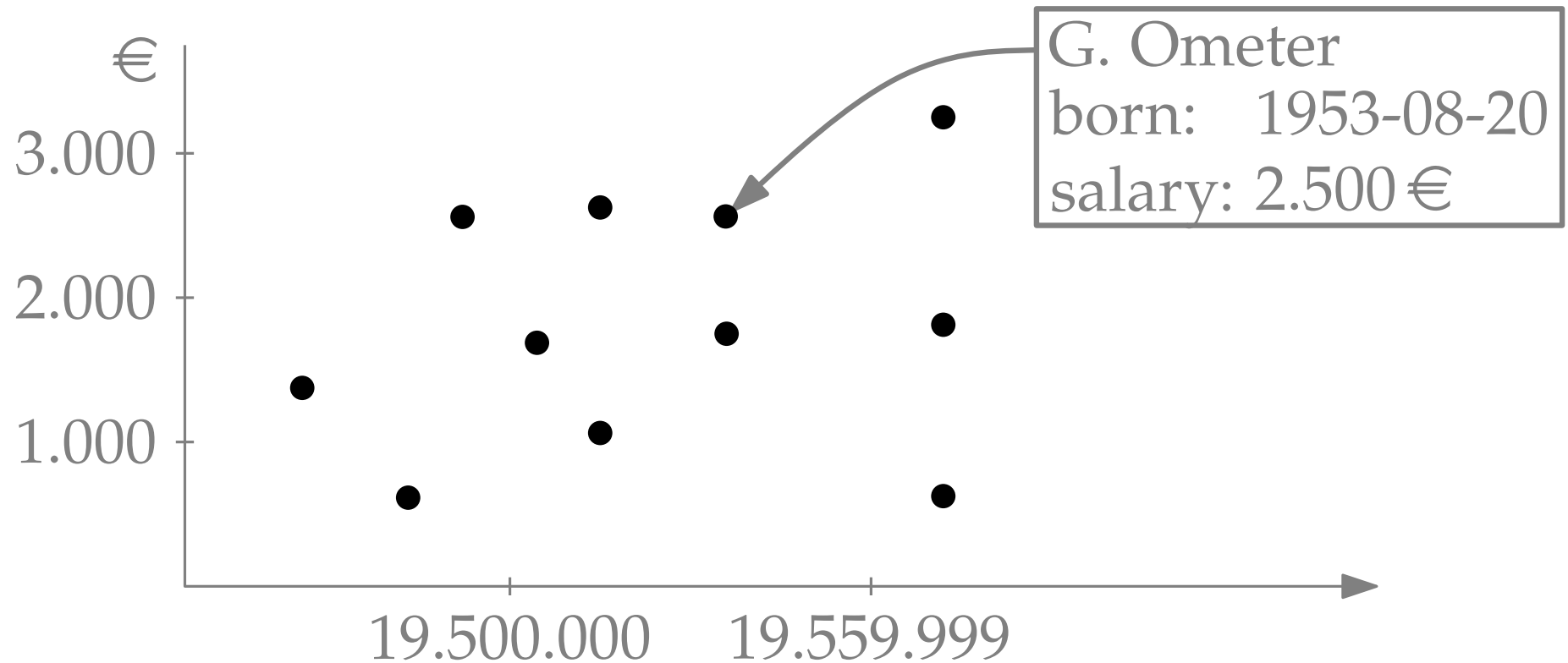
Orthogonal Range Queries

Example: Personnel management in a company



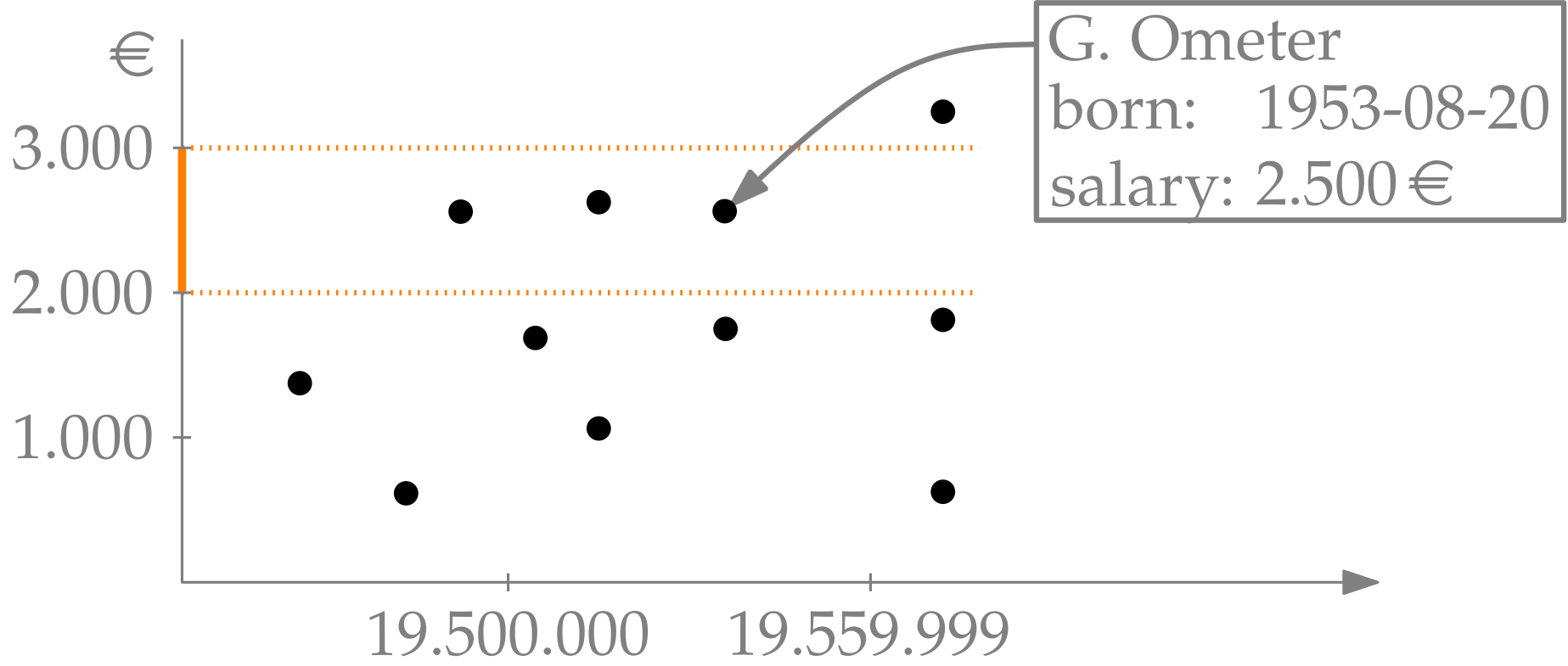
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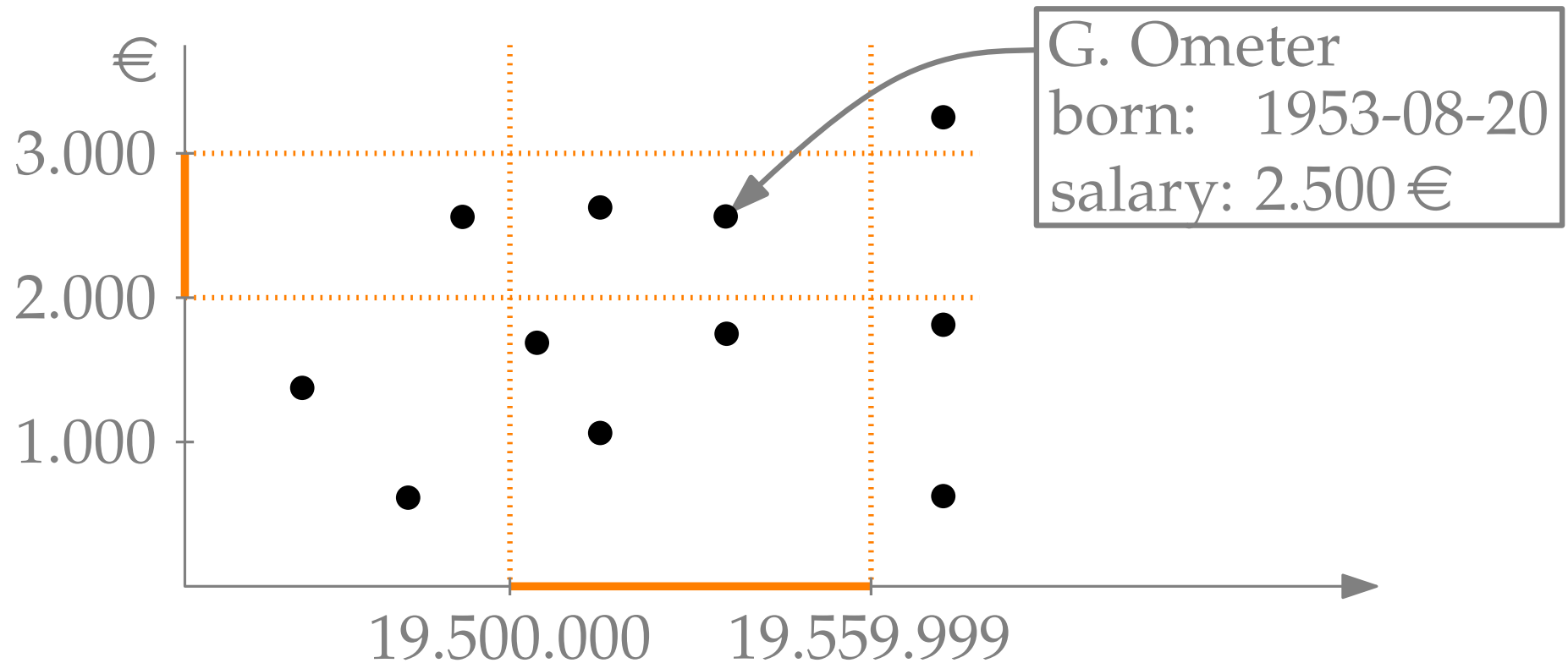
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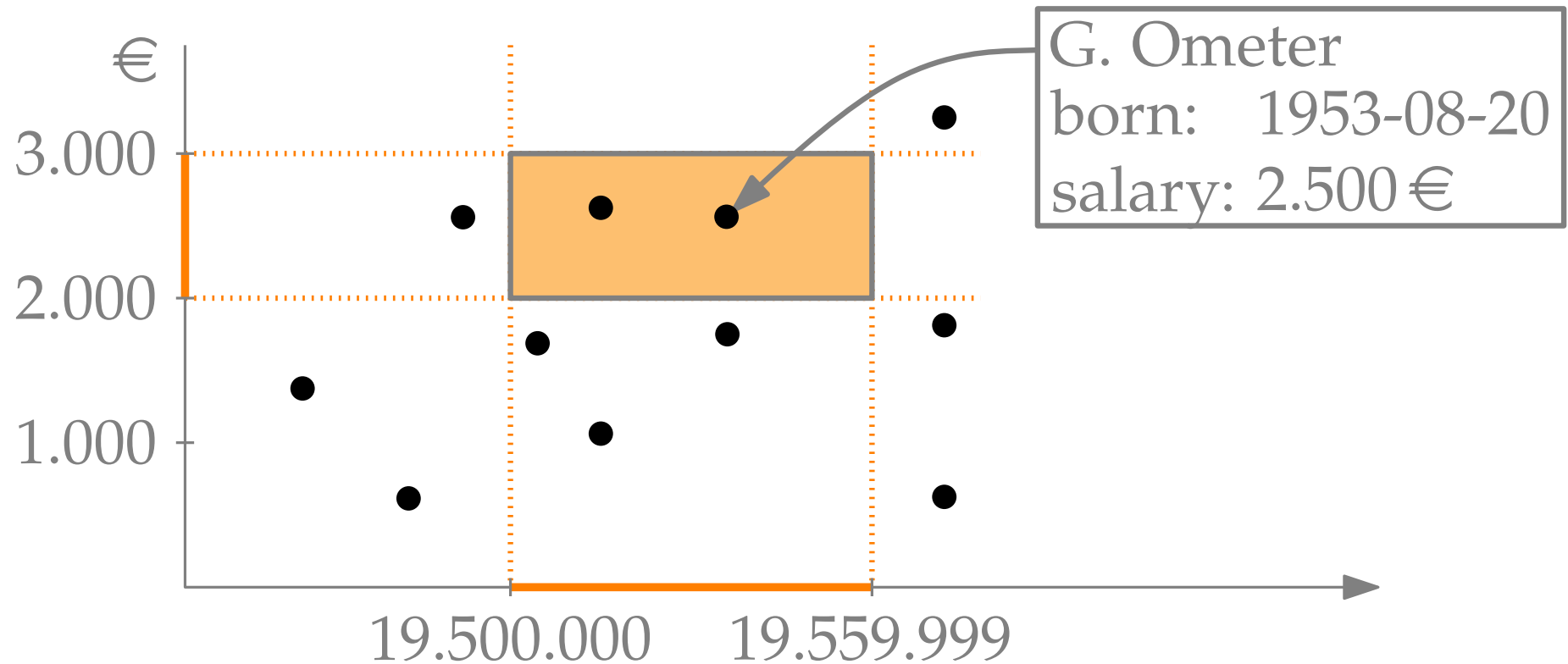
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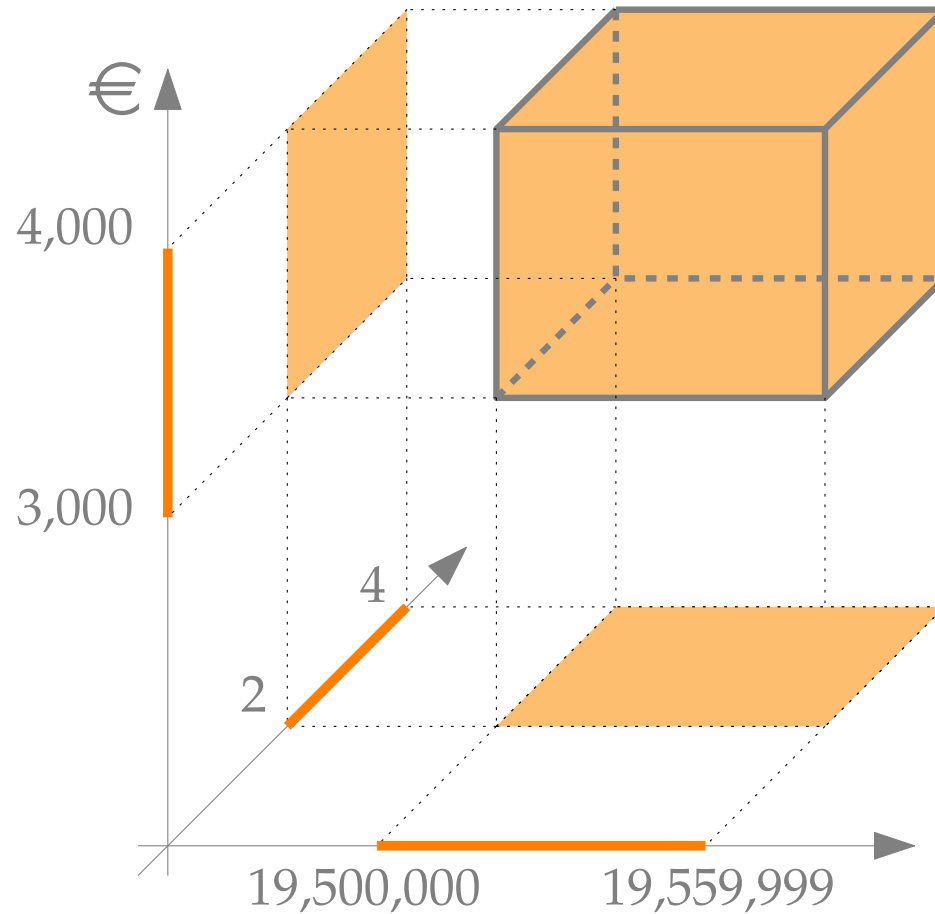
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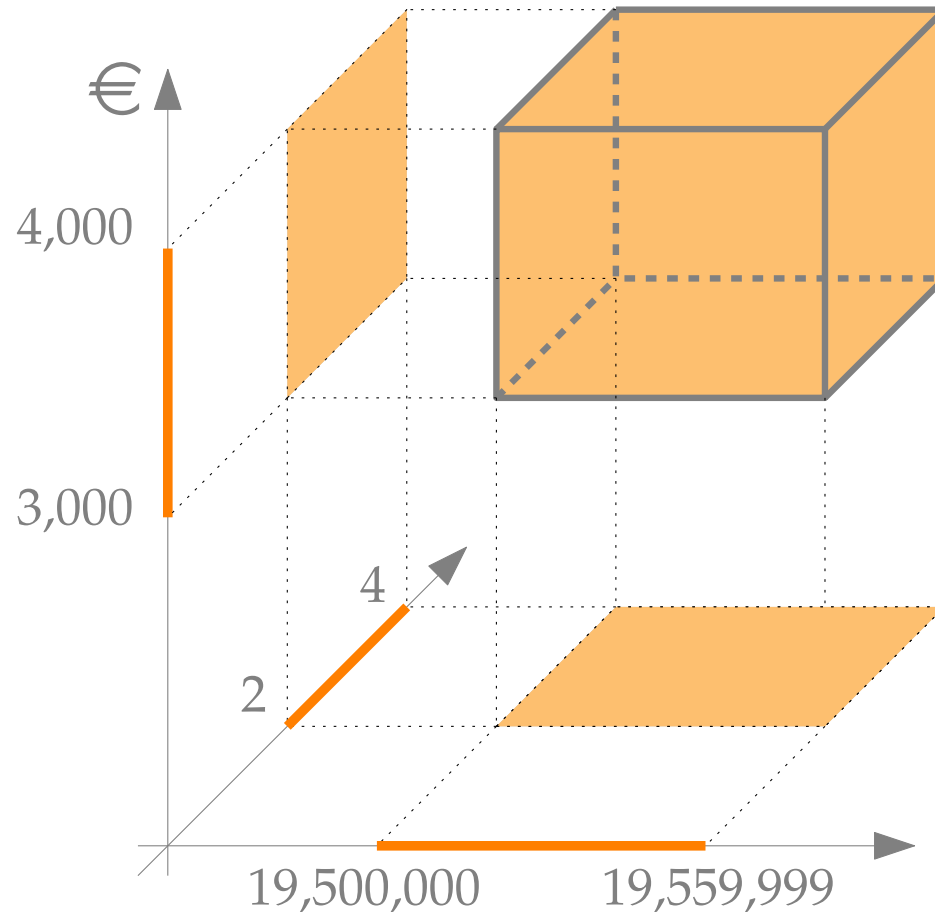
Orthogonal Range Queries

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Orthogonal Range Queries

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Typical queries for data bases!

1D Range Searching

Task: Preprocess a finite set $P \subset \mathbb{R}$ such that for any interval $[x, x']$ the set $P \cap [x, x']$ can be reported quickly.

1D Range Searching

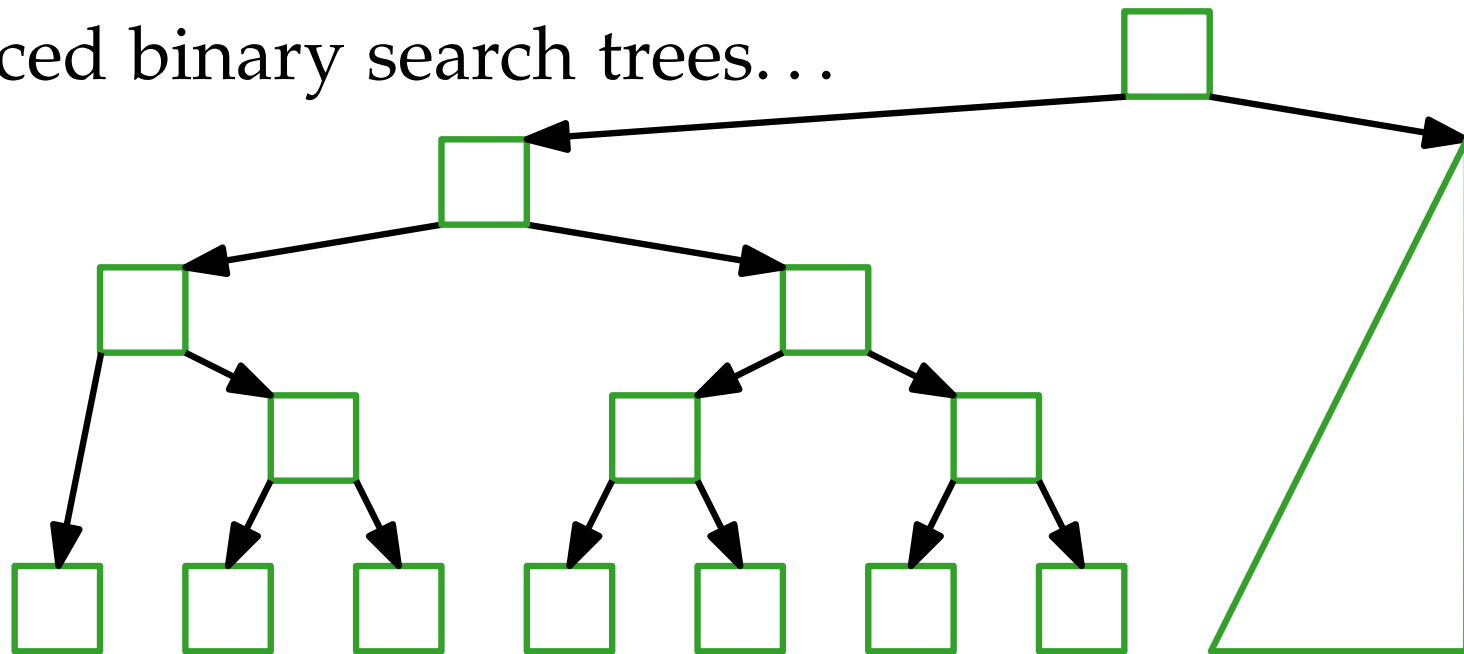
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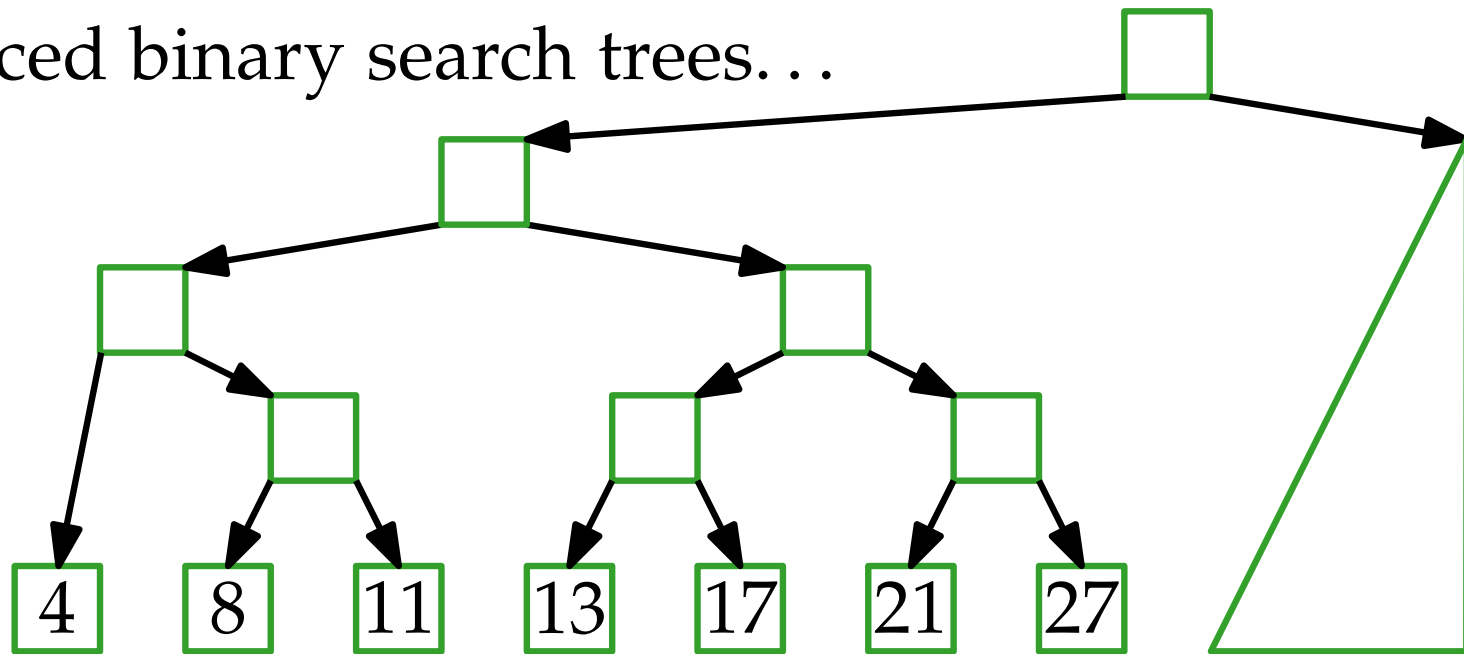
Solution: balanced binary search trees...



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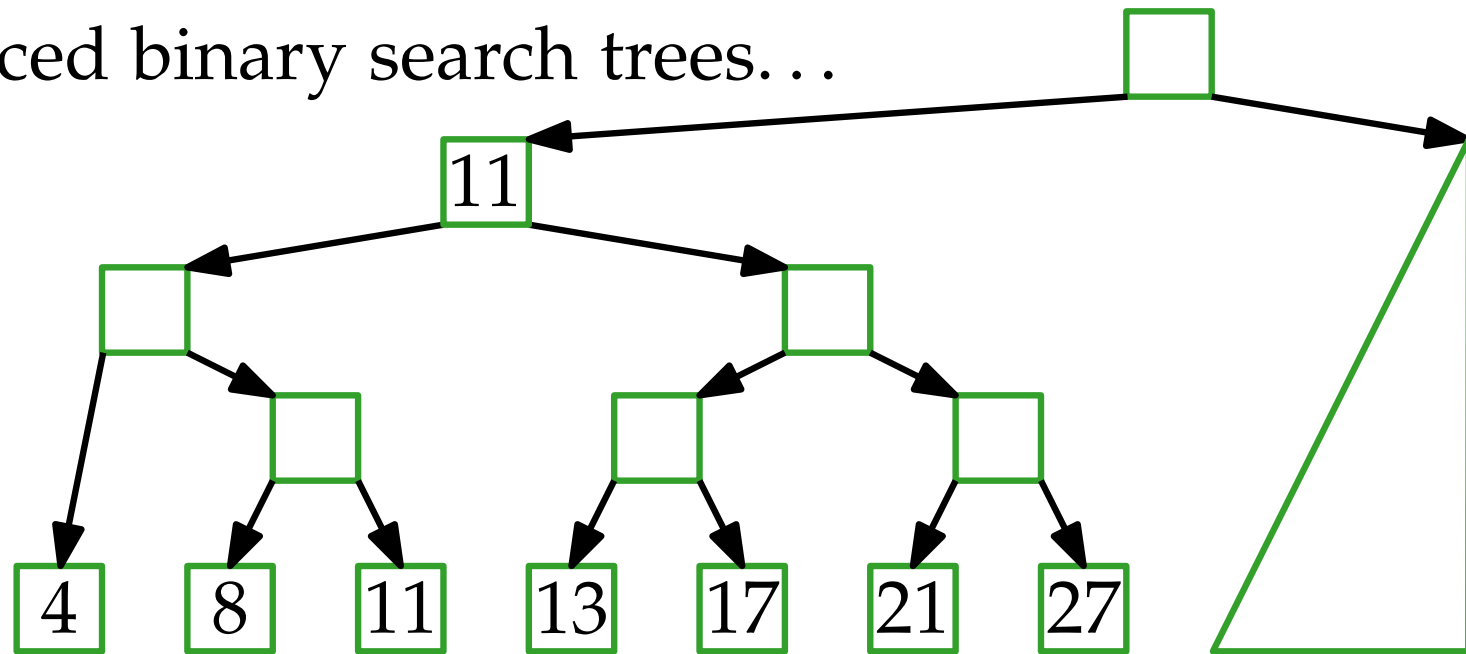


Small changes: – keys only in leaves

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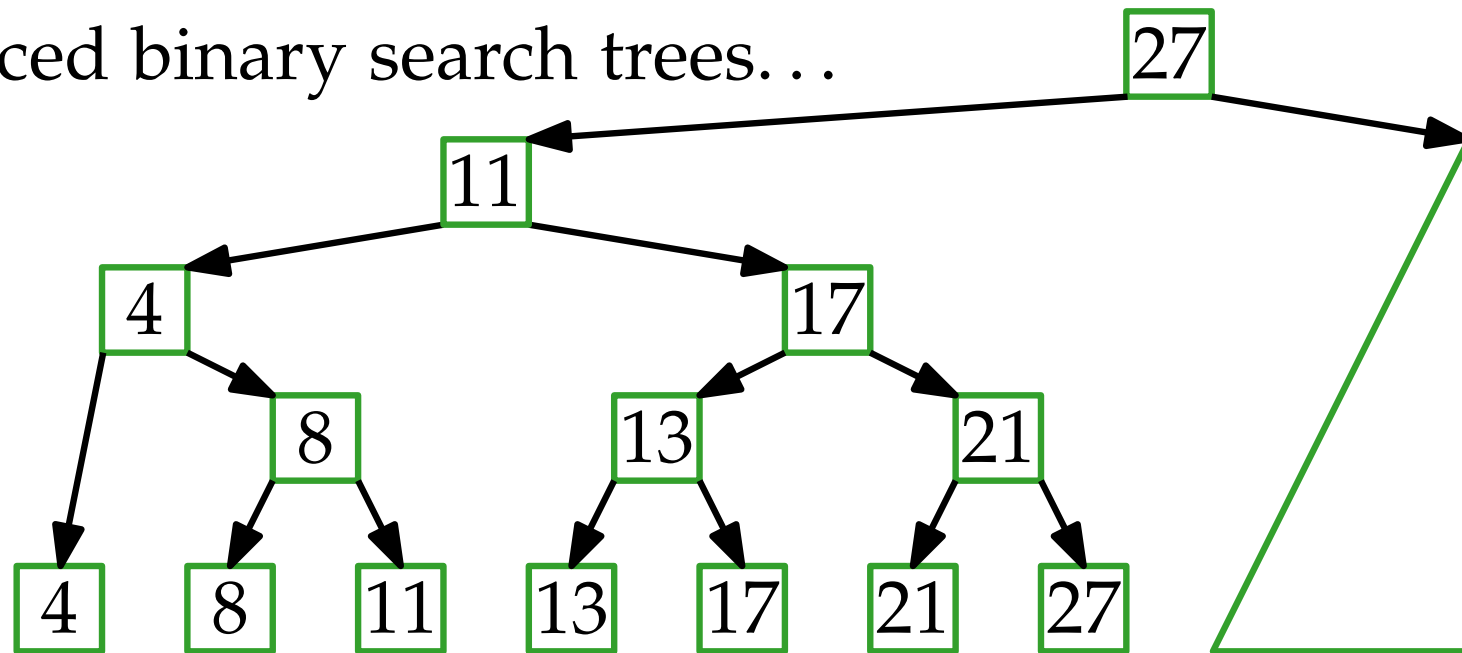
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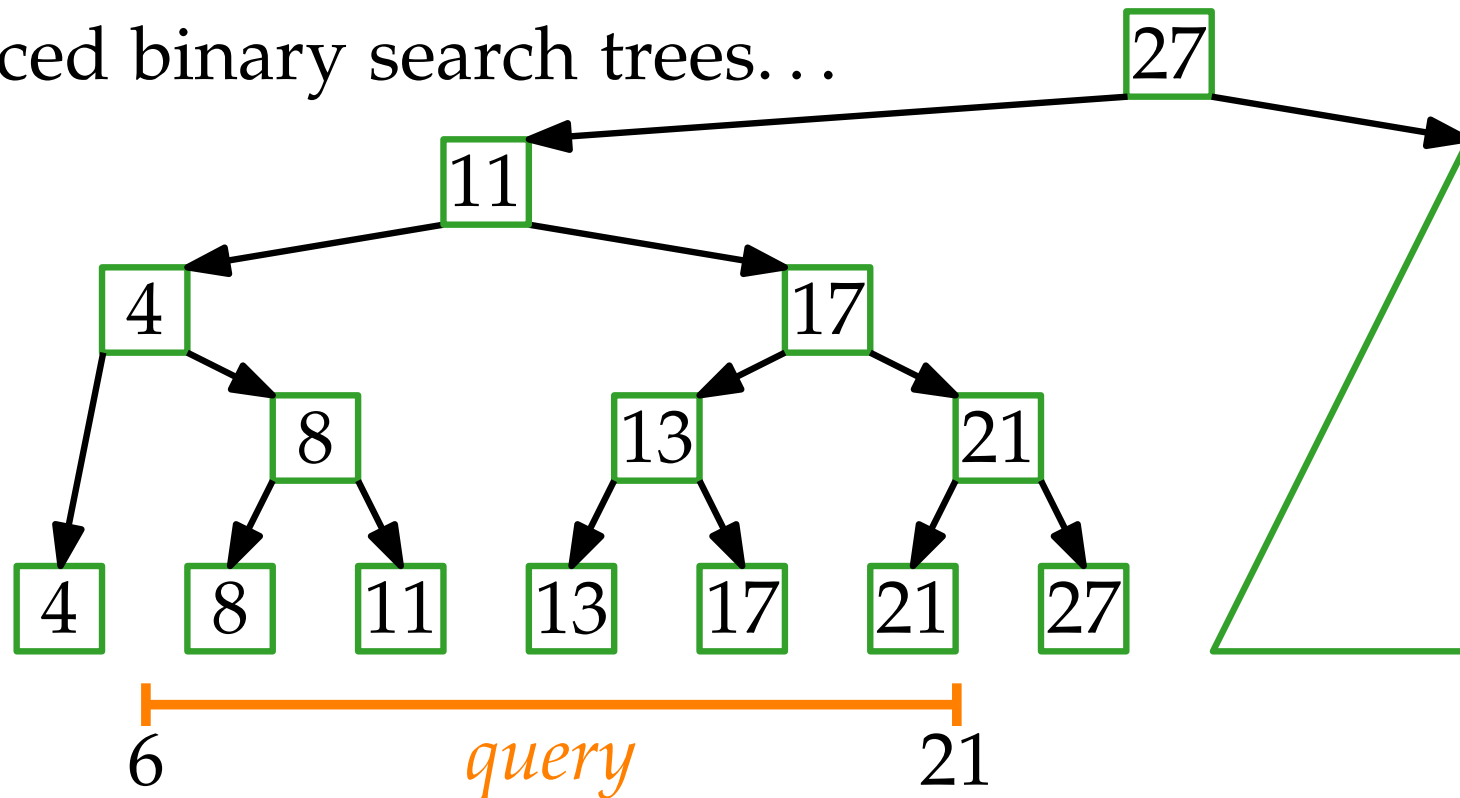
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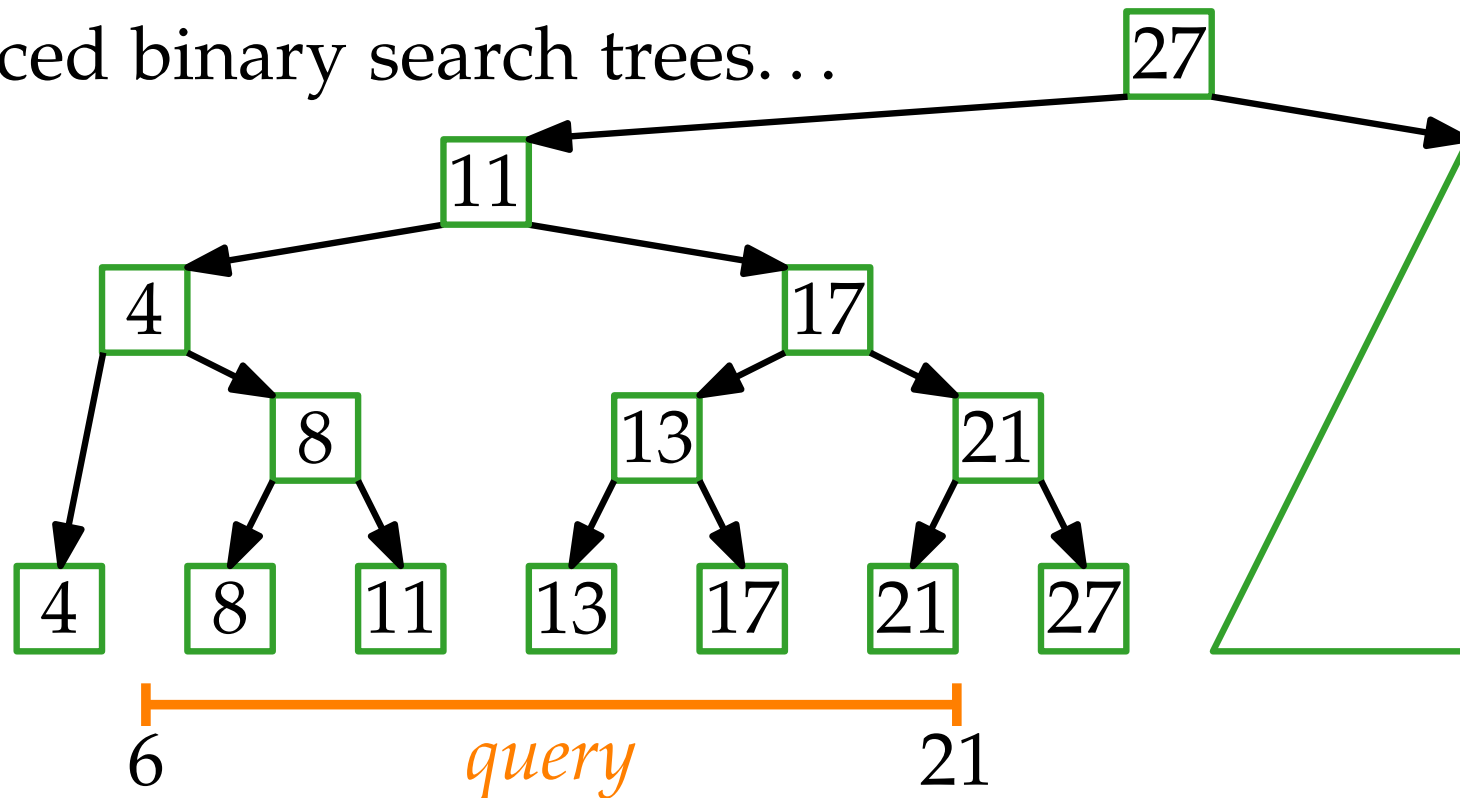
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① Search $x = 6$.



Small changes:

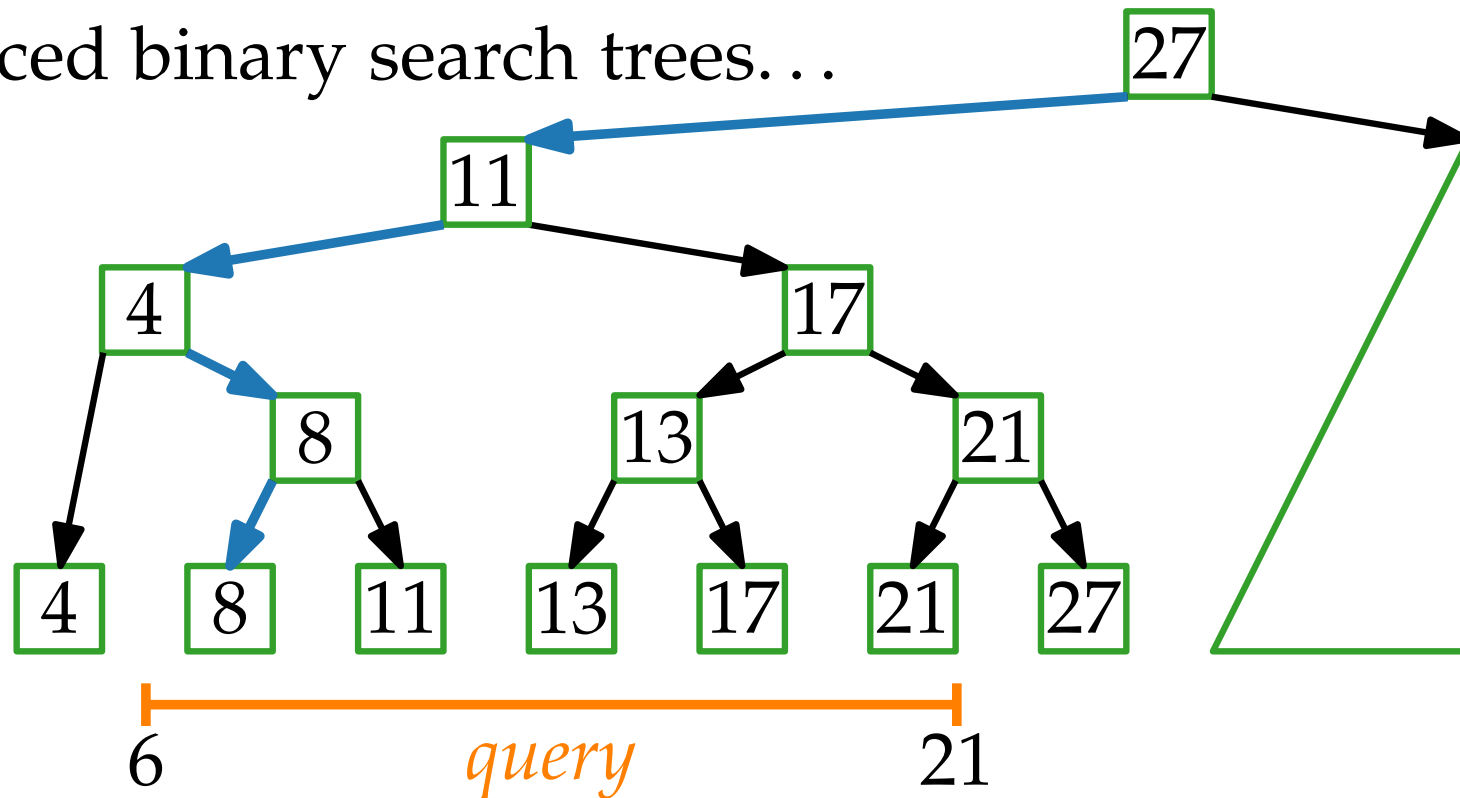
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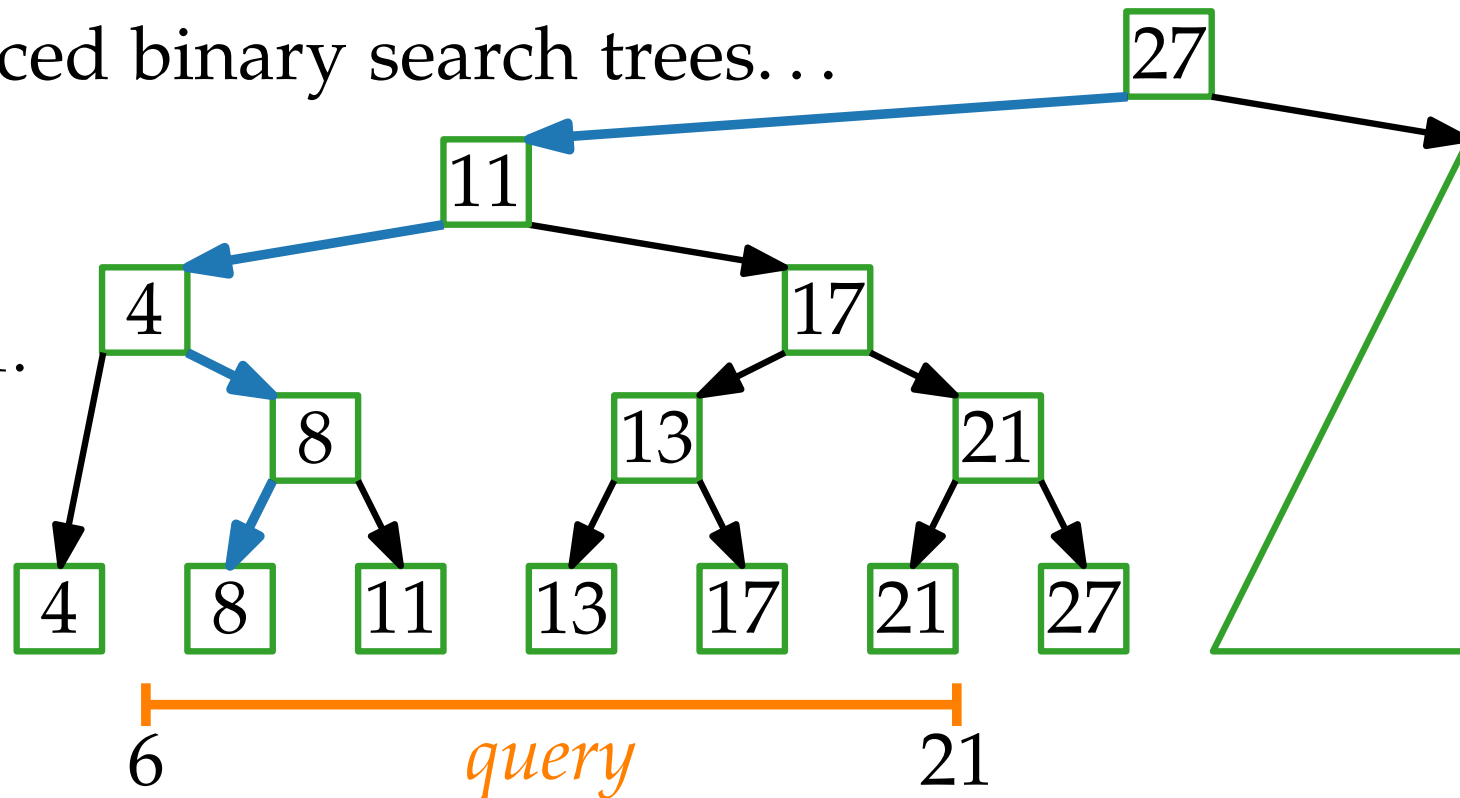
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1. Search $x = 6$.
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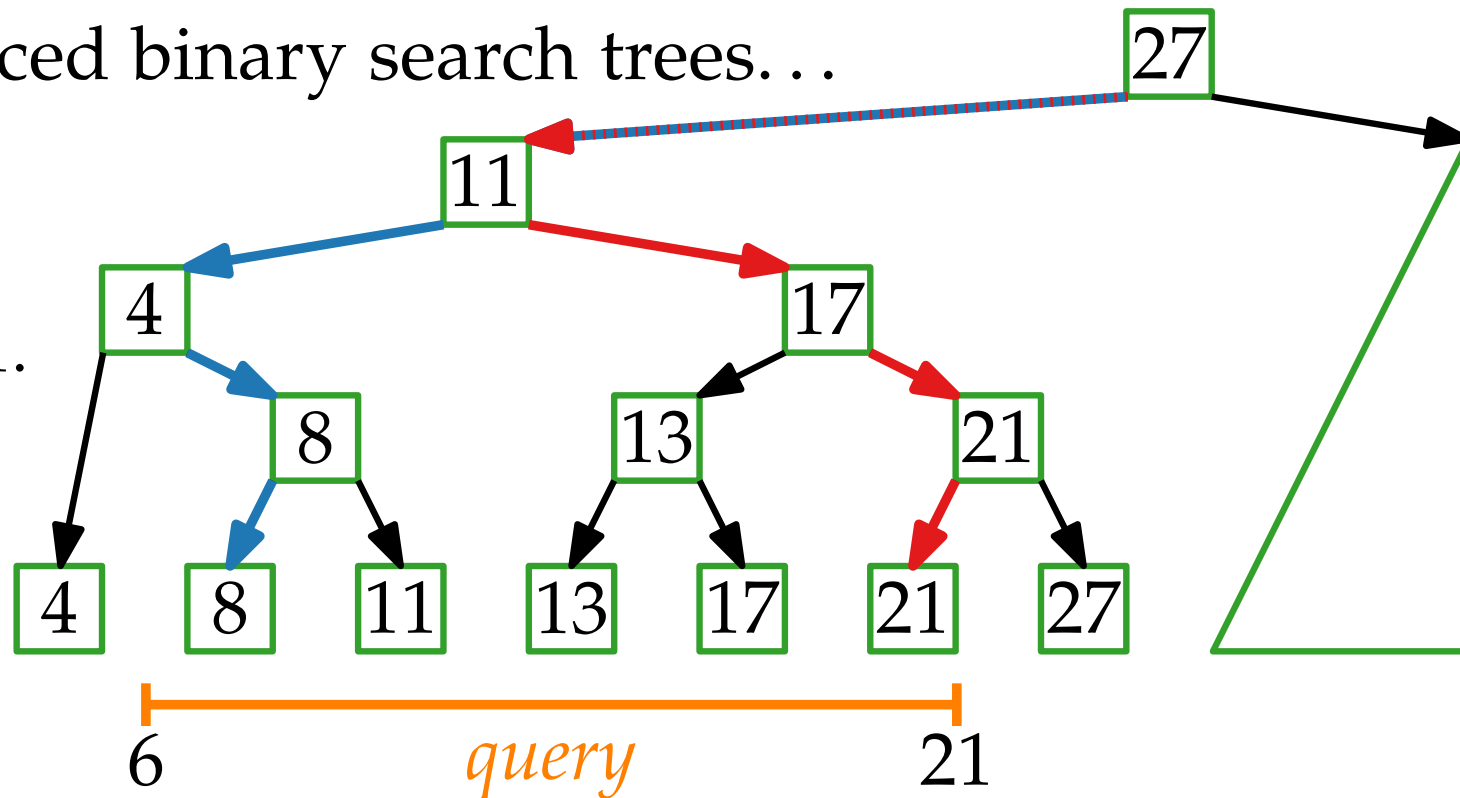
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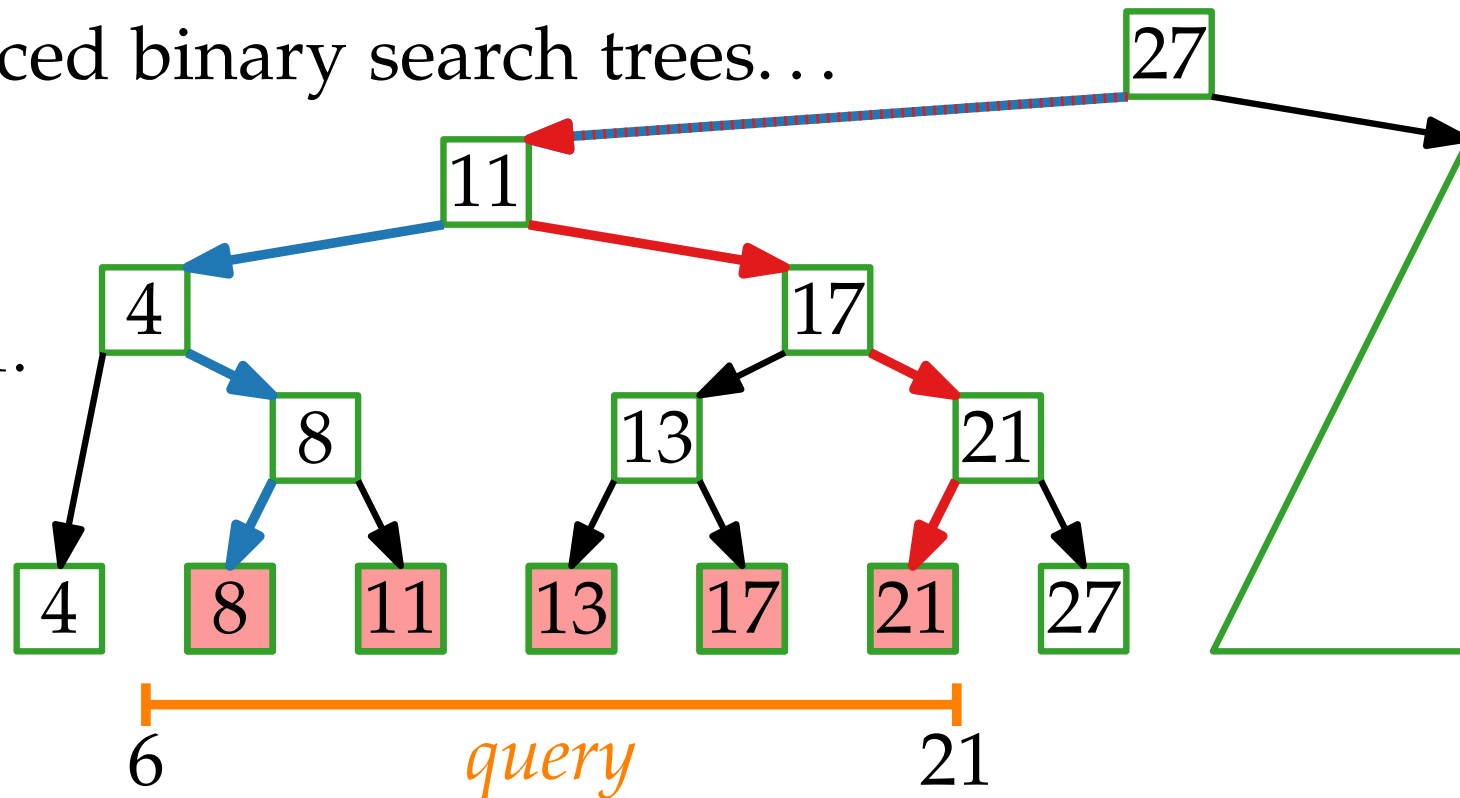
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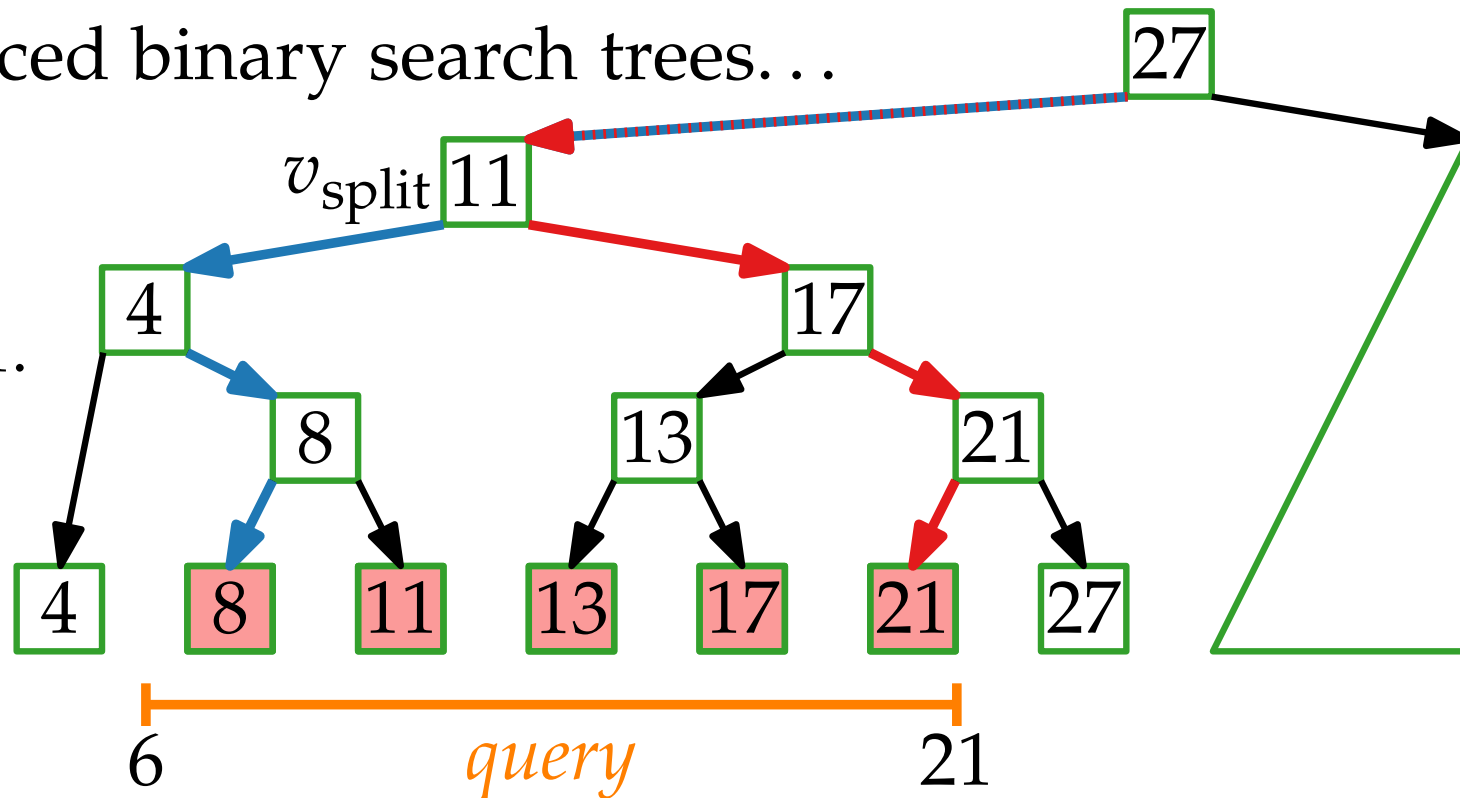
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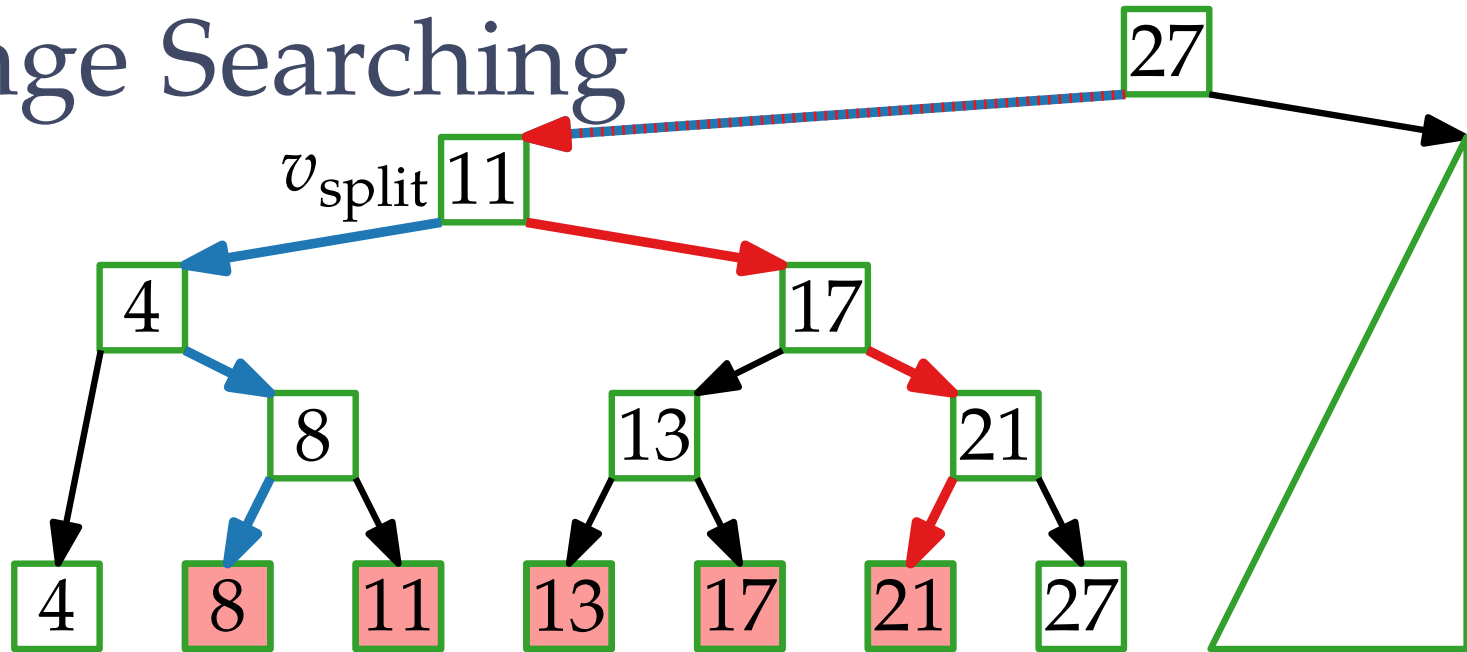
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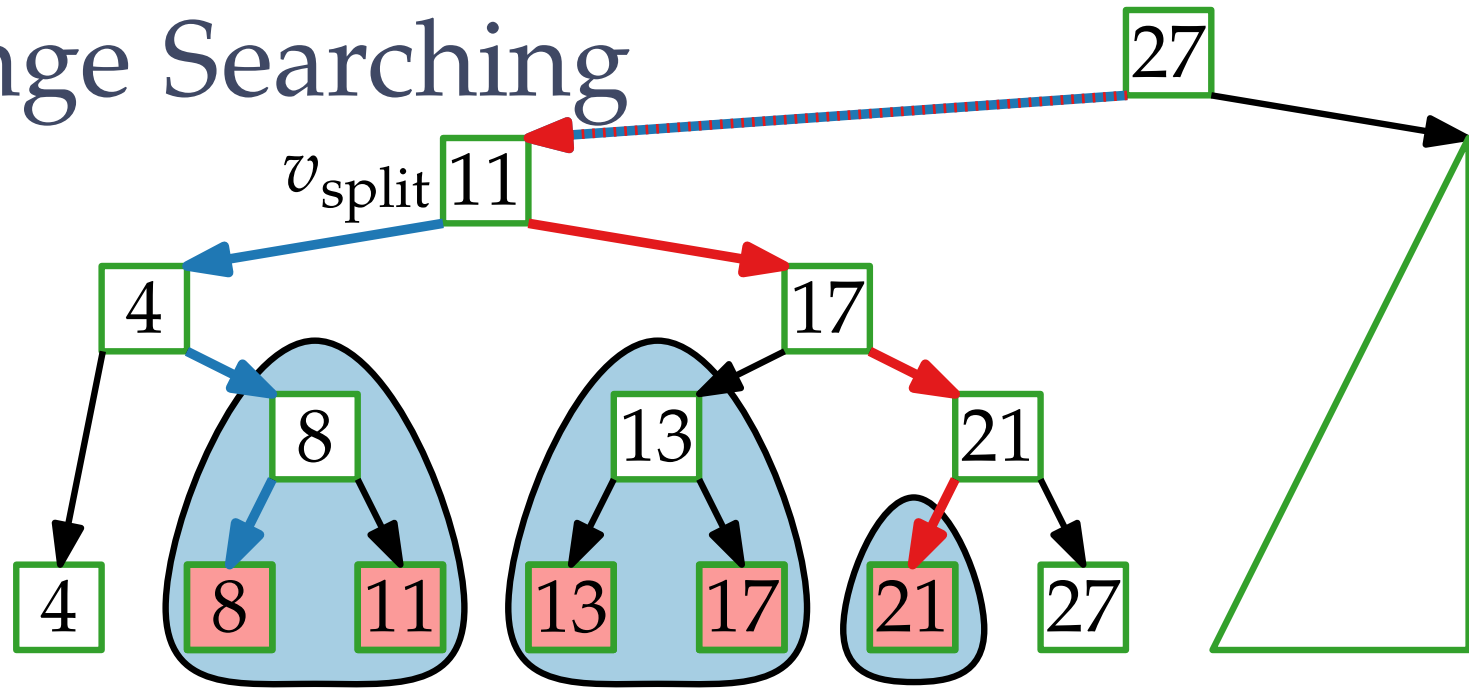
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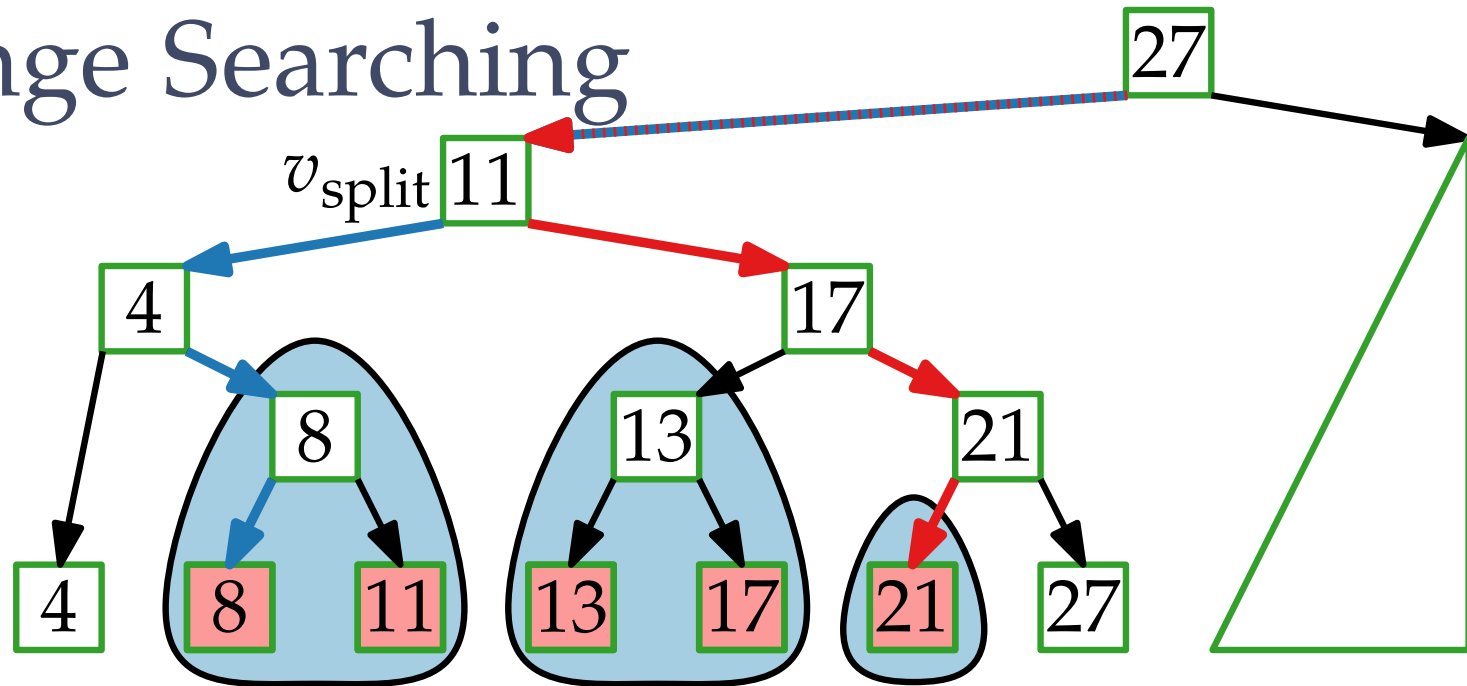
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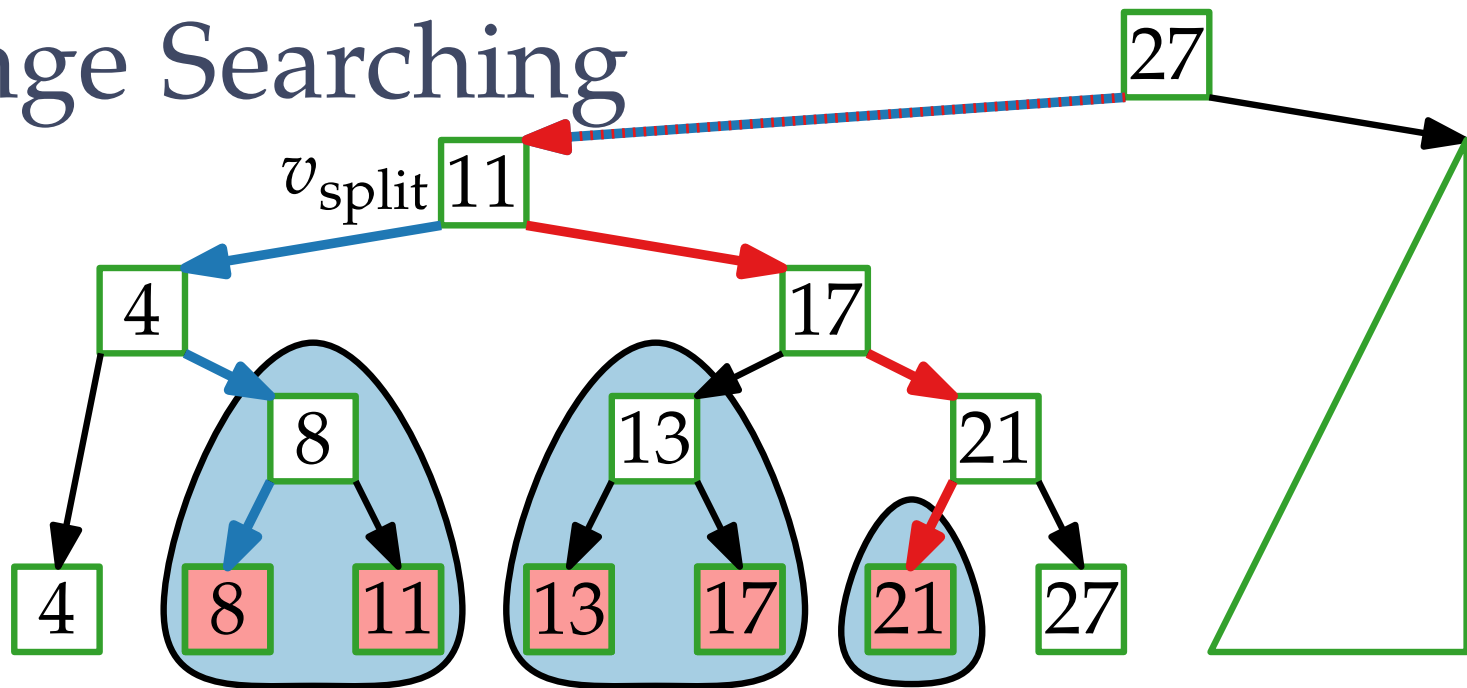


1D Range Searching



Observe: The result of a query is the disjoint union of at most $2h$ canonical subsets

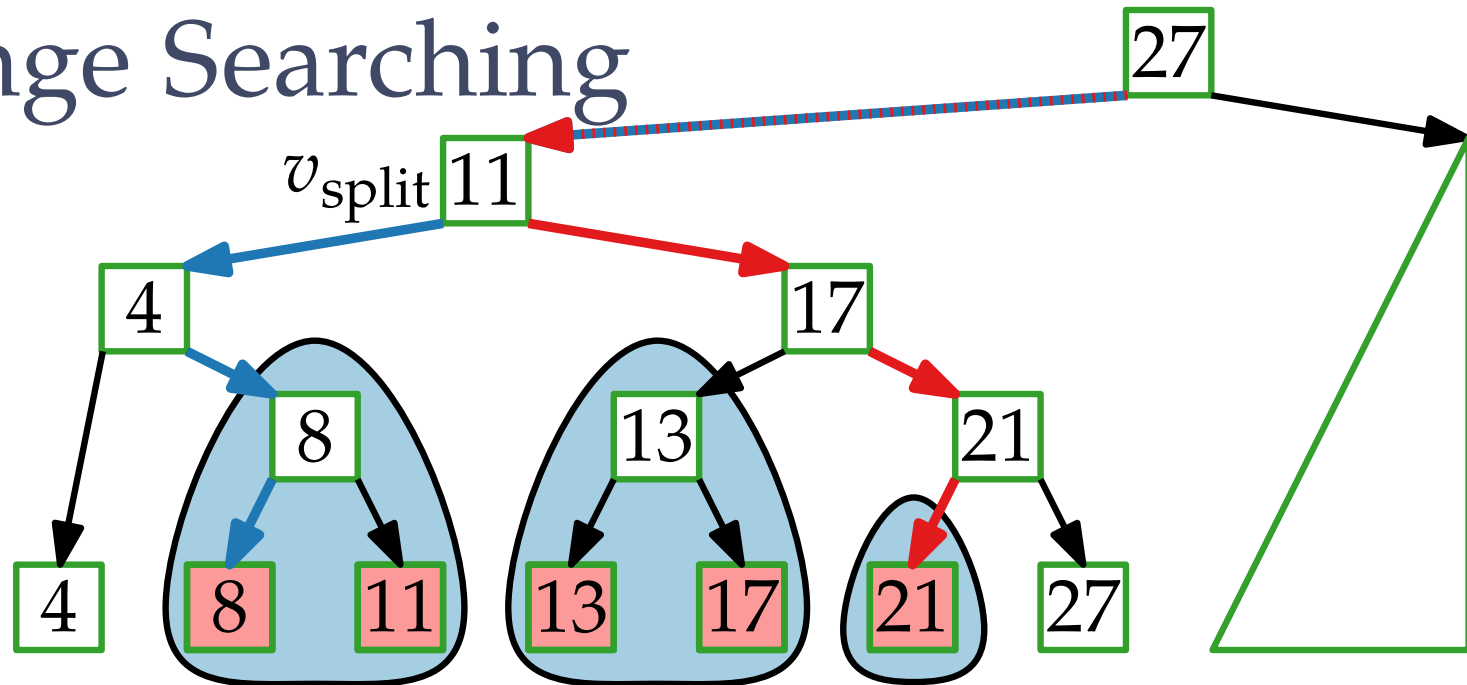
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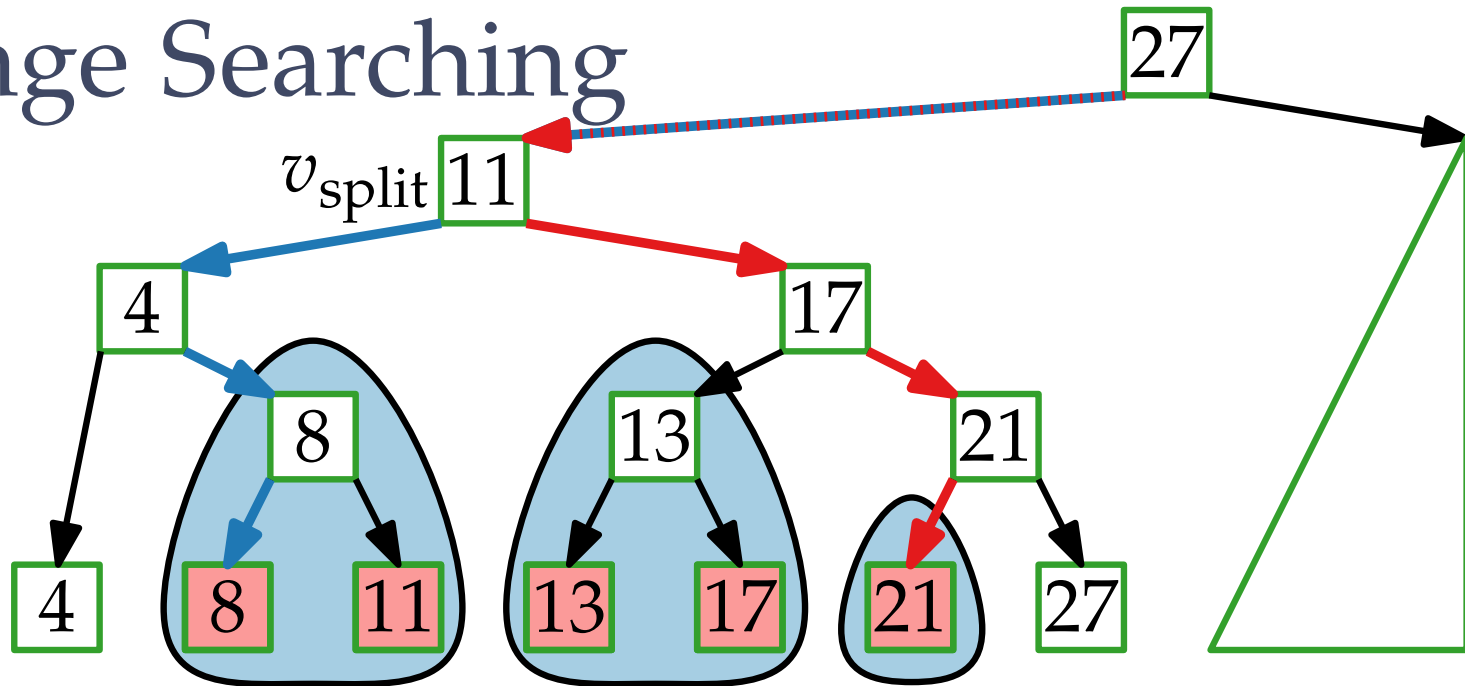
1D Range Searching



Observe: The result of a query is the disjoint union of at most $2h$ canonical subsets, where

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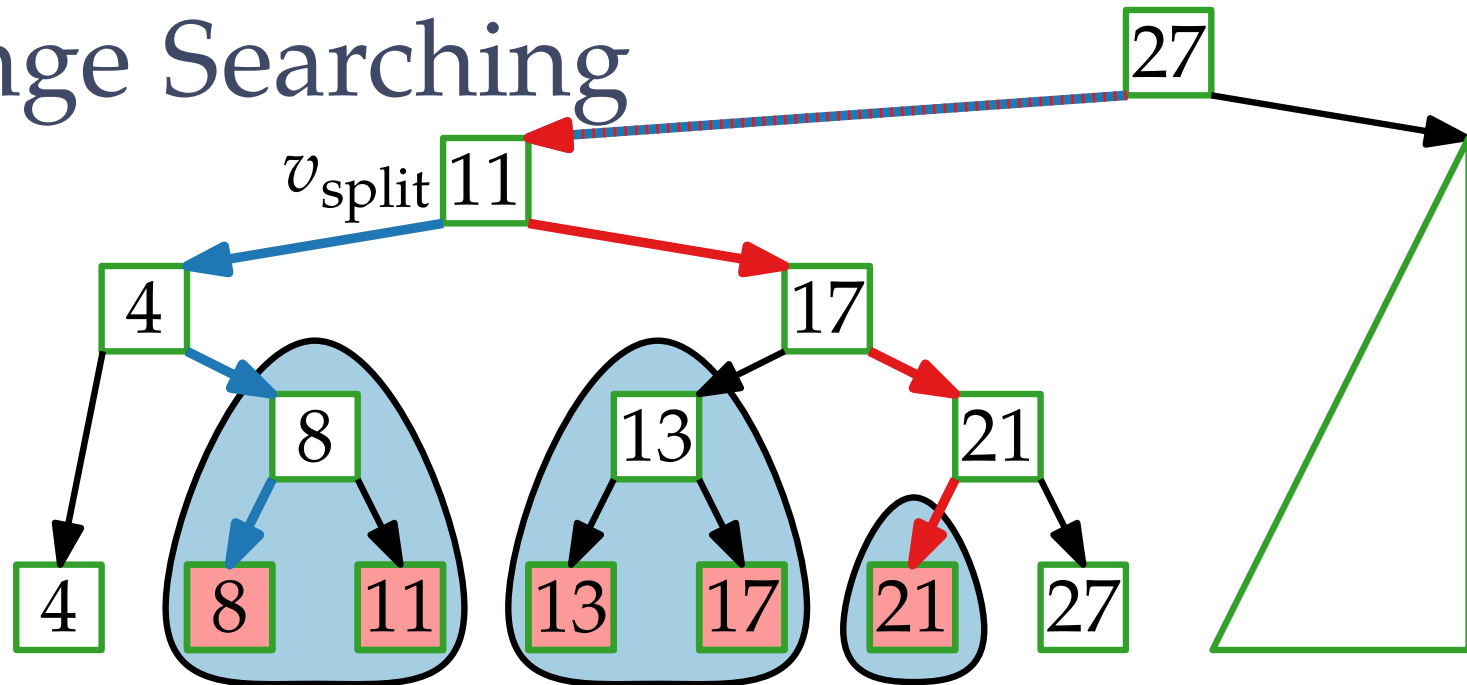


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Theorem. A set of n real numbers can be preprocessed in $O(n \log n)$ time and $O(n)$ space such that 1D range queries take time.

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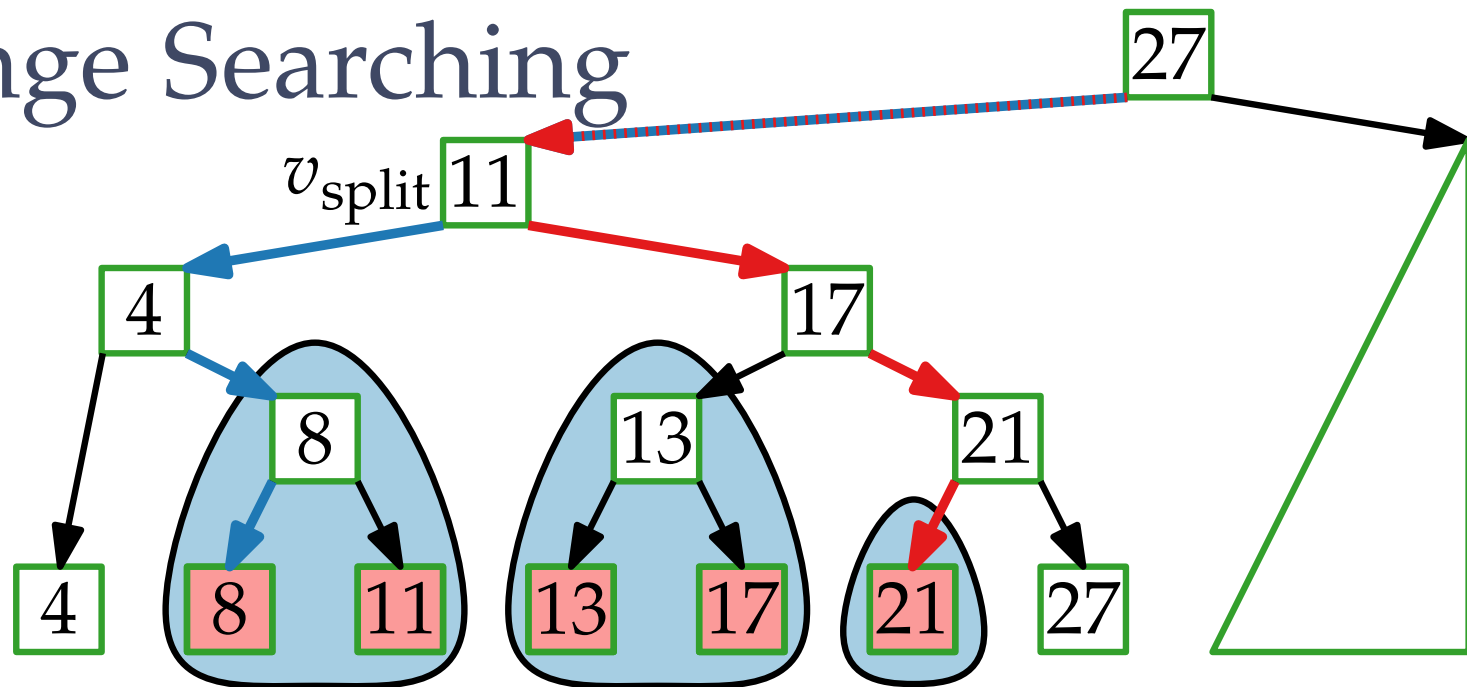


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output sensitive!

Extensions to 2D

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Computational Geometry

Lecture 5: Orthogonal Range Queries or Fast Access to Data Bases

Part II: Kd-Trees: Construction

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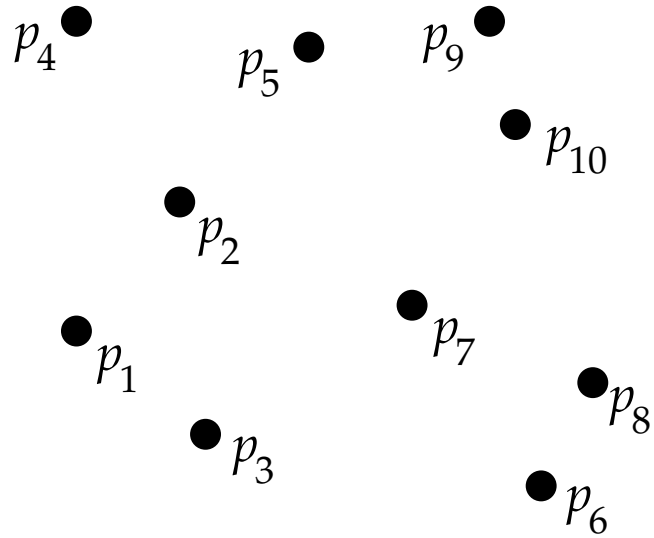
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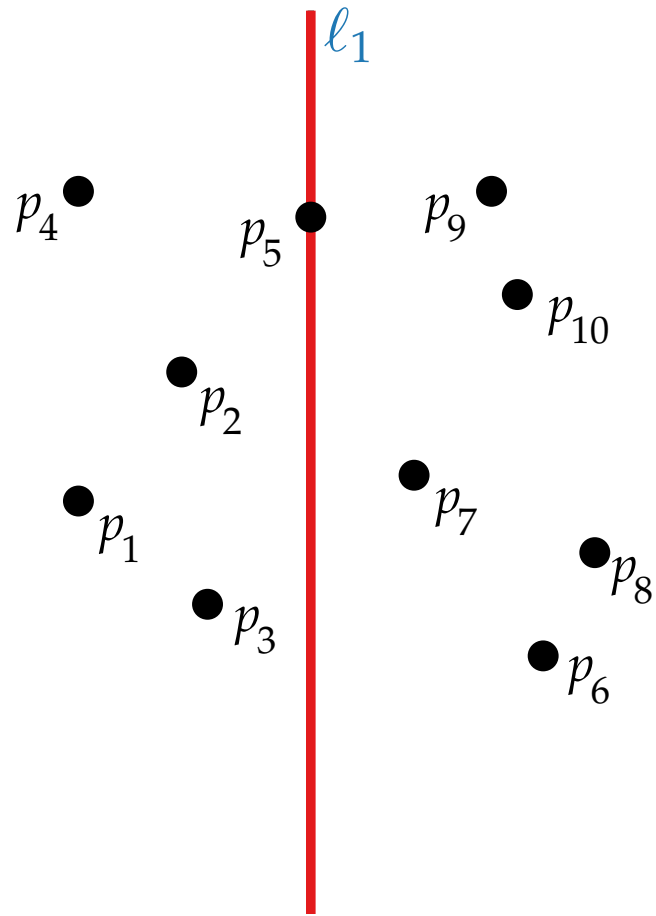
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Kd-Trees: Example

[dBCvKO'08]

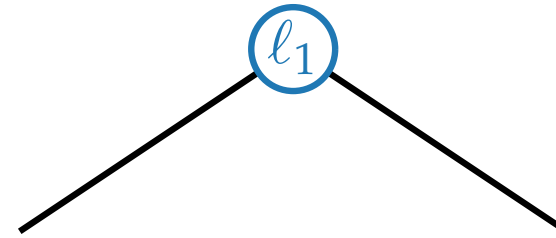
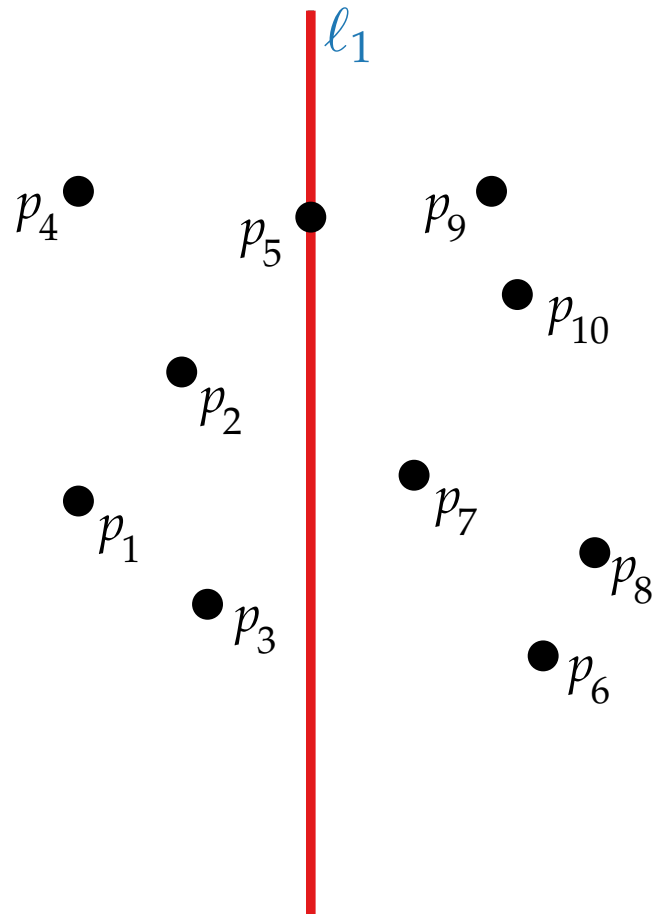


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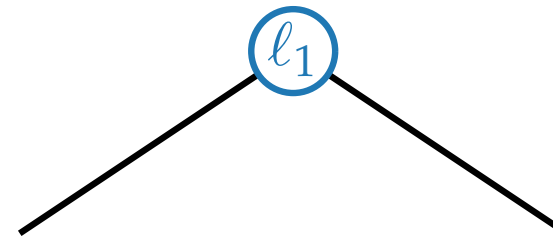
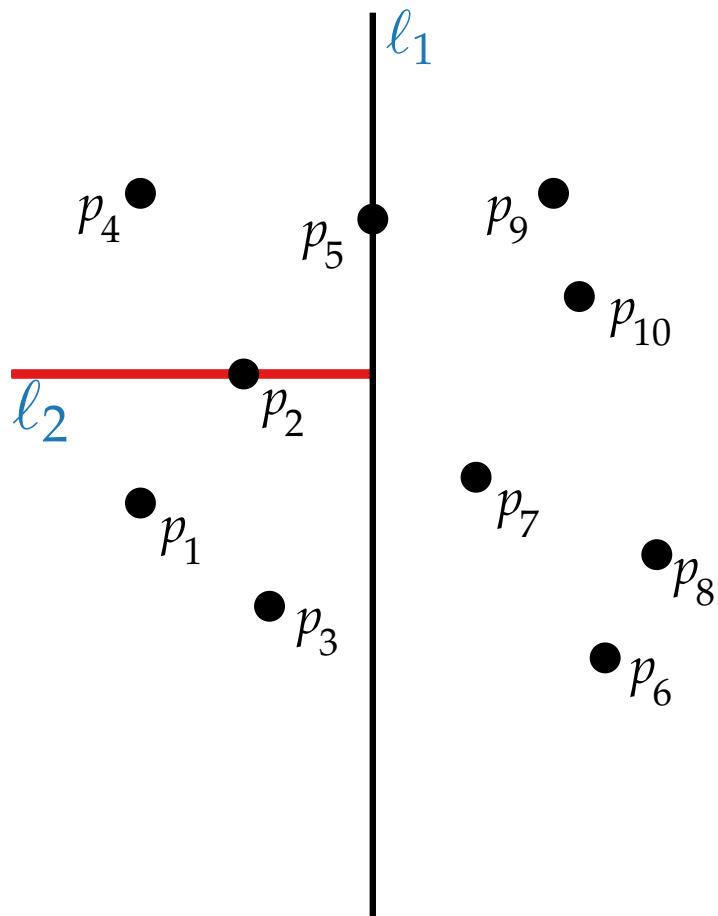
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[dBCvKO'08]

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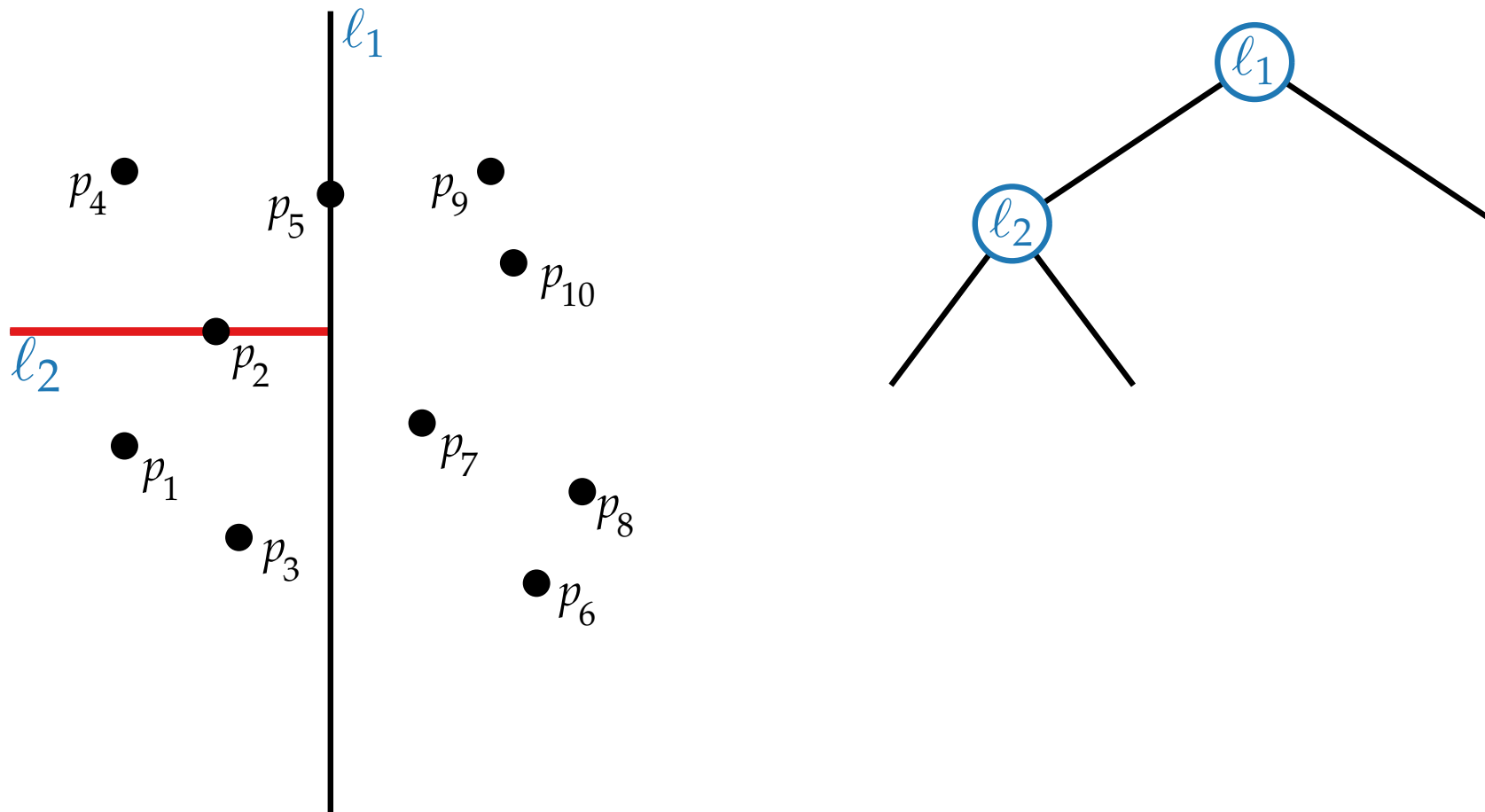
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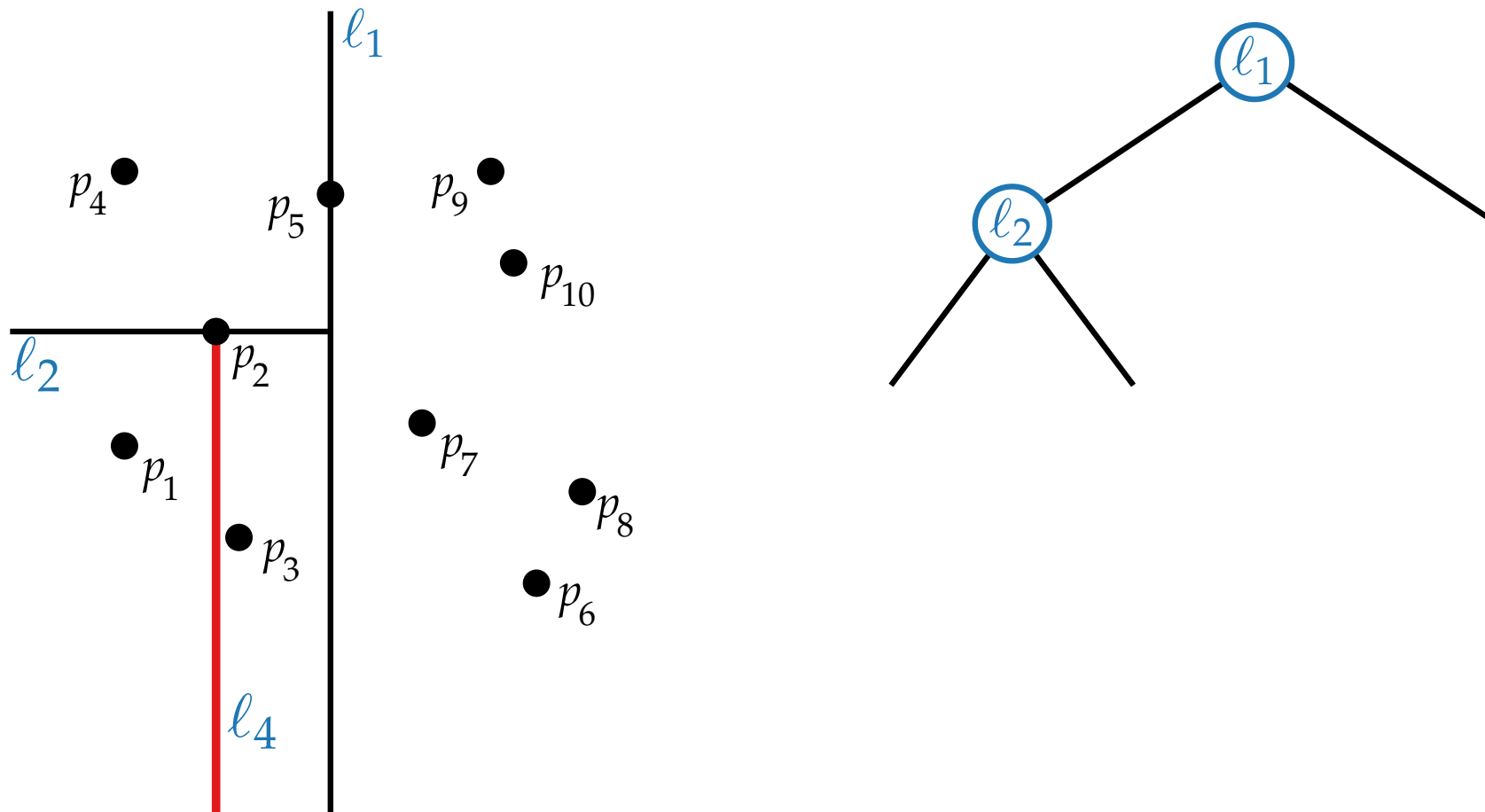
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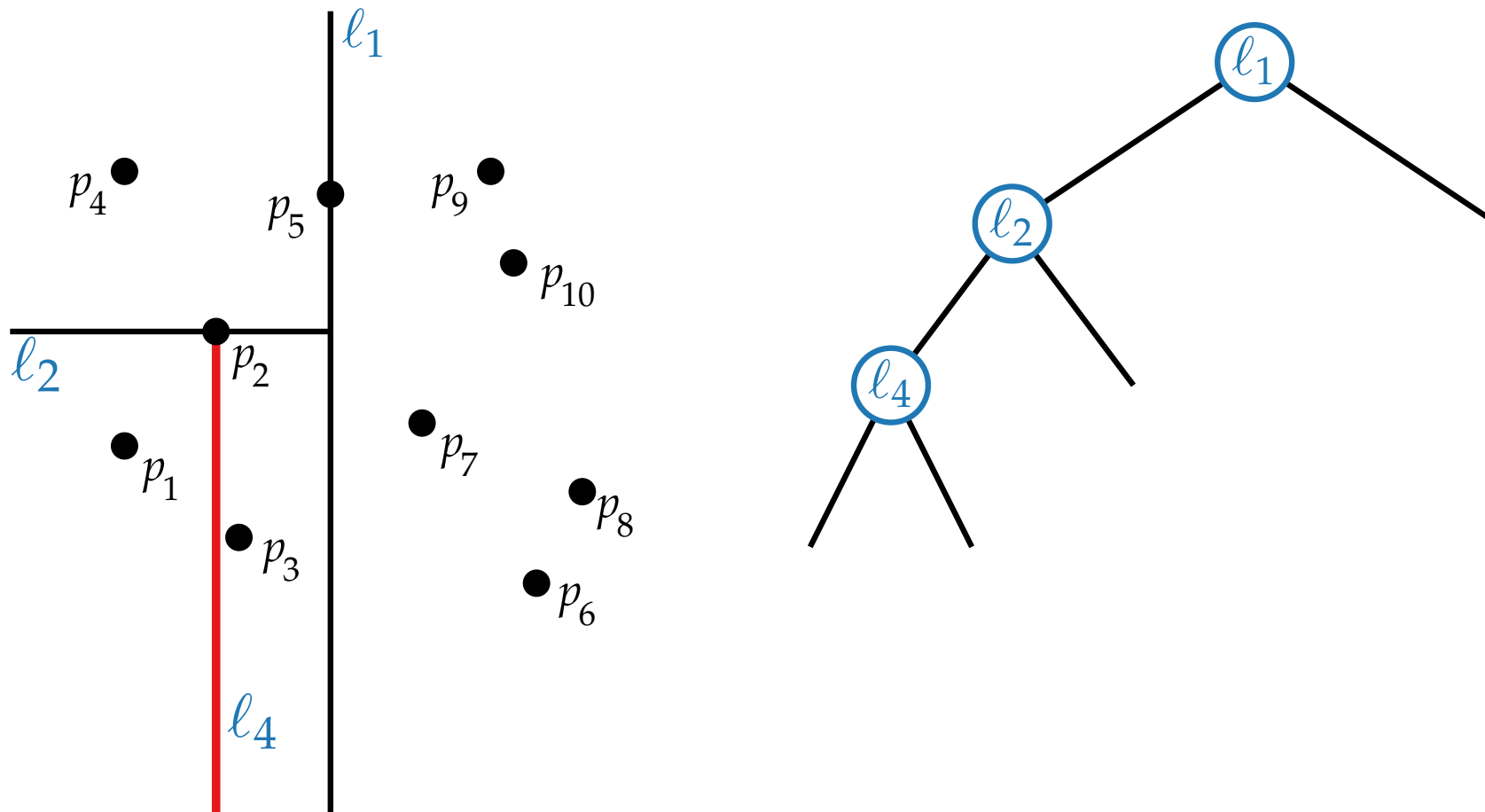
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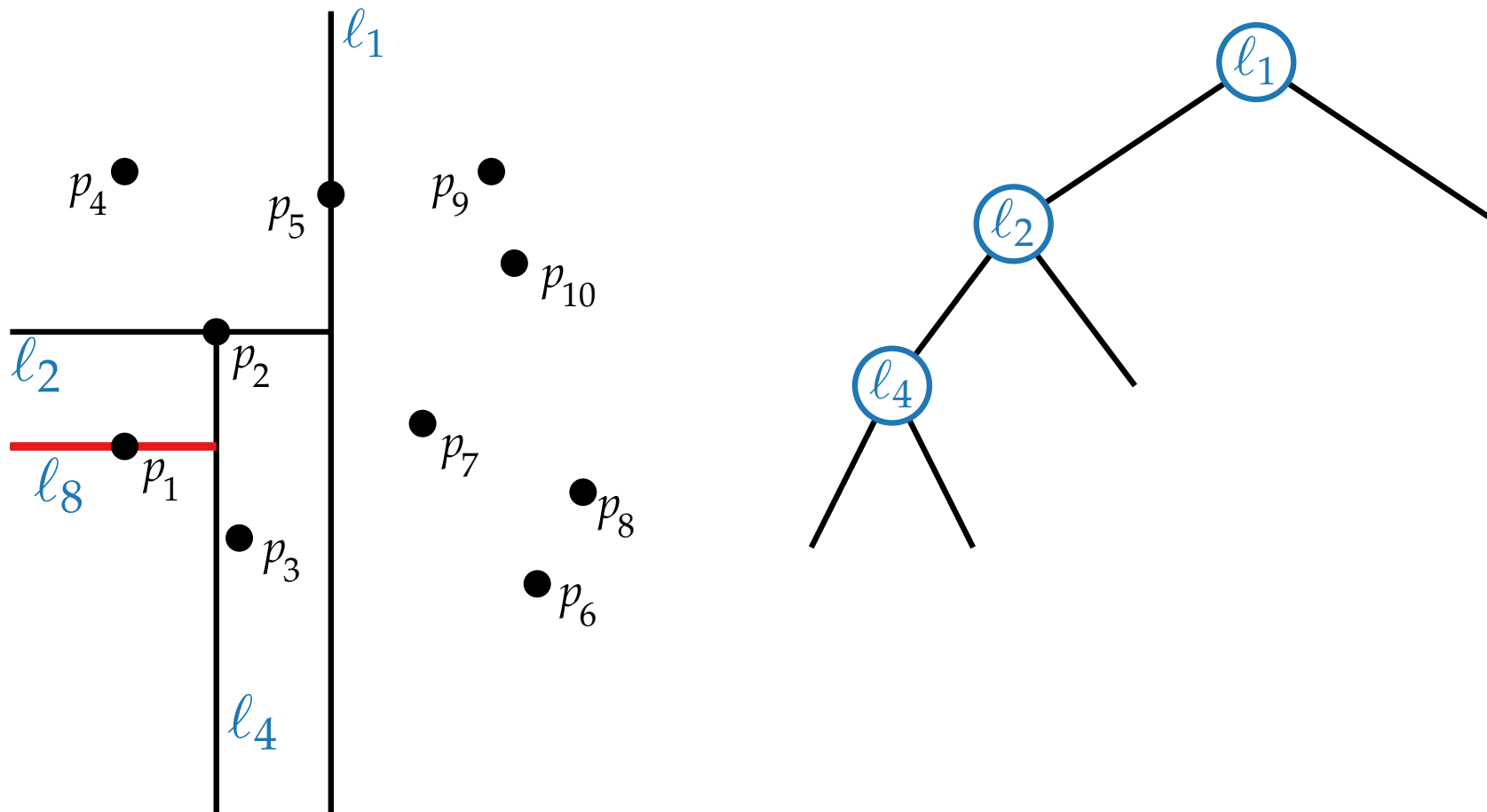
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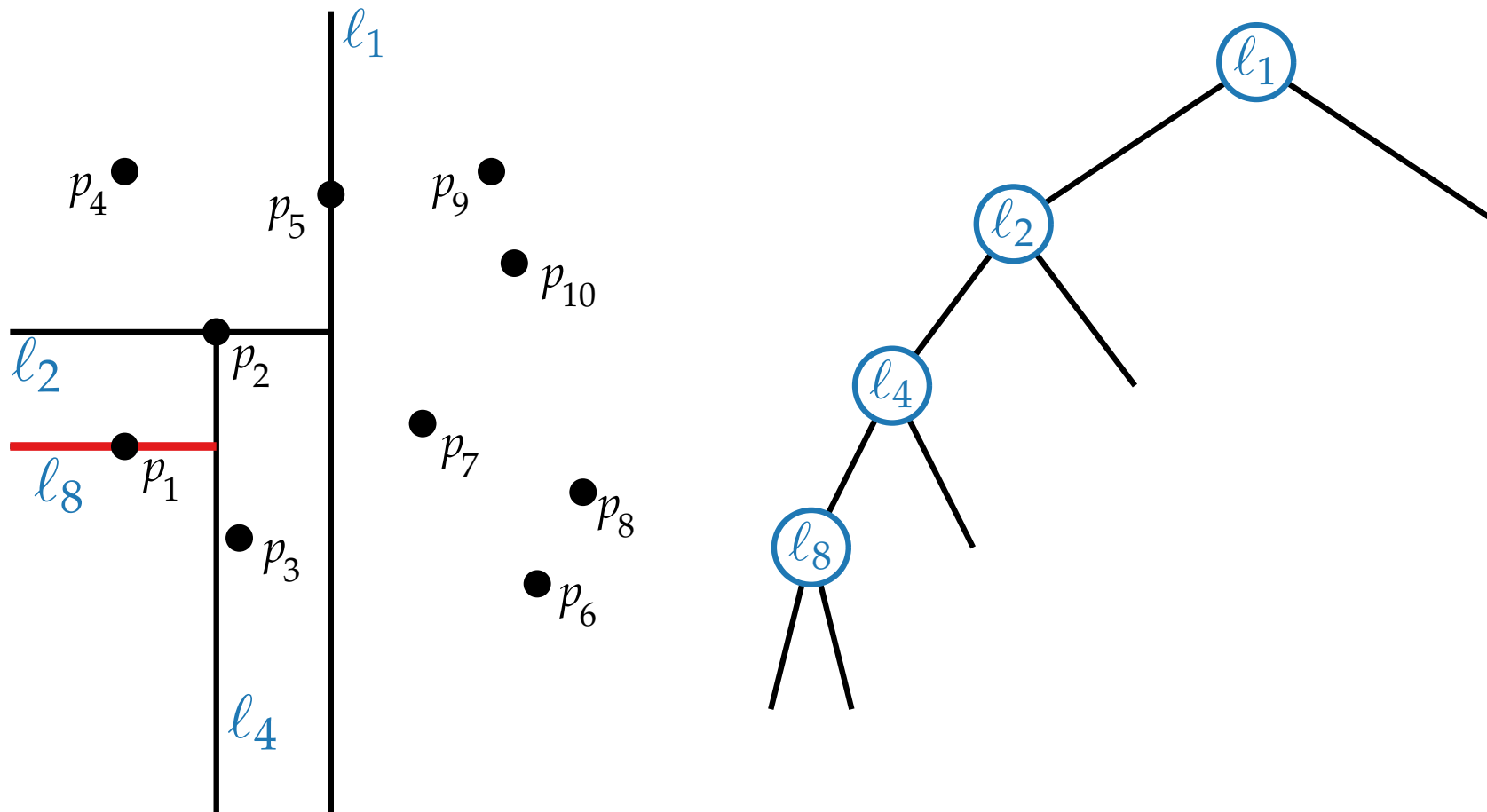
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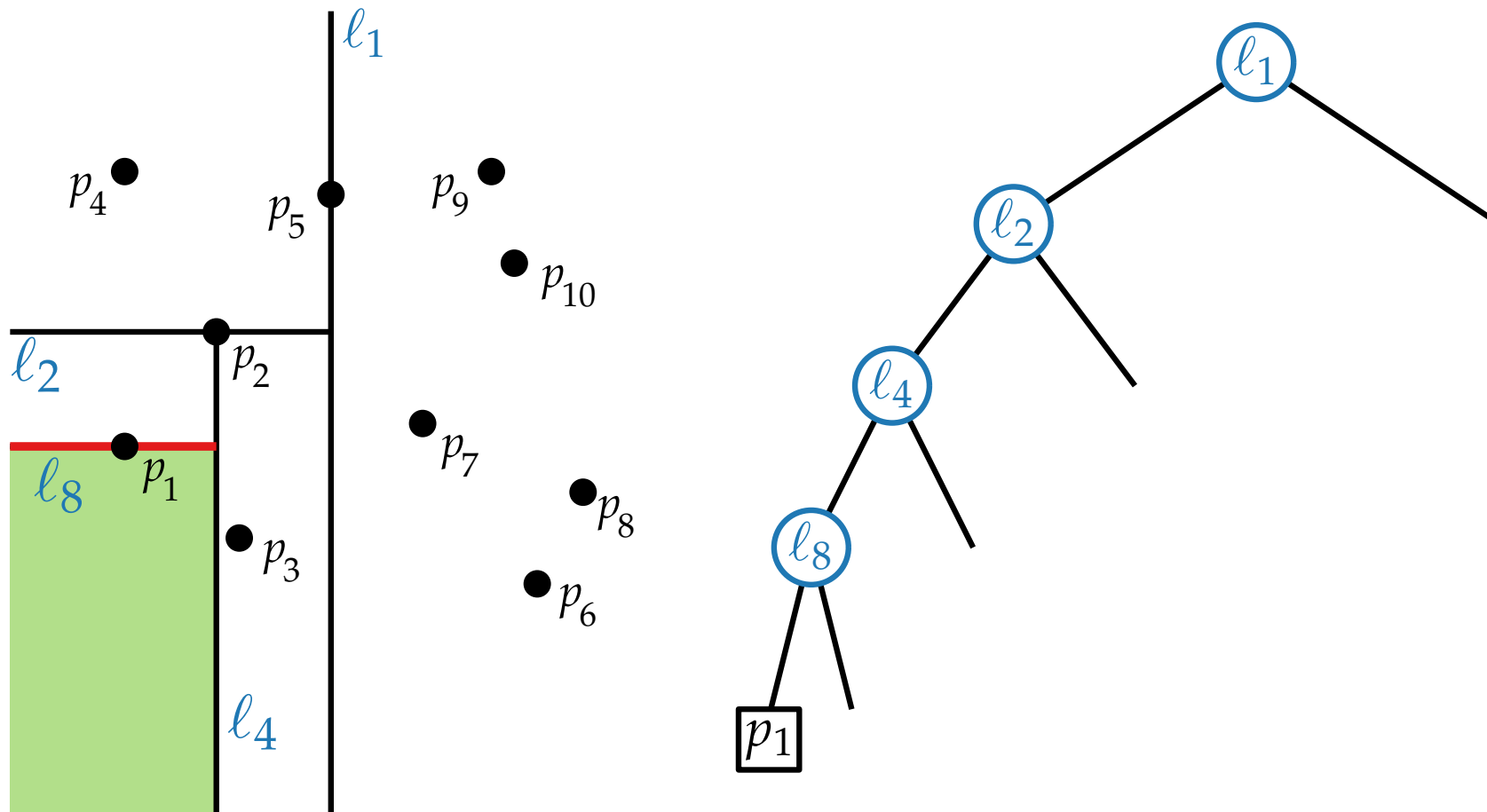
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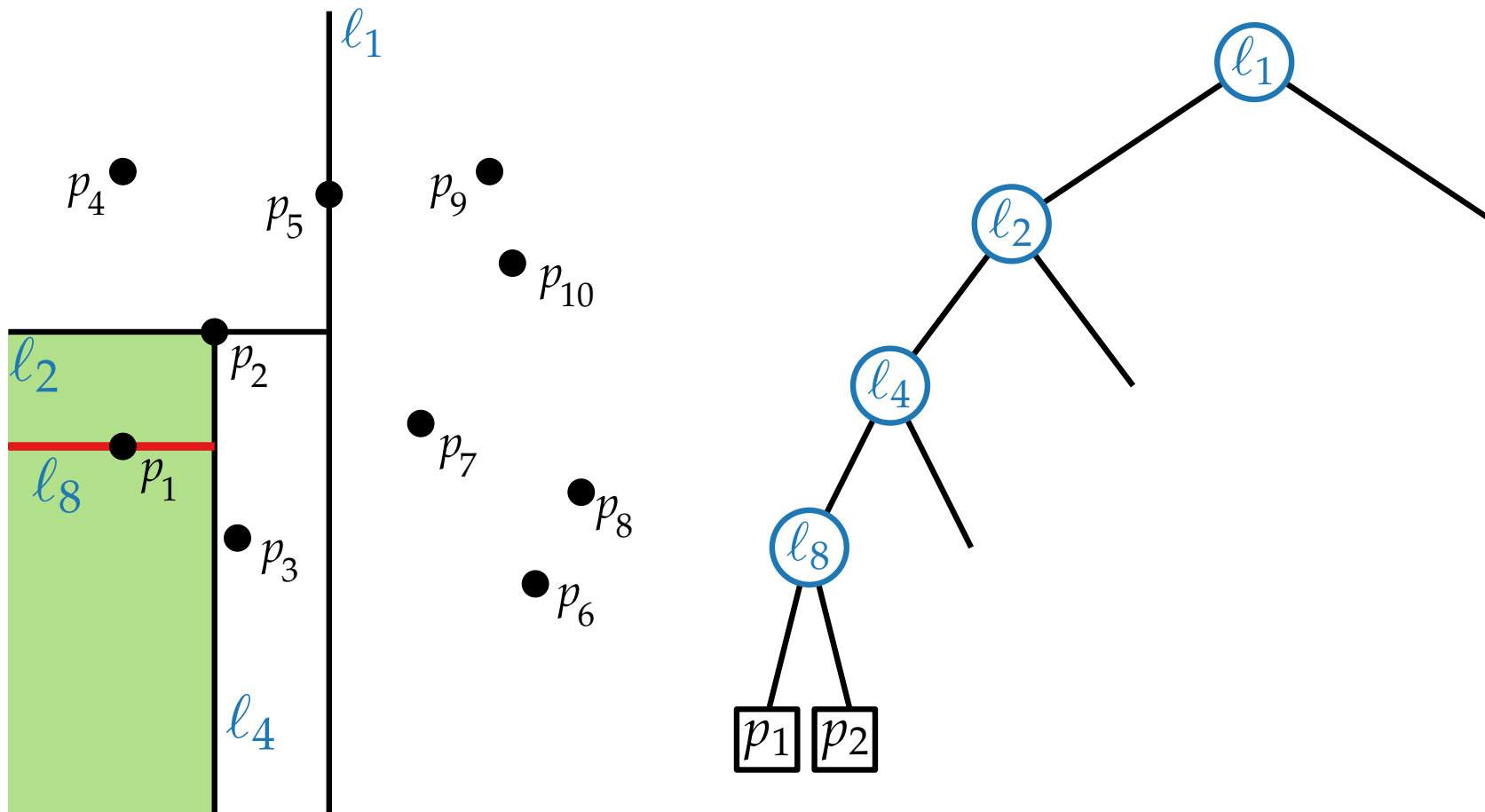
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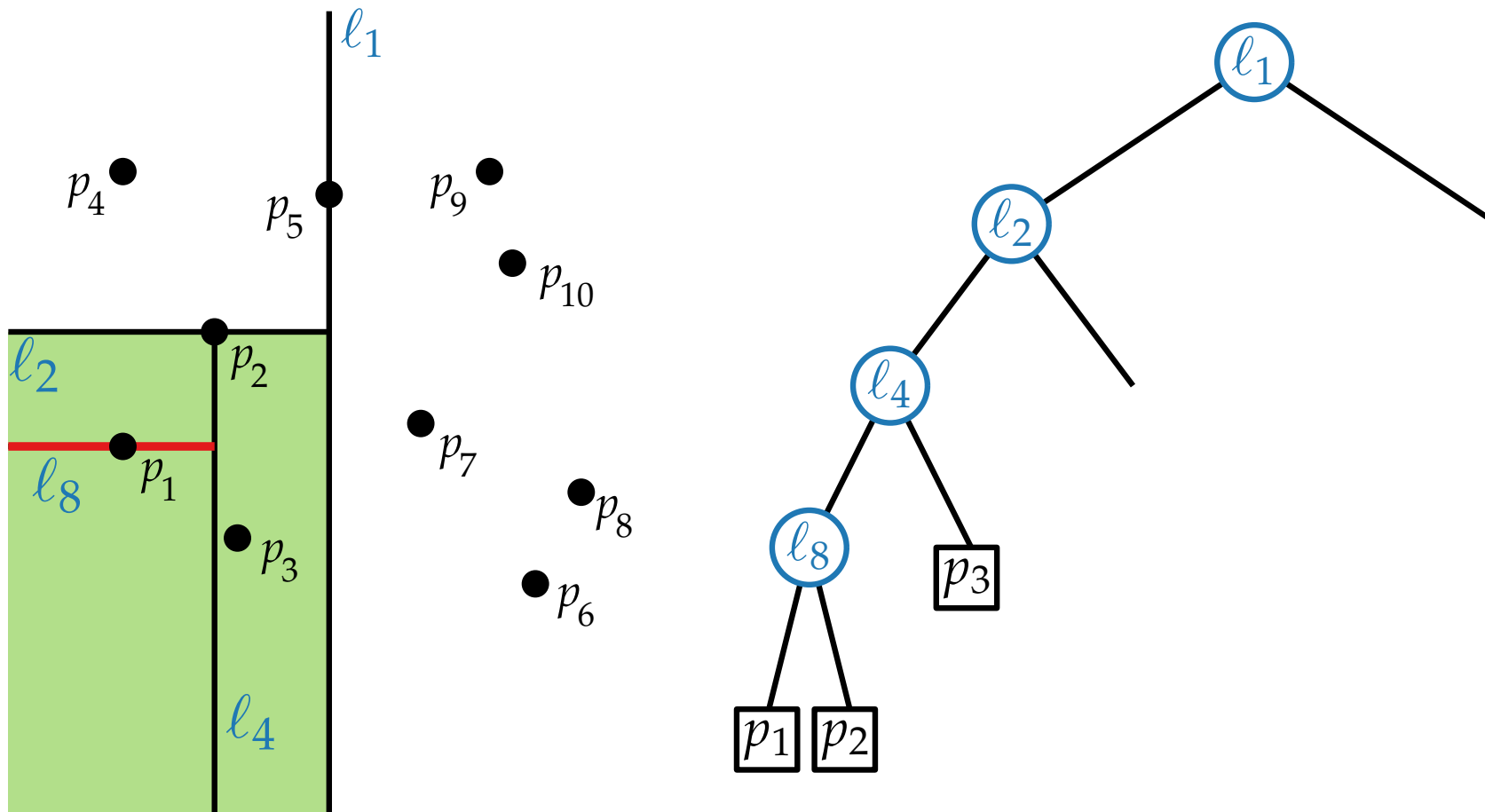
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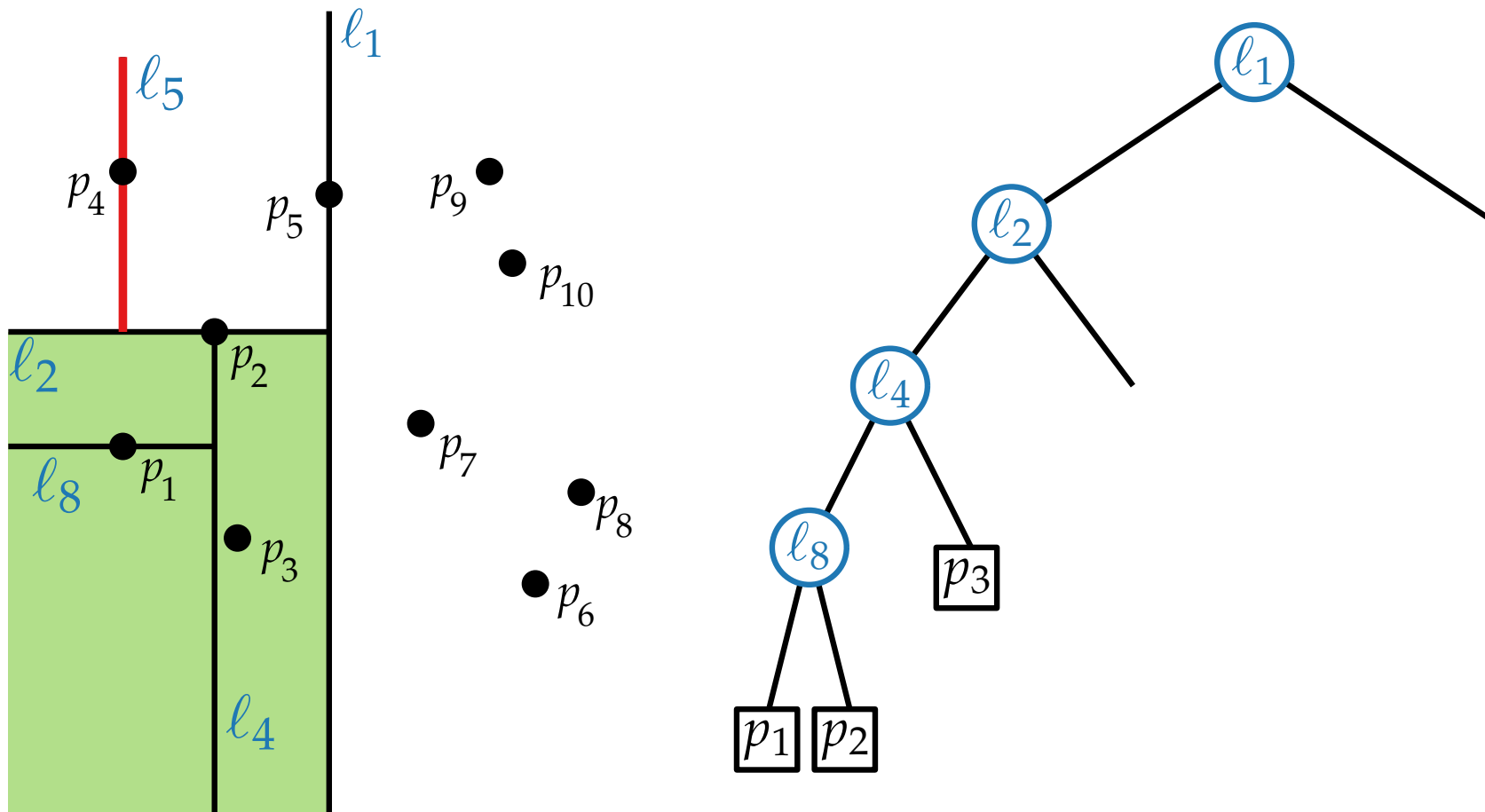
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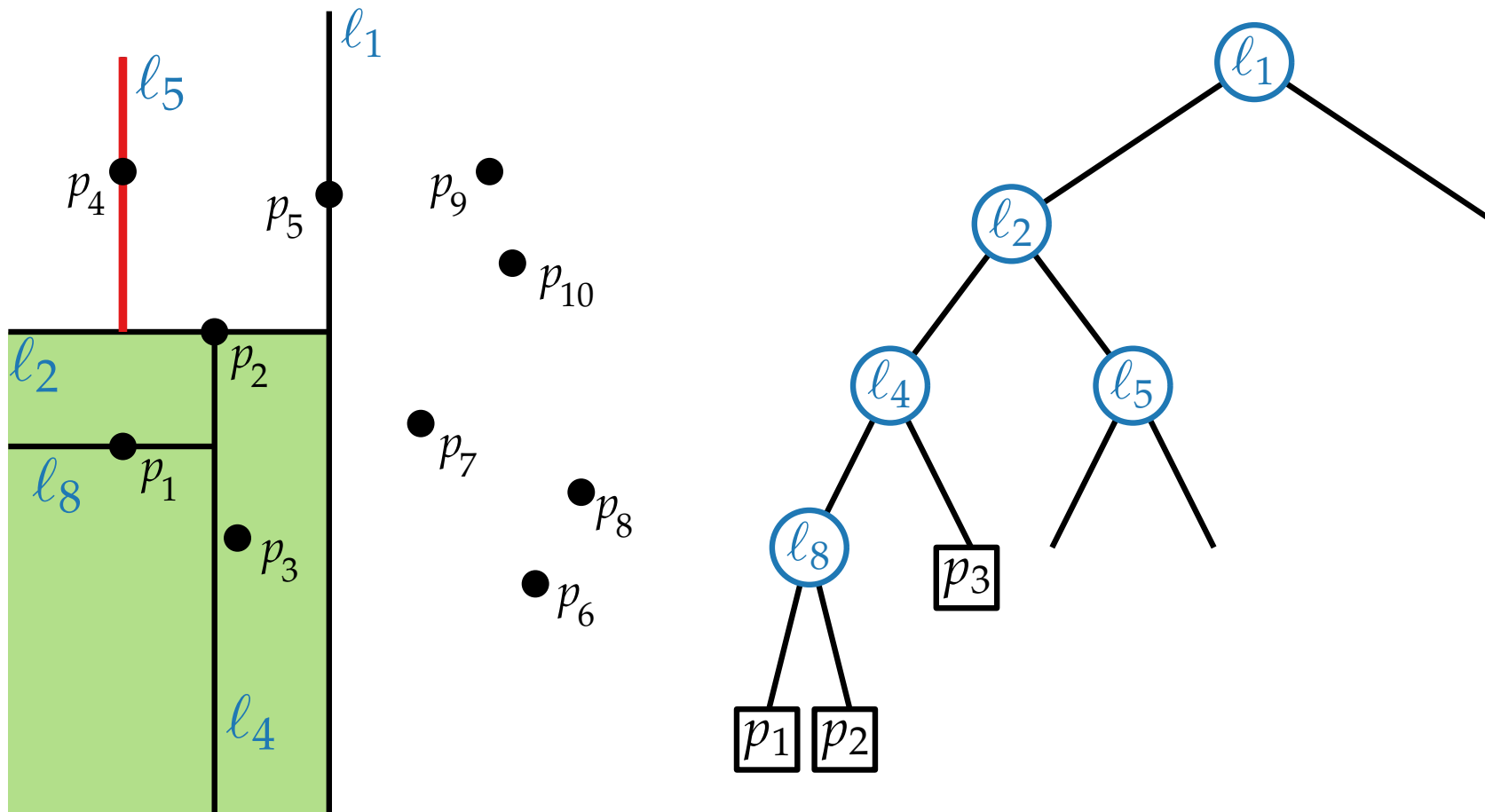
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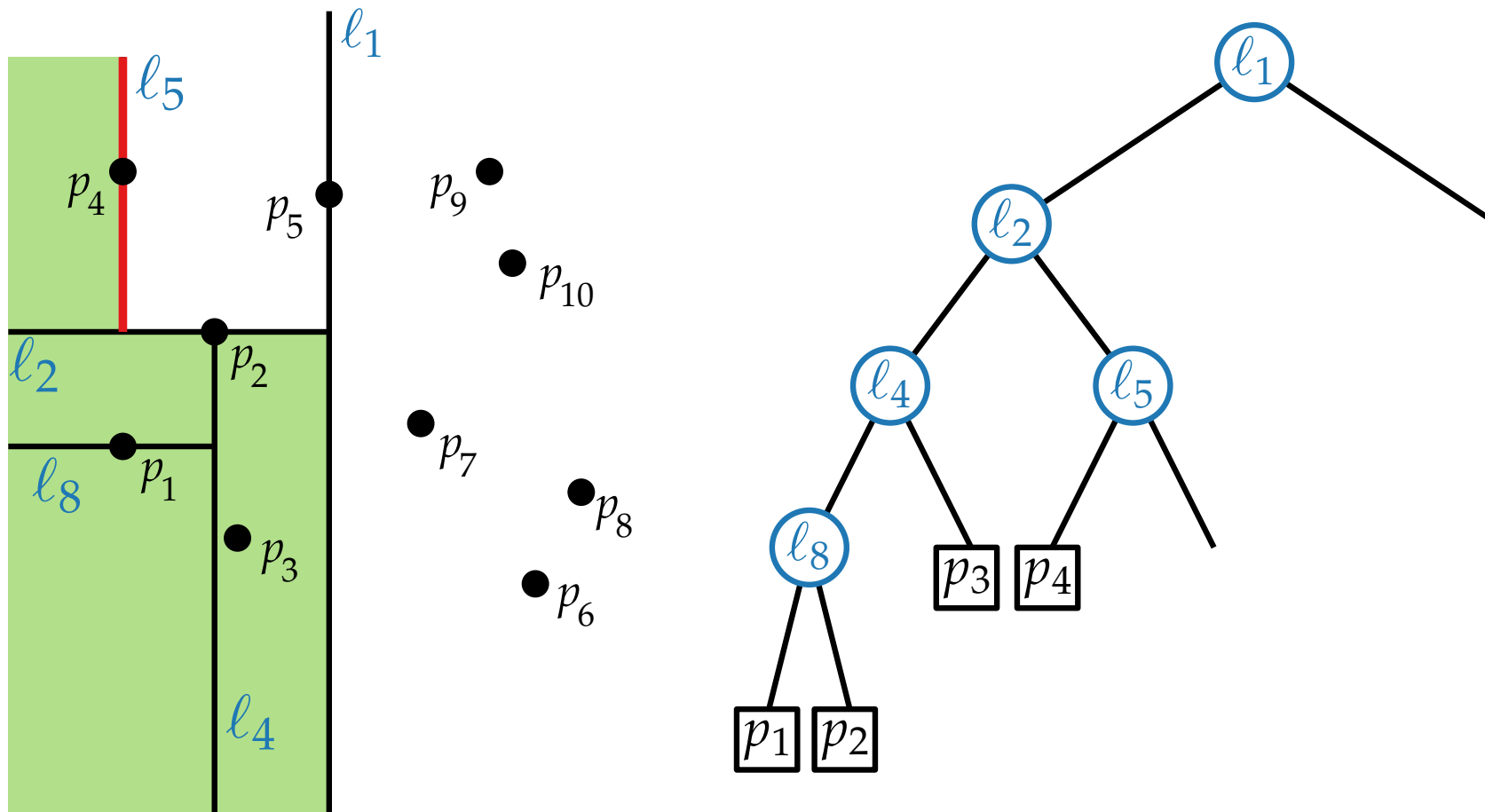
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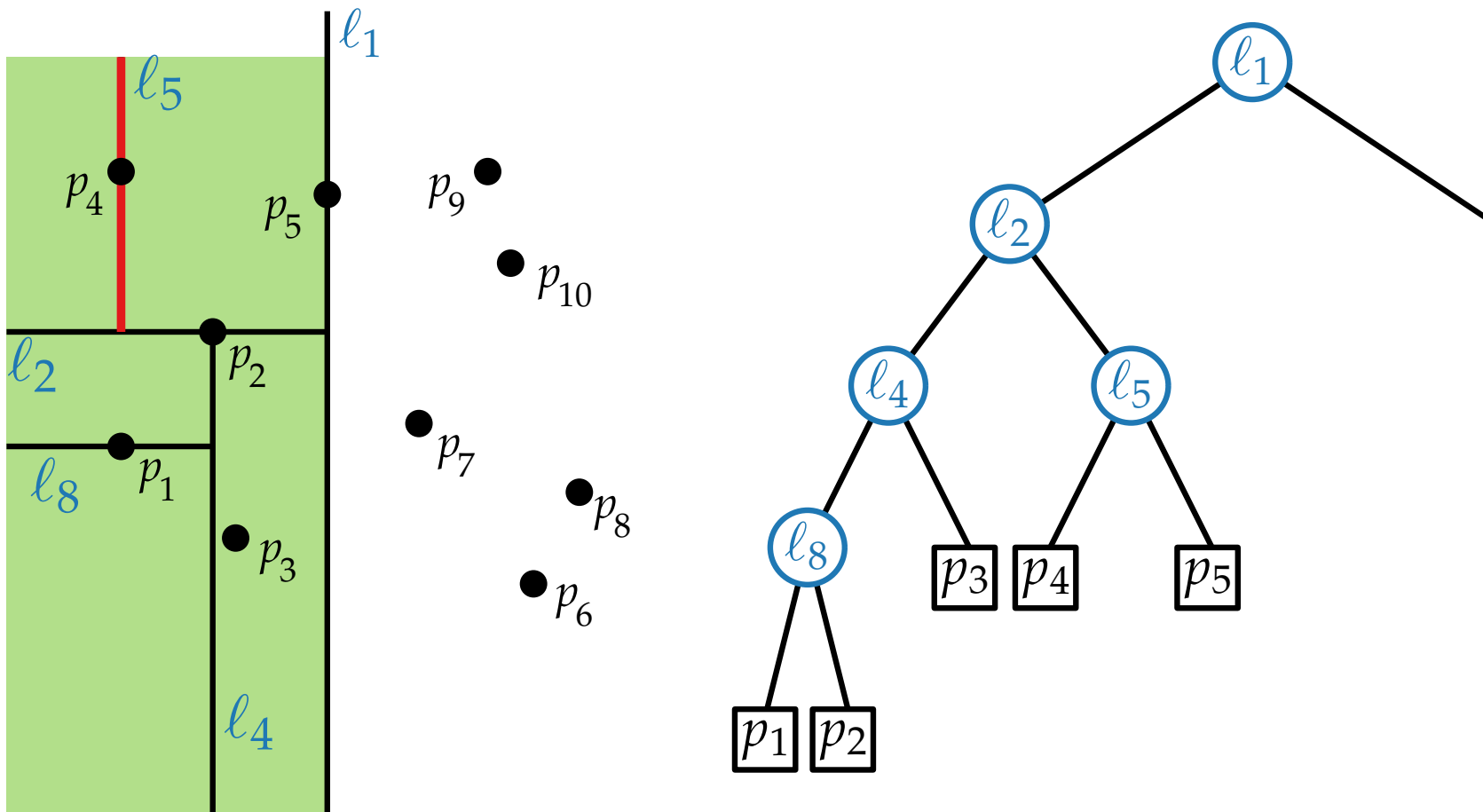
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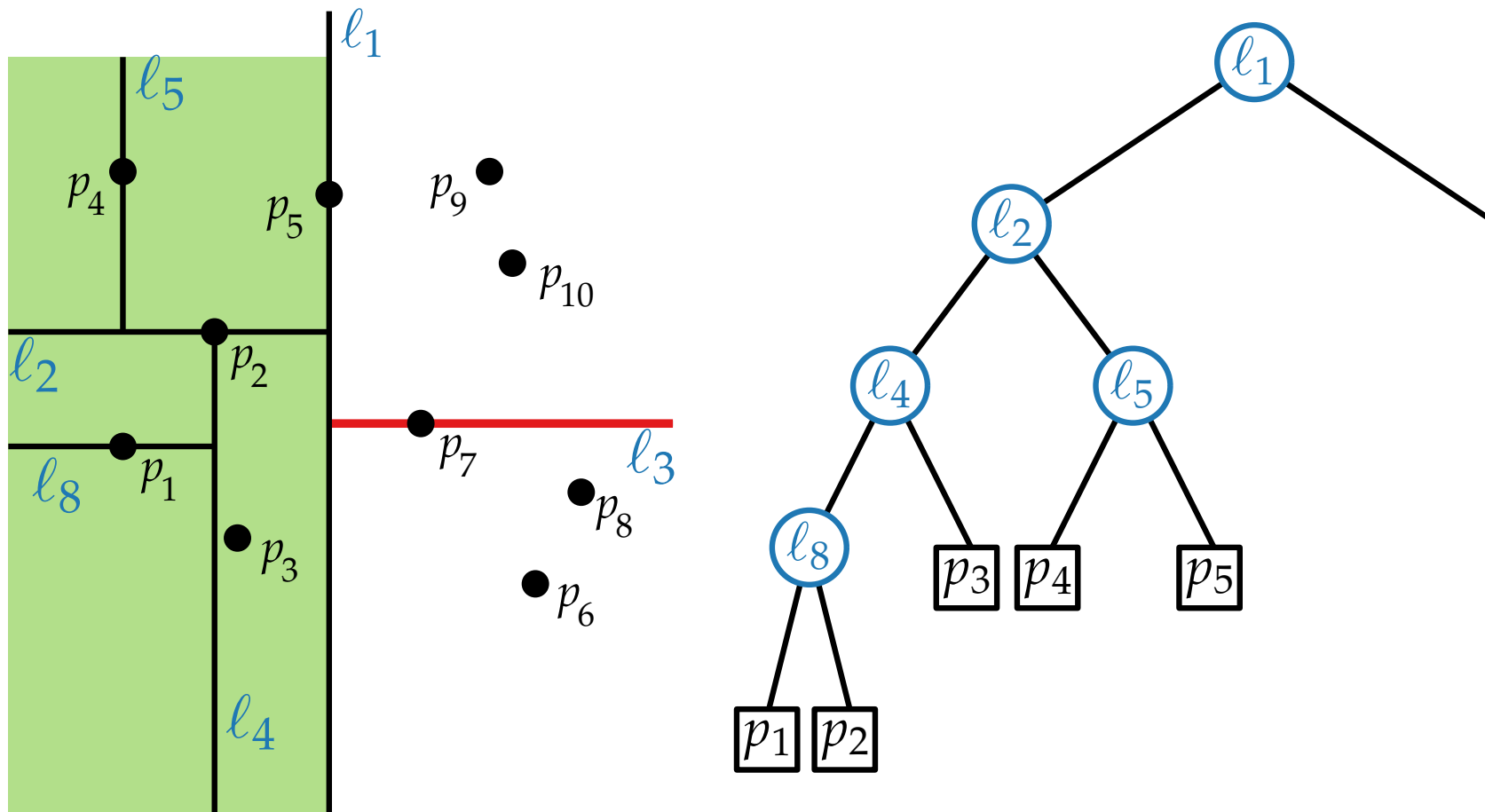
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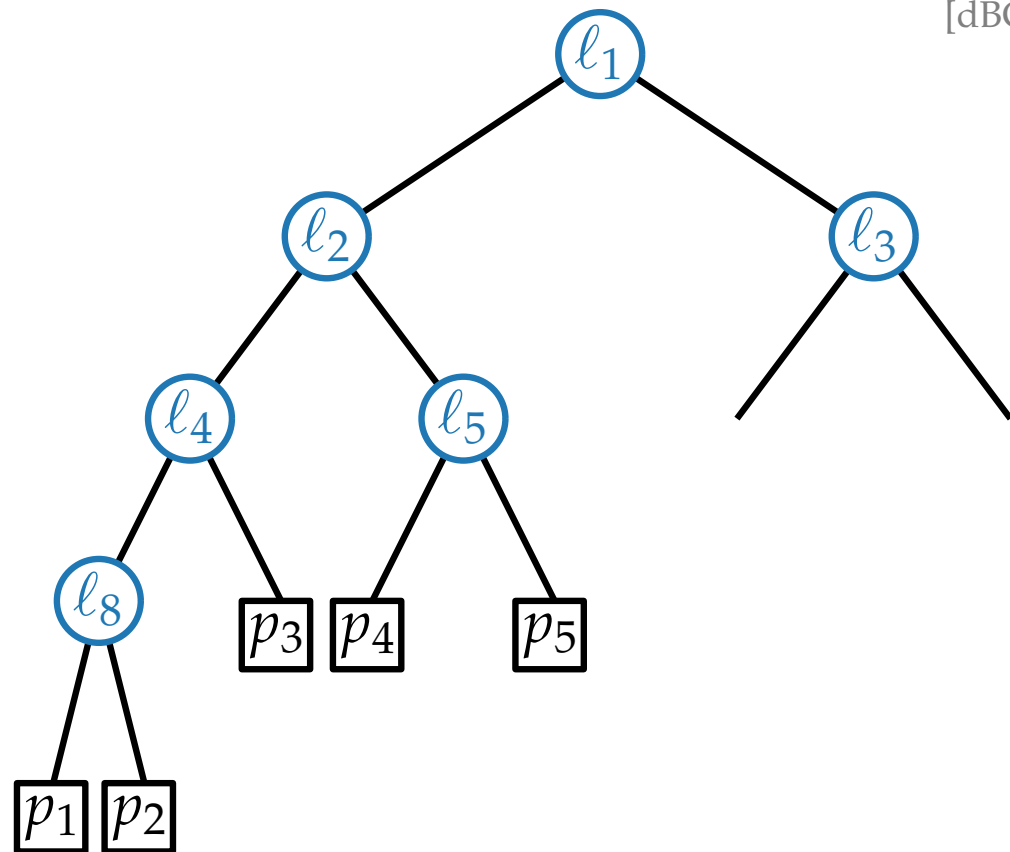
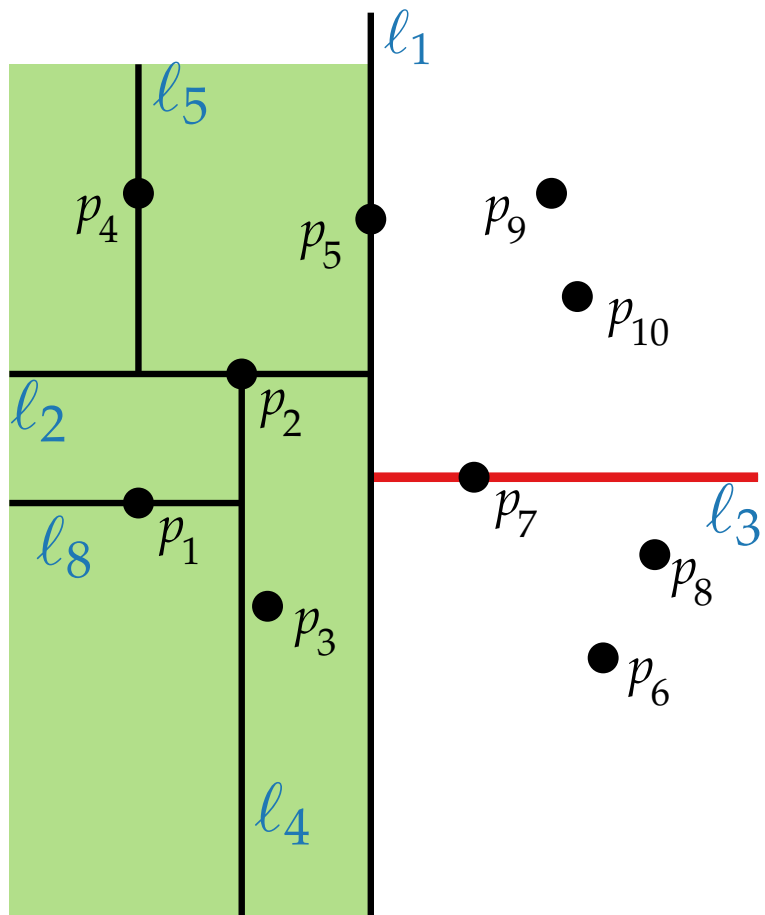
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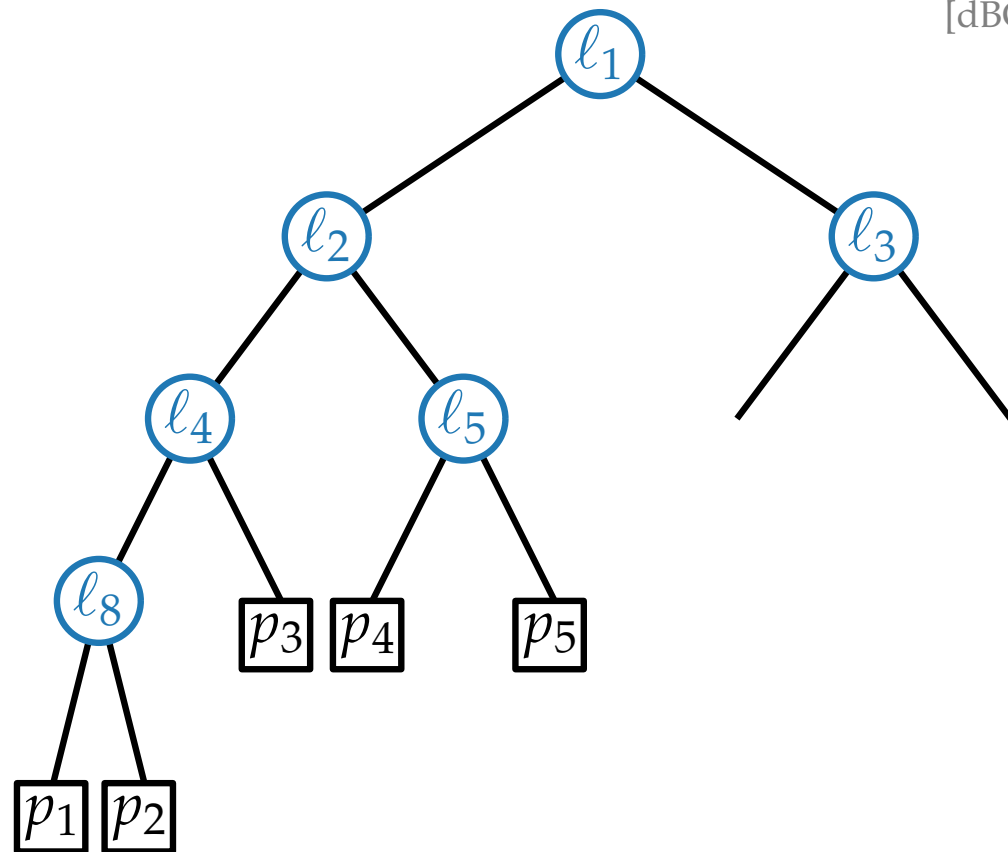
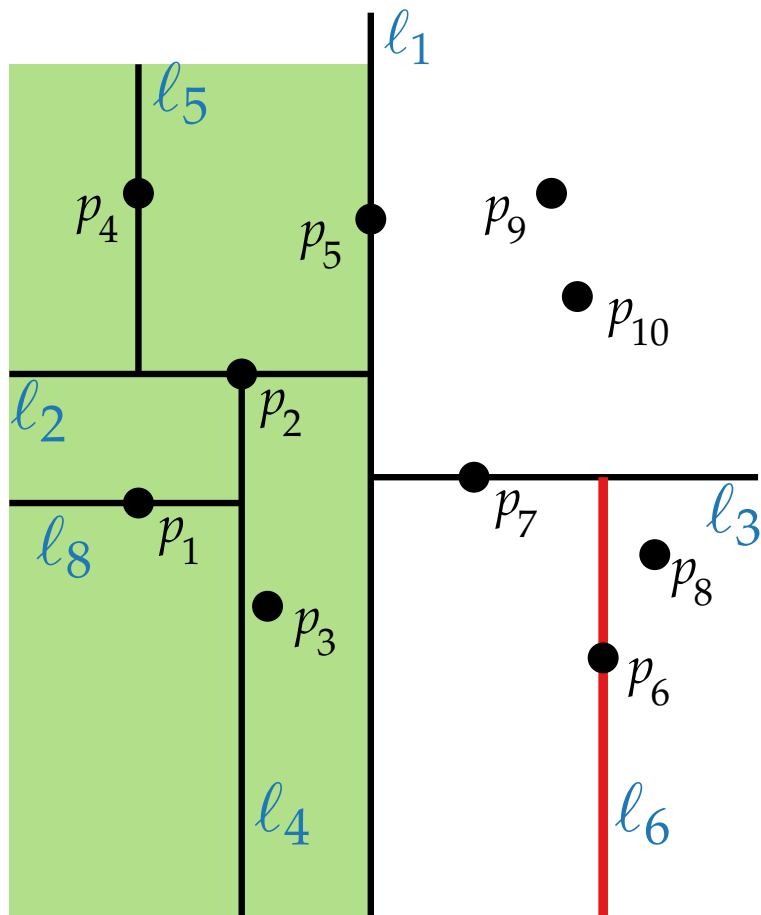
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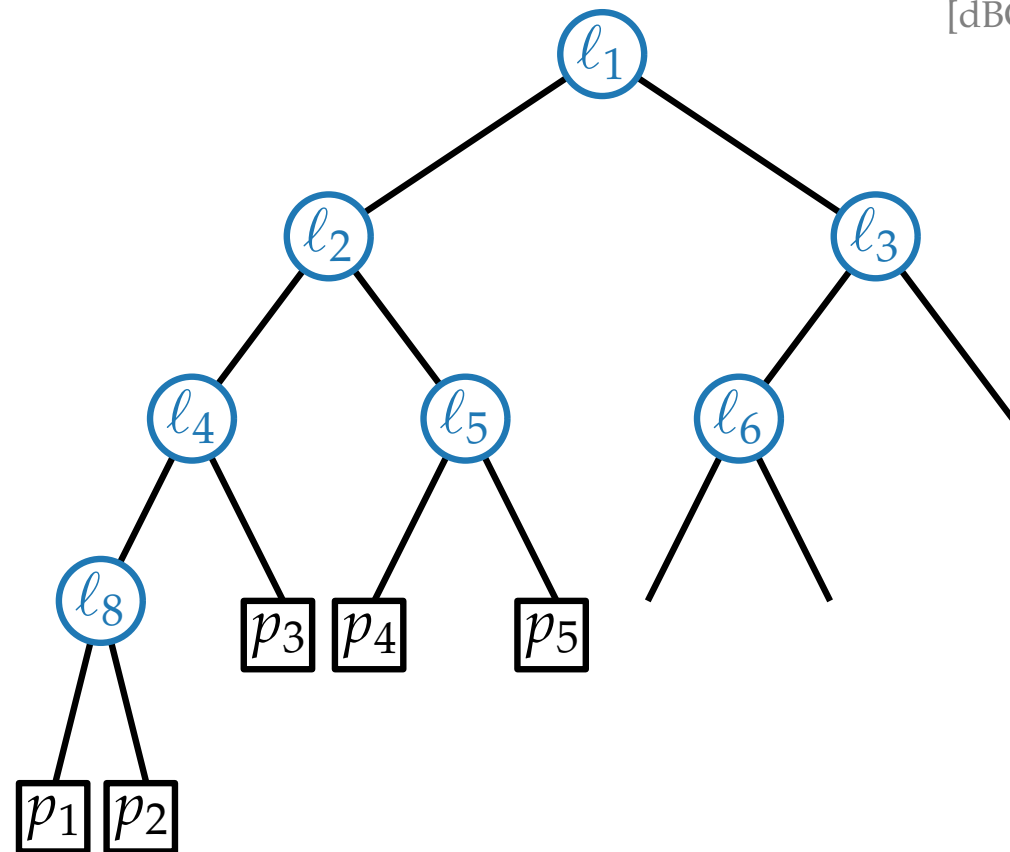
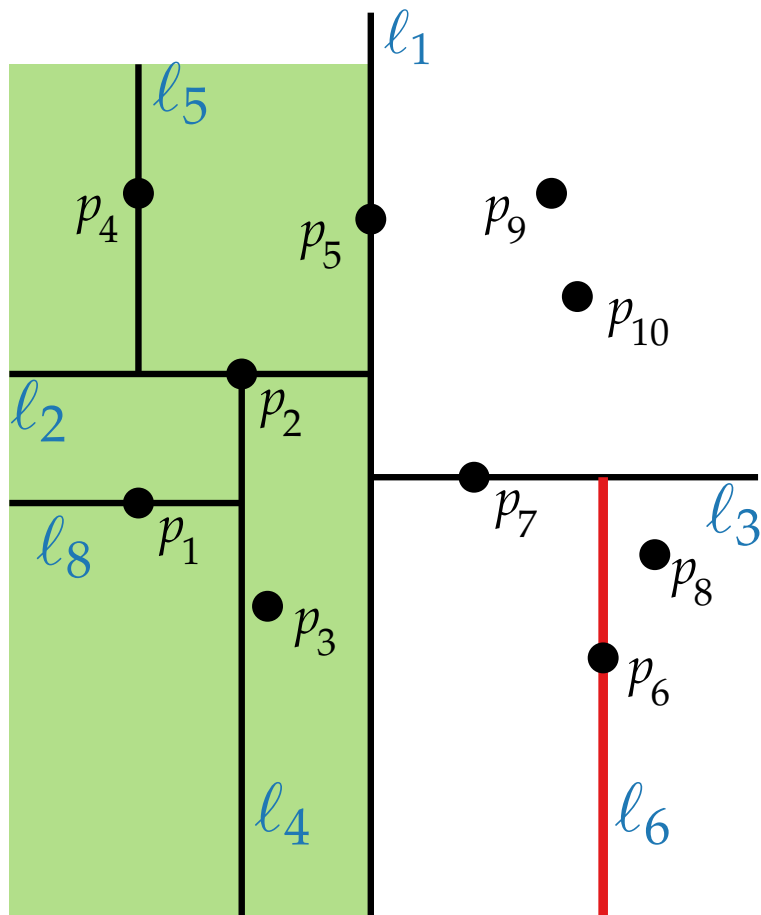
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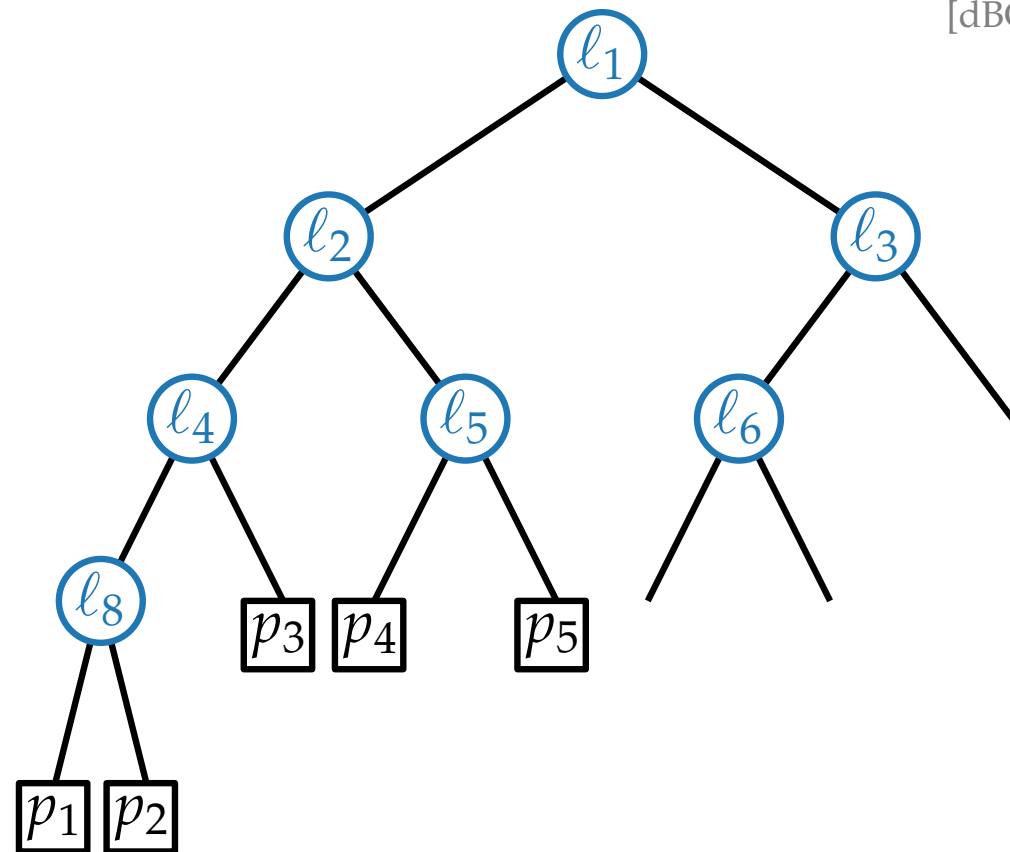
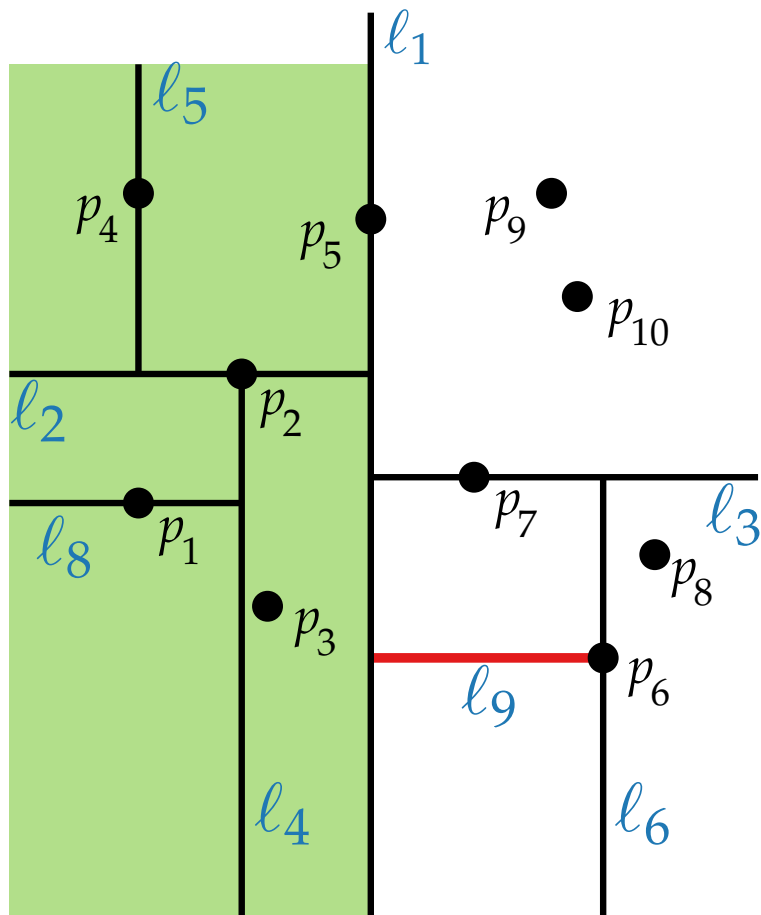
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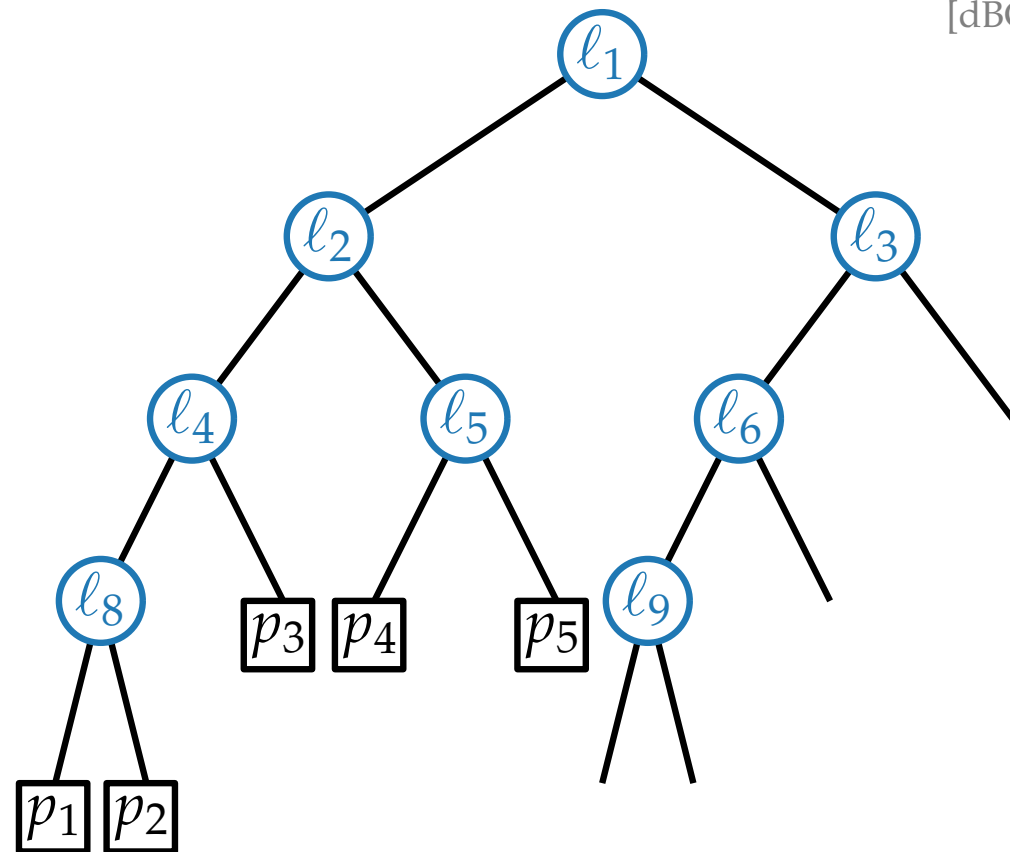
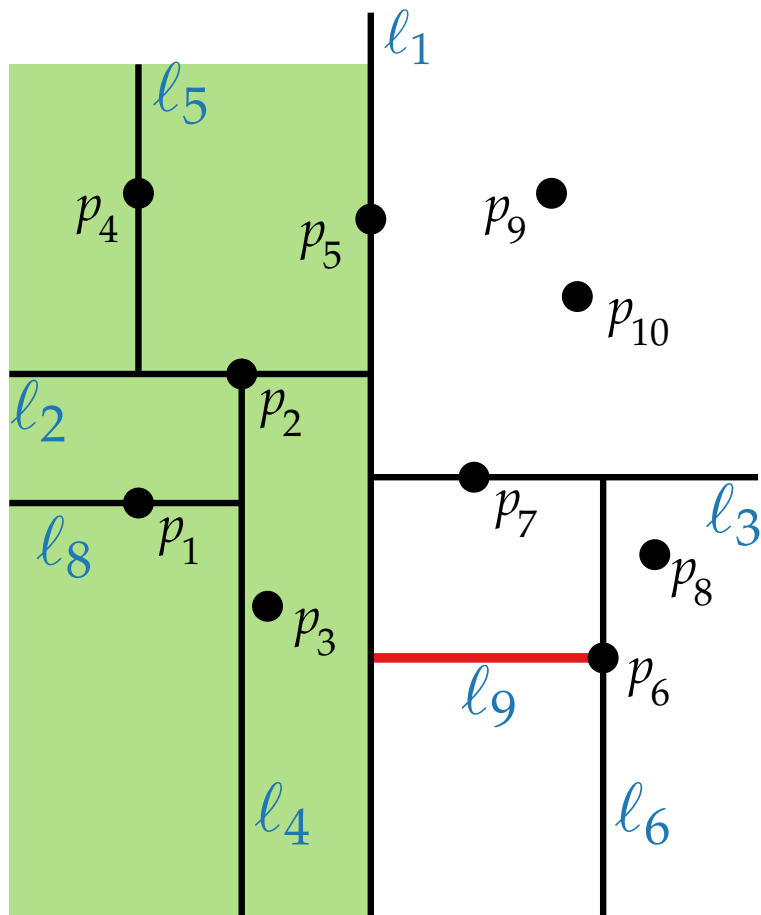
Kd-Trees: Example



[dBCvKO'08]

- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
Vertical left.

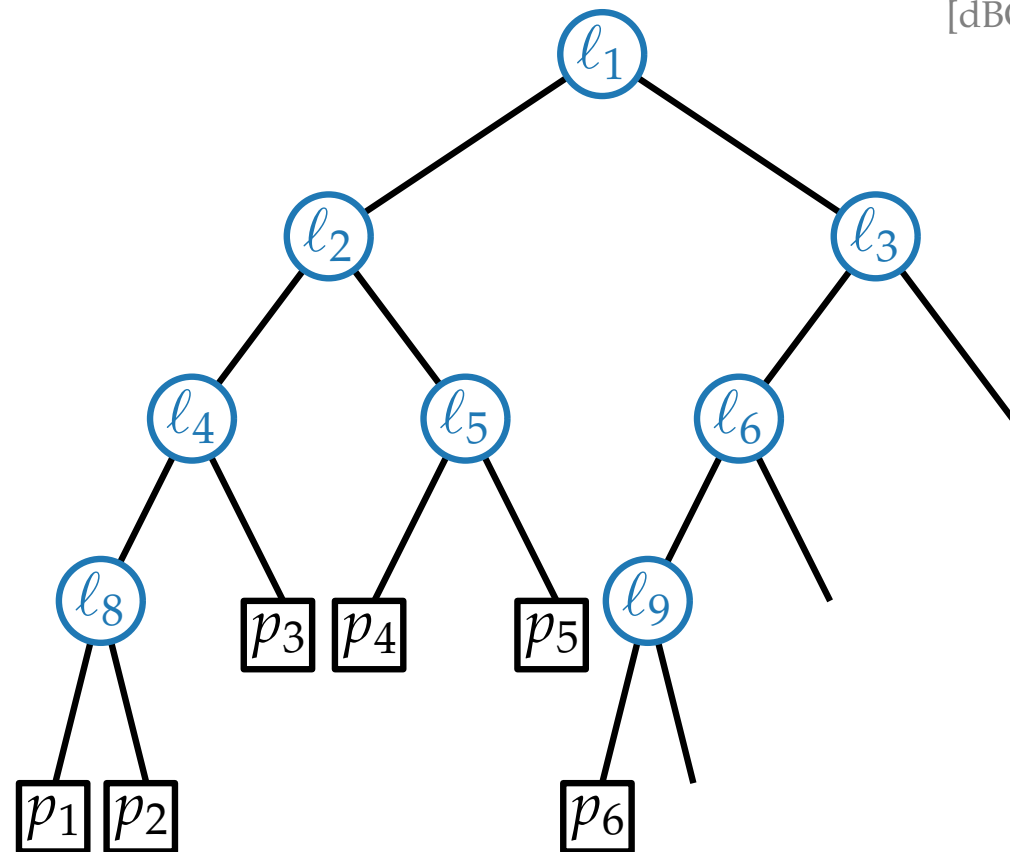
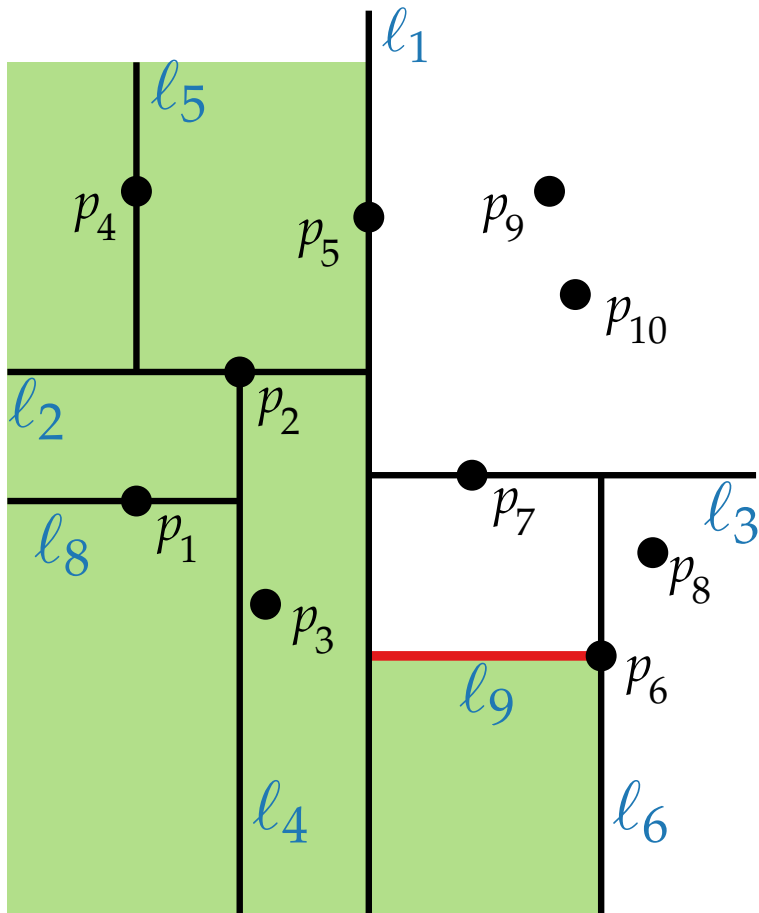
Kd-Trees: Example



[dBCvKO'08]

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Vertical left.

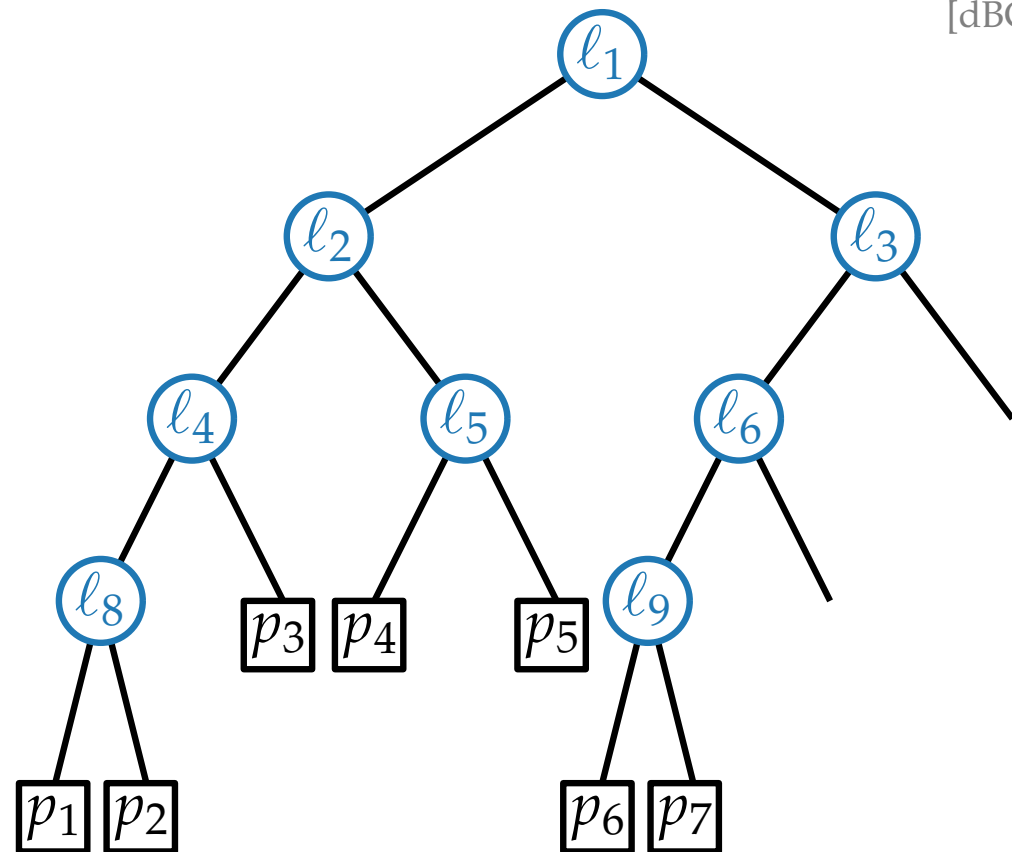
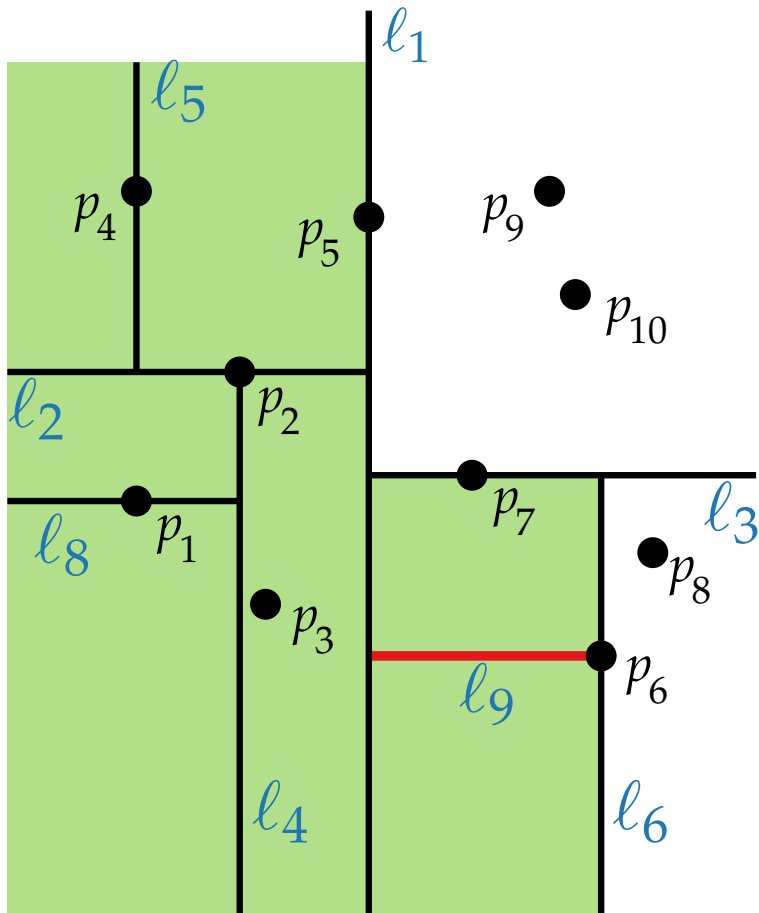
Kd-Trees: Example



[dBCvKO'08]

- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
Vertical left.

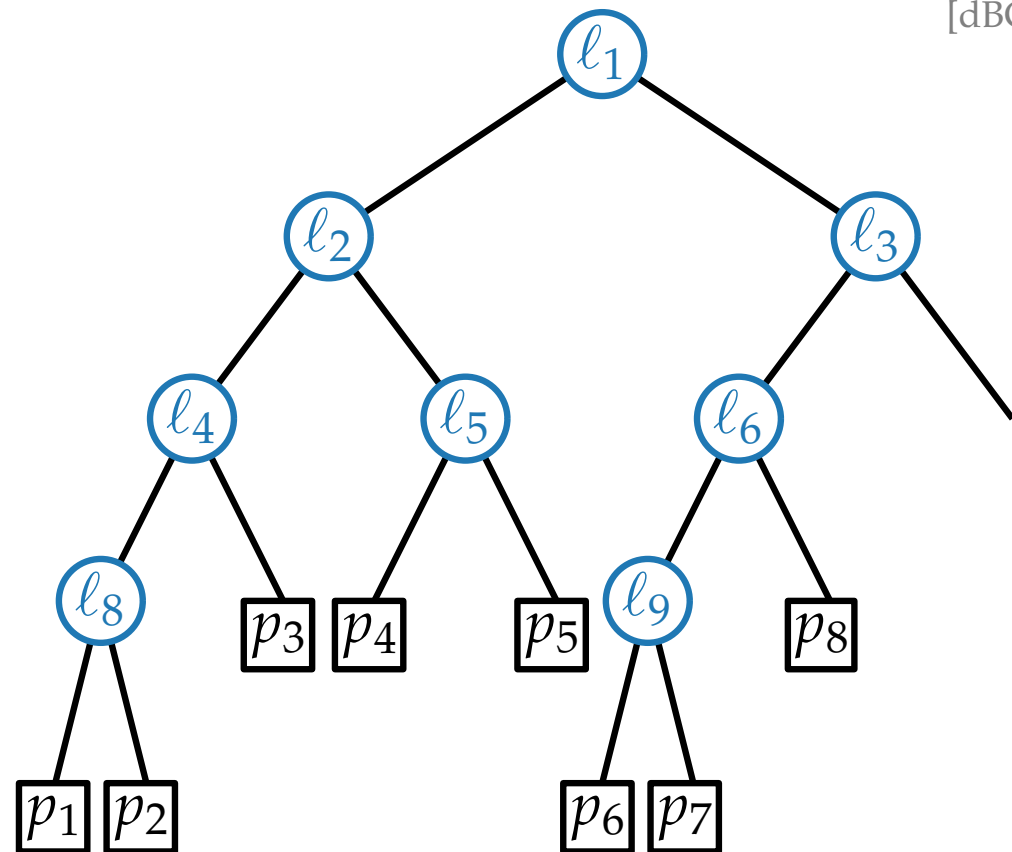
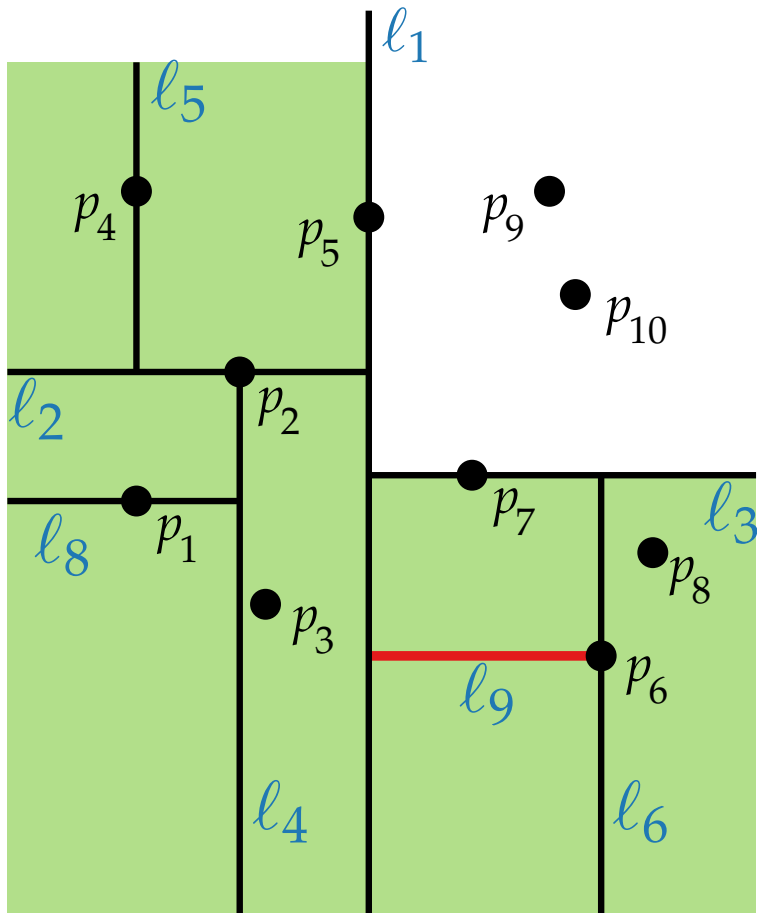
Kd-Trees: Example



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- Horizontal split lines/segm. belong to the region below.
Vertical left.

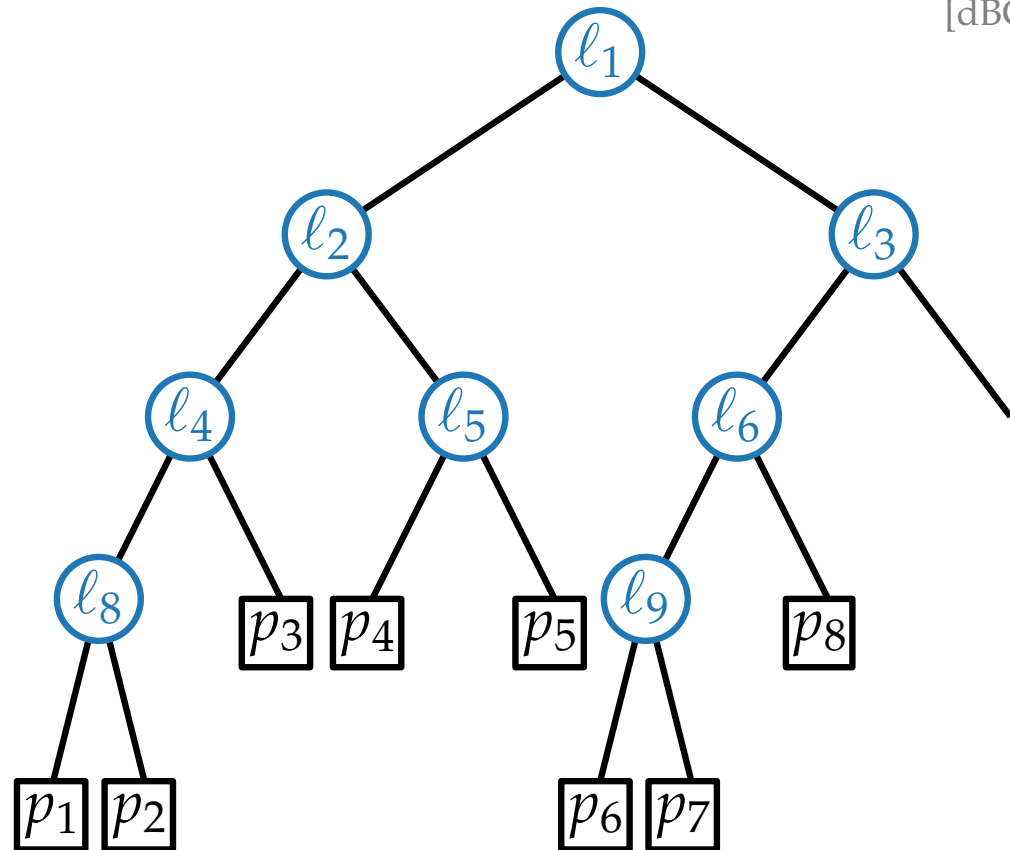
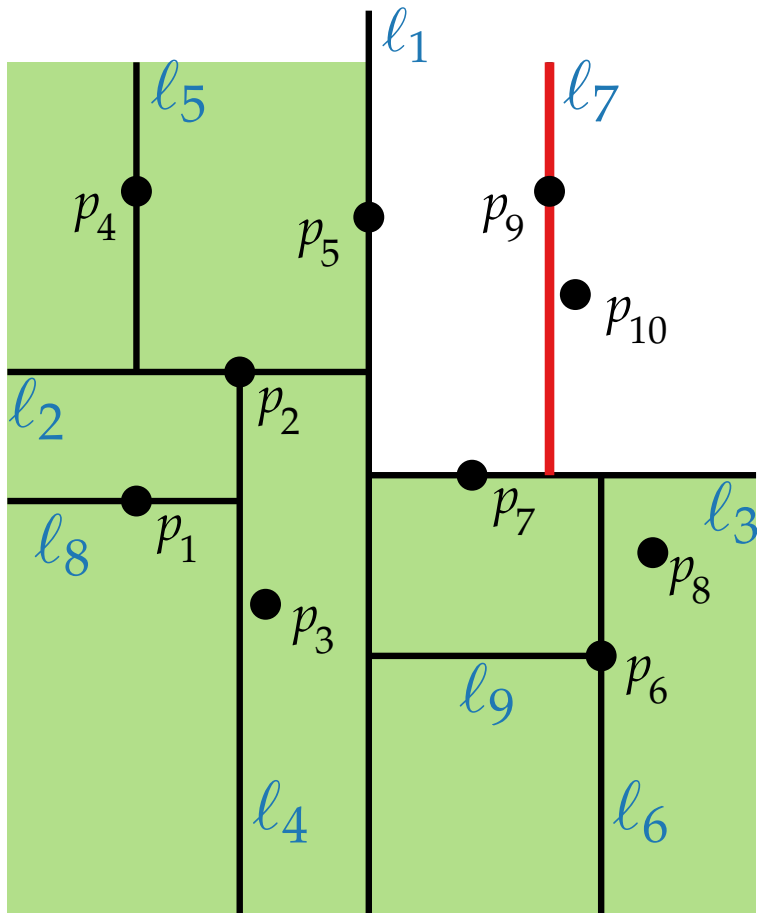
Kd-Trees: Example



[dBCvKO'08]

- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
Vertical left.

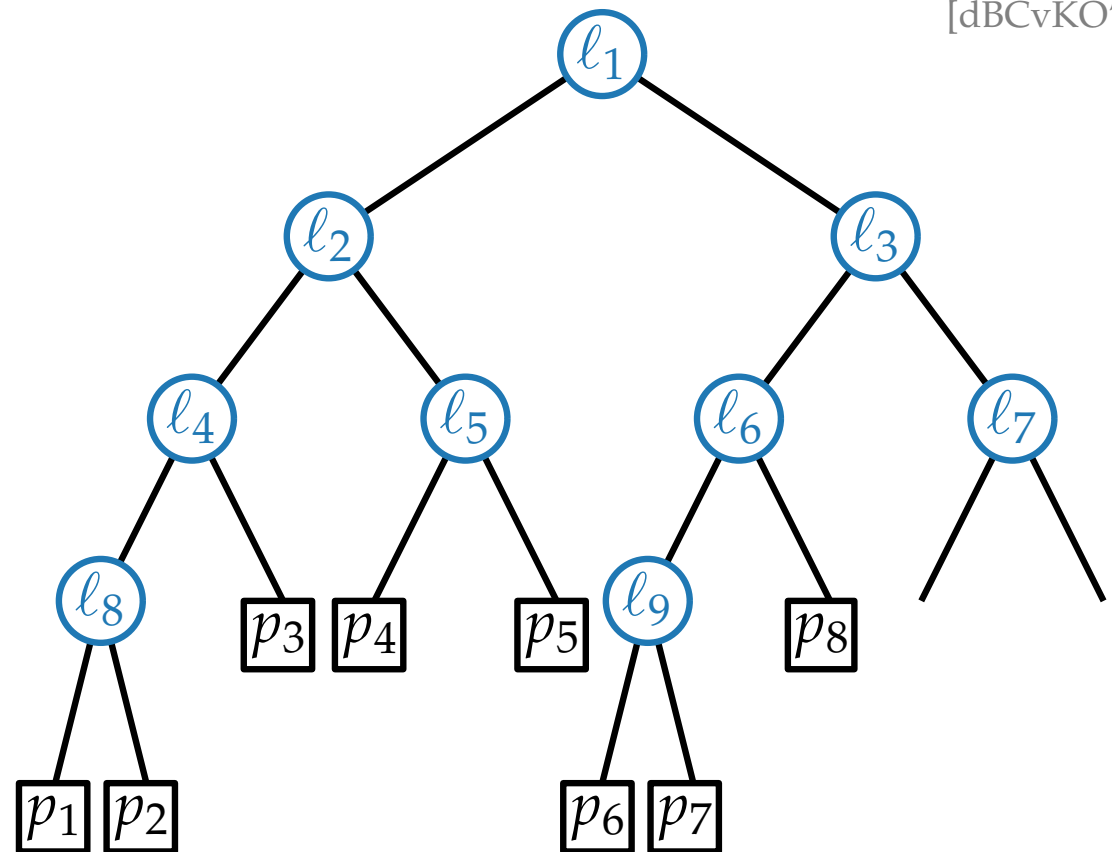
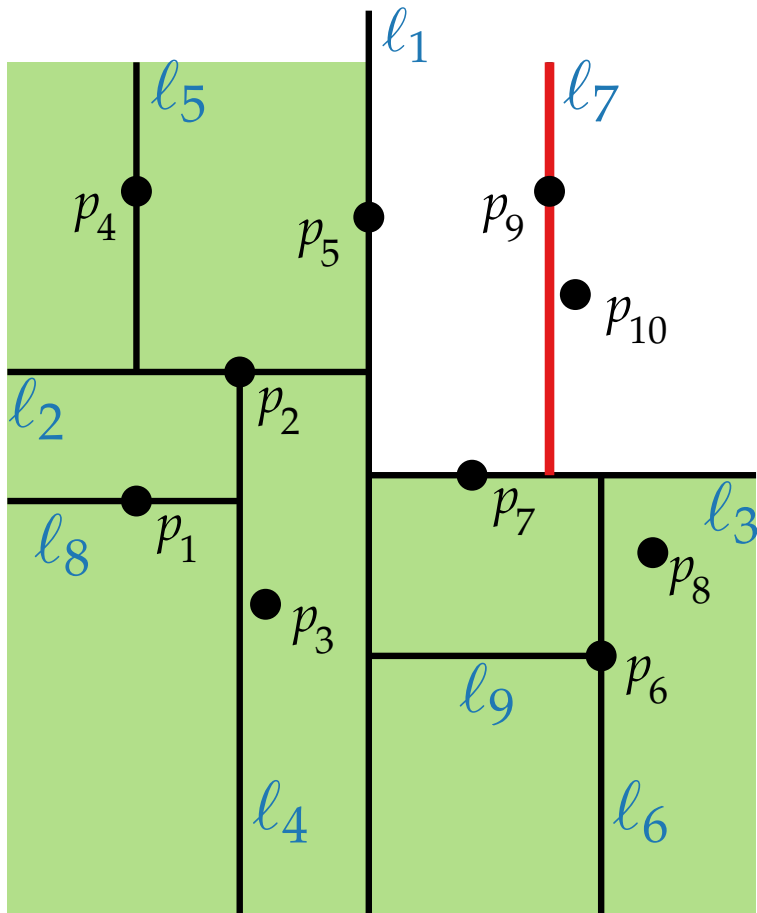
Kd-Trees: Example



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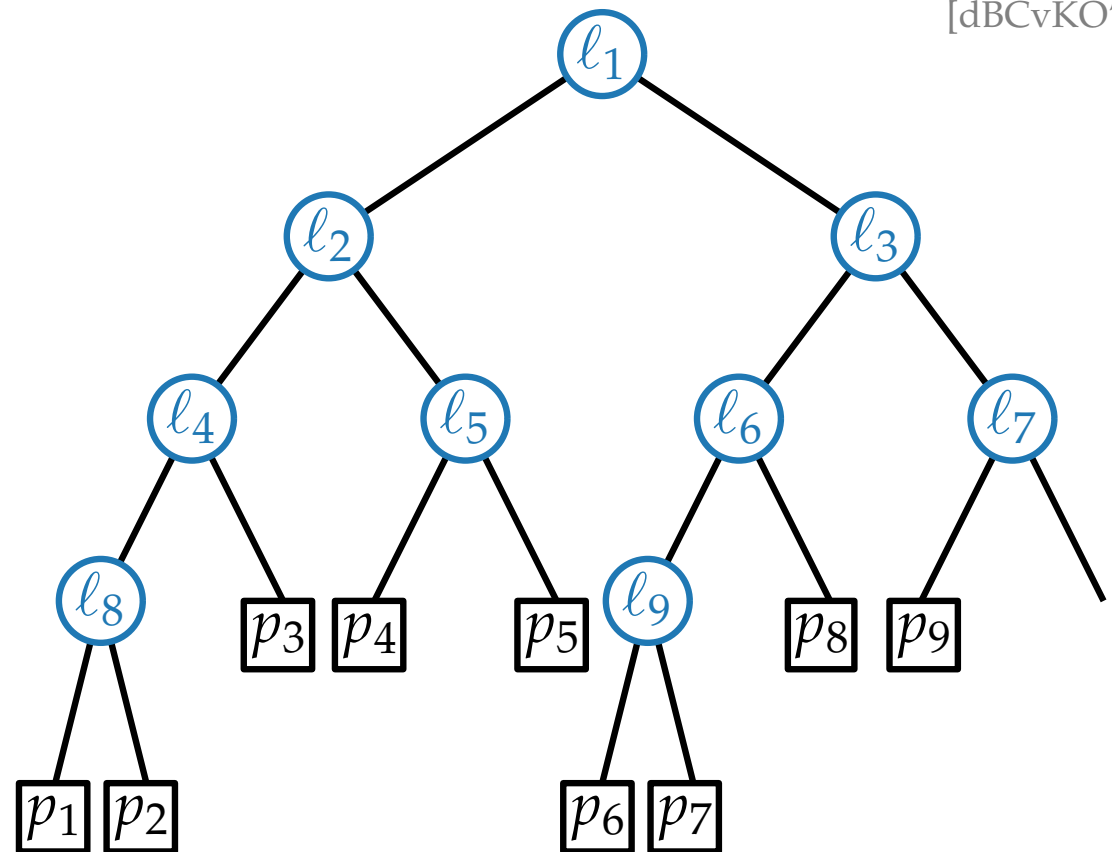
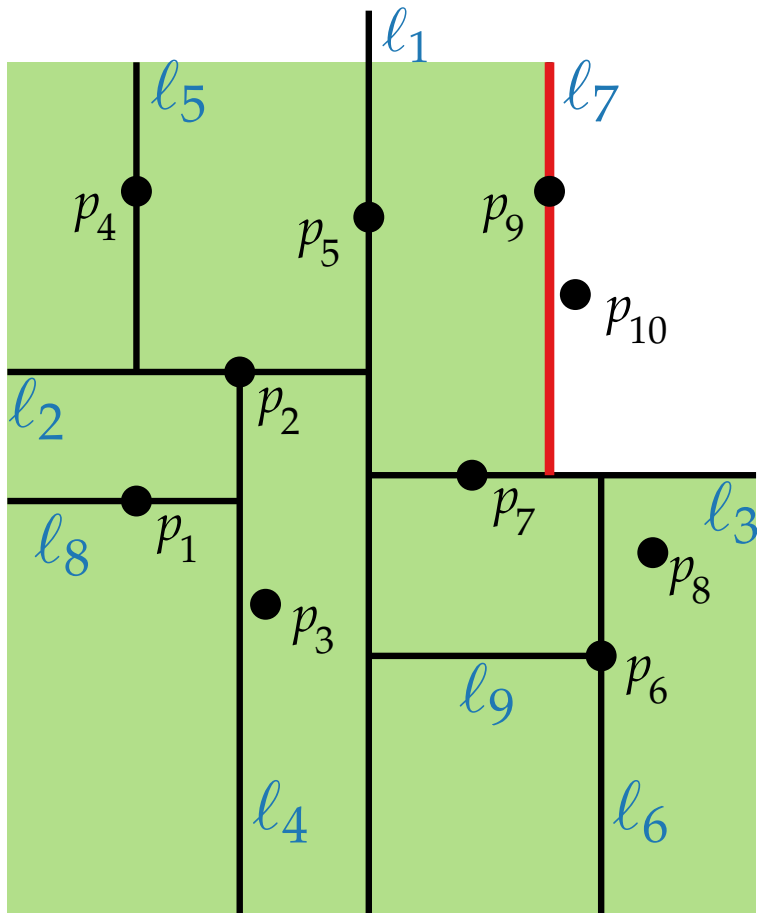
Kd-Trees: Example



[dBCvKO'08]

- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
Vertical left.

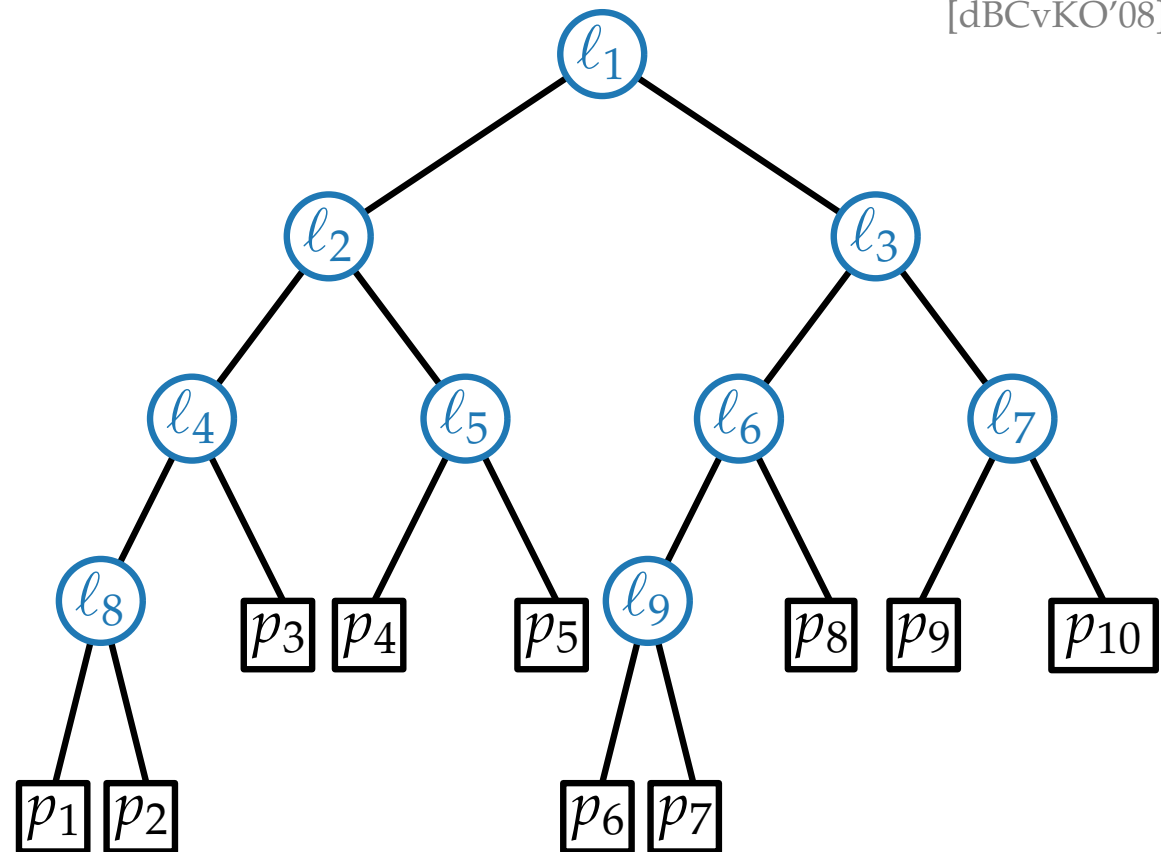
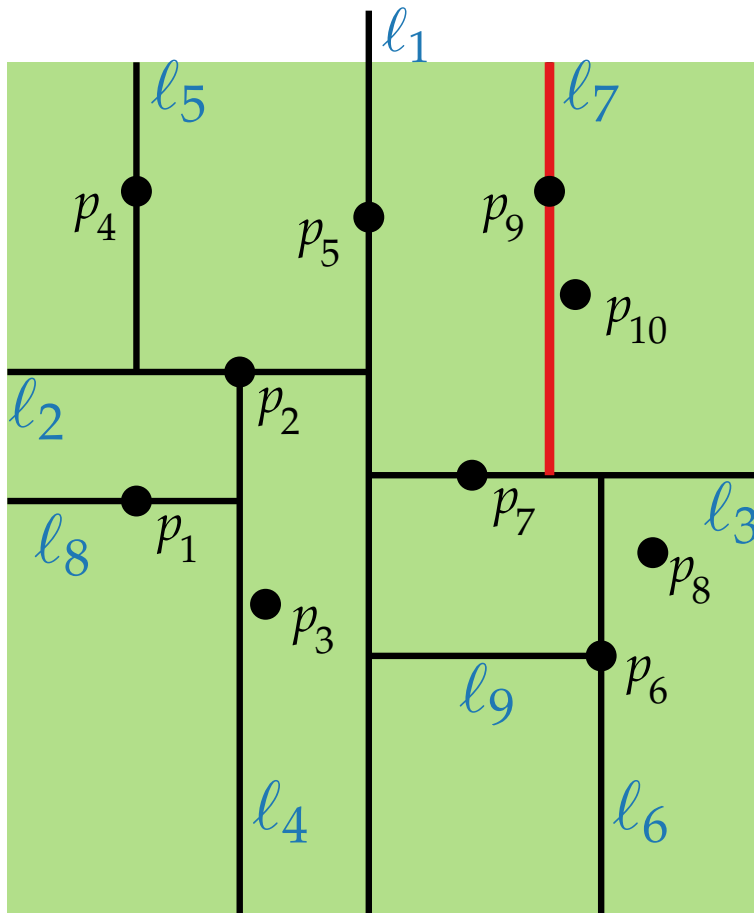
Kd-Trees: Example



[dBCvKO'08]

- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
Vertical left.

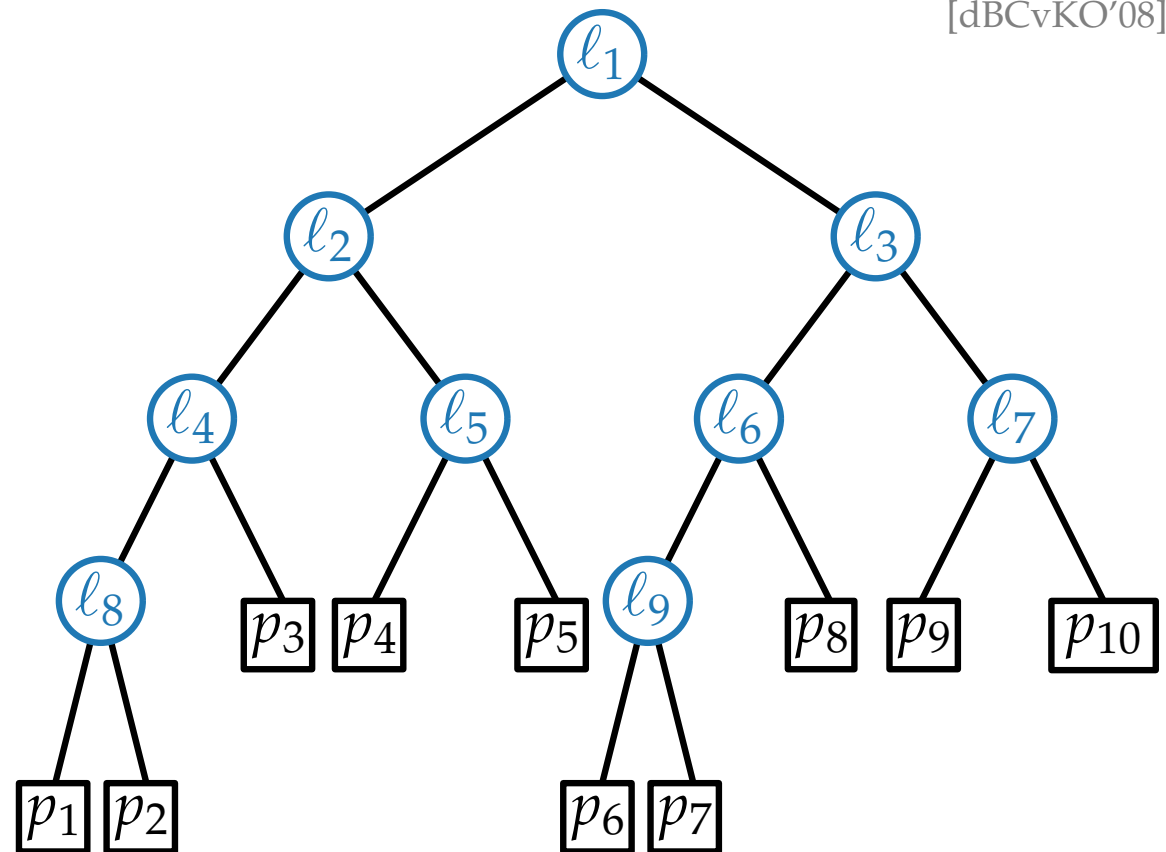
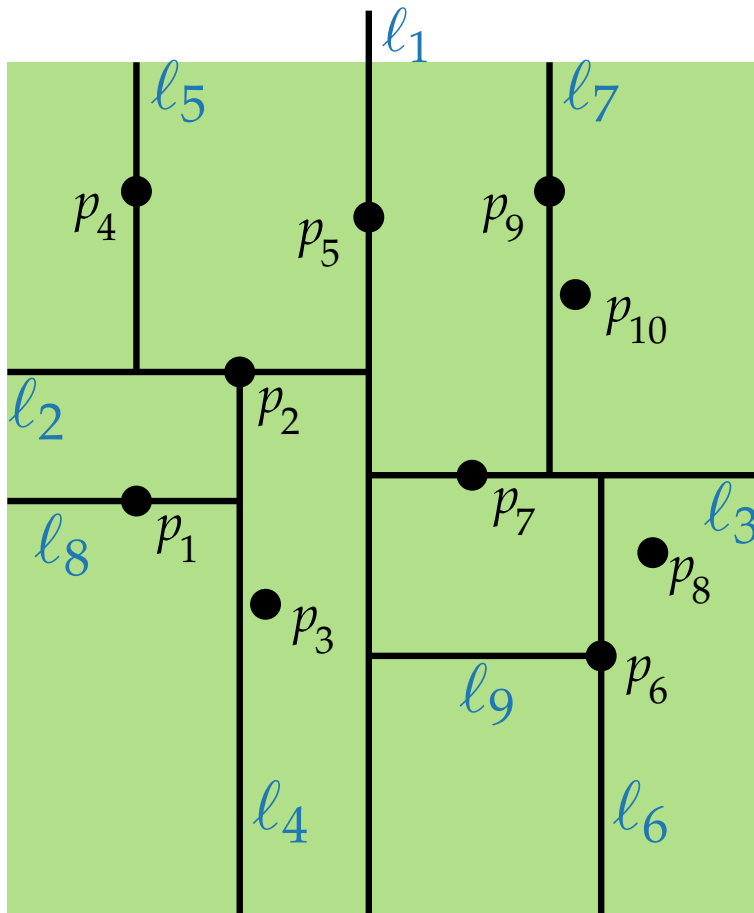
Kd-Trees: Example



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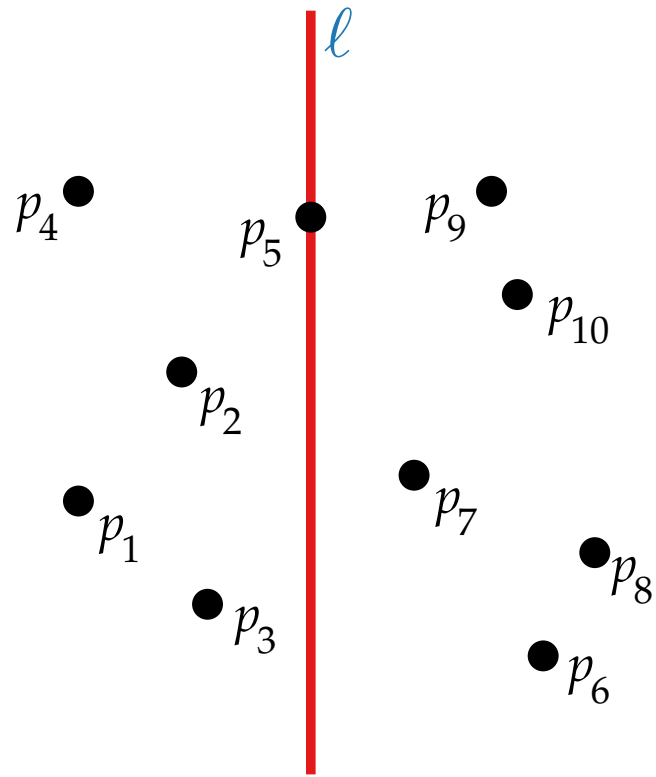
Kd-Trees: Example



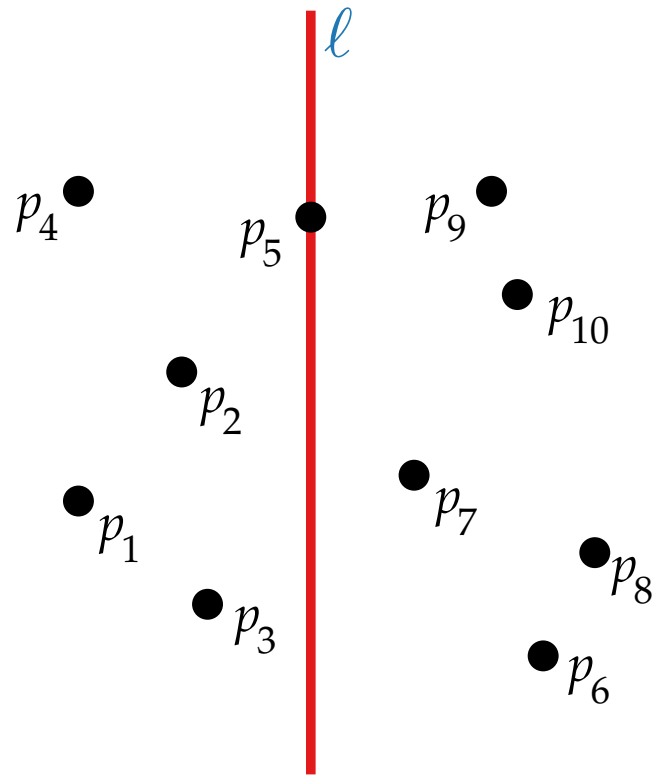
[dBCvKO'08]

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Kd-Trees: Construction

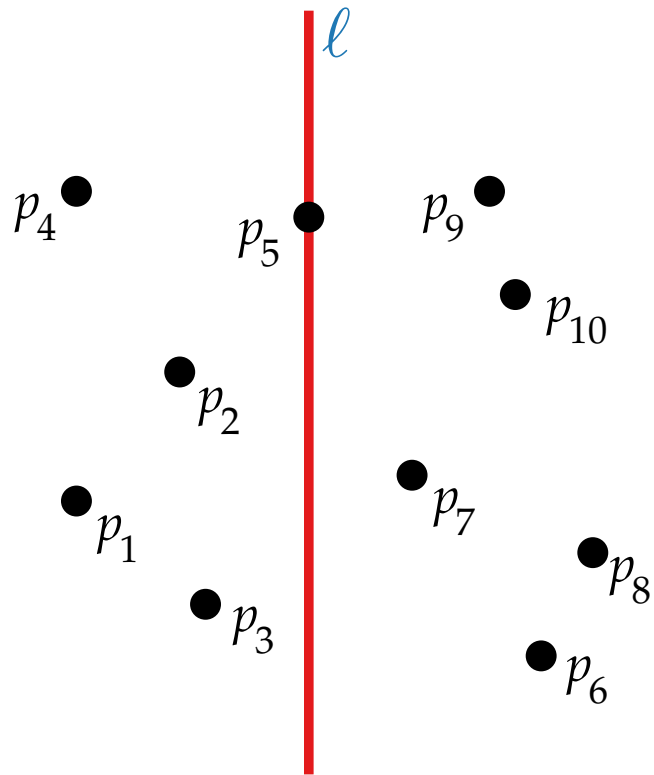


Kd-Trees: Construction



Pseudo-code:

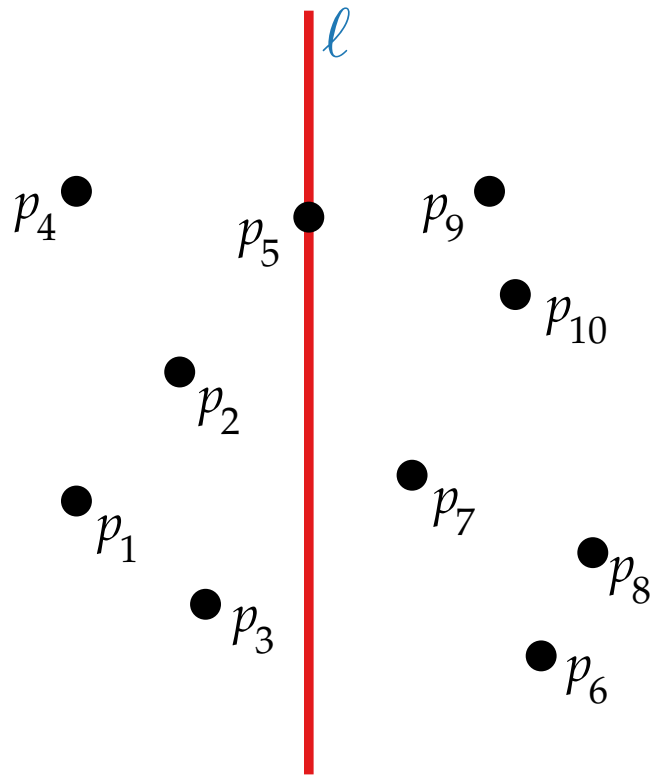
Kd-Trees: Construction



Pseudo-code:

```
BuildKdTree(points  $P$ , int  $depth$ )
```

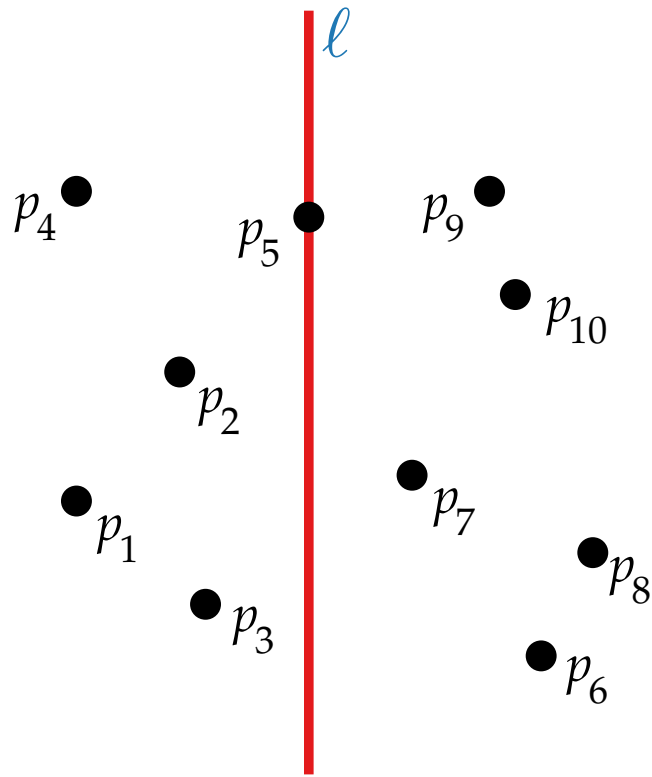
Kd-Trees: Construction



Pseudo-code:

```
BuildKdTree(points  $P$ , int  $depth$ )  
  if  $|P| = 1$  then  
    return (leaf storing the pt in  $P$ )  
  else
```


Kd-Trees: Construction

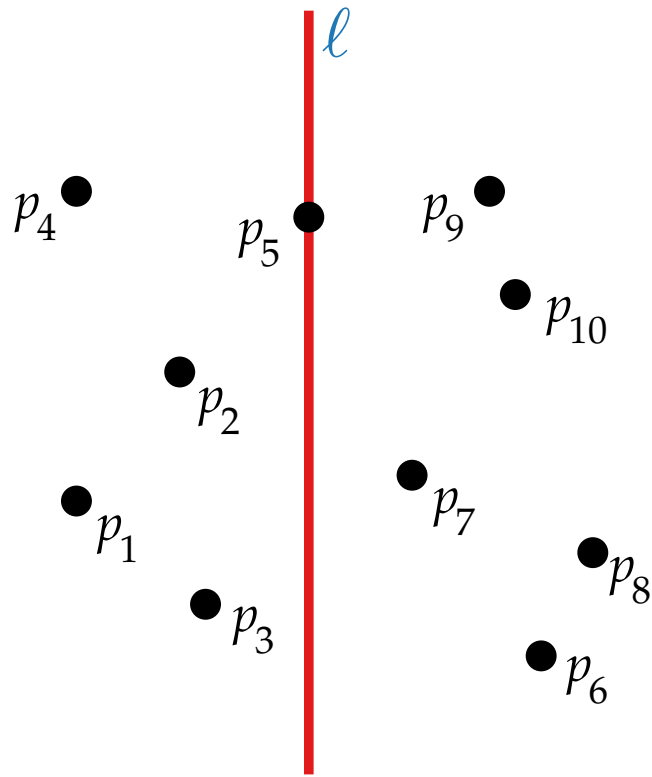


Pseudo-code:

```

BuildKdTree(points  $P$ , int  $depth$ )
  if  $|P| = 1$  then
    | return (leaf storing the pt in  $P$ )
  else
    if  $depth$  is even then
      |
    else
      |
  
```

Kd-Trees: Construction

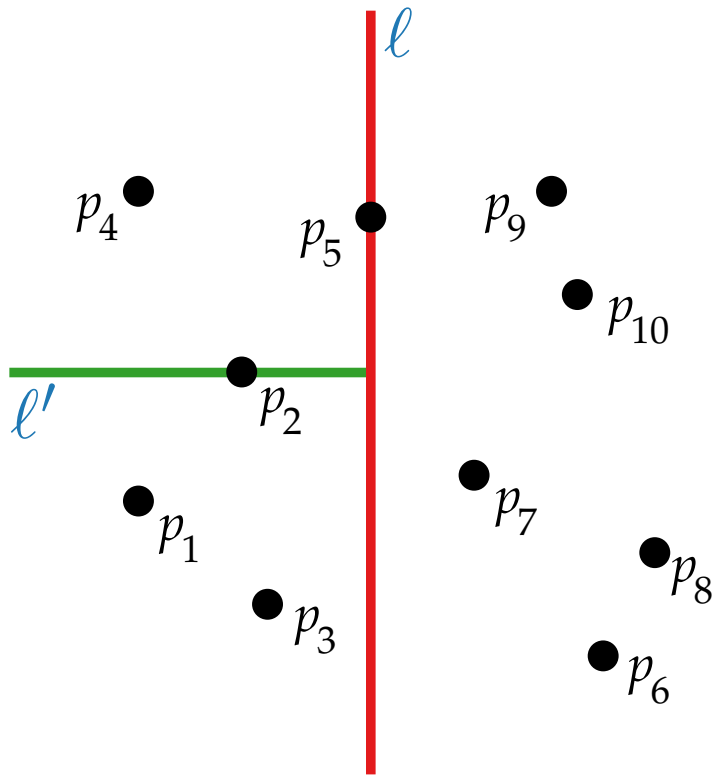


Pseudo-code:

```

BuildKdTree(points  $P$ , int  $depth$ )
  if  $|P| = 1$  then
    | return (leaf storing the pt in  $P$ )
  else
    if  $depth$  is even then
      | split  $P$  with the vertical line
      |    $\ell: x = x_{\text{median}(P)}$  into
      |    $P_1$  (pts left of or on  $\ell$ ) and
      |    $P_2 = P \setminus P_1$ 
    else
      |
  
```

Kd-Trees: Construction

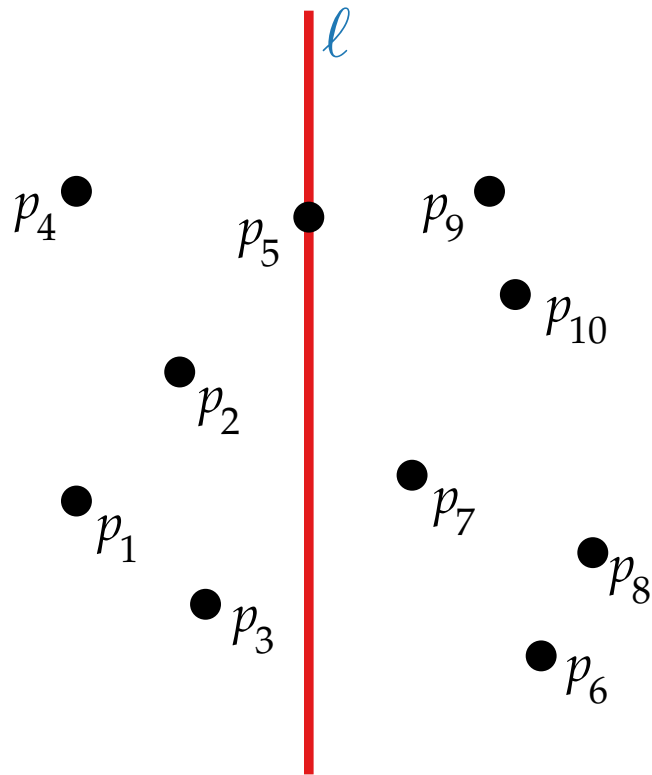


Pseudo-code:

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BuildKdTree(points  $P$ , int  $depth$ )
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```

Kd-Trees: Construction

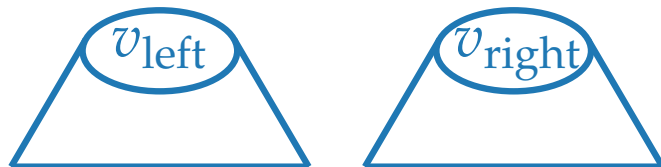
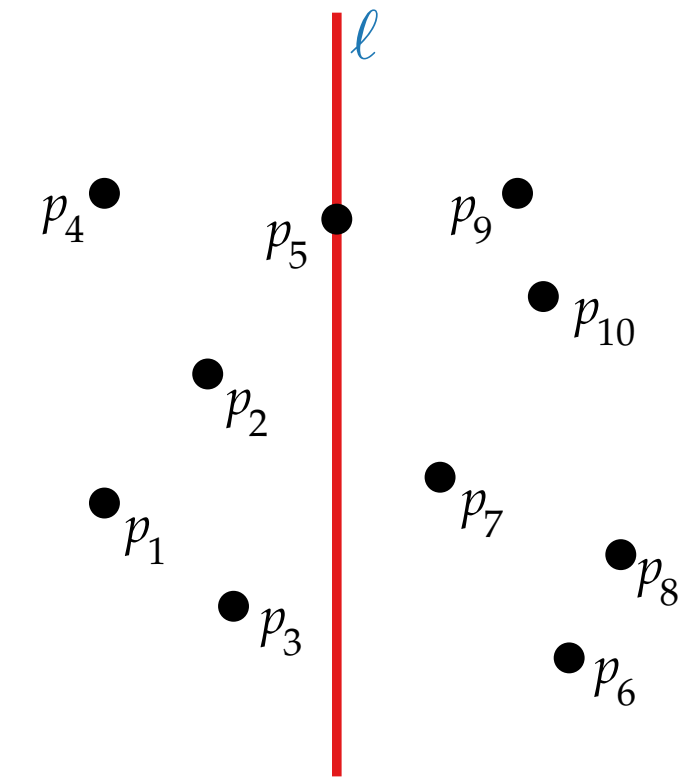


Pseudo-code:

```

BuildKdTree(points  $P$ , int  $depth$ )
  if  $|P| = 1$  then
    return (leaf storing the pt in  $P$ )
  else
    if  $depth$  is even then
      split  $P$  with the vertical line
         $\ell: x = x_{\text{median}(P)}$  into
         $P_1$  (pts left of or on  $\ell$ ) and
         $P_2 = P \setminus P_1$ 
    else
      split  $P$  horizontally...
     $v_{\text{left}} \leftarrow \text{BuildKdTree}(P_1, depth + 1)$ 
     $v_{\text{right}} \leftarrow \text{BuildKdTree}(P_2, depth + 1)$ 
  
```

Kd-Trees: Construction

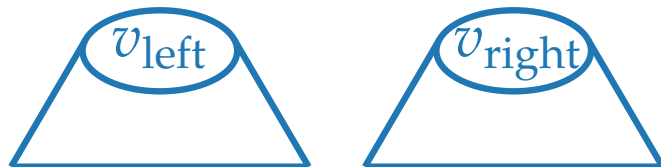
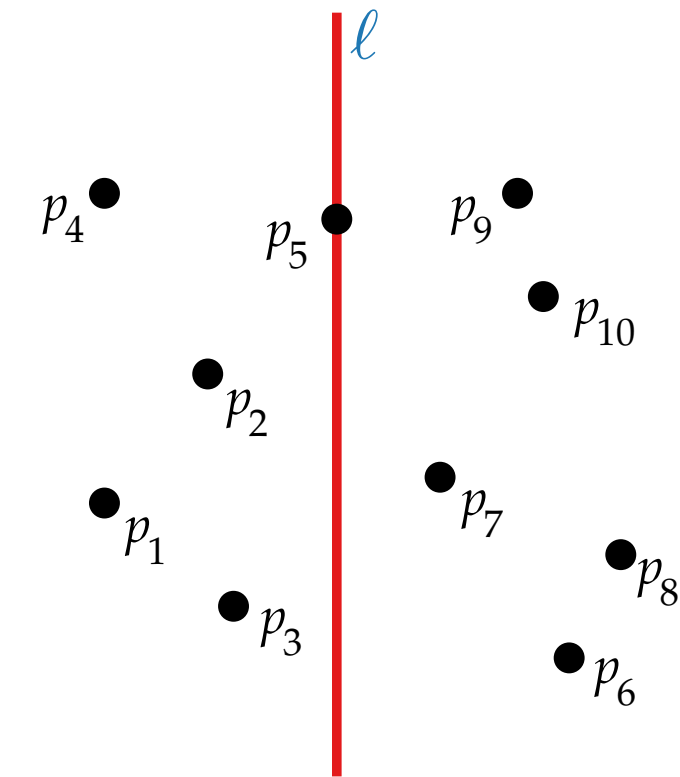


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Kd-Trees: Construction

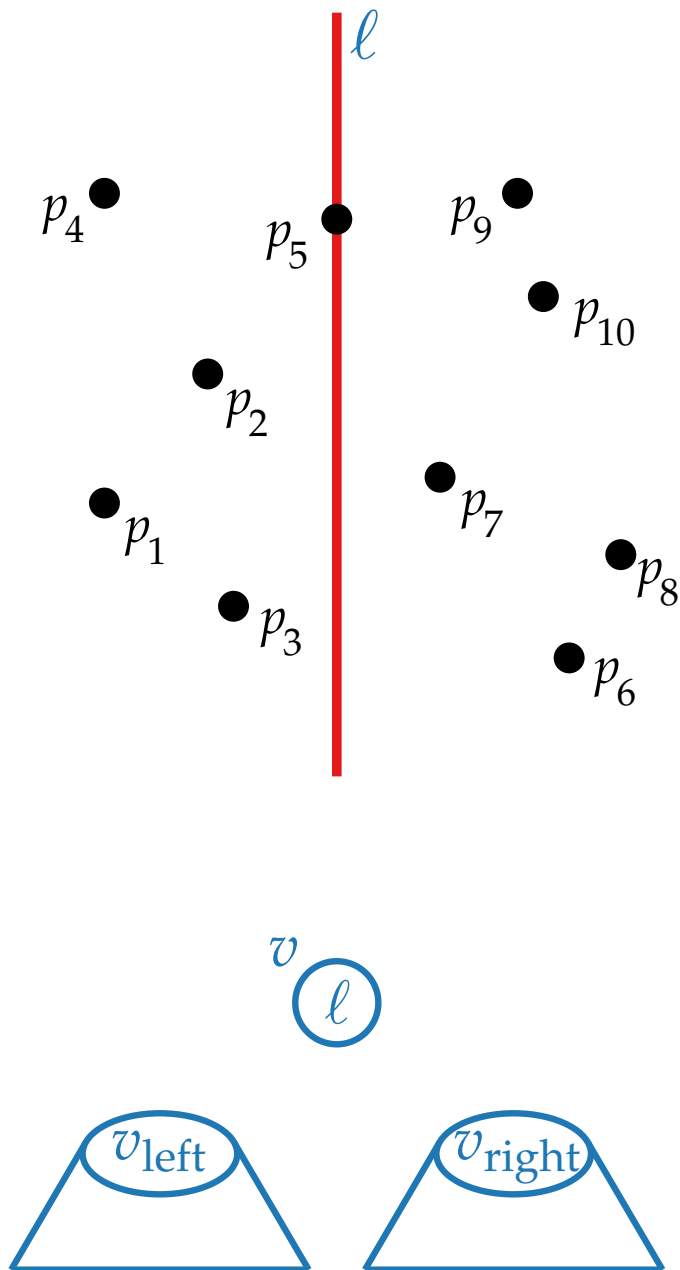


Pseudo-code:

```

BuildKdTree(points  $P$ , int  $depth$ )
  if  $|P| = 1$  then
    return (leaf storing the pt in  $P$ )
  else
    if  $depth$  is even then
      split  $P$  with the vertical line
         $l: x = x_{\text{median}(P)}$  into
         $P_1$  (pts left of or on  $l$ ) and
         $P_2 = P \setminus P_1$ 
    else
      split  $P$  horizontally...
     $v_{\text{left}} \leftarrow \text{BuildKdTree}(P_1, depth + 1)$ 
     $v_{\text{right}} \leftarrow \text{BuildKdTree}(P_2, depth + 1)$ 
    create a node  $v$  storing  $l$ 
  
```

Kd-Trees: Construction

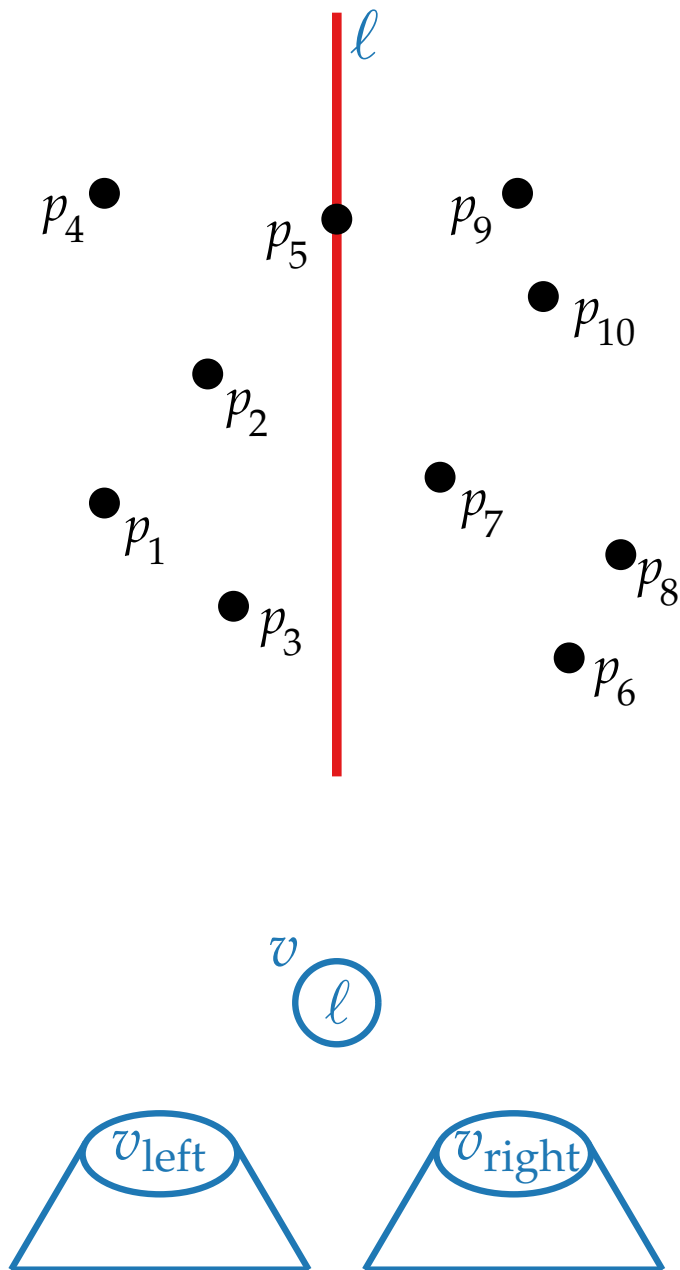


Pseudo-code:

```

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  if  $|P| = 1$  then
    return (leaf storing the pt in  $P$ )
  else
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    create a node  $v$  storing  $\ell$ 
  
```

Kd-Trees: Construction

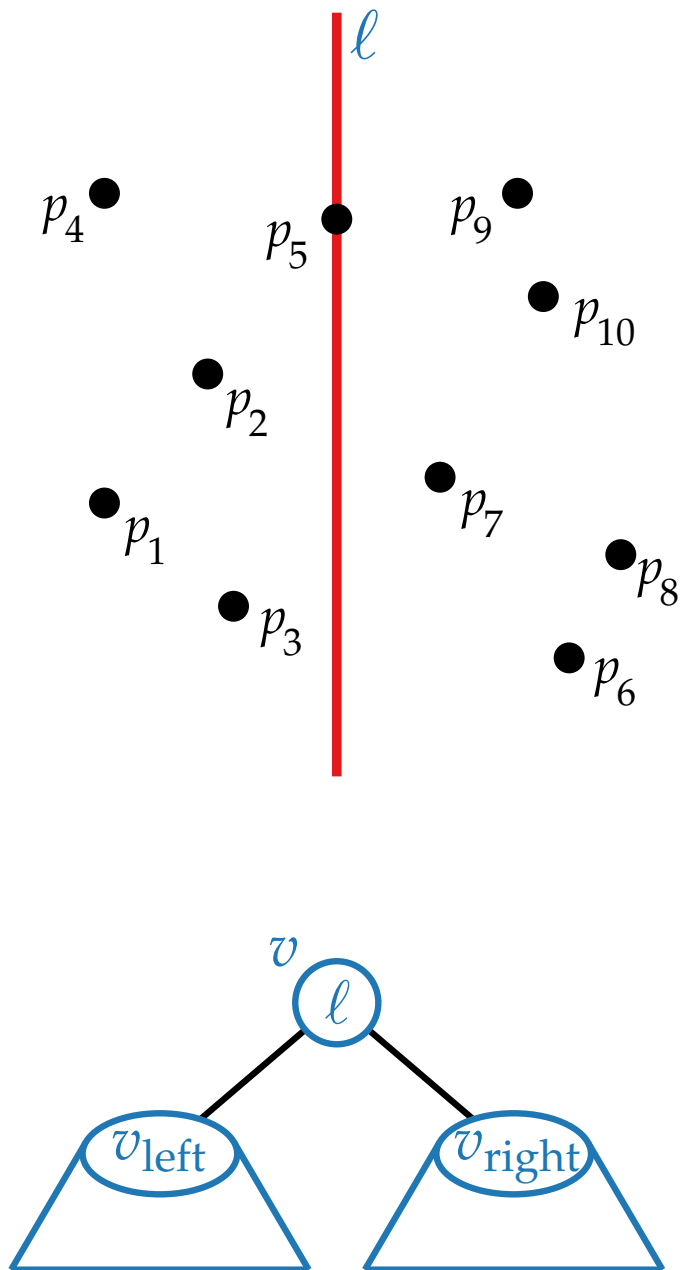


Pseudo-code:

```

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     $v_{\text{right}} \leftarrow \text{BuildKdTree}(P_2, depth + 1)$ 
    create a node  $v$  storing  $\ell$ 
    make  $v_{\text{left}}$  and  $v_{\text{right}}$  the children of  $v$ 
  
```


Kd-Trees: Construction

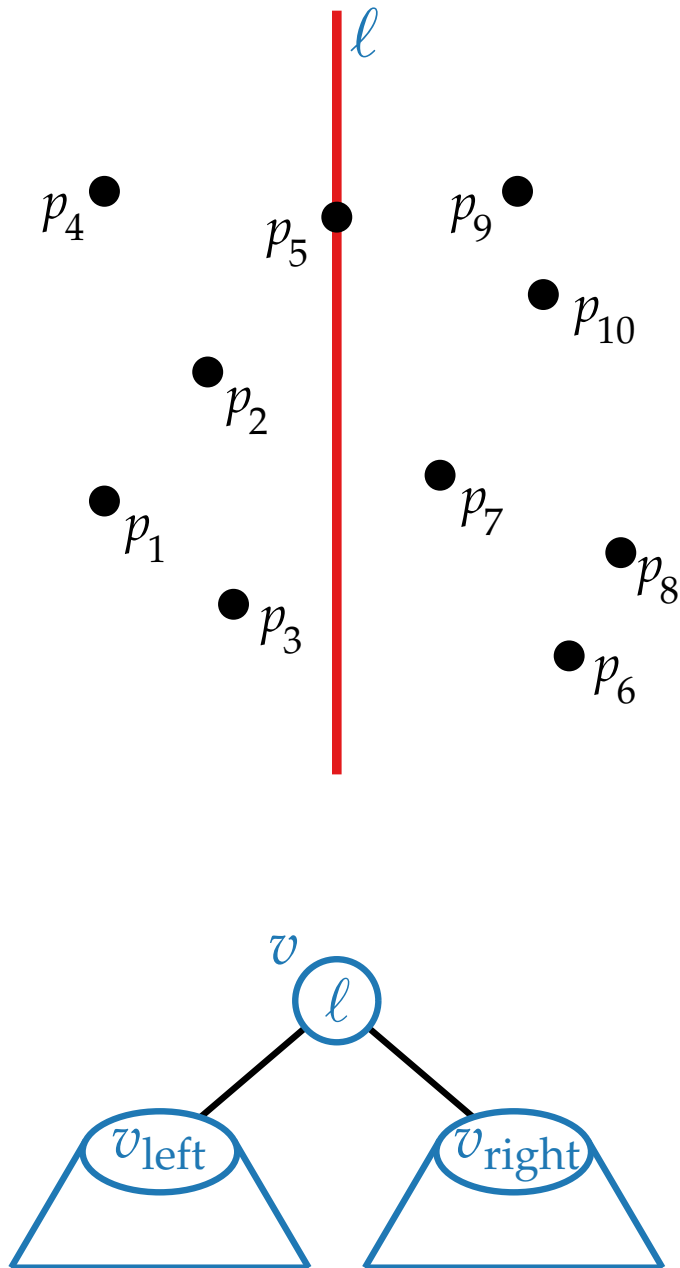


Pseudo-code:

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  if  $|P| = 1$  then
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Kd-Trees: Construction



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    create a node  $v$  storing  $\ell$ 
    make  $v_{\text{left}}$  and  $v_{\text{right}}$  the children of  $v$ 
    return ( $v$ )
  
```

Computational Geometry

Lecture 5: Orthogonal Range Queries or Fast Access to Data Bases

Part III: Kd-Trees: Analysis & Querying

Kd-Trees: Analysis

Construction time?

Kd-Trees: Analysis

Construction time?

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases}$$

Kd-Trees: Analysis

Construction time?

$$T(n) = \left\{ \begin{array}{ll} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{array} \right\} = O(n \log n)$$

Kd-Trees: Analysis

Construction time?

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases} \quad \begin{matrix} \text{see Mergesort!} \\ \swarrow \\ \blacktriangledown \\ \end{matrix} \quad \equiv O(n \log n)$$

Kd-Trees: Analysis

Construction time?

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases} \stackrel{\text{see Mergesort!}}{=} O(n \log n)$$

Lemma. A kd-tree for a set of n pts in the plane takes $O(n \log n)$ time to construct and uses $O(n)$ storage.

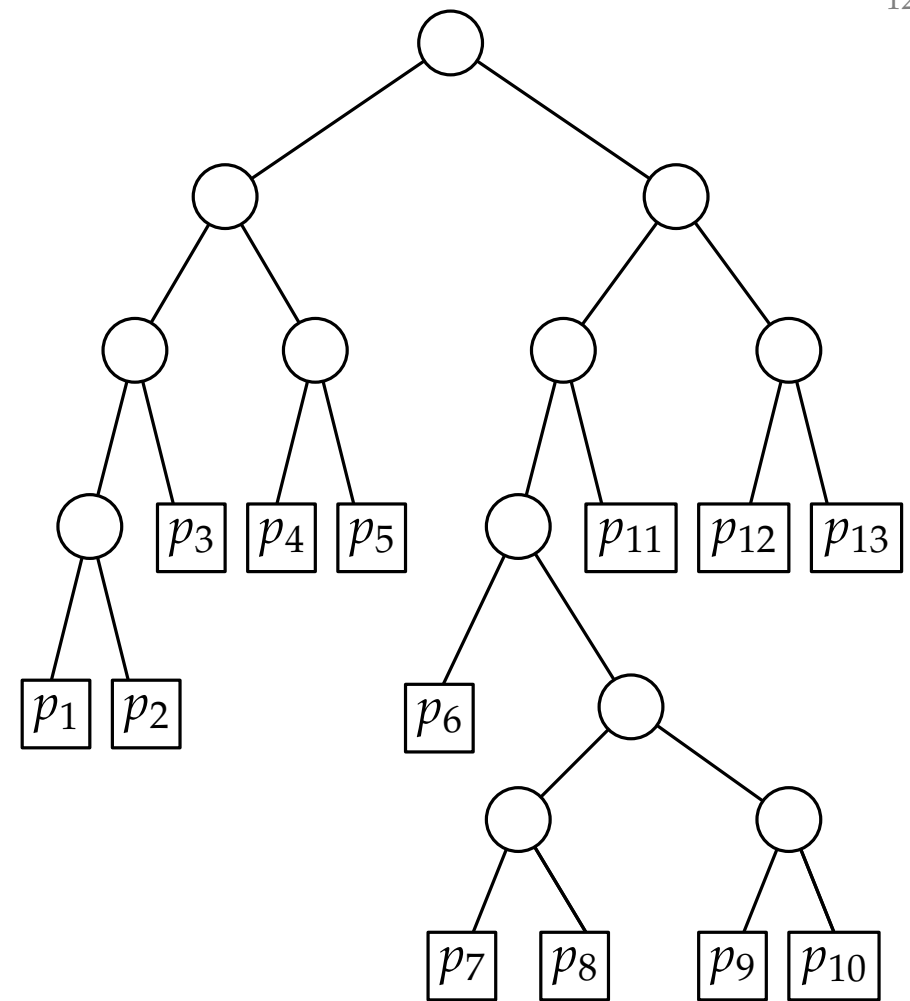
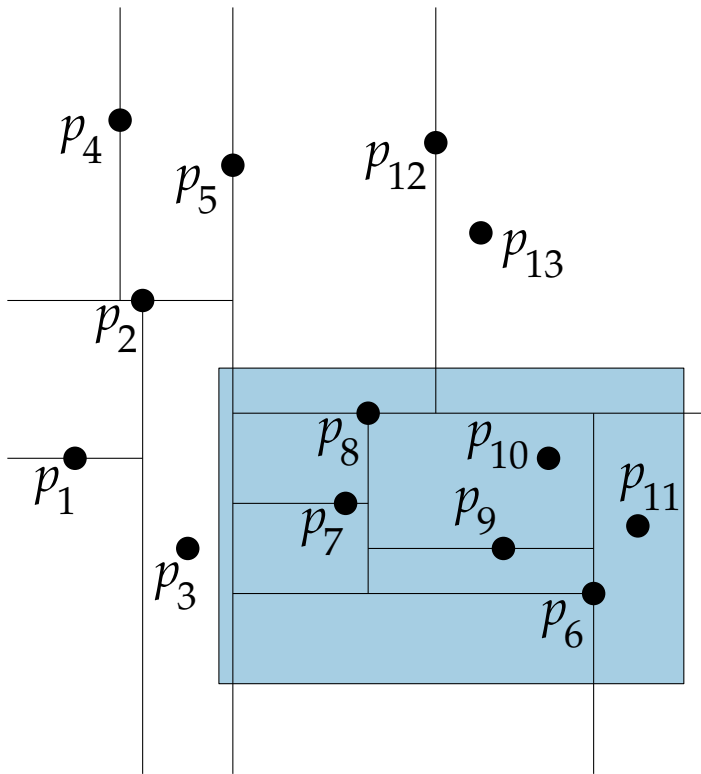
Kd-Trees: Analysis

Construction time?

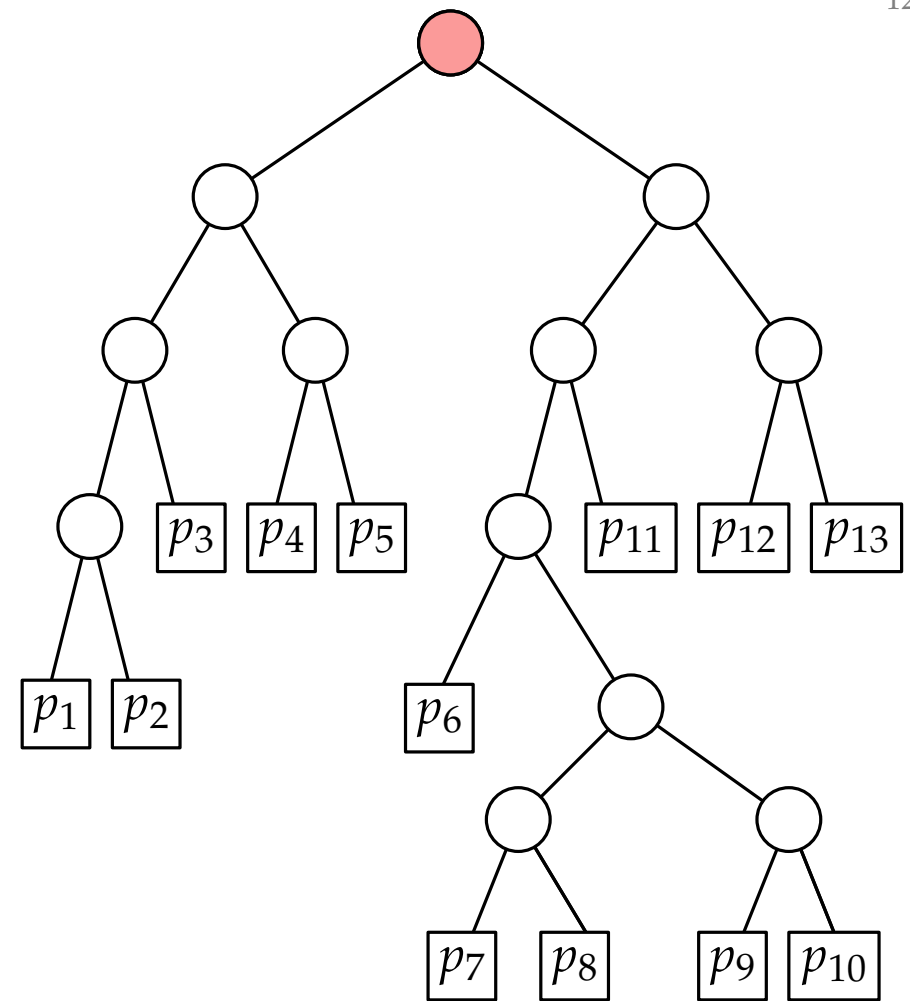
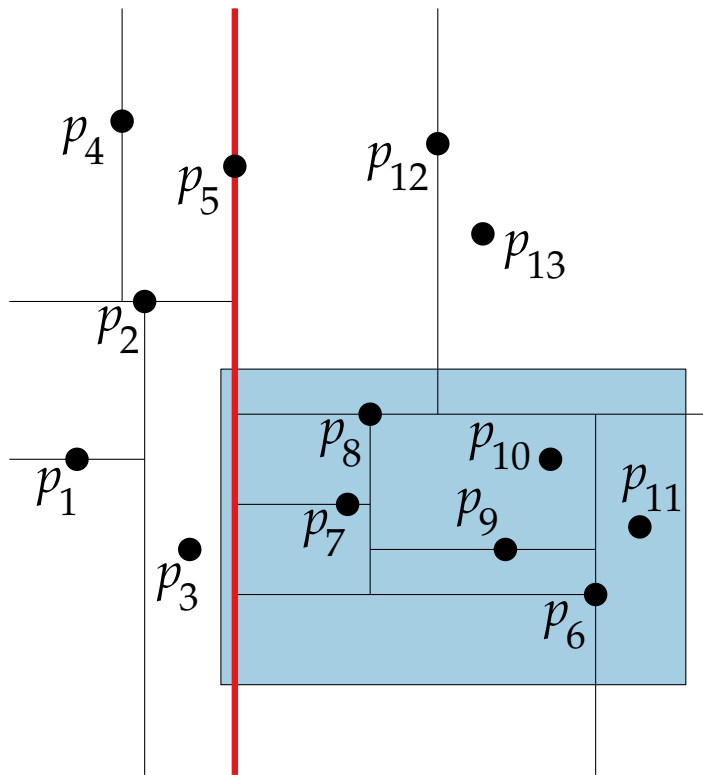
$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases} \stackrel{\text{see Mergesort!}}{=} O(n \log n)$$

Lemma. A kd-tree for a set of n pts in the plane takes $O(n \log n)$ time to construct and uses $O(n)$ storage.

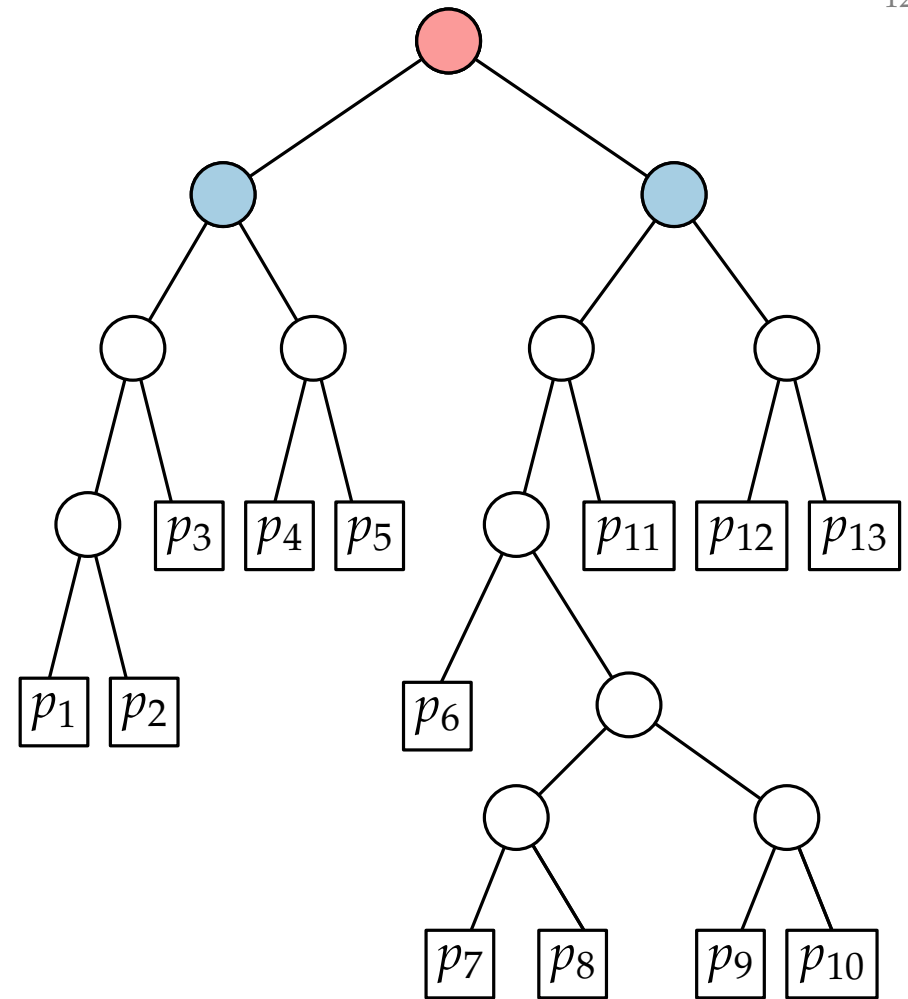
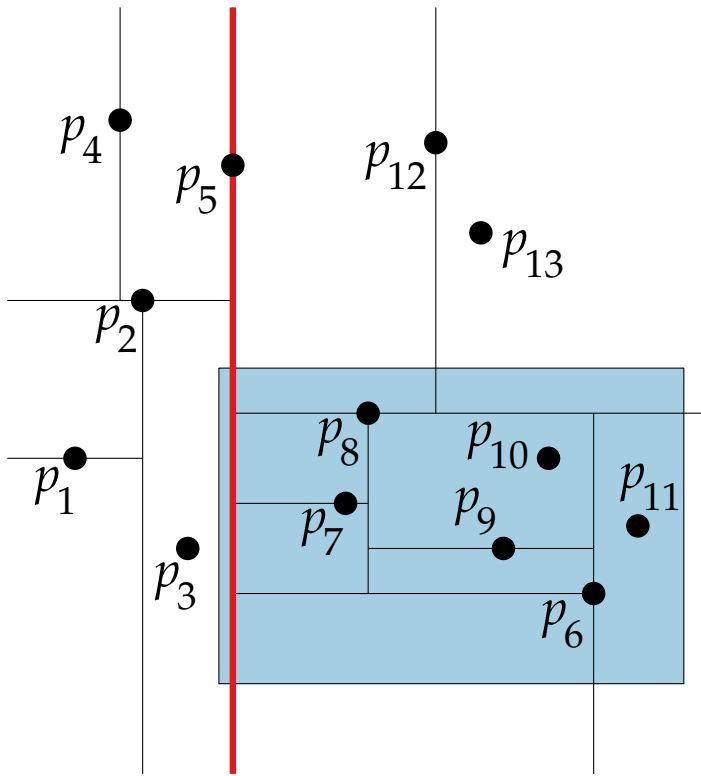
Kd-Trees: Querying



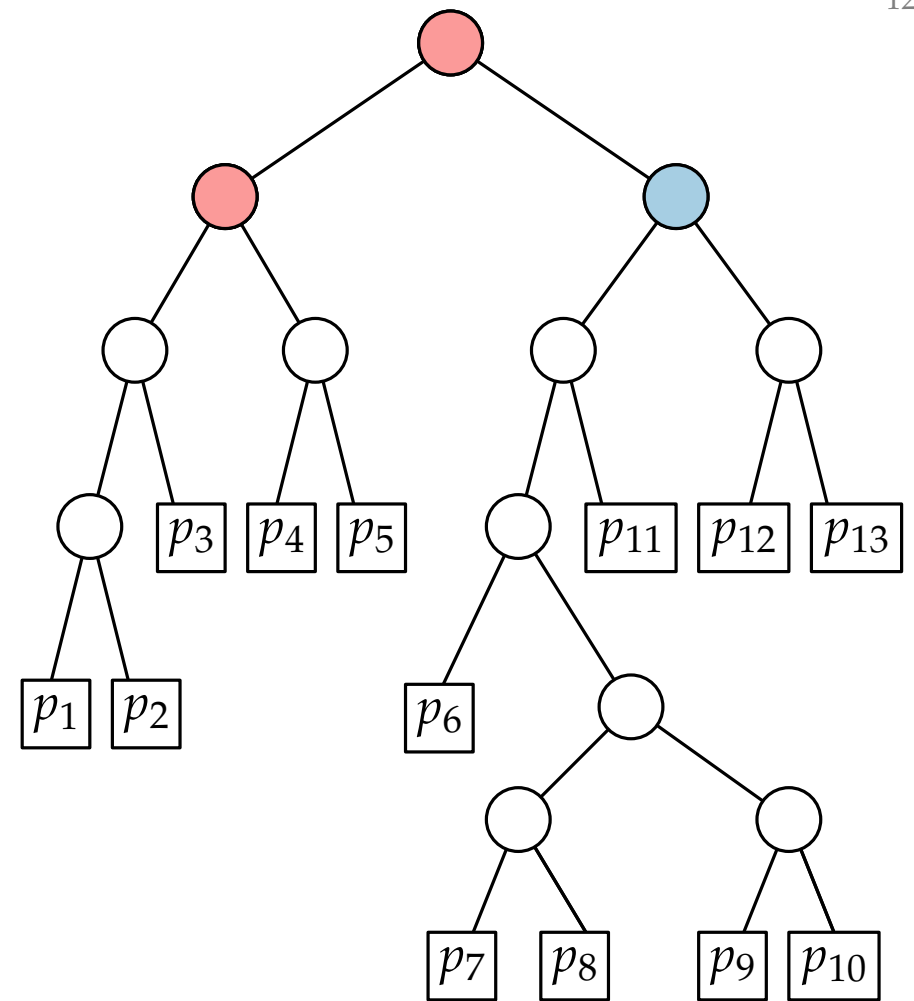
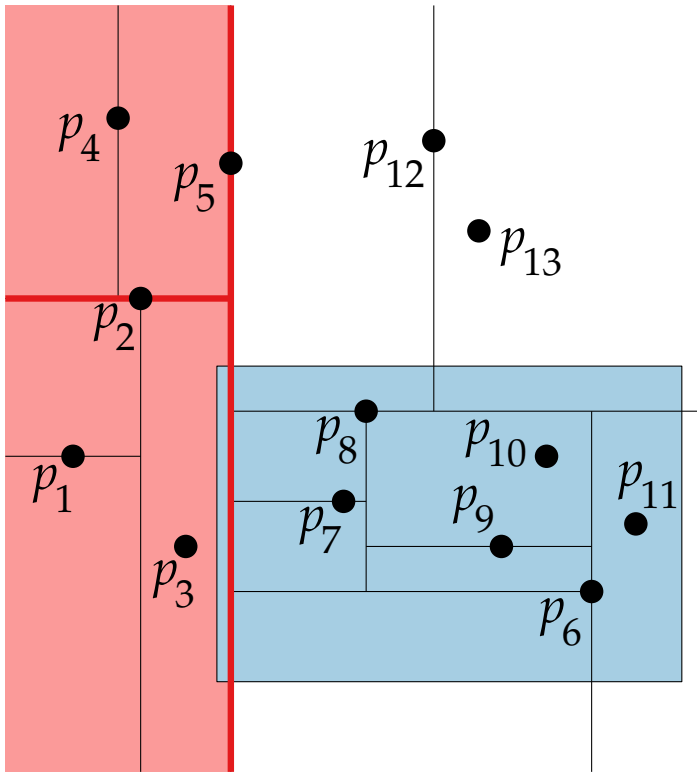
Kd-Trees: Querying



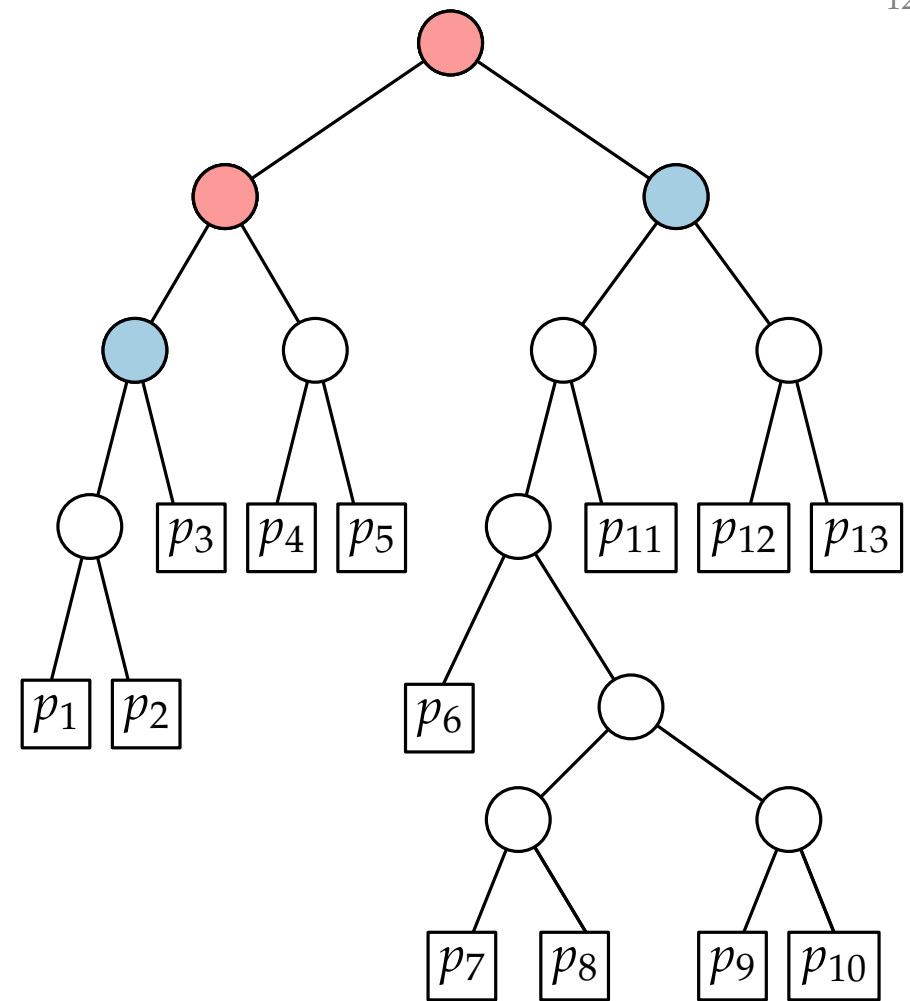
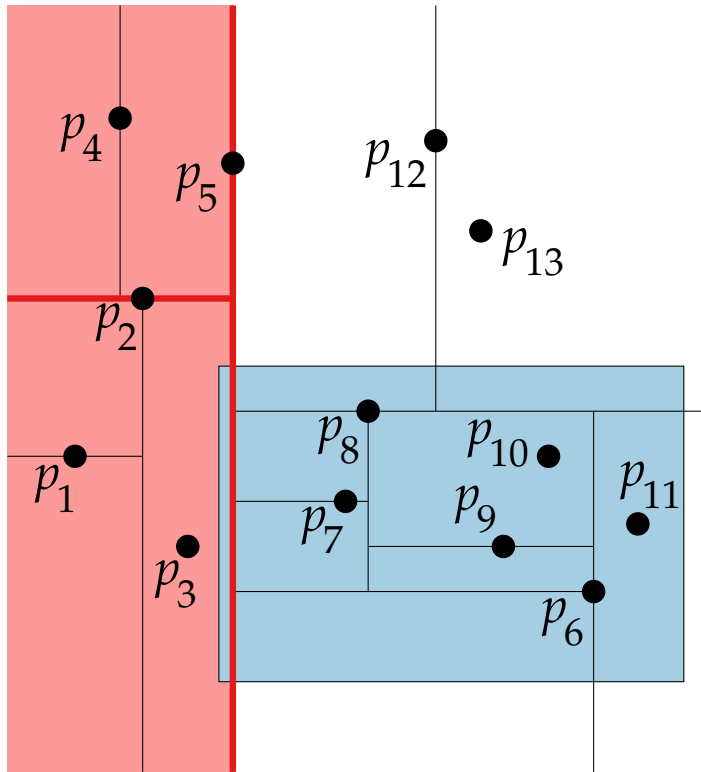
Kd-Trees: Querying



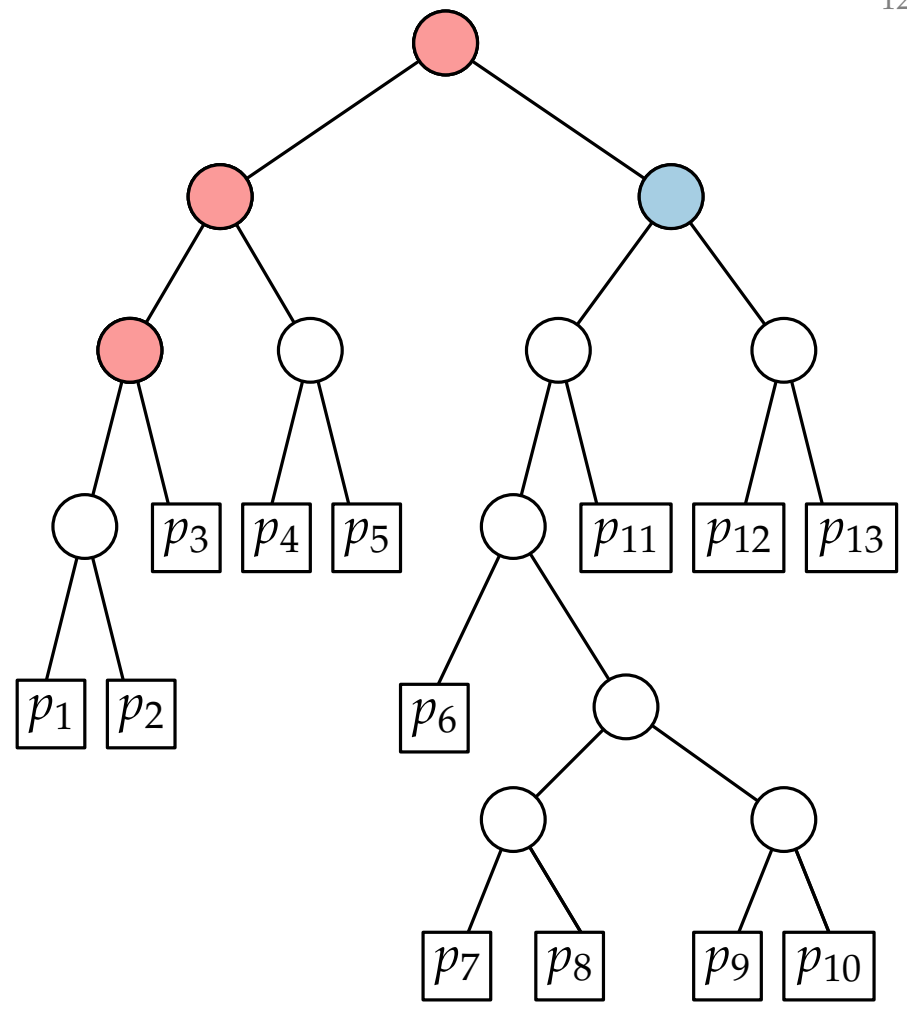
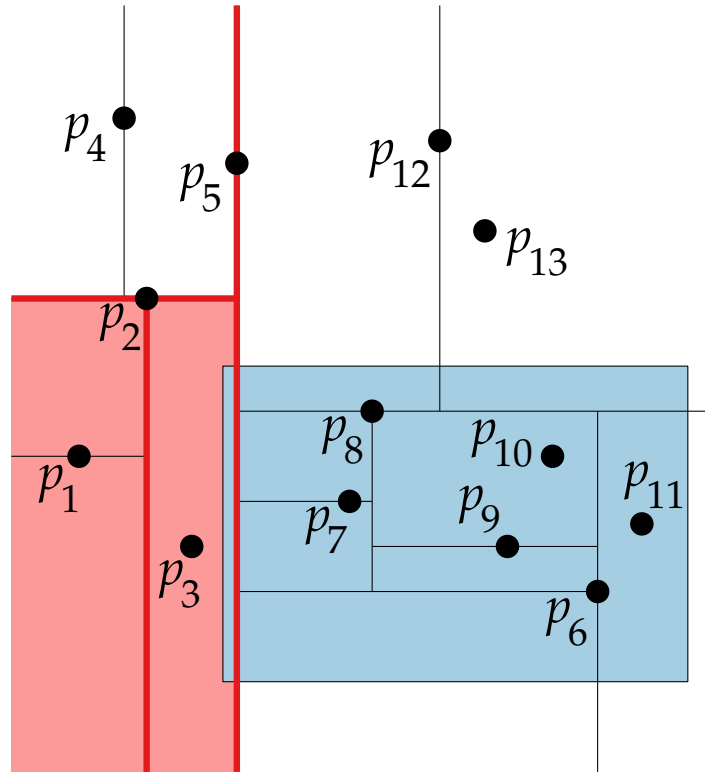
Kd-Trees: Querying



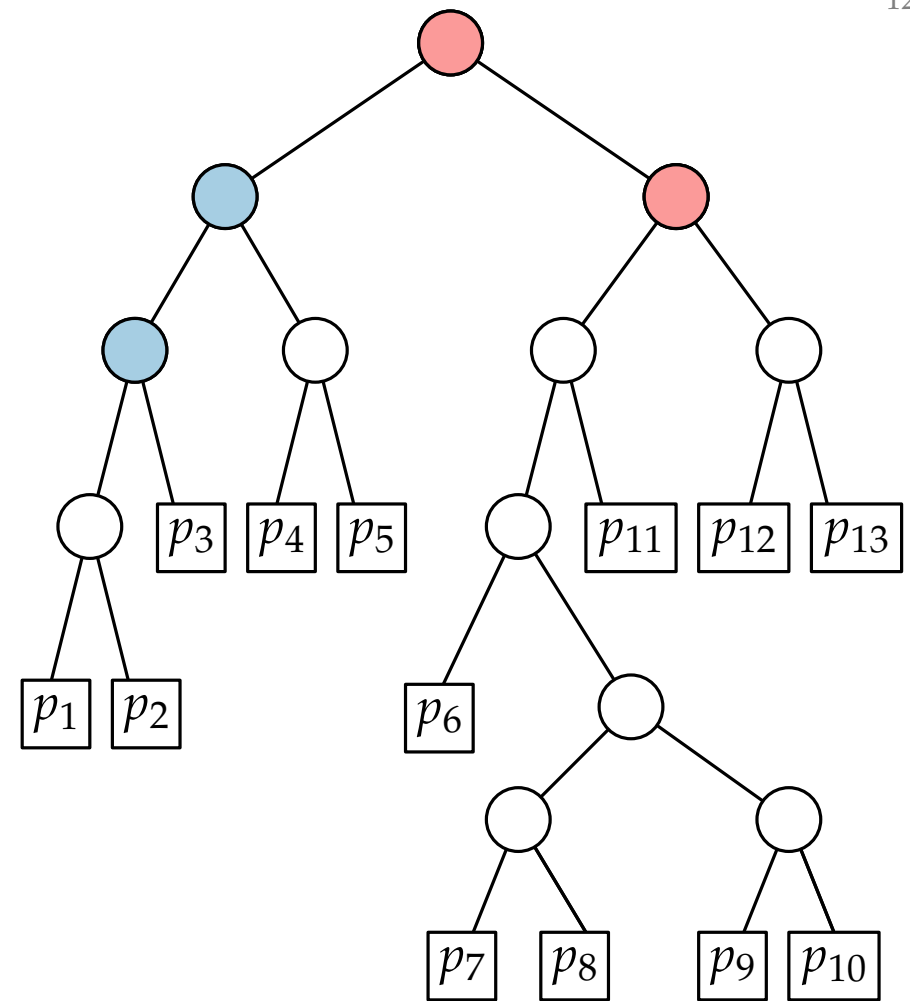
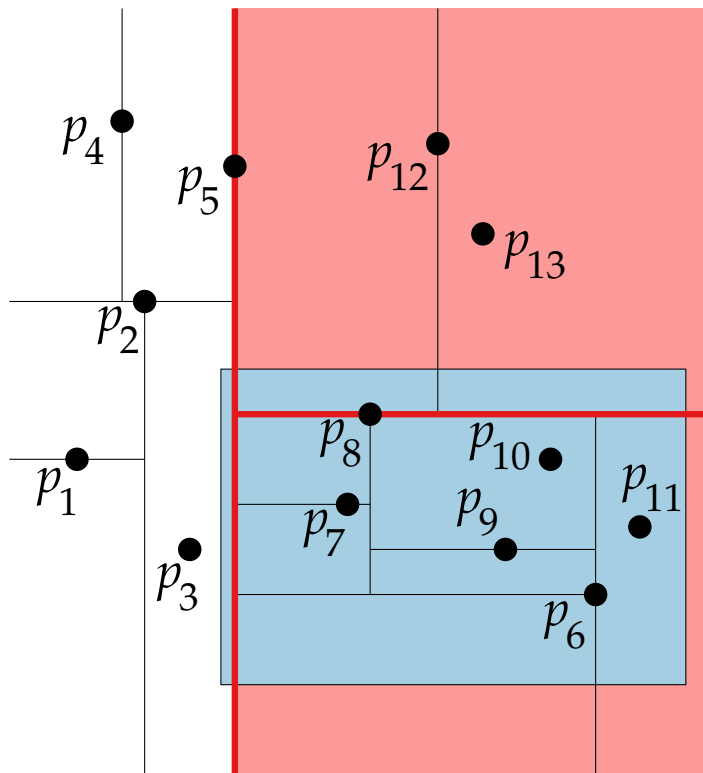
Kd-Trees: Querying



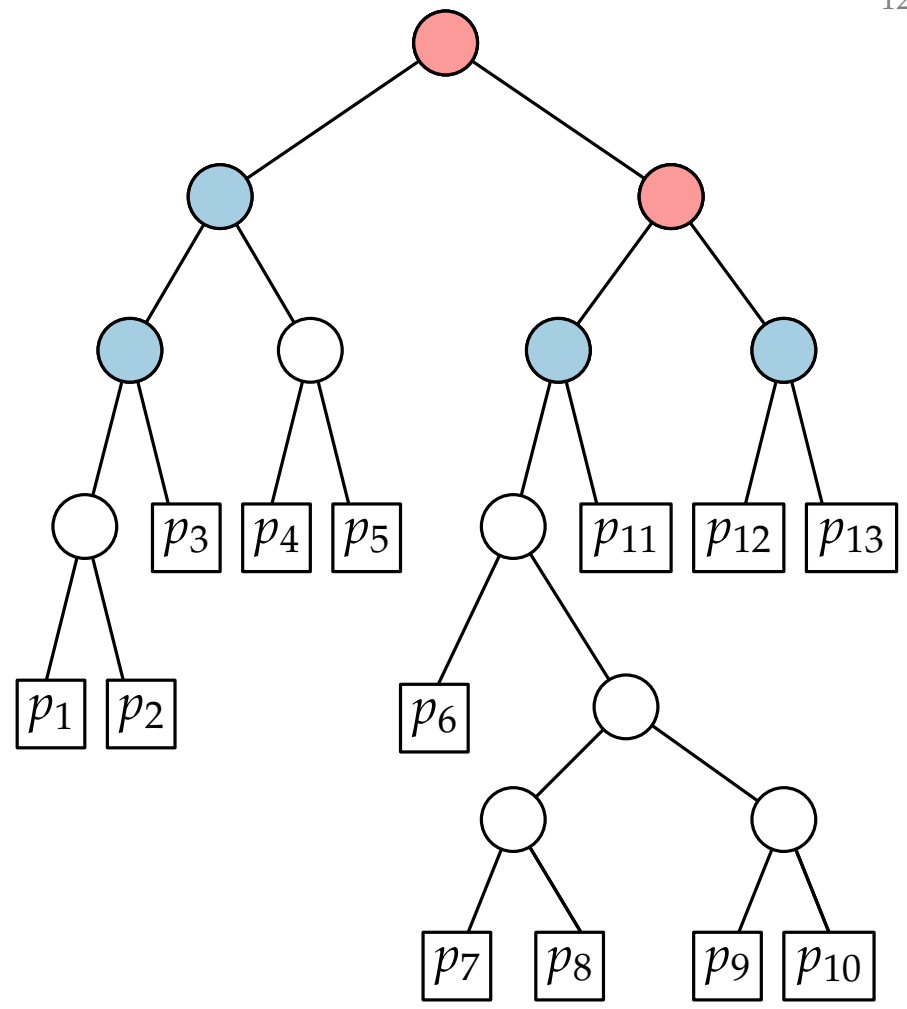
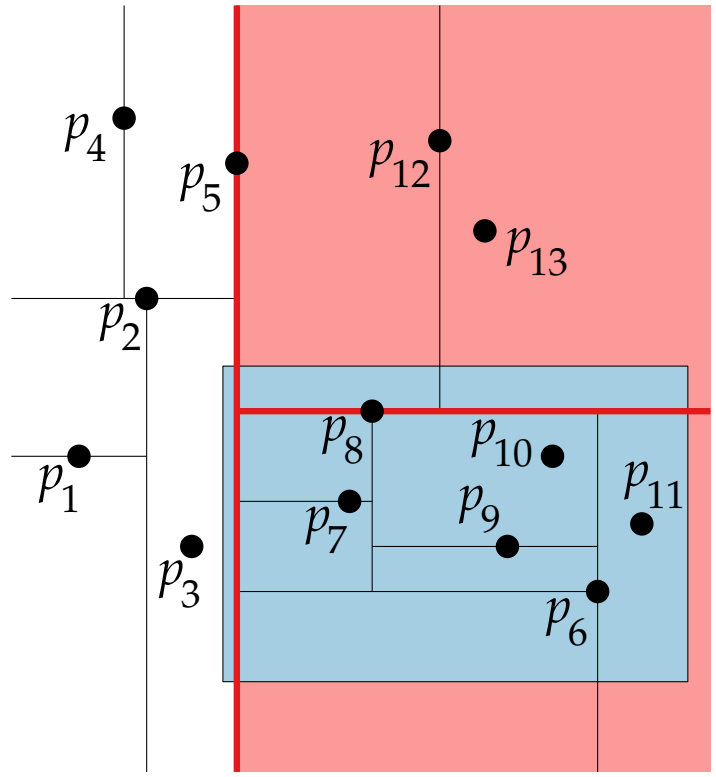
Kd-Trees: Querying



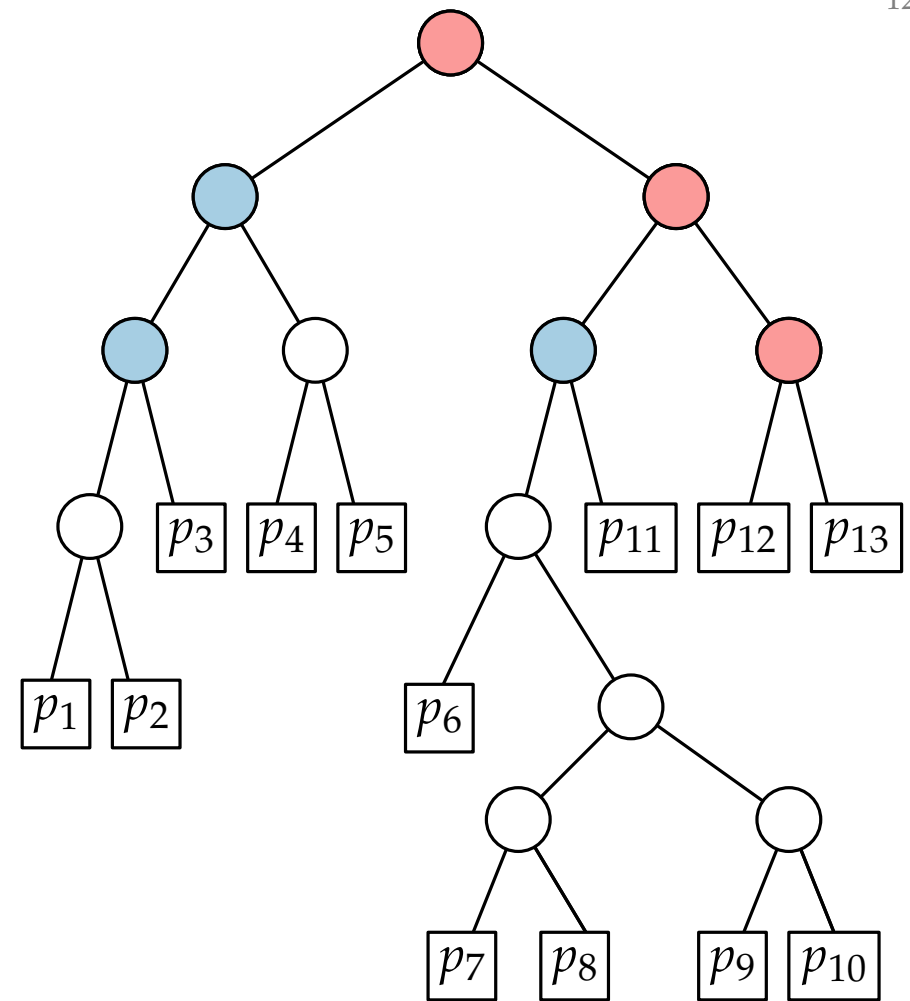
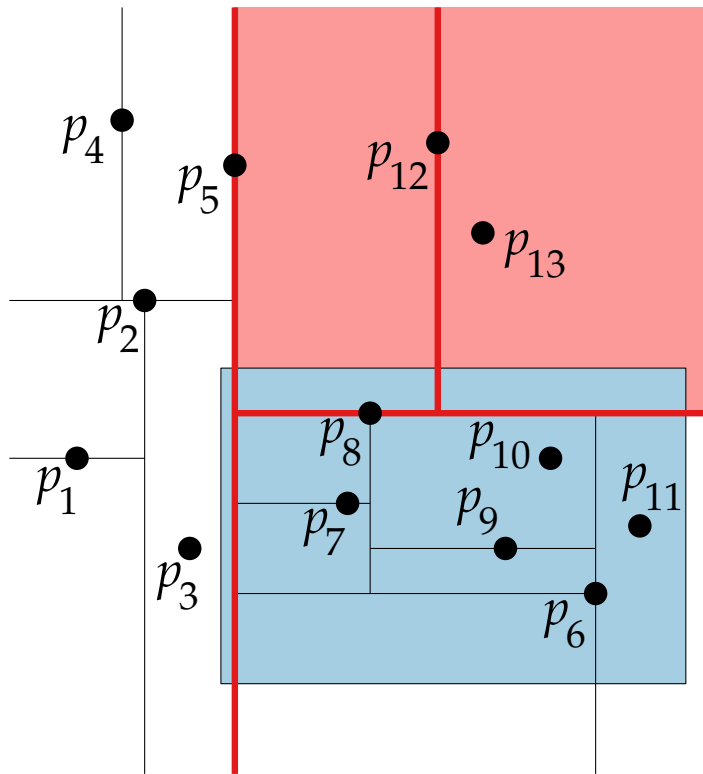
Kd-Trees: Querying



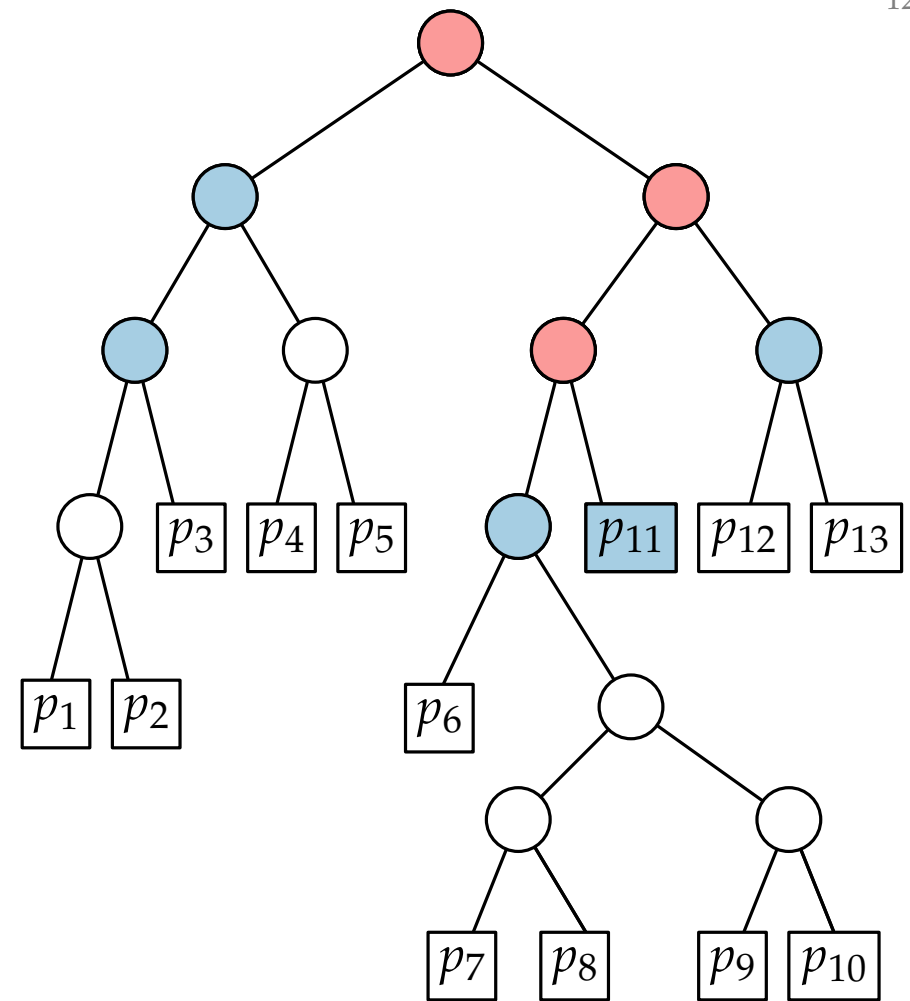
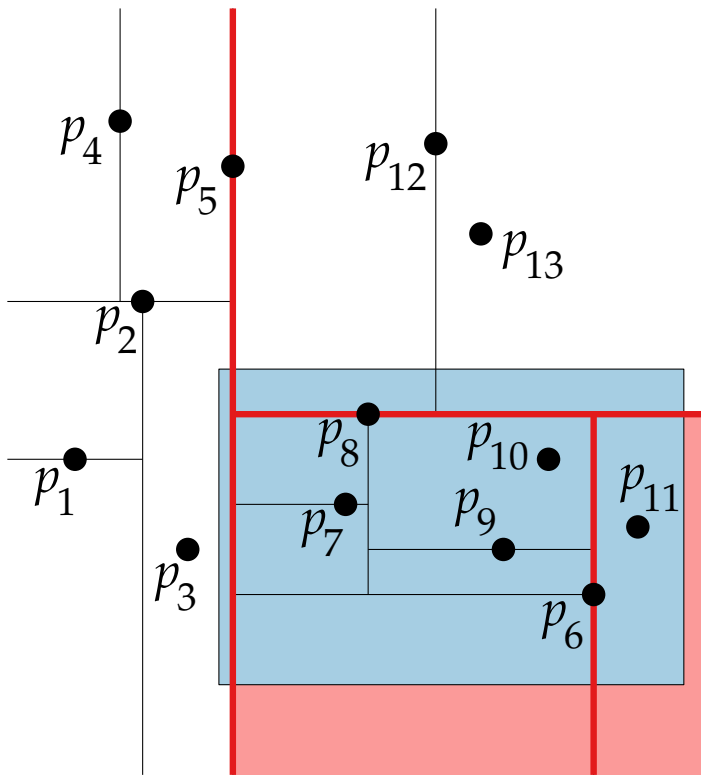
Kd-Trees: Querying



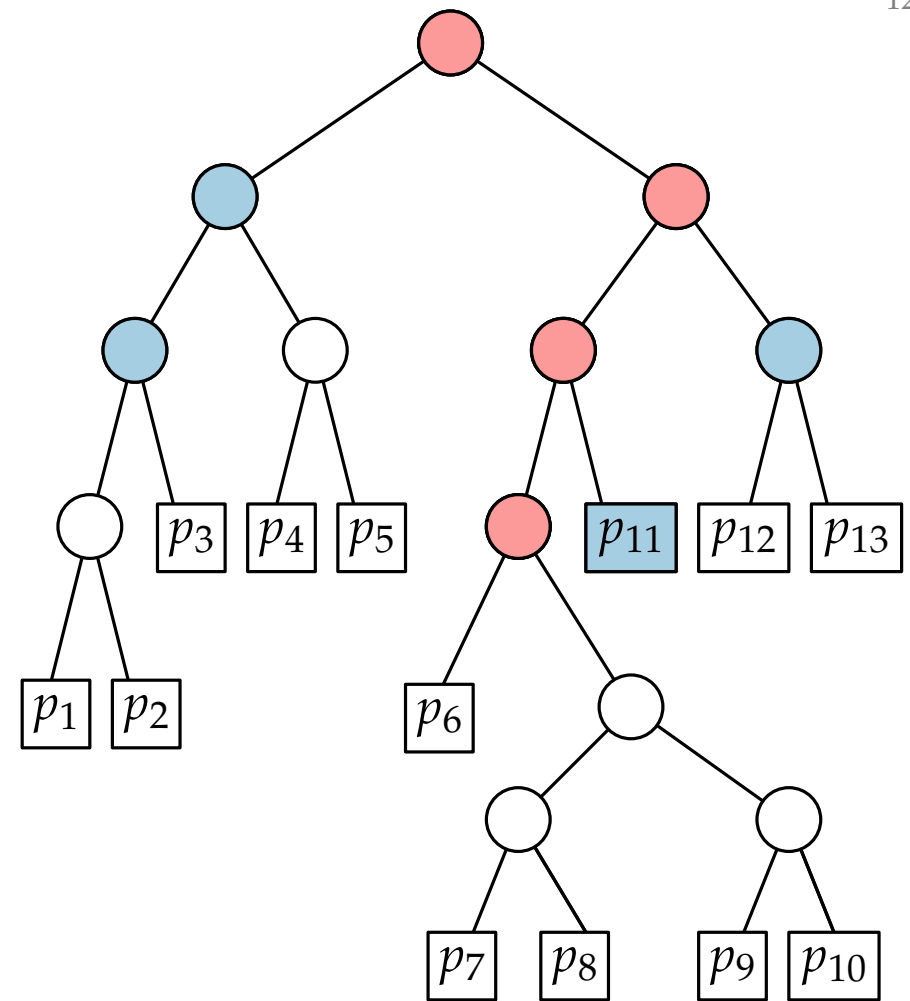
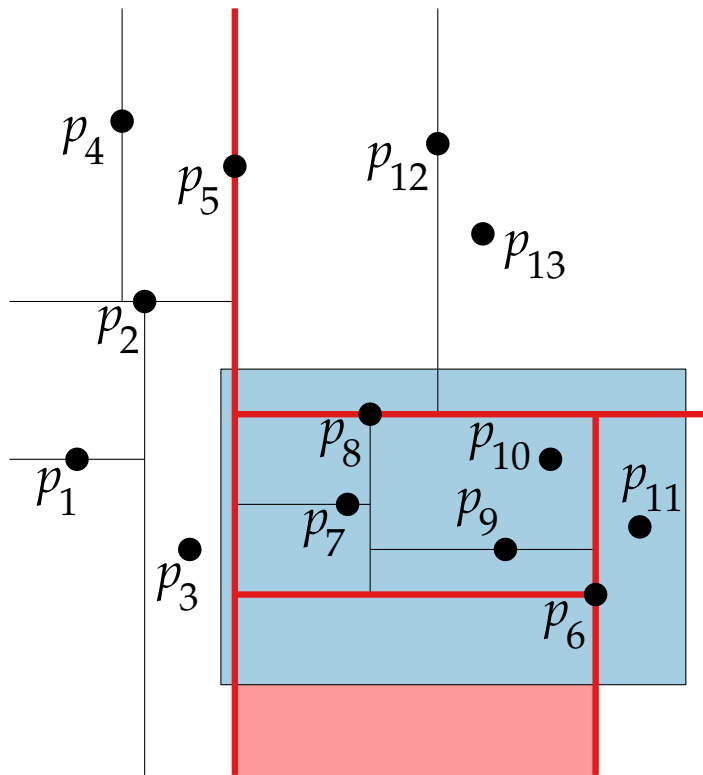
Kd-Trees: Querying



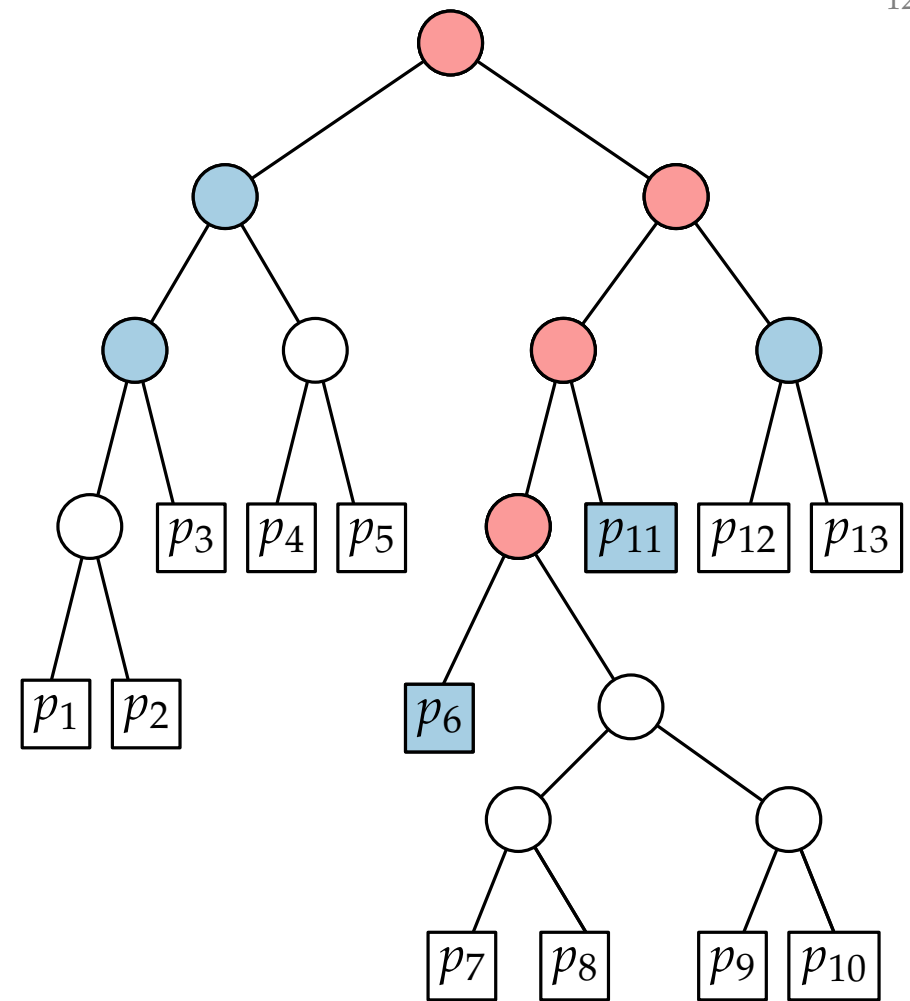
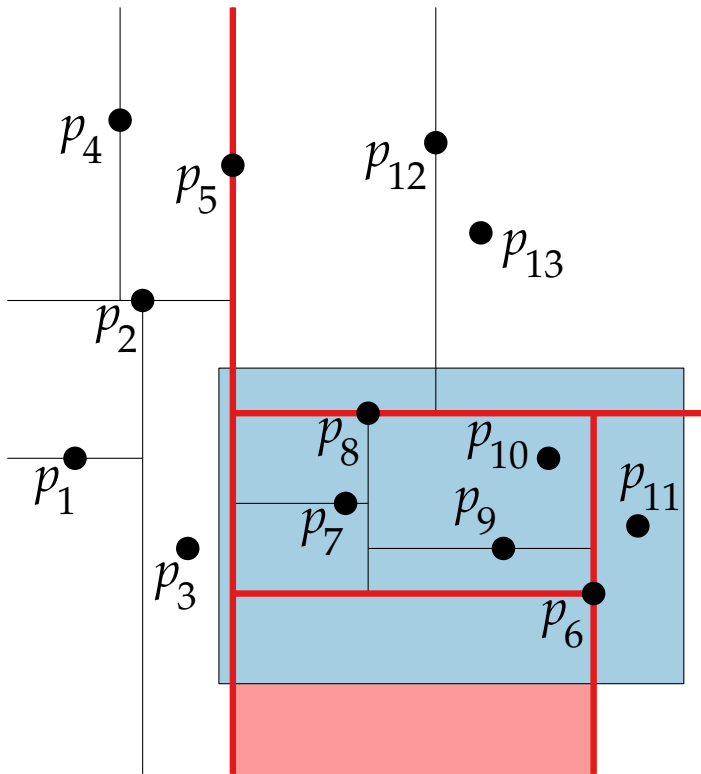
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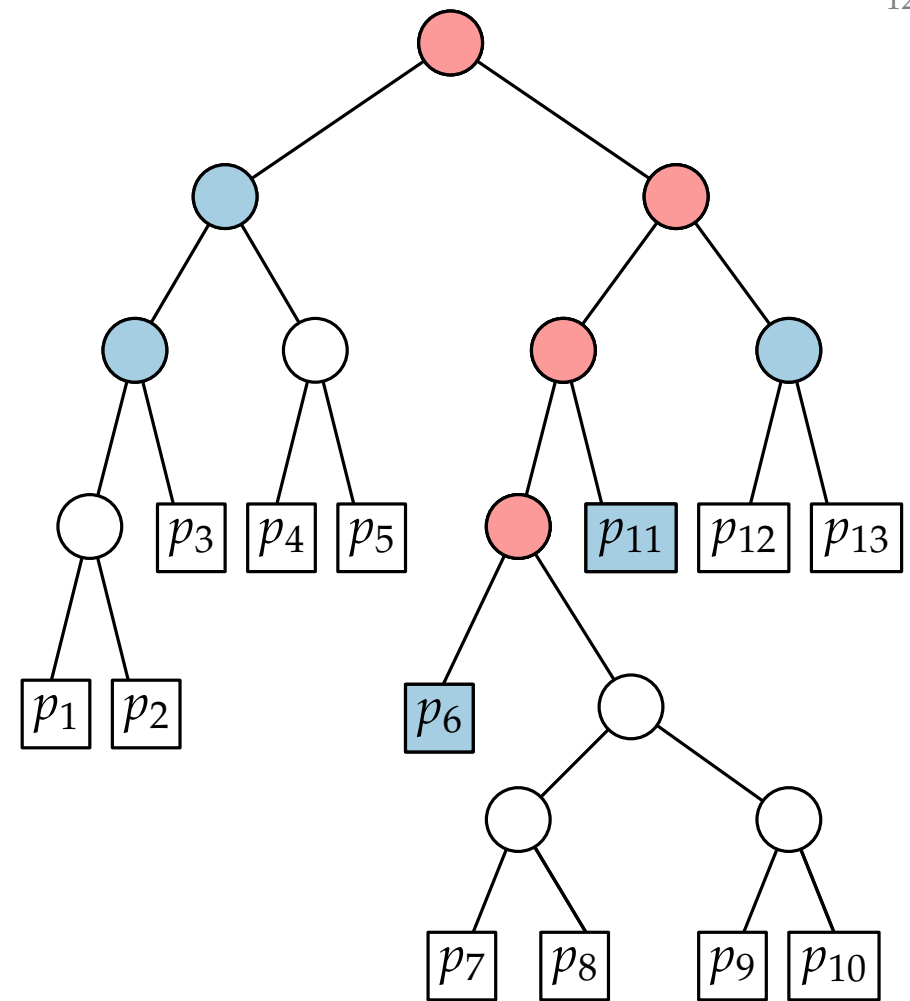
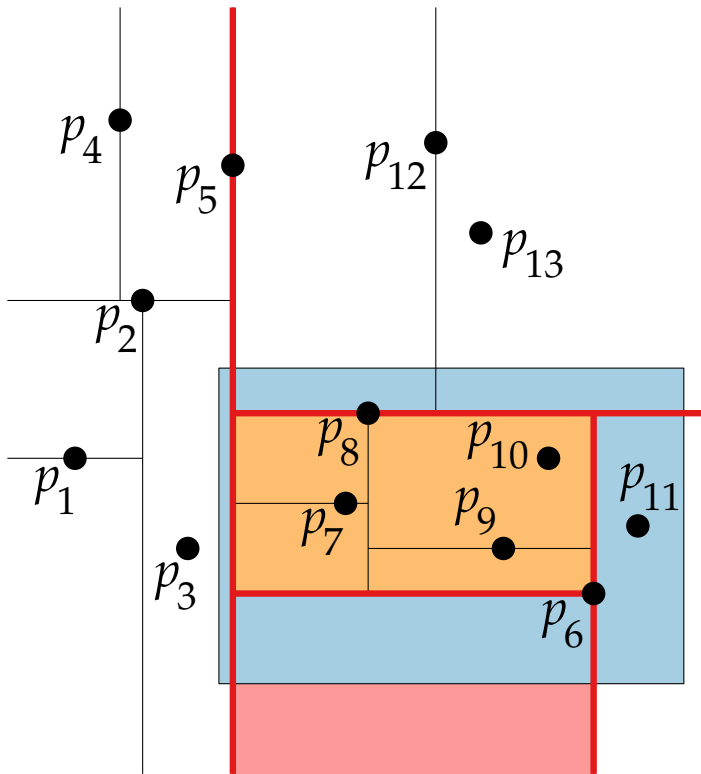
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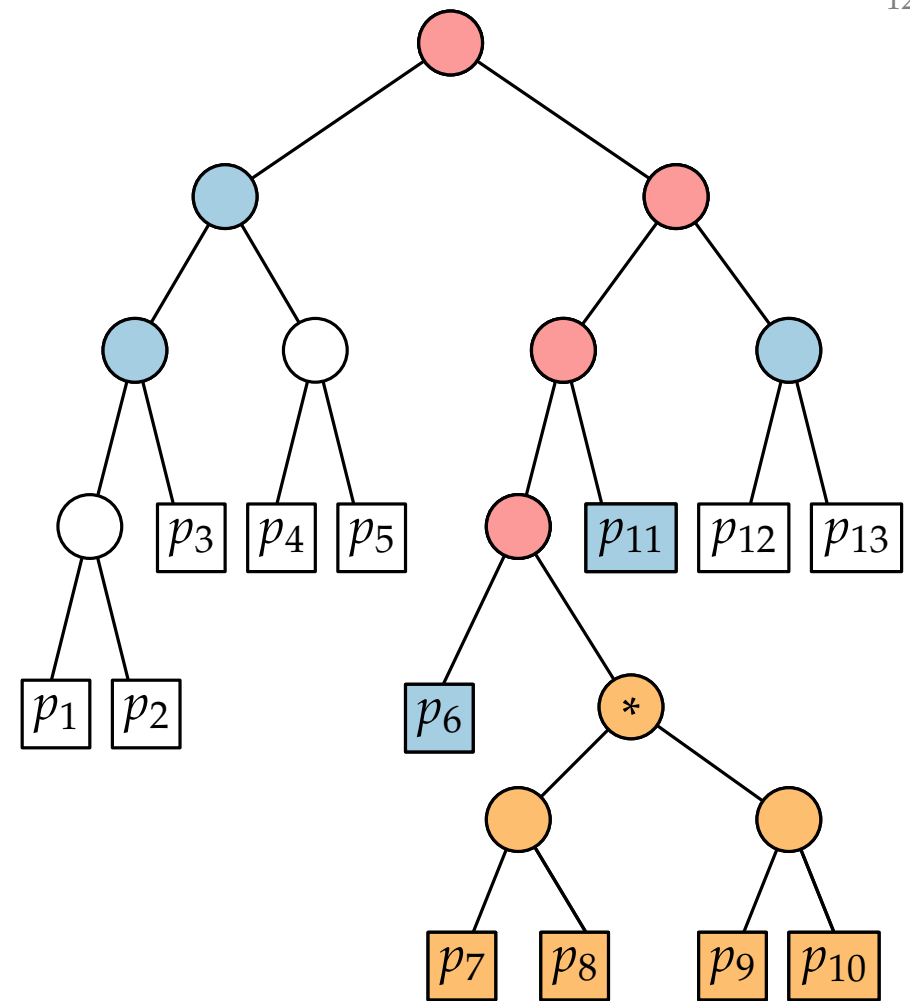
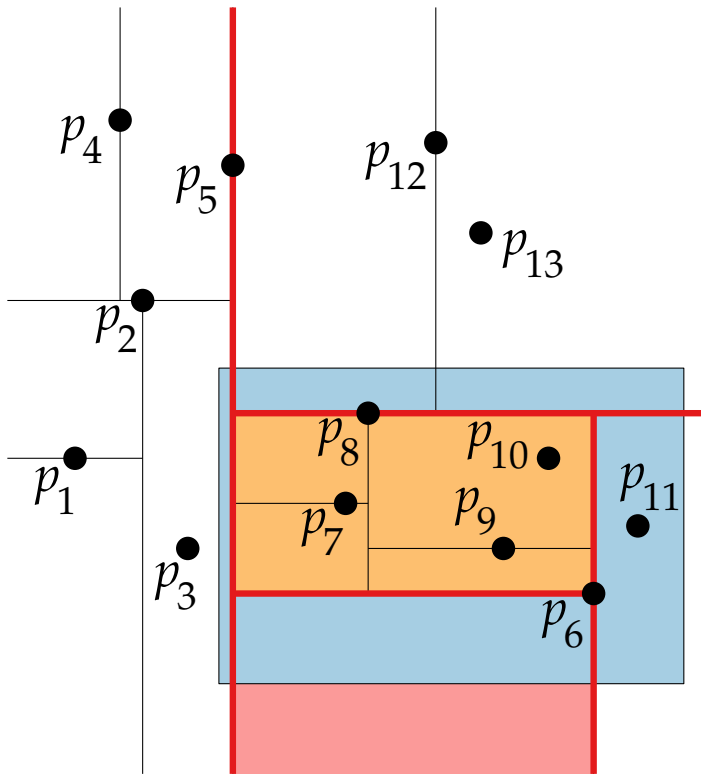
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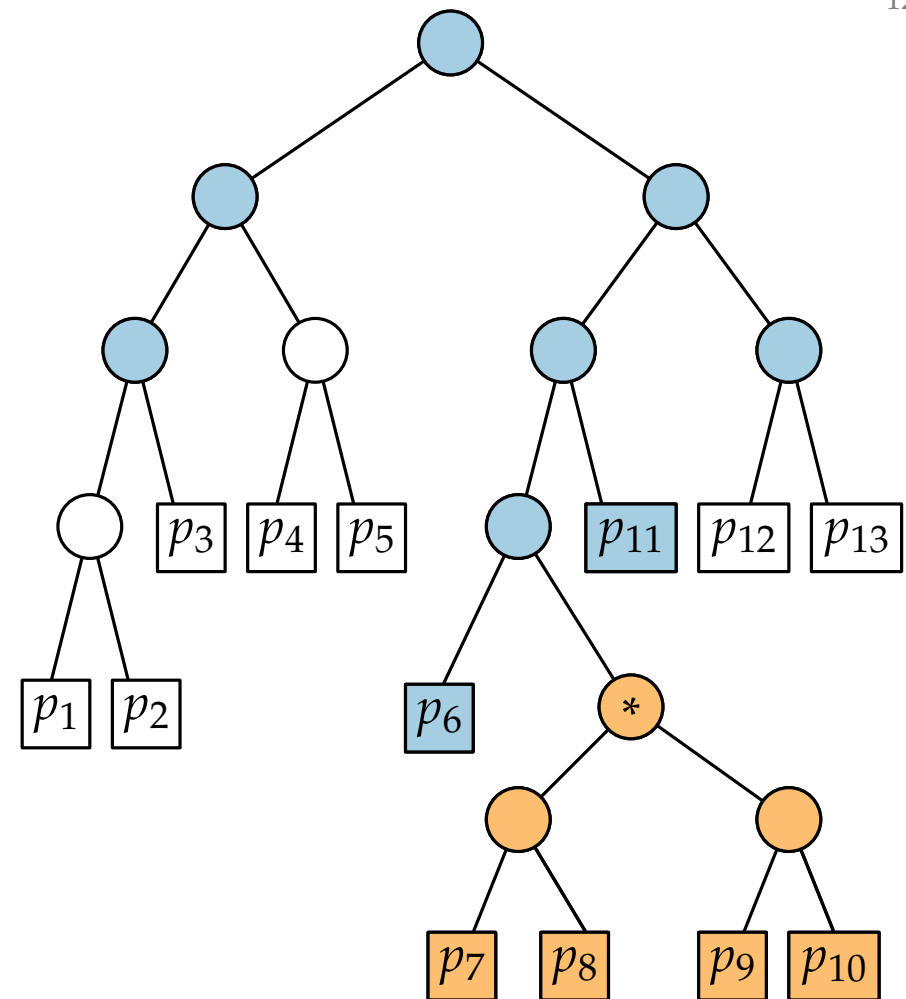
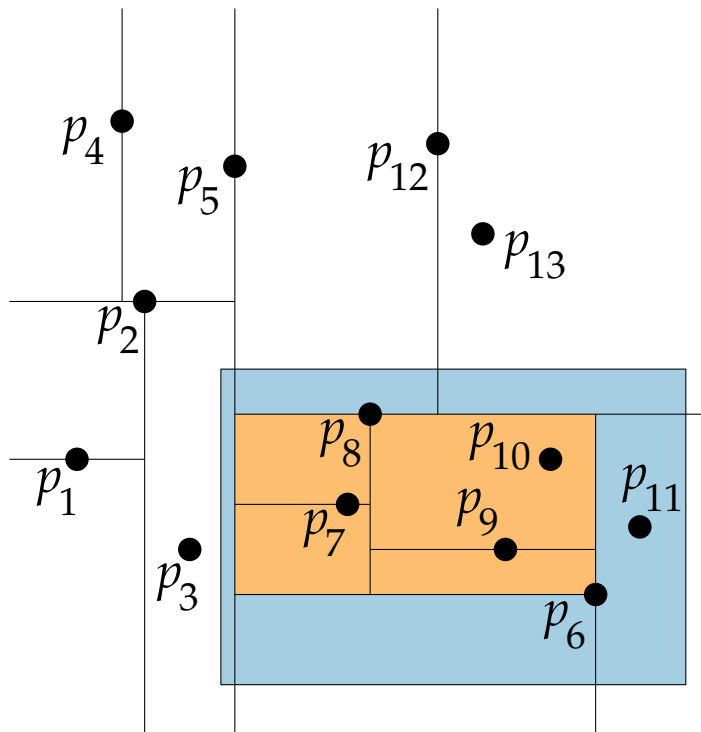
Kd-Trees: Querying



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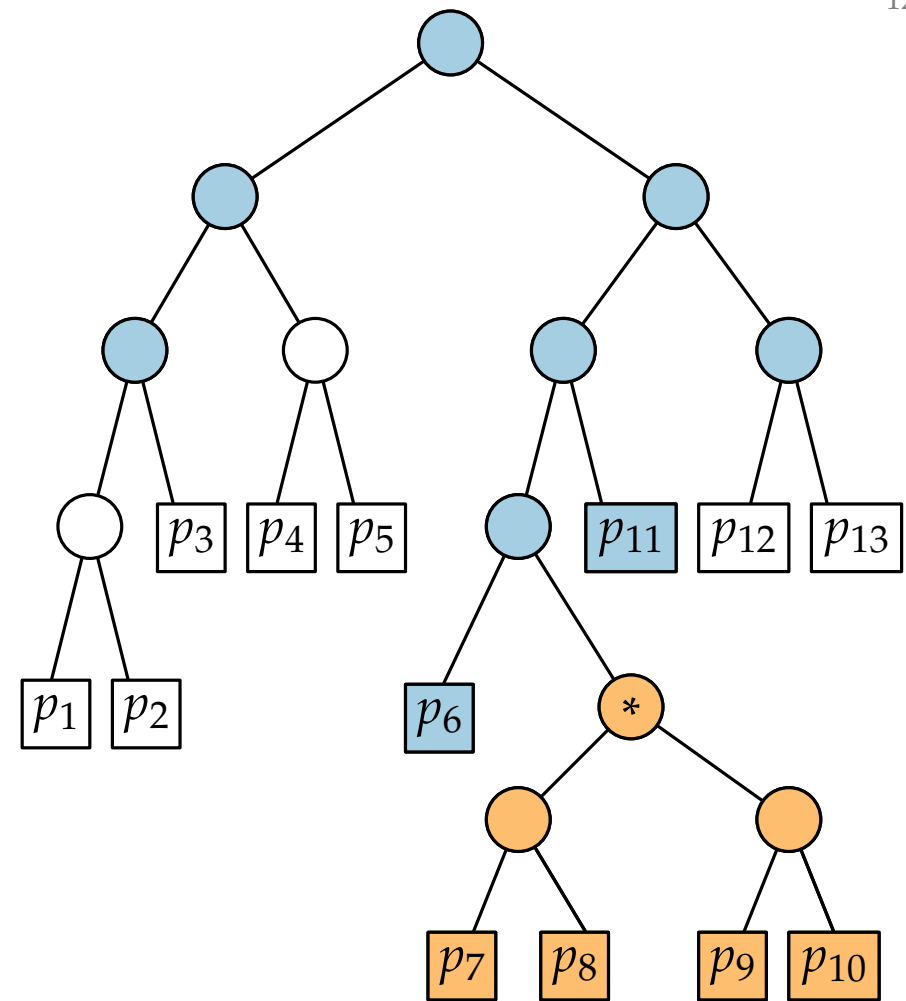
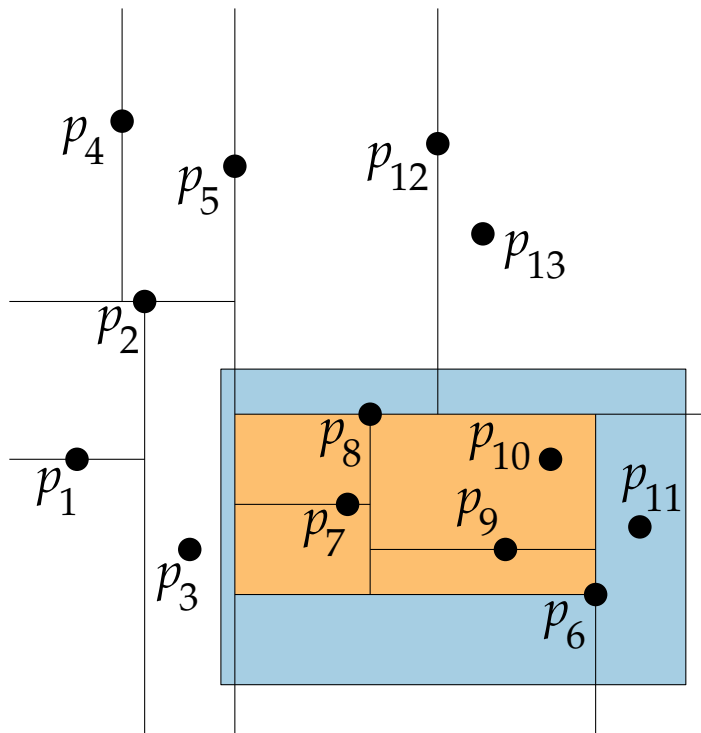


Kd-Trees: Querying



Lemma. Querying a kd-tree for n pts in the plane with an axis-parallel rectangle R takes $O(k + \sqrt{n})$ time, where $k = |\text{output}|$.

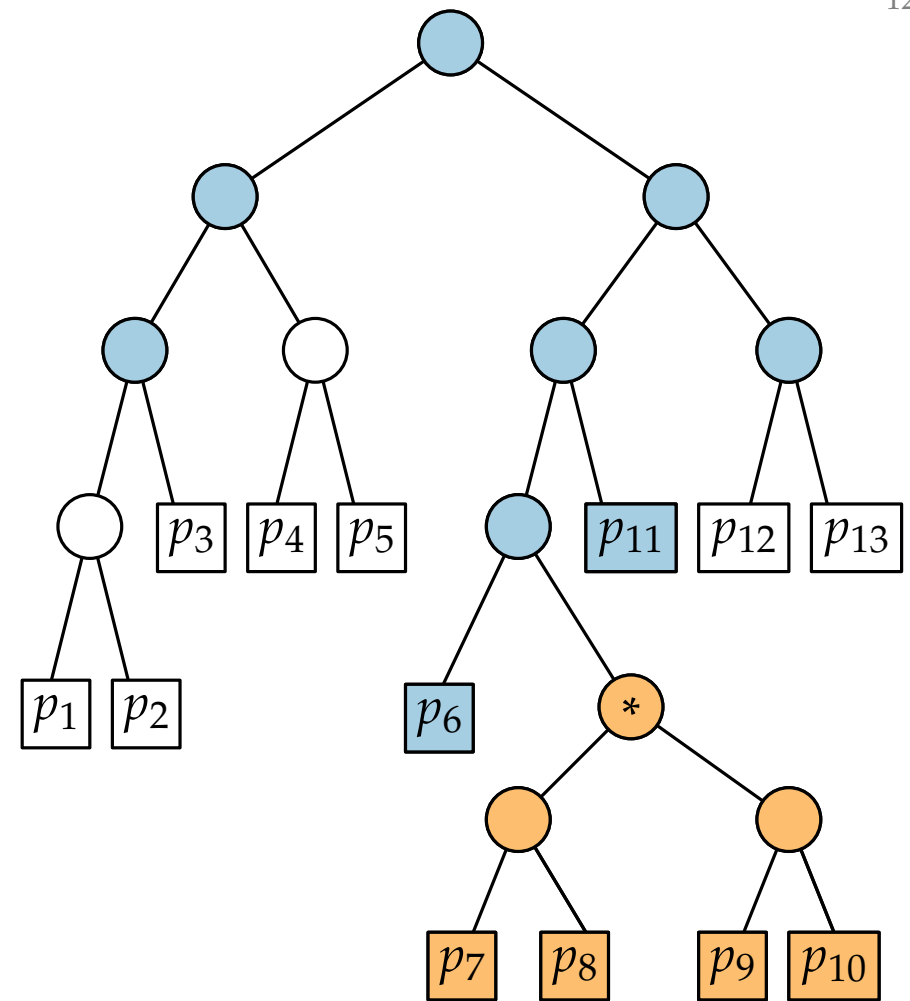
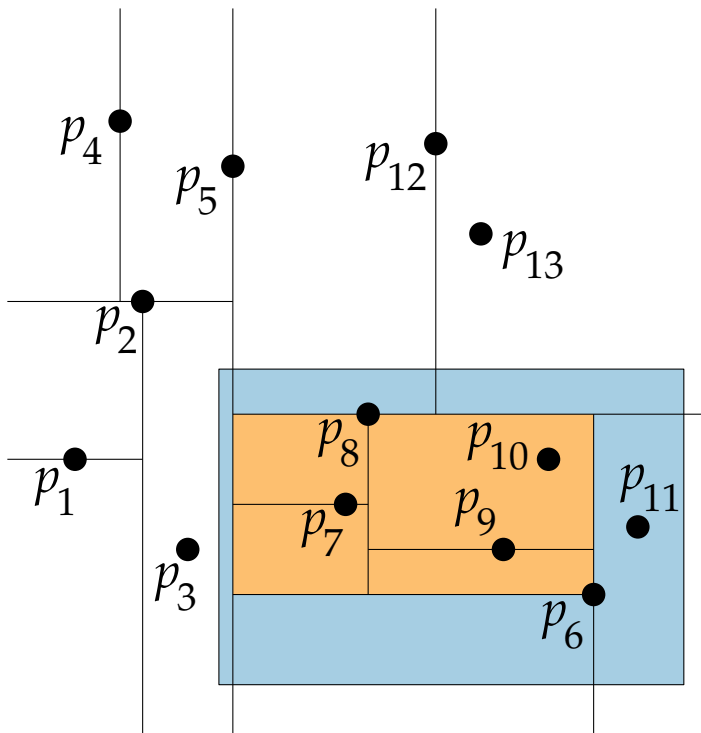
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Kd-Trees: Querying

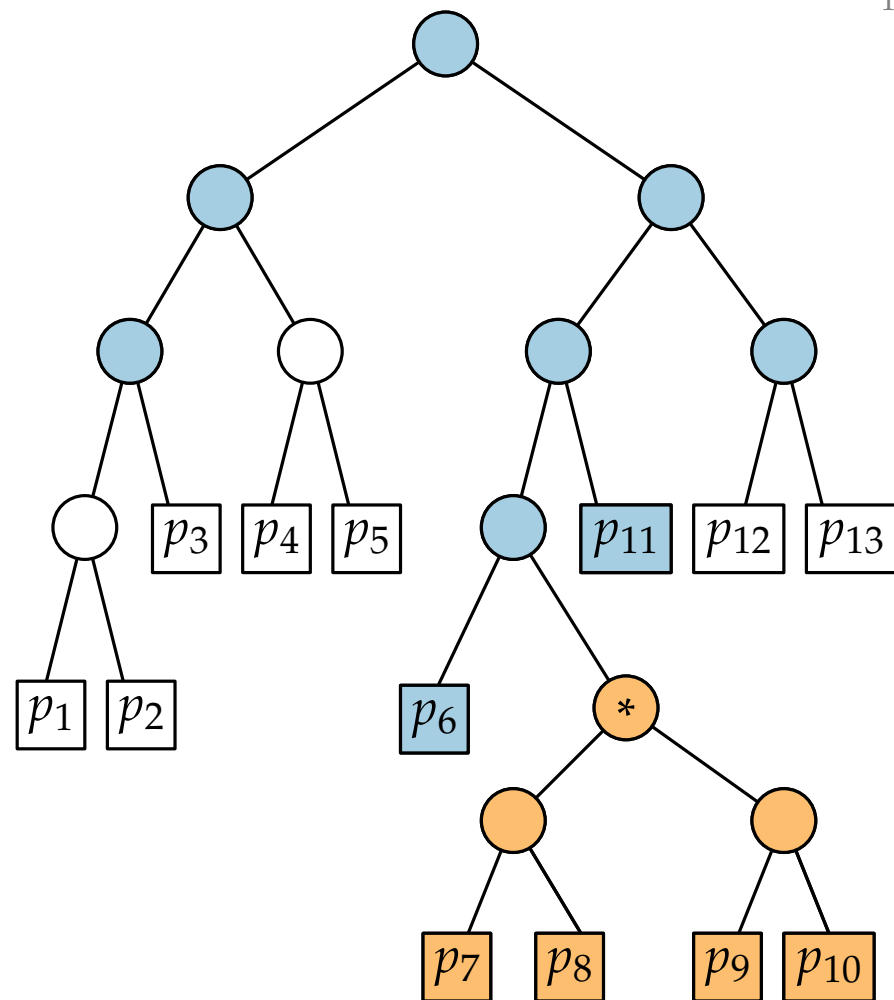
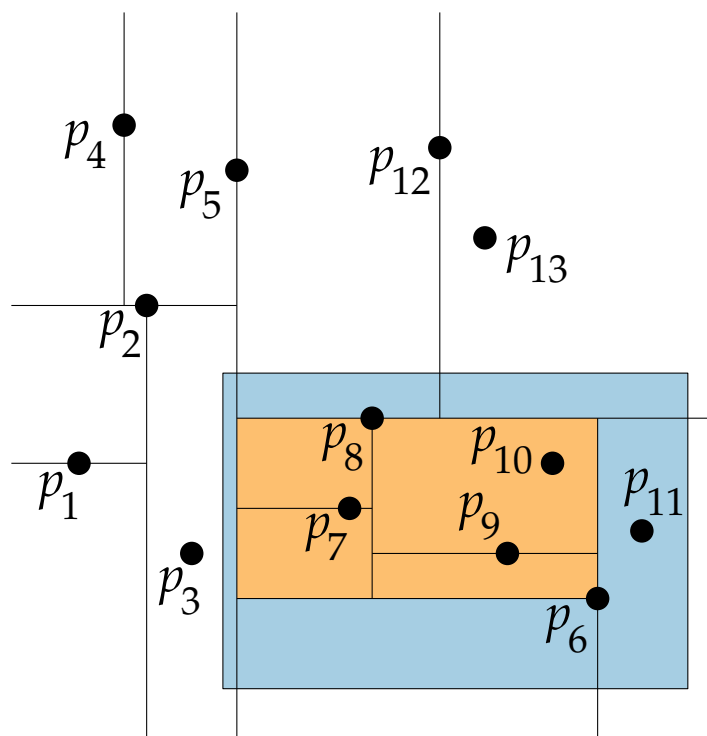


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$$Q(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2 + 2Q(\lceil n/4 \rceil) & \text{else.} \end{cases}$$

Kd-Trees: Querying

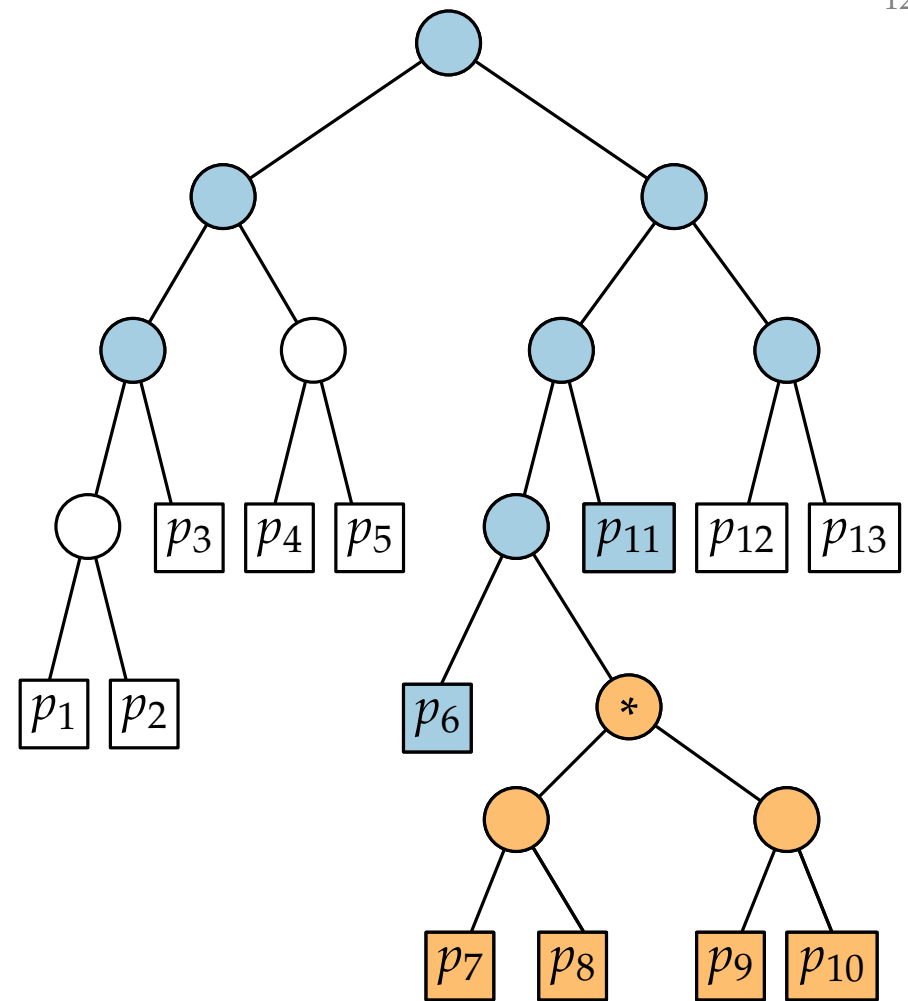
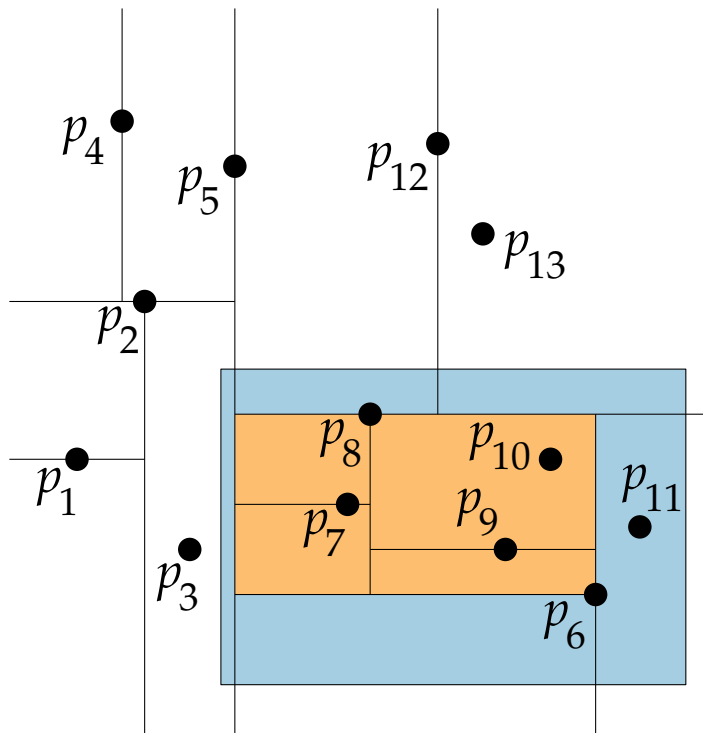


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Kd-Trees: Querying

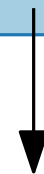


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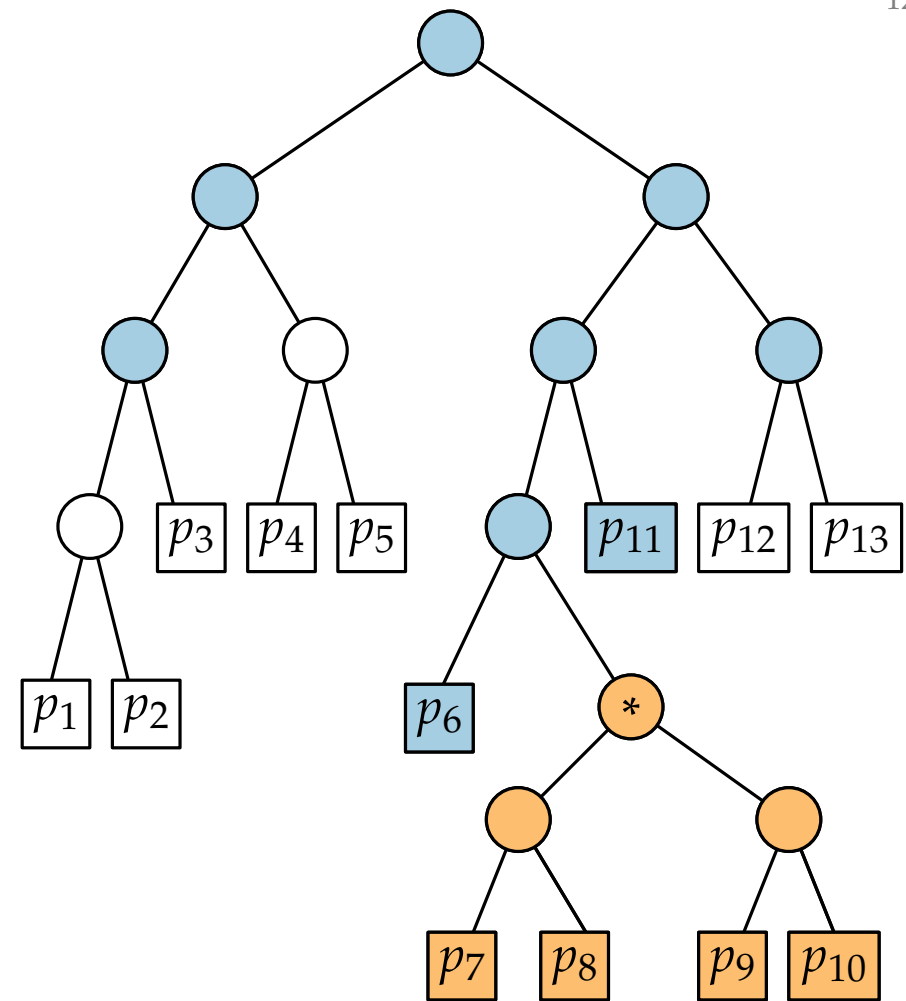
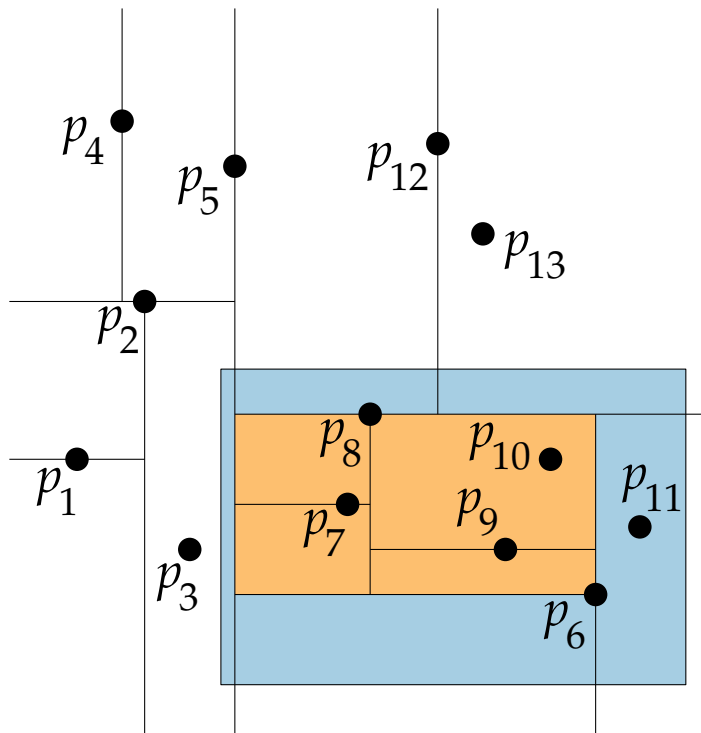
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in \mathbb{R}^d

$$O(k + n^{1-1/d})$$

Computational Geometry

Lecture 5: Orthogonal Range Queries or Fast Access to Data Bases

Part IV: Range Trees: Querying & Construction

Extensions to 2D

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

Solutions:

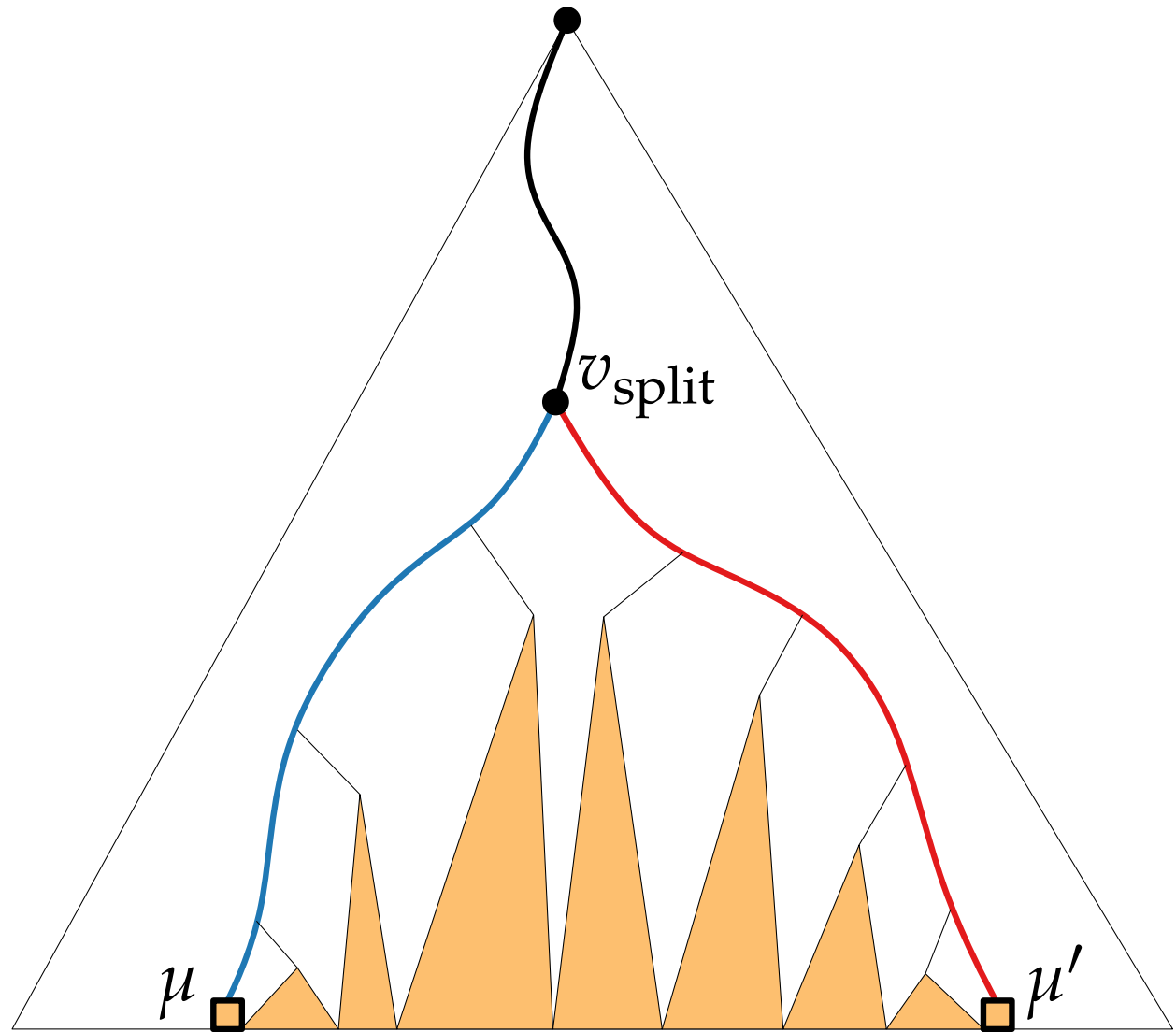
- *one* tree;
query path alternates between x - and y -coord. } *kd-tree*
- first-level tree for x -coordinates;
many second-level trees for y -coord. } *range tree*

Assume: *General position!*

Here: no two points have the same x - or y -coordinate.

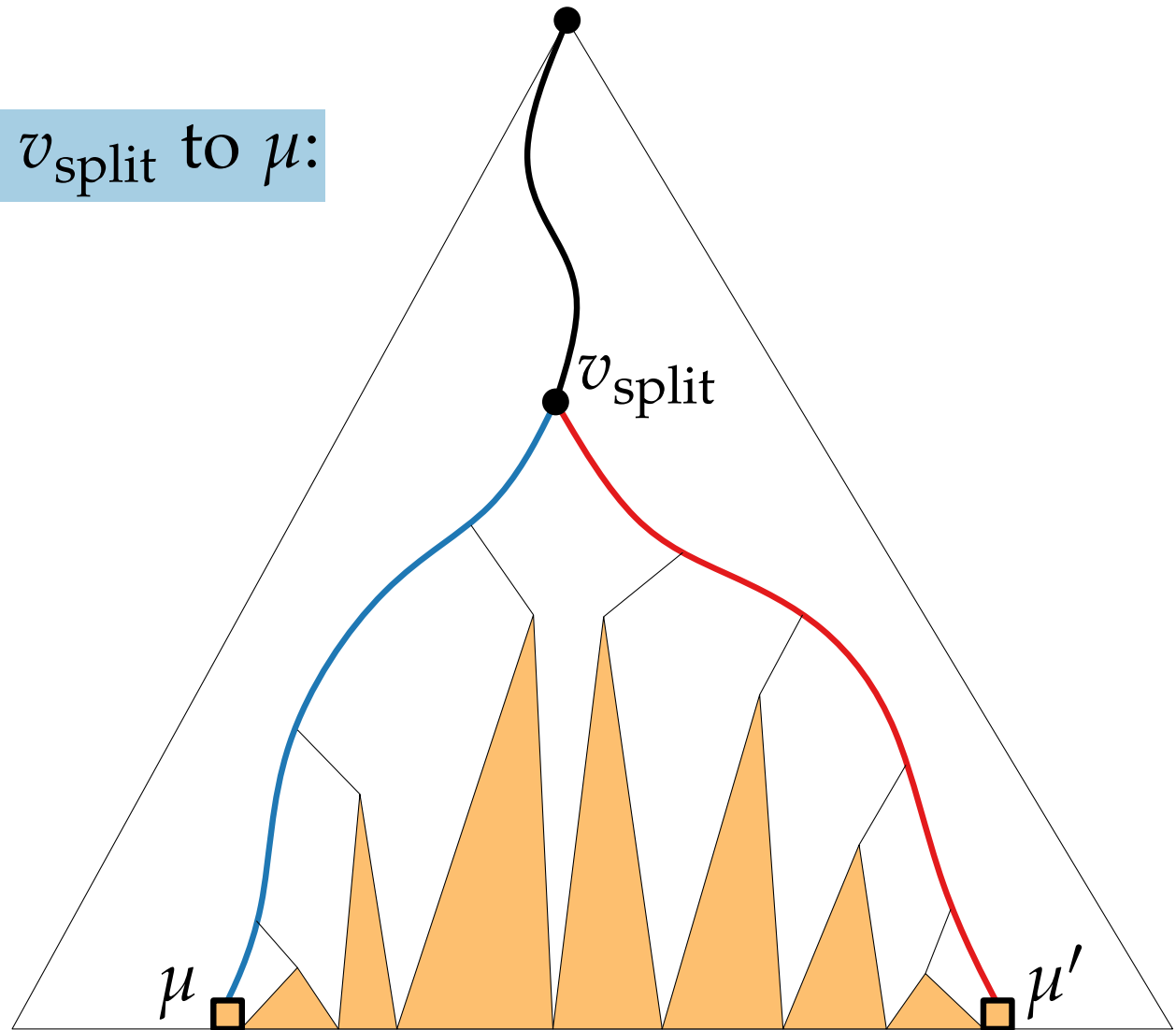
Range Trees: Query Algorithm

1. Search in main tree for x-coordinate



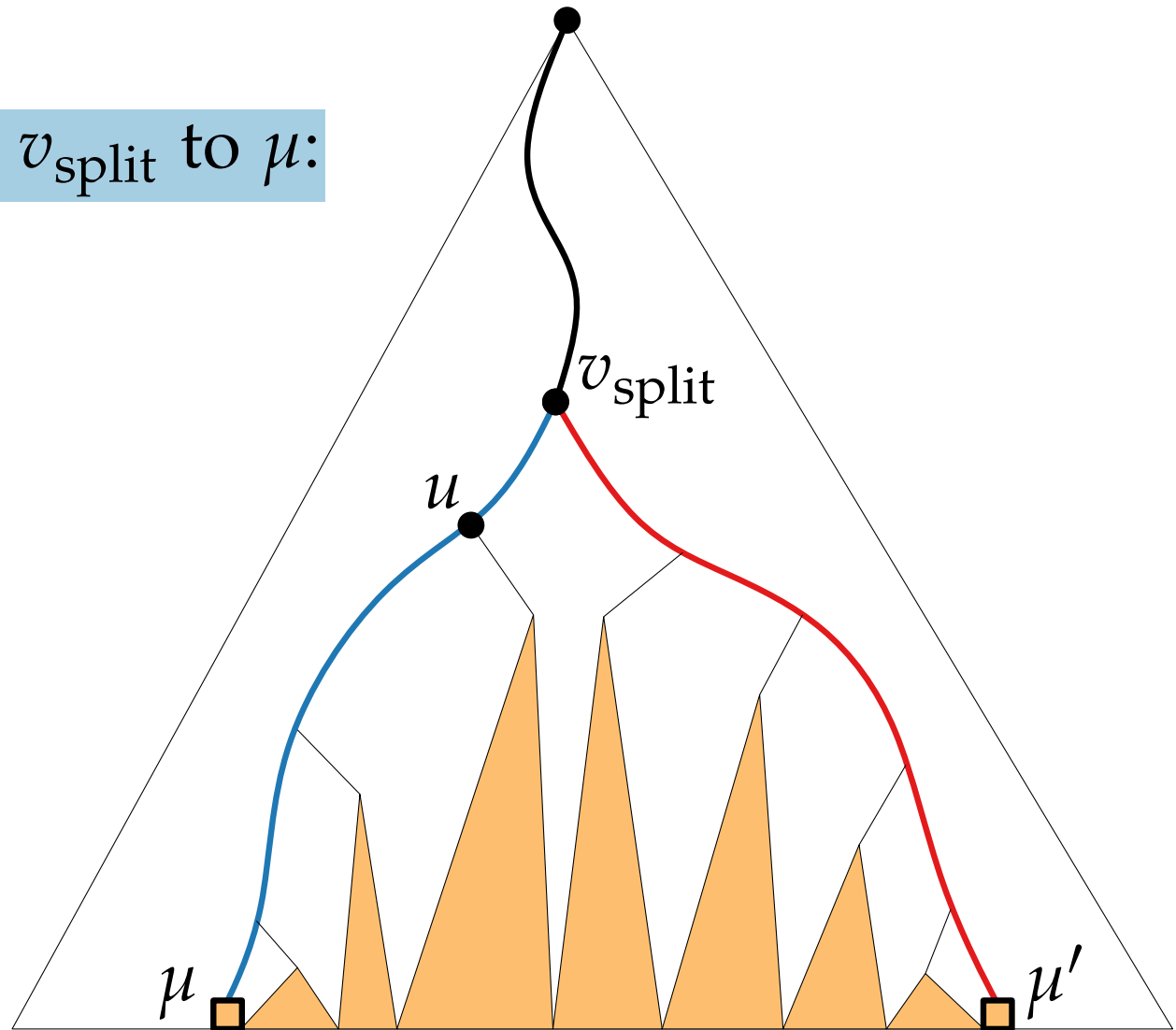
Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u
on the path from v_{split} to μ :



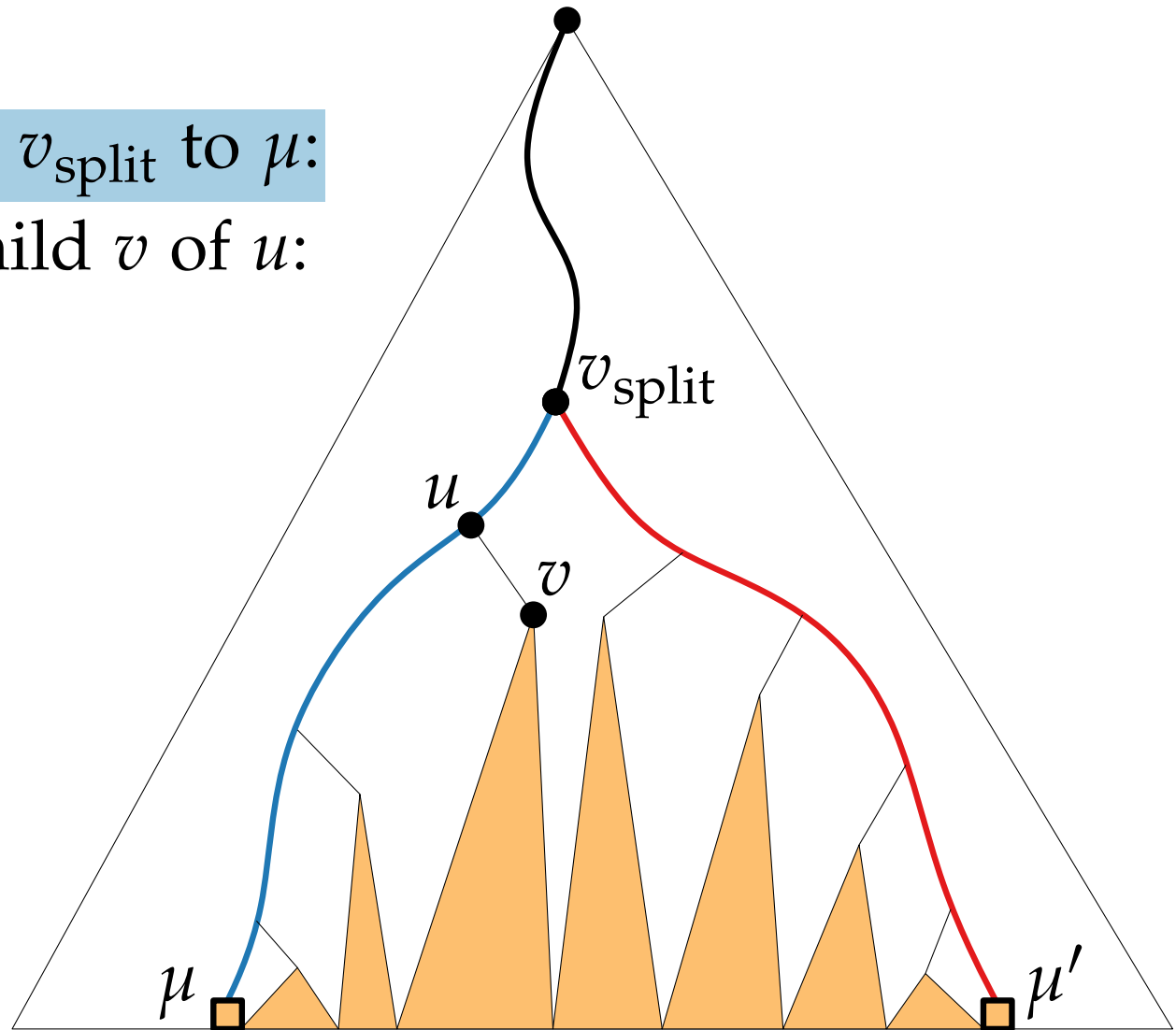
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Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u
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For the right child v of u :

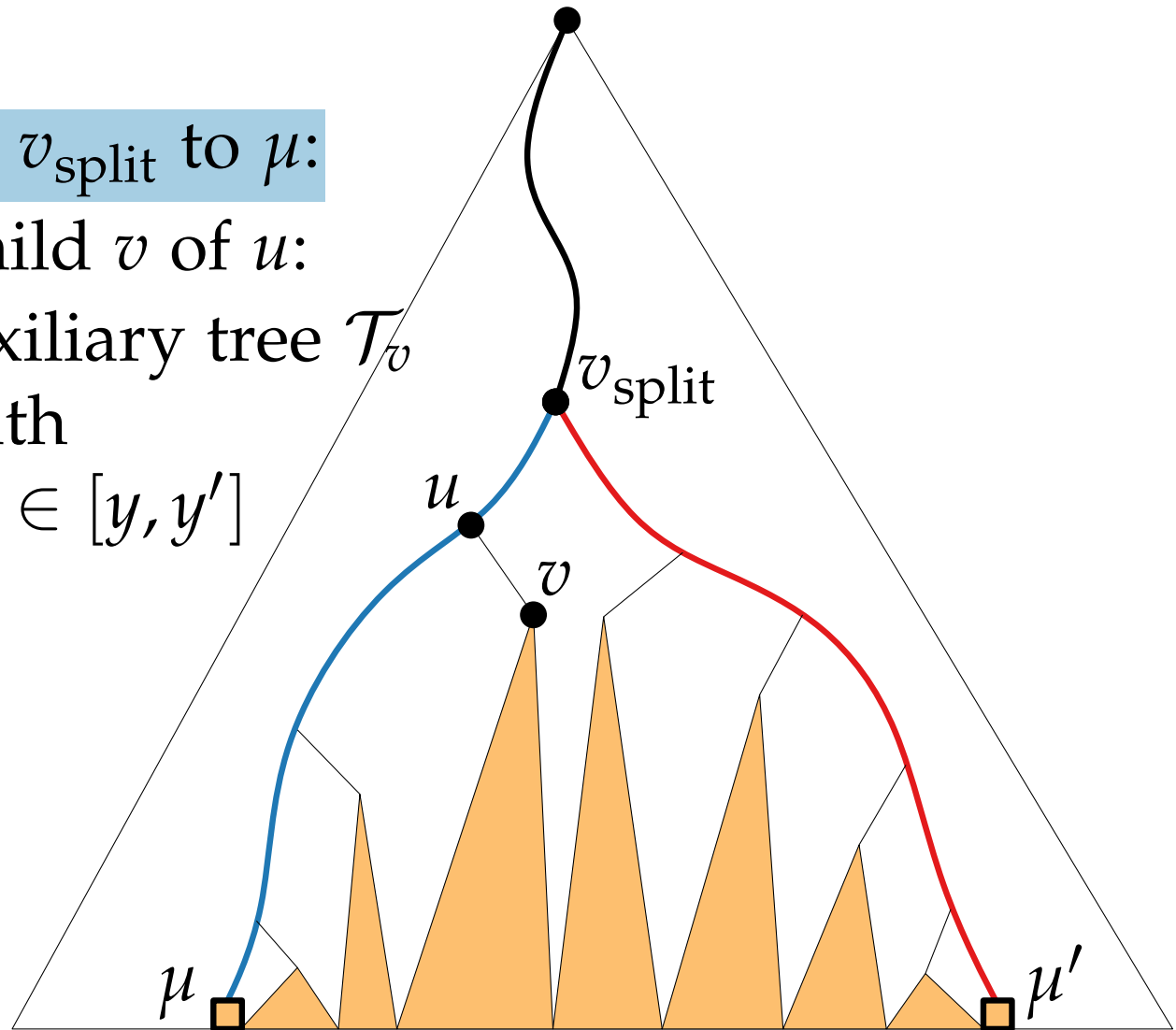


Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
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For the right child v of u :

Search in auxiliary tree \mathcal{T}_v
for points with
y-coordinate $\in [y, y']$

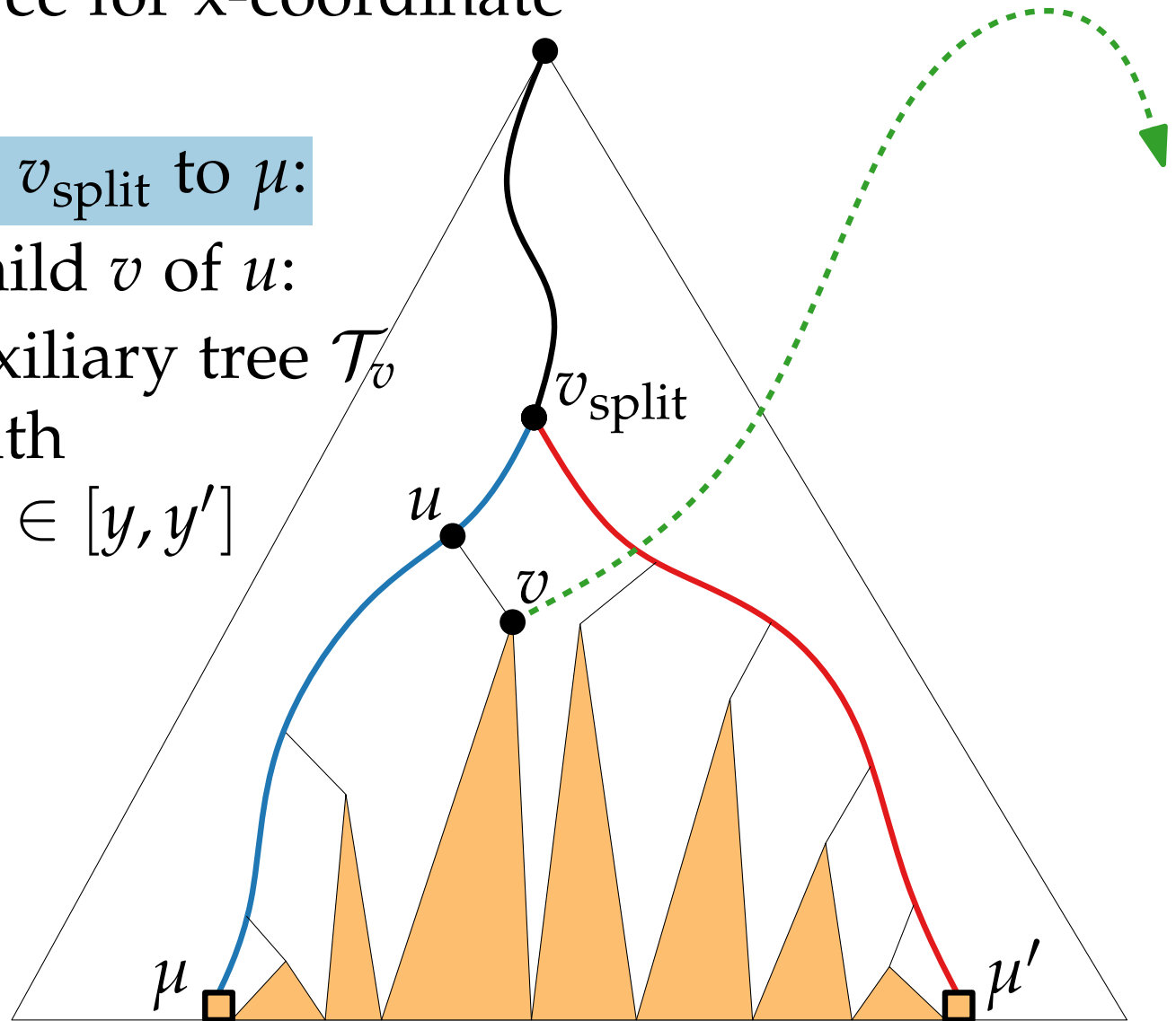


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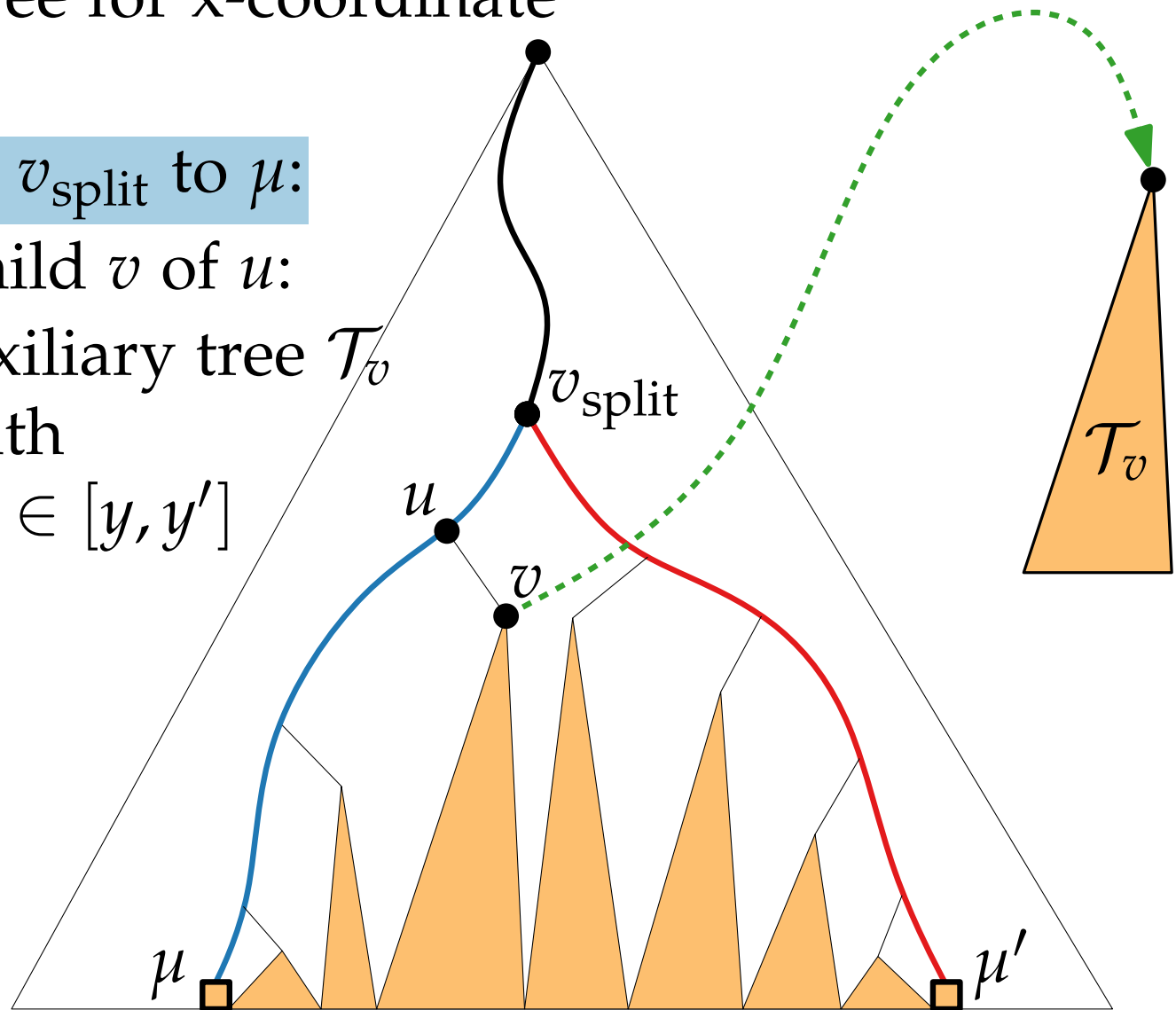


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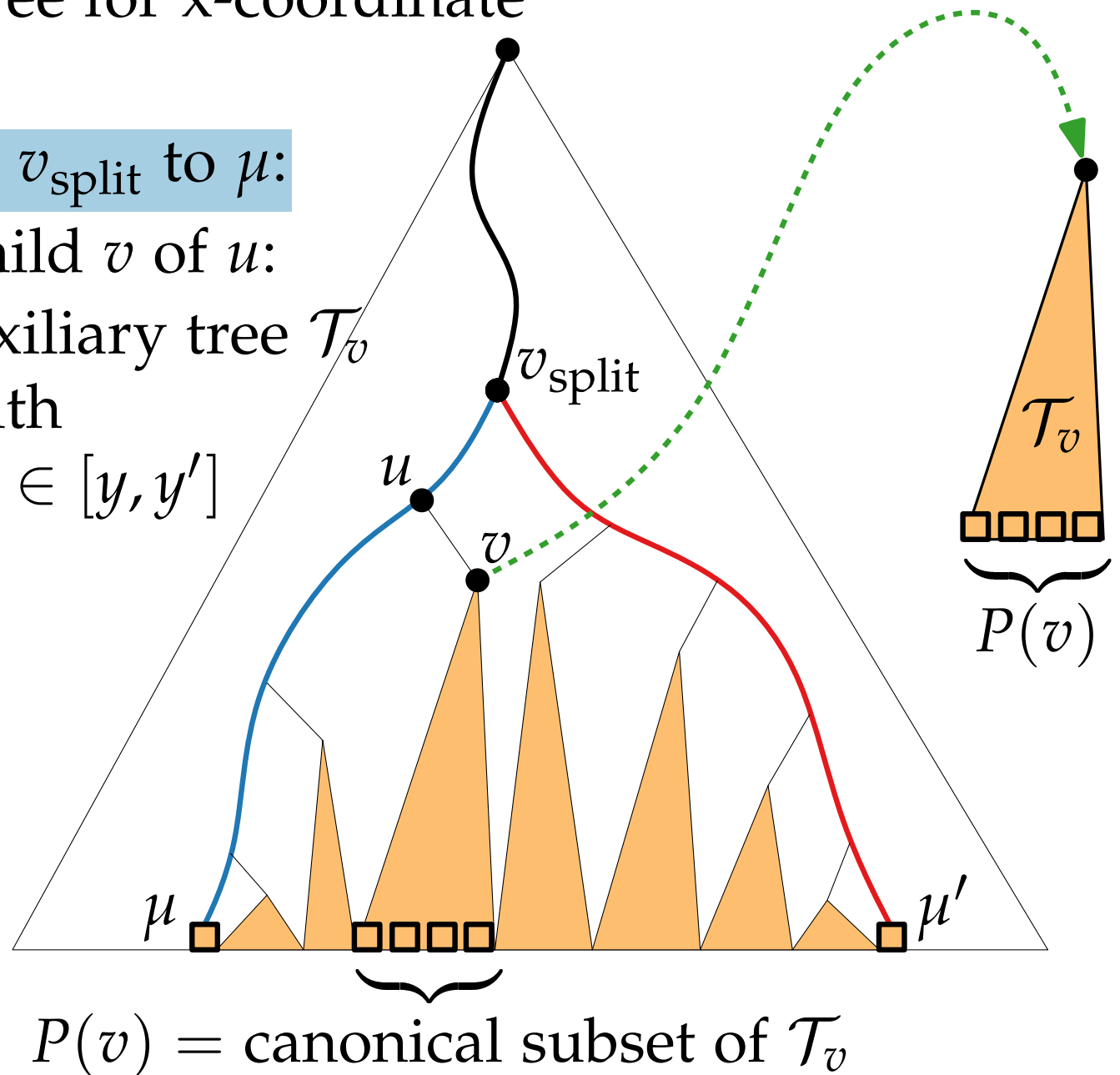


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Range Trees: Query Algorithm

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2. For each node u

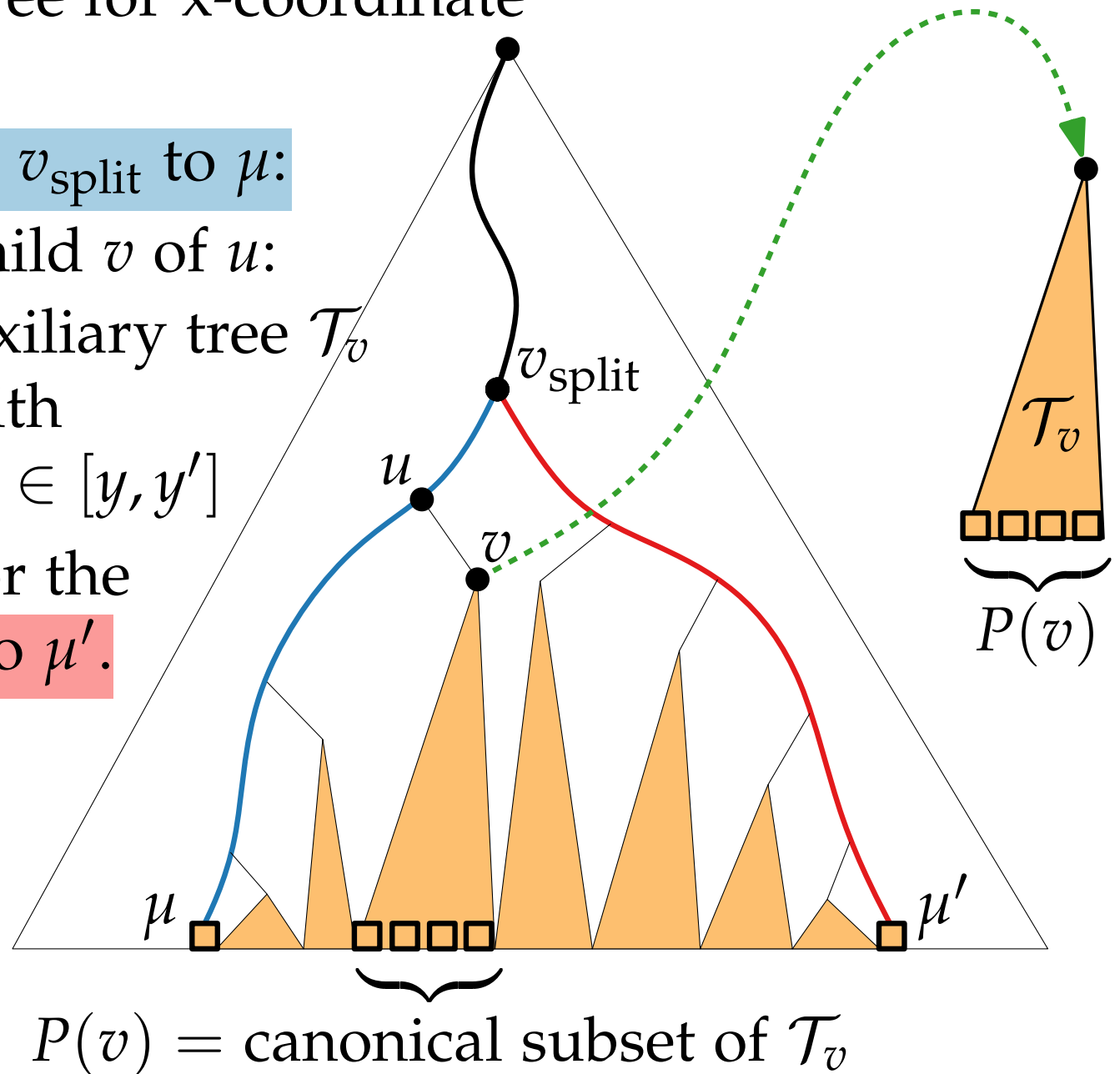
on the path from v_{split} to μ :

For the right child v of u :

Search in auxiliary tree \mathcal{T}_v
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y-coordinate $\in [y, y']$

3. Symmetrically for the

path from v_{split} to μ' .



Range Trees: Construction

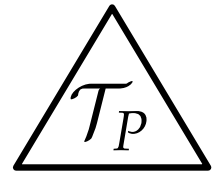
Build2DRangeTree(point[] P)

construct 2nd-level tree \mathcal{T}_P on P (y -order)

Range Trees: Construction

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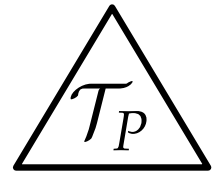
if $P = \{p\}$ **then**

 | create leaf v :

else



return v



Range Trees: Construction

```
Build2DRangeTree(point[ ] P)
```

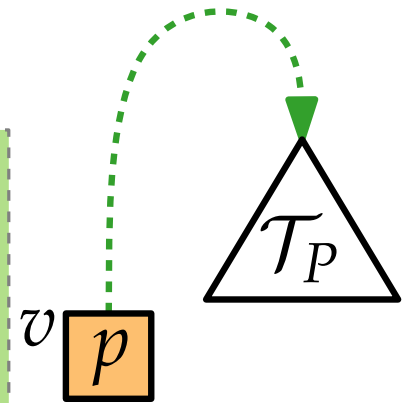
```
  construct 2nd-level tree  $\mathcal{T}_P$  on  $P$  ( $y$ -order)
```

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  if  $P = \{p\}$  then
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```
    | create leaf  $v$ :
```

```
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```
  return  $v$ 
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Range Trees: Construction

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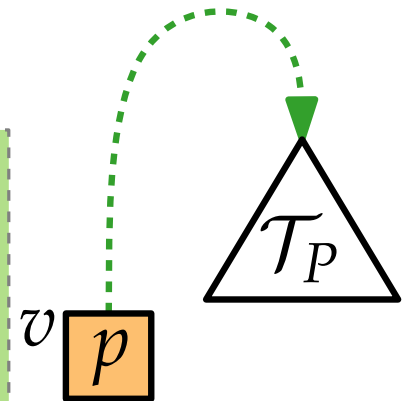
```
  else
```

```
     $x_{\text{mid}} =$  median  $x$ -coordinate of  $P$ 
```

```
     $P_{\text{left}} =$  pts in  $P$  with  $x$ -coordinate  $\leq x_{\text{mid}}$ 
```

```
     $P_{\text{right}} =$  pts in  $P$  with  $x$ -coordinate  $> x_{\text{mid}}$ 
```

```
  return  $v$ 
```



Range Trees: Construction

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construct 2nd-level tree \mathcal{T}_P on P (y -order)

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else

x_{mid} = median x-coordinate of P

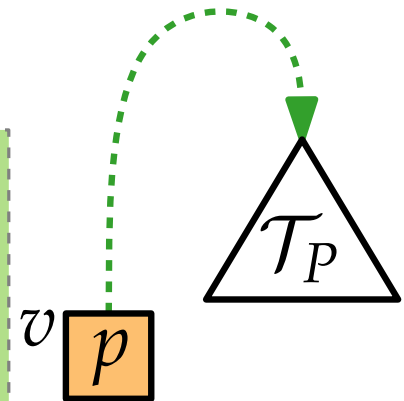
P_{left} = pts in P with x-coordinate $\leq x_{\text{mid}}$

P_{right} = $>$

v_{left} = **Build2DRangeTree**(P_{left})

v_{right} = **Build2DRangeTree**(P_{right})

return v



Range Trees: Construction

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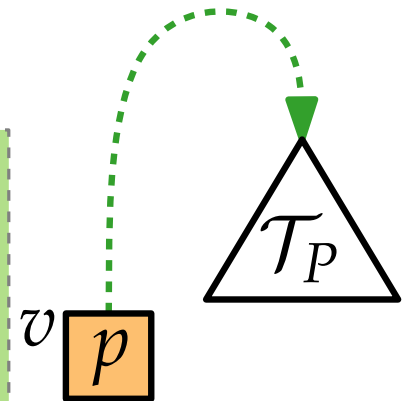
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 create node v :

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Range Trees: Construction

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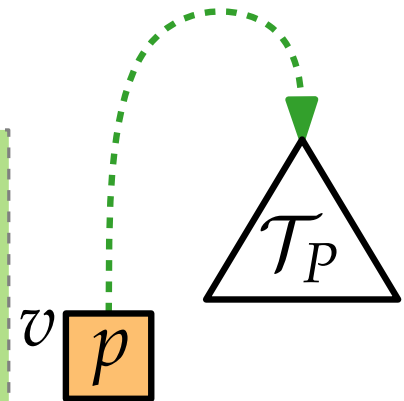
v_{left} = Build2DRangeTree(P_{left})

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 create node v :

v
 x_{mid}

return v



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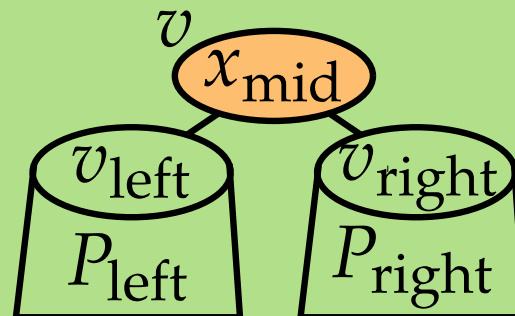
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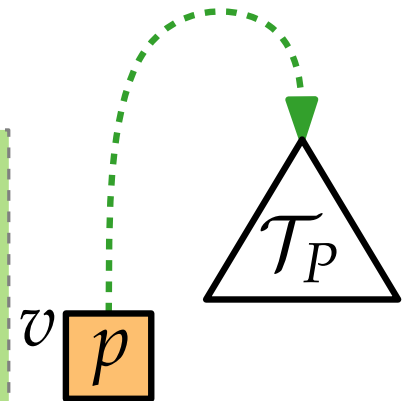
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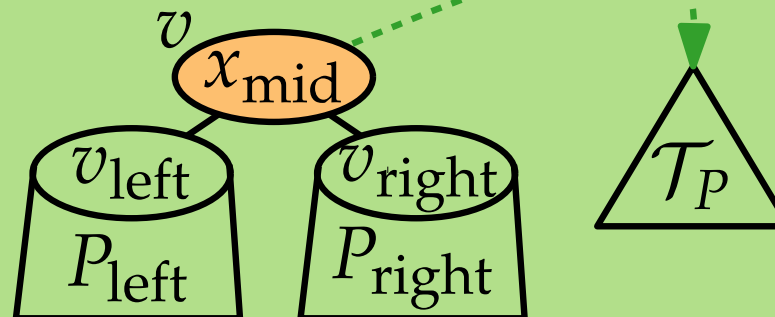
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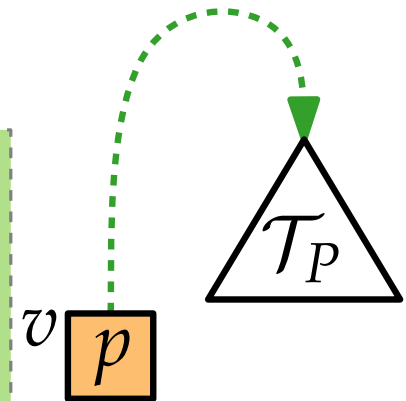
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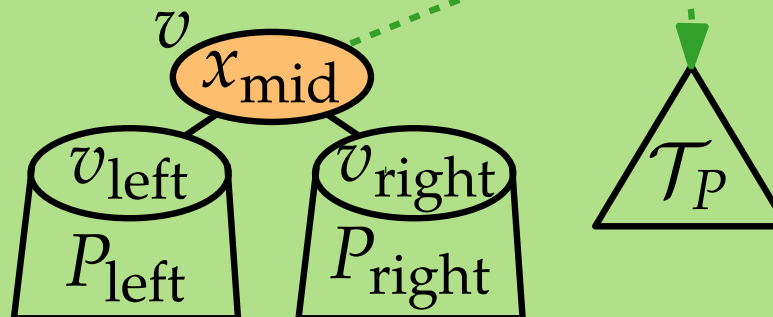
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v_{right} = Build2DRangeTree(P_{right})

create node v :



return v

v p

Running time?

Range Trees: Construction

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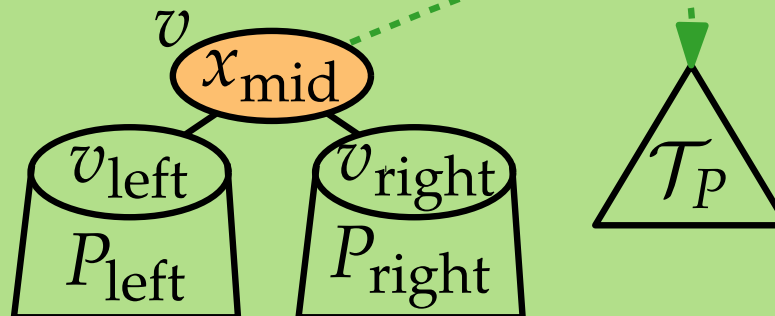
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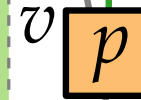
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Running time?

$O(n \log n)$:- (



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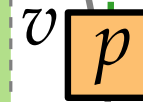
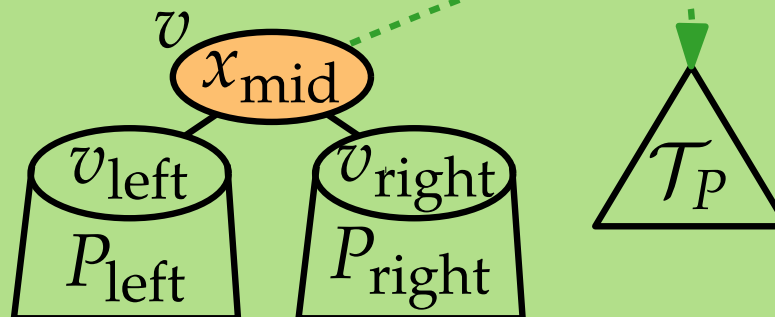
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create node v :

return v



Running time?

$O(n \log n)$:- (

Better:

Pre-sort once,
then build tree
bottom-up
in linear time.

Range Trees: Construction

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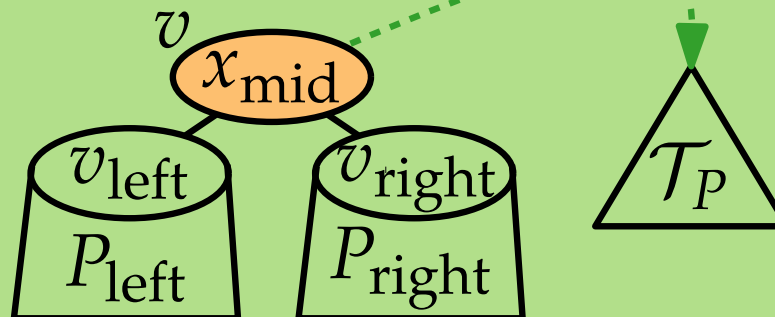
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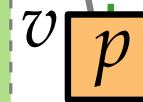
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Running time?

$O(n \log n)$:- (

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in linear time.



Total
construction
time $O(n \log n)$

Computational Geometry

Lecture 5: Orthogonal Range Queries or Fast Access to Data Bases

Part V: Range Trees: Analysis

Range Trees: Space Consumption

Each node v of the 1st-level tree has a pointer to a 2nd-level tree \mathcal{T}_v with $|\mathcal{T}_v| = \Theta(|P(v)|)$.

Range Trees: Space Consumption

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Q: What's the *total* space consumption of all 2nd-level trees?

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What's your guess:

- $\Theta(n^2)$,
- $\Theta(n \log n)$,
- $\Theta(n \log^2 n)$, or
- $\Theta(n)$?

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What's your guess:

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- $\Theta(n \log^2 n)$, or
- $\Theta(n)$?

How many trees will contain a given point?

Range Trees: Space Consumption

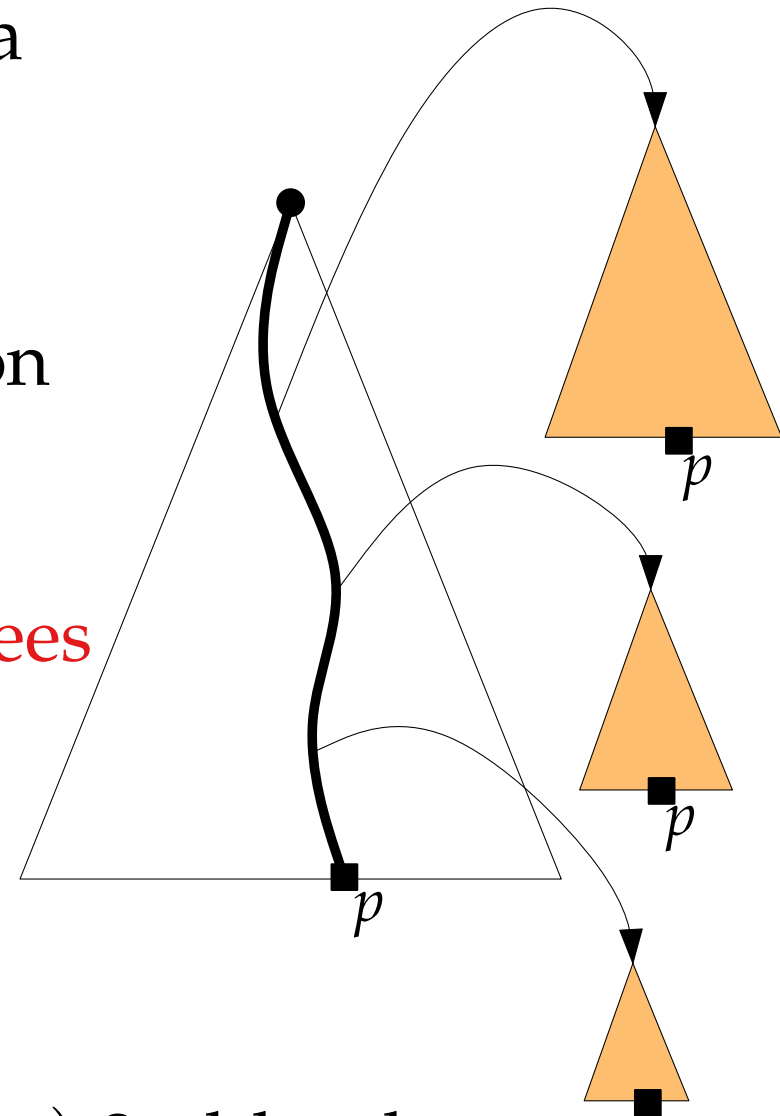
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- $\Theta(n \log n)$,
- $\Theta(n \log^2 n)$, or
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How many trees will contain a given point?



A: Each $p \in P$ is stored in $h = \Theta(\log n)$ 2nd-level trees. p

Range Trees: Space Consumption

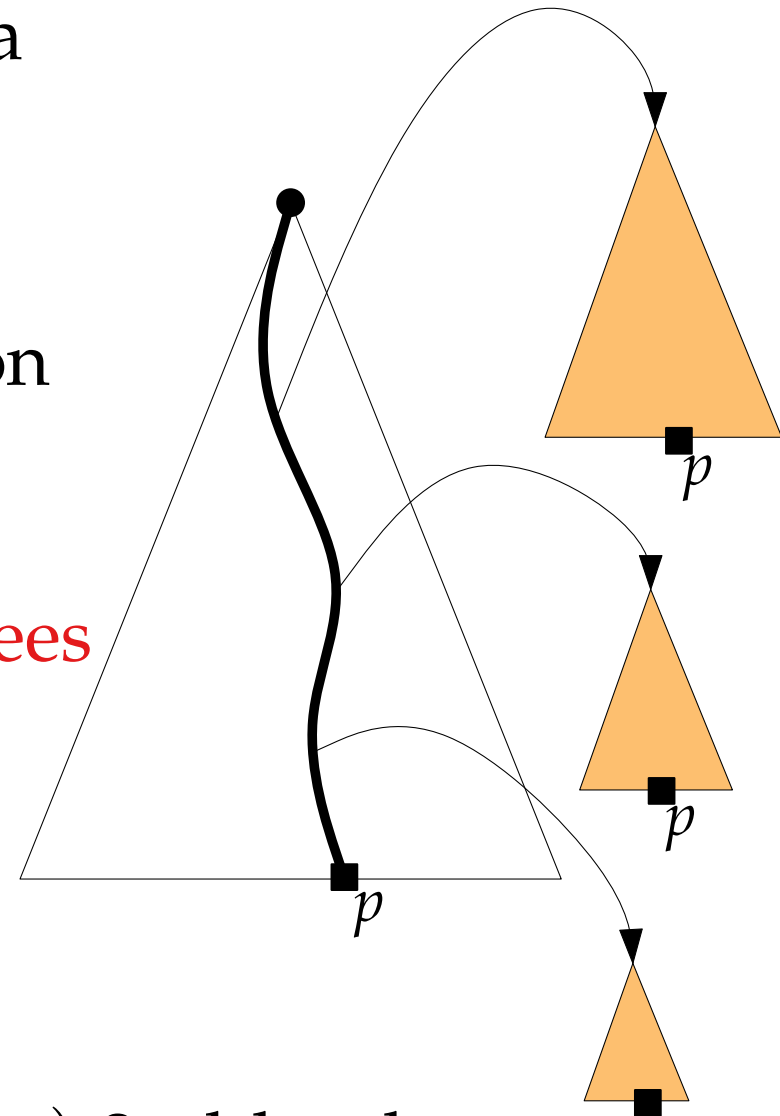
Each node v of the 1st-level tree has a pointer to a 2nd-level tree \mathcal{T}_v with $|\mathcal{T}_v| = \Theta(|P(v)|)$.

Q: What's the *total* space consumption of all 2nd-level trees?

What's your guess:

- $\Theta(n^2)$,
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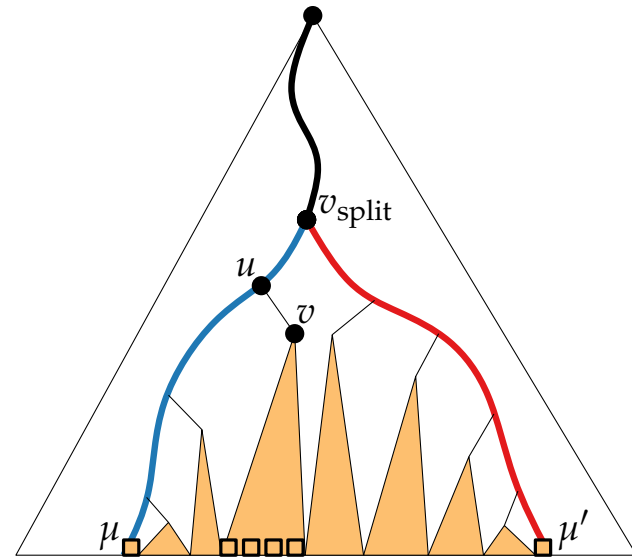


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$\Rightarrow \Theta(n \log n)$ space

Range Trees: Query time

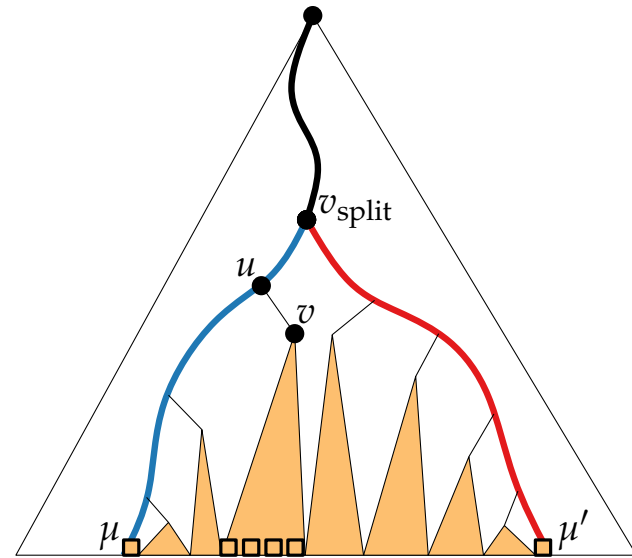
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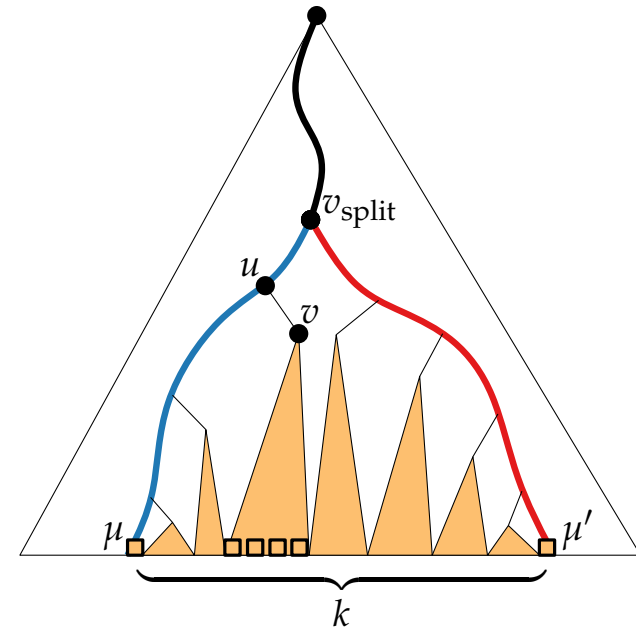
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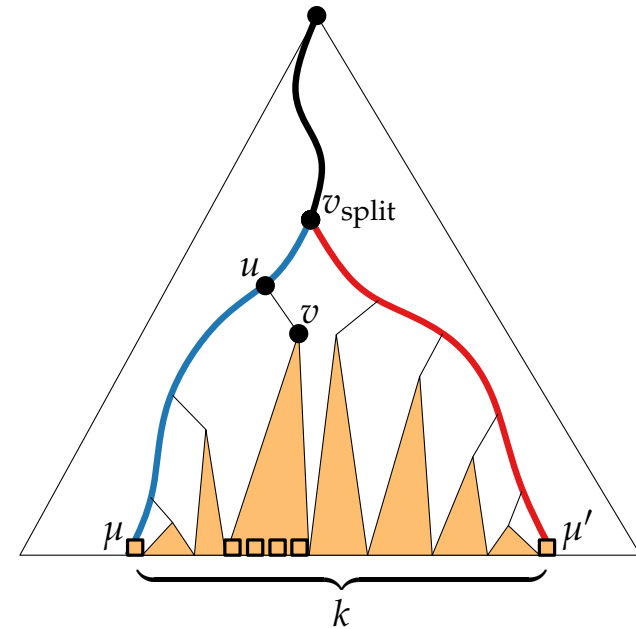
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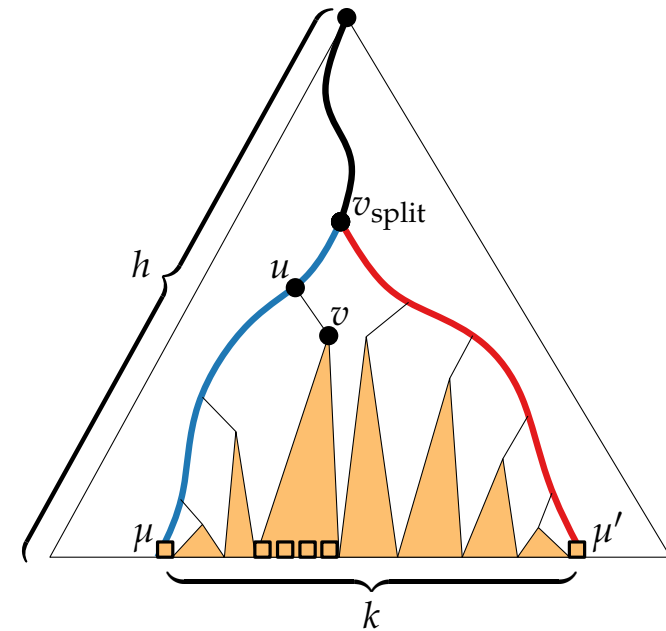
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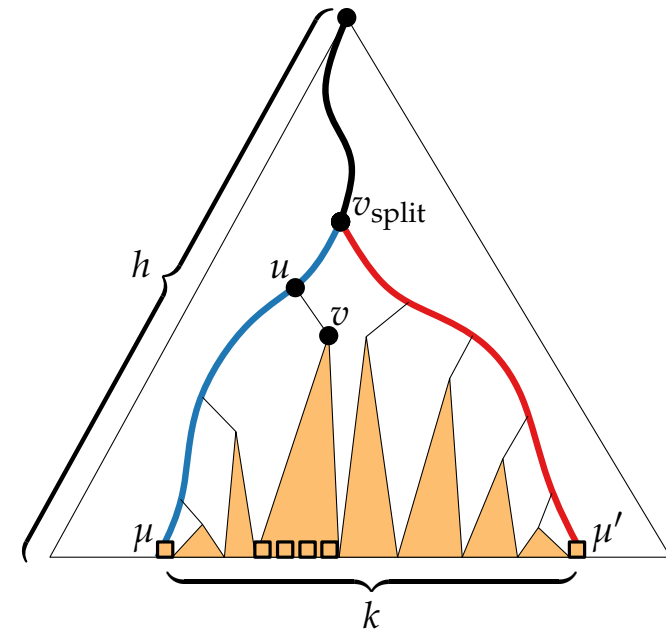
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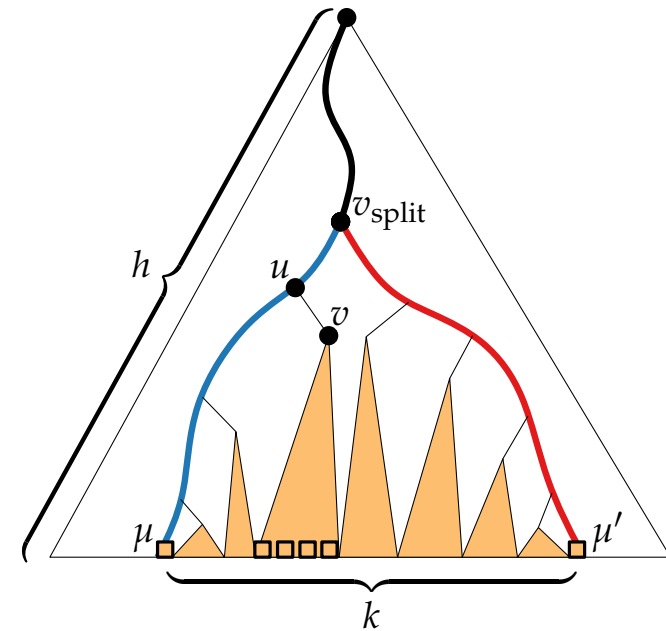
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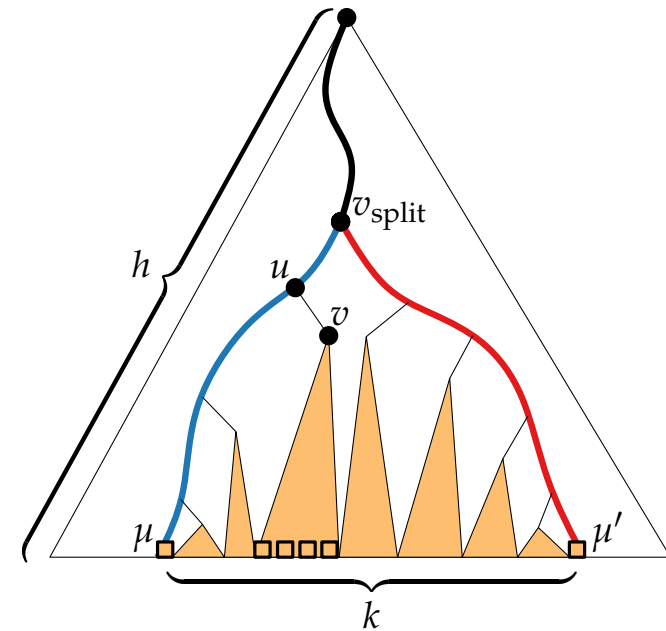
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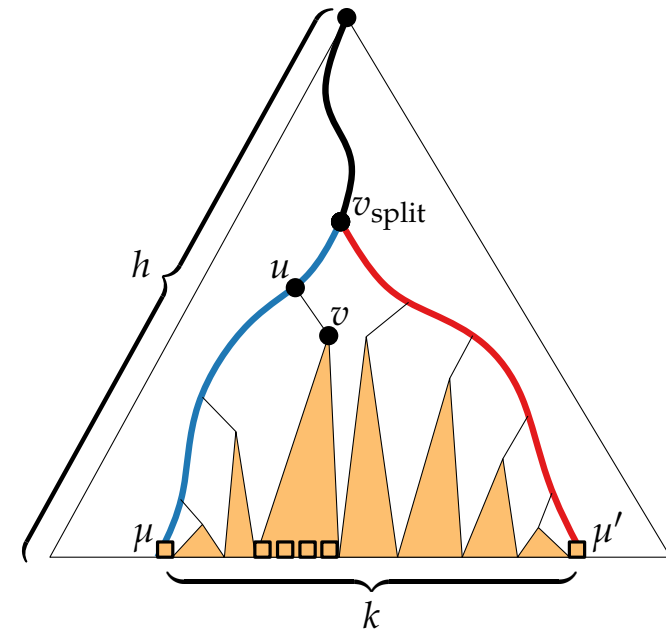
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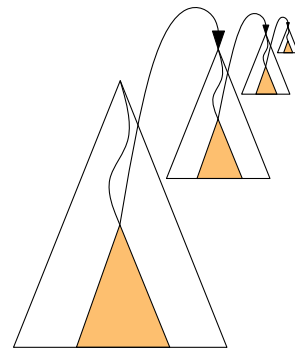
\mathbb{R}^d ?

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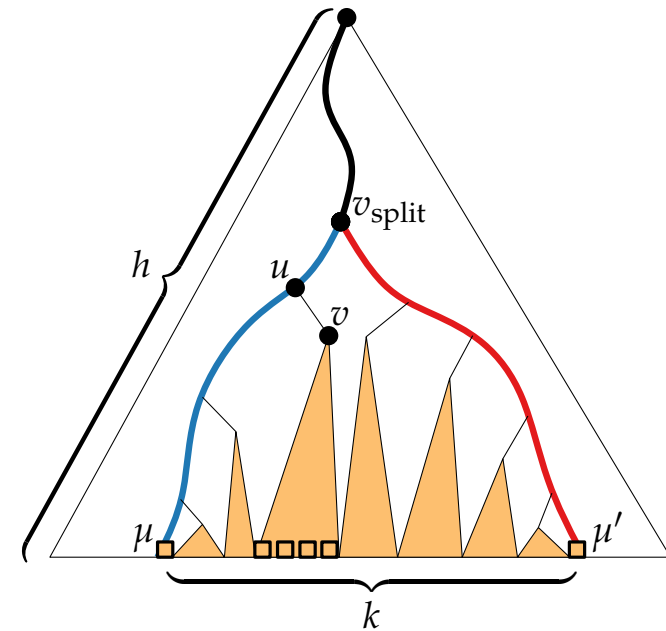


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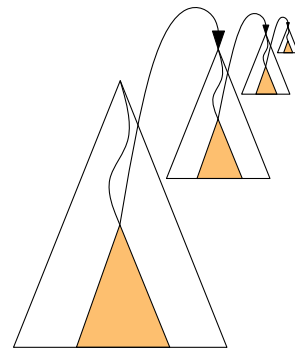
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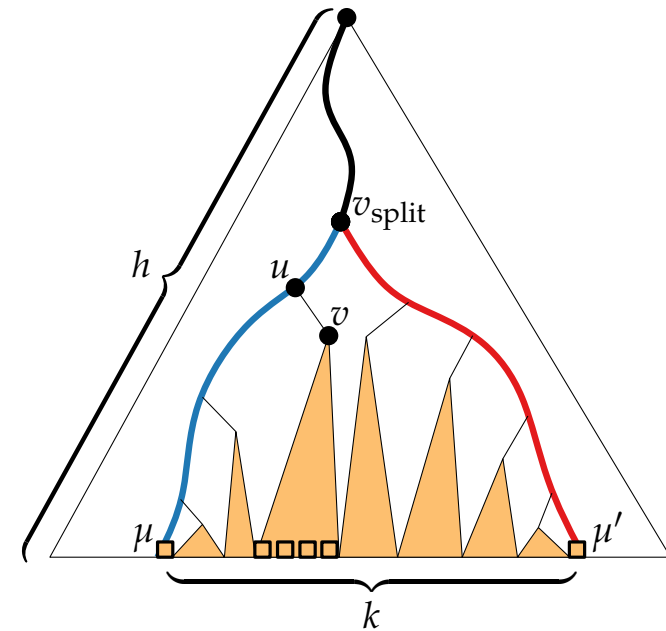
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Range Trees: Query time

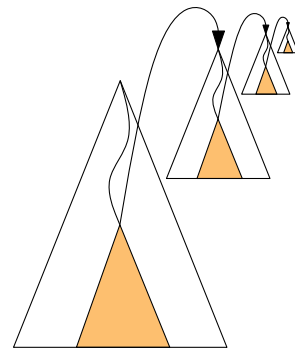
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See Chapter 5.4 in Comp. Geom A&A

Comparison

	kd-tree	range tree
construction time	$O(n \log n)$	$O(n \log n)$
storage	$O(n)$	$O(n \log n)$
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Note: *trade-off* between space and query time

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Show:

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This removes our assumption about the input points being in general position.

We can use kd-trees and range trees for *any* set of points; no matter how many points have the same x - or y -coord.

Computational Geometry

Lecture 5: Orthogonal Range Queries or Fast Access to Data Bases

Part VI: Fractional Cascading

Fractional Cascading

Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

A	3	10	19	23	30	37	59	62	70	80	100	105
-----	---	----	----	----	----	----	----	----	----	----	-----	-----

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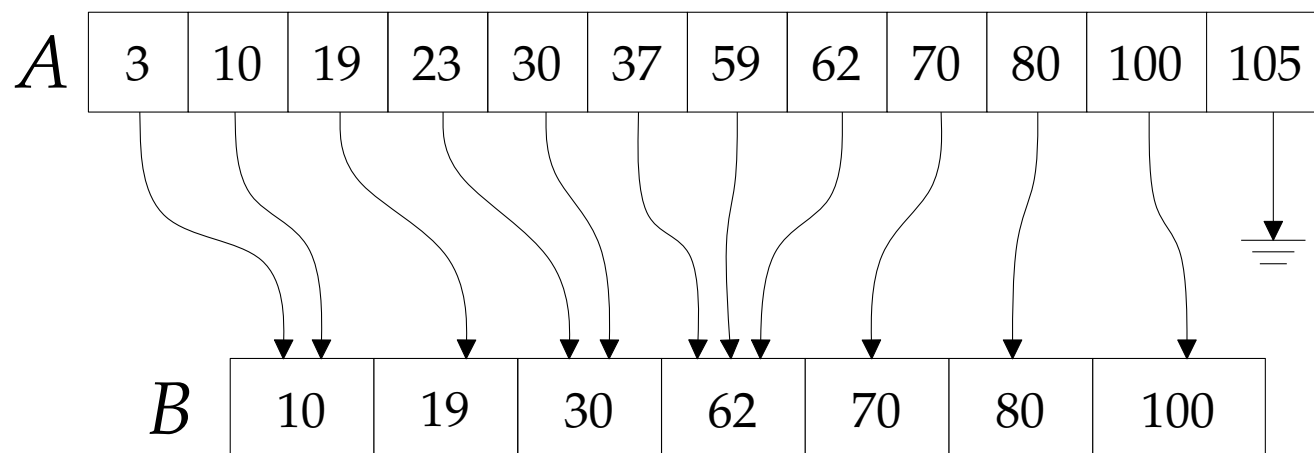
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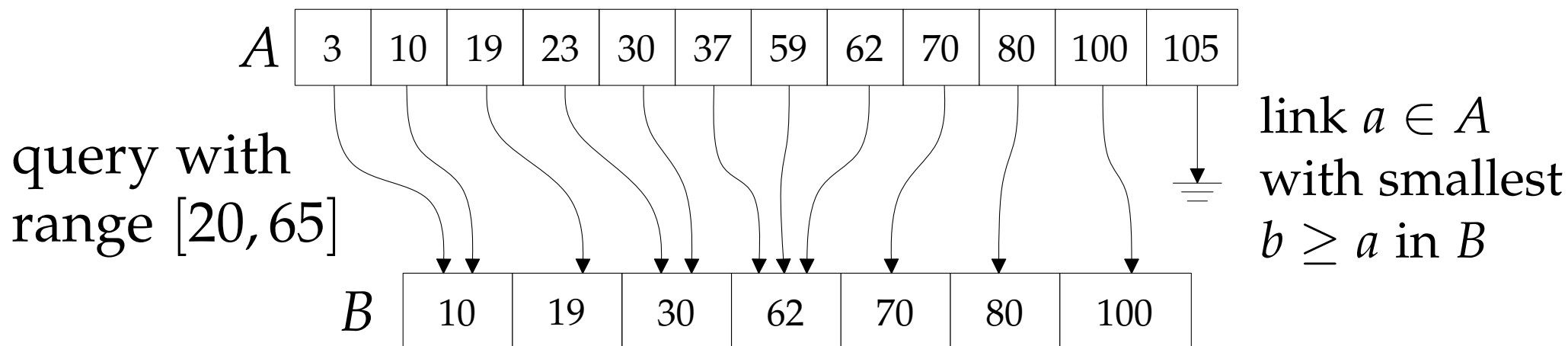
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link $a \in A$
with smallest
 $b \geq a$ in B

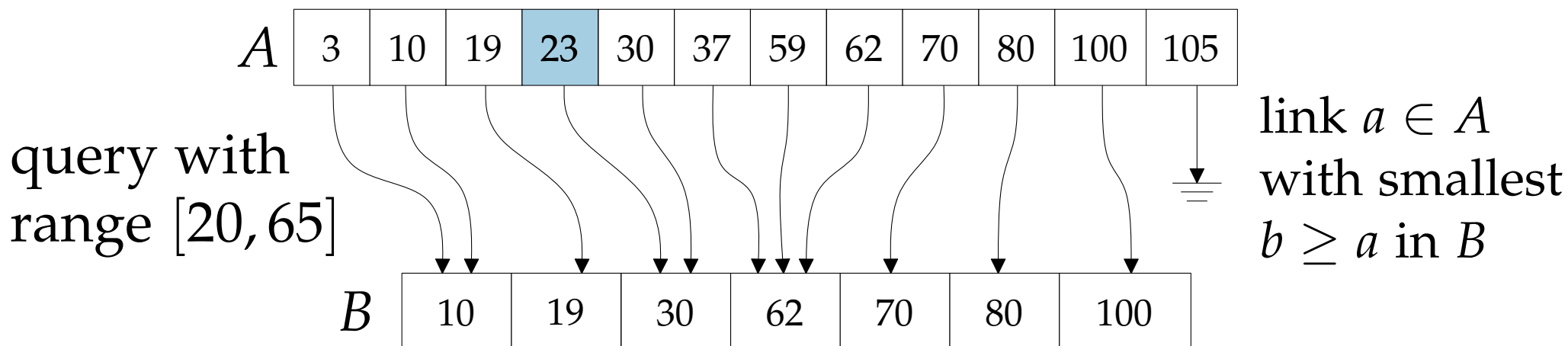
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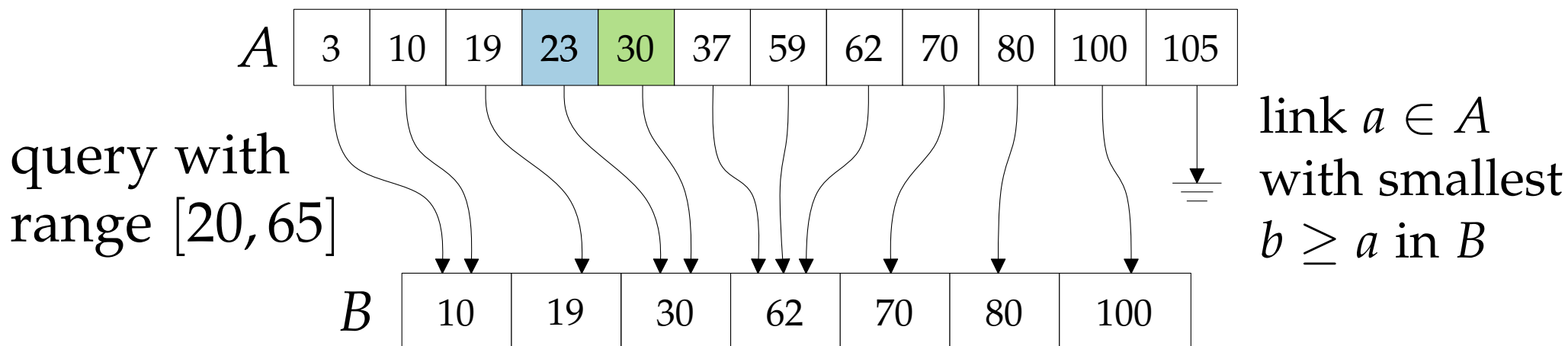
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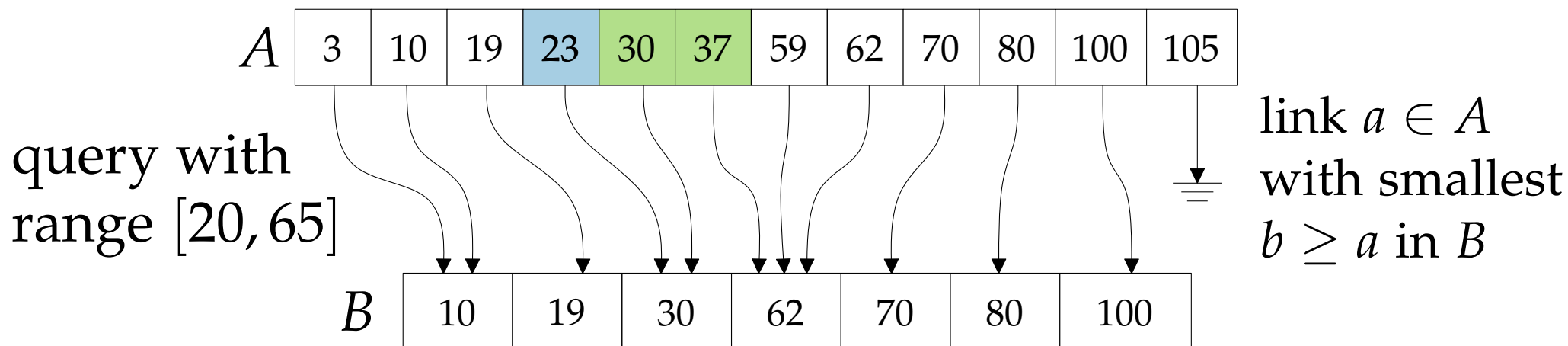
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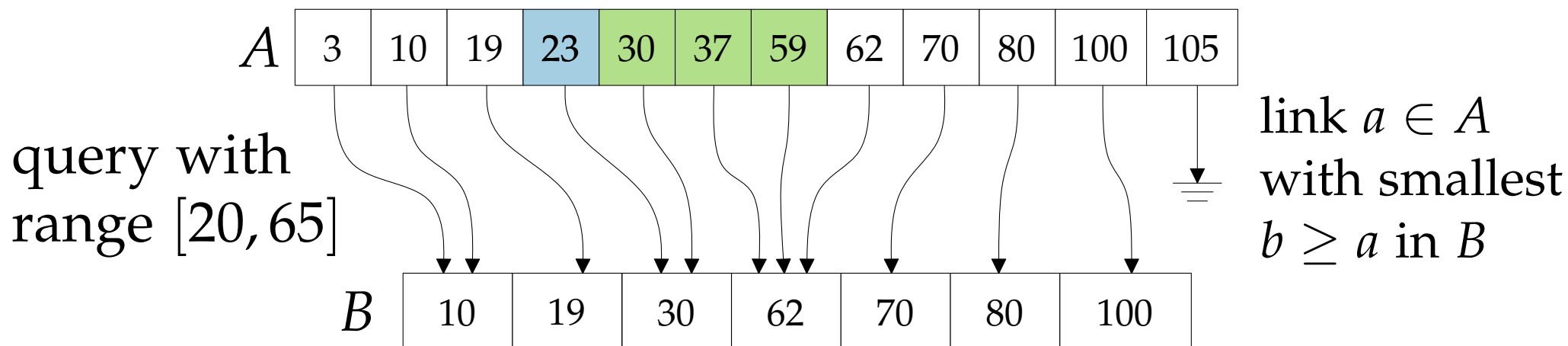
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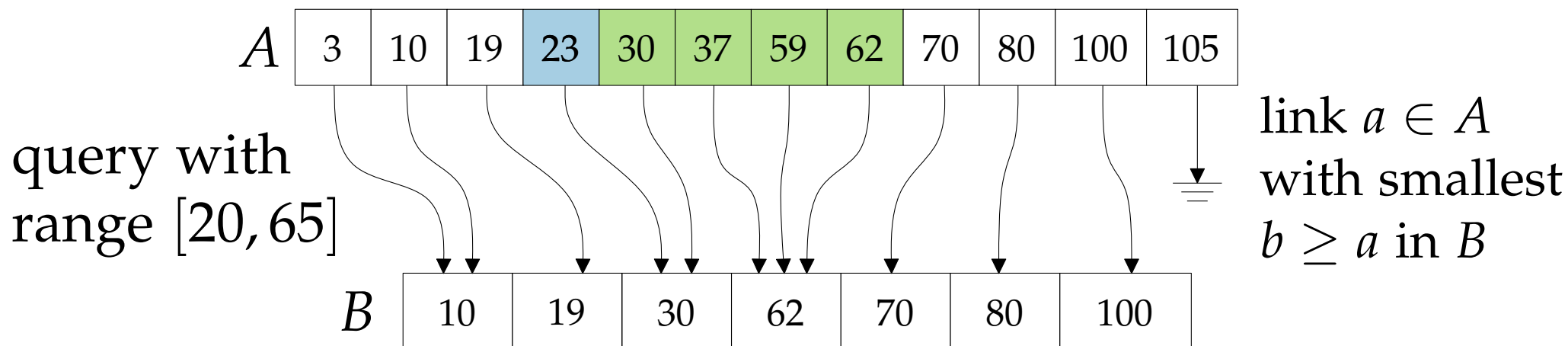
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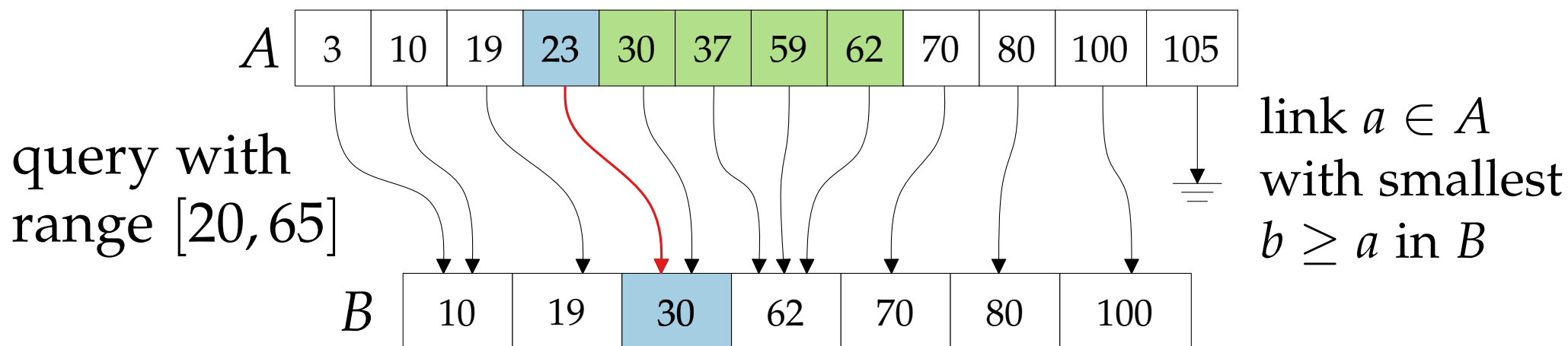
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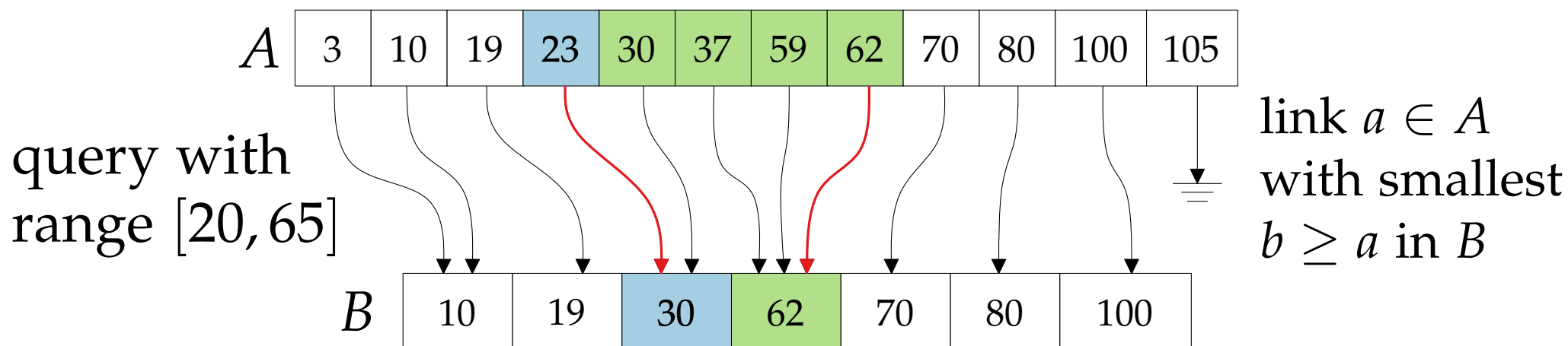
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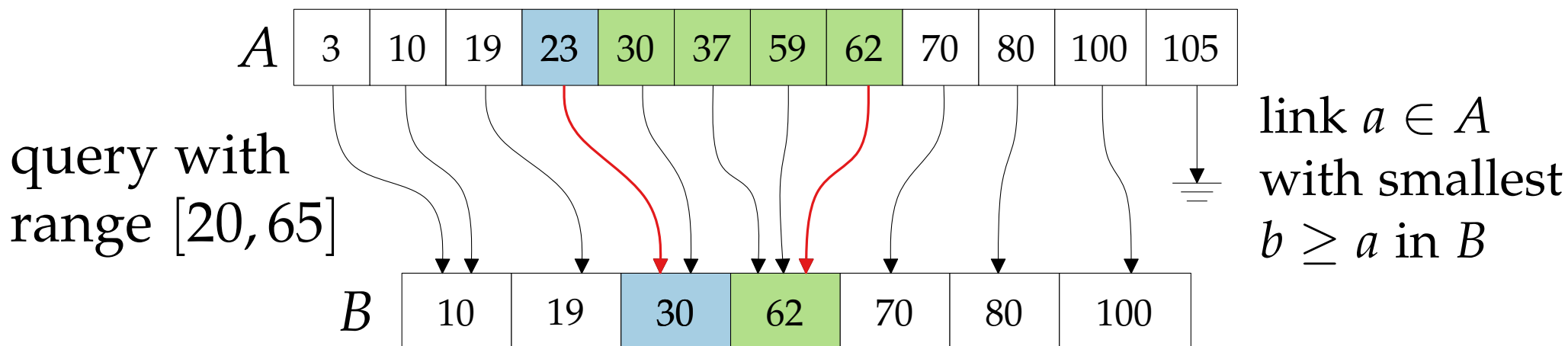
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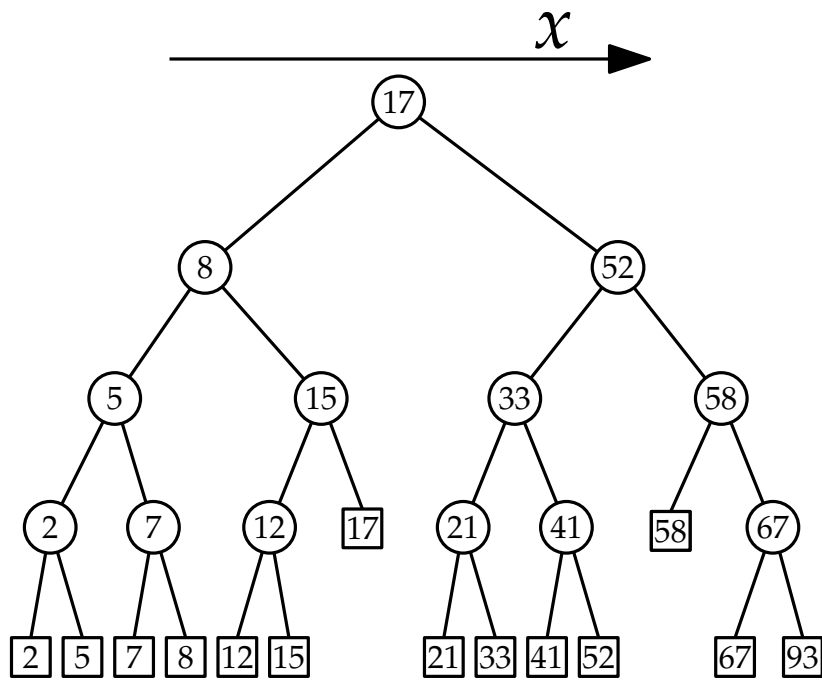
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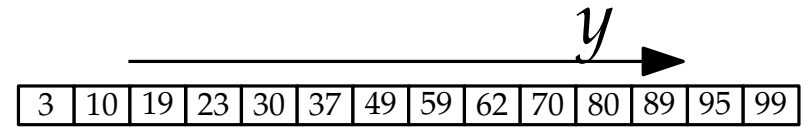
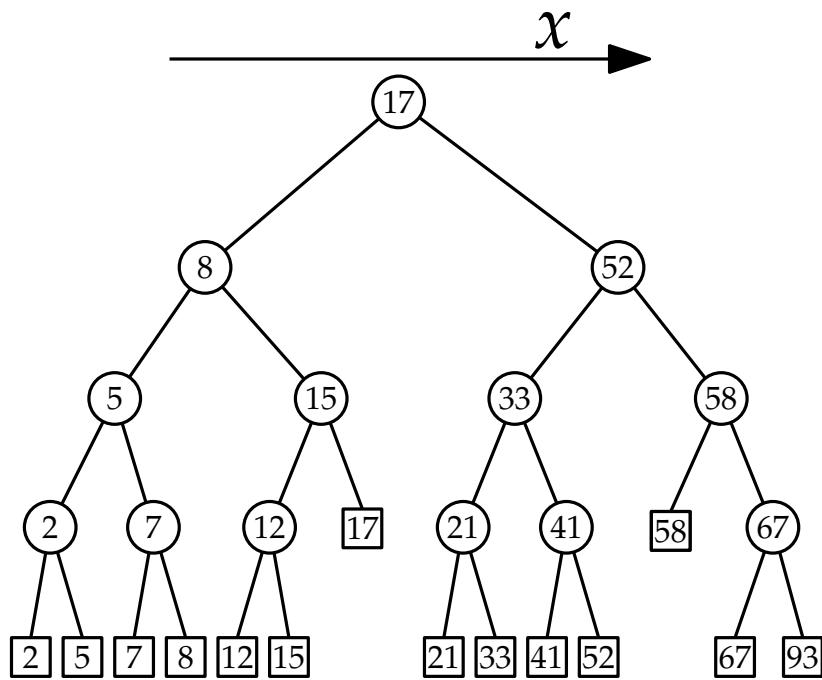
Task 2: Speed up 2d range queries:
 $O(k + \log^2 n) \rightarrow O(k + \log n)$ time!



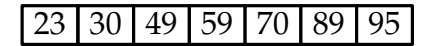
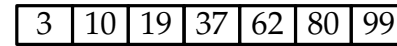
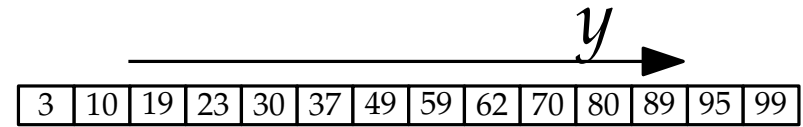
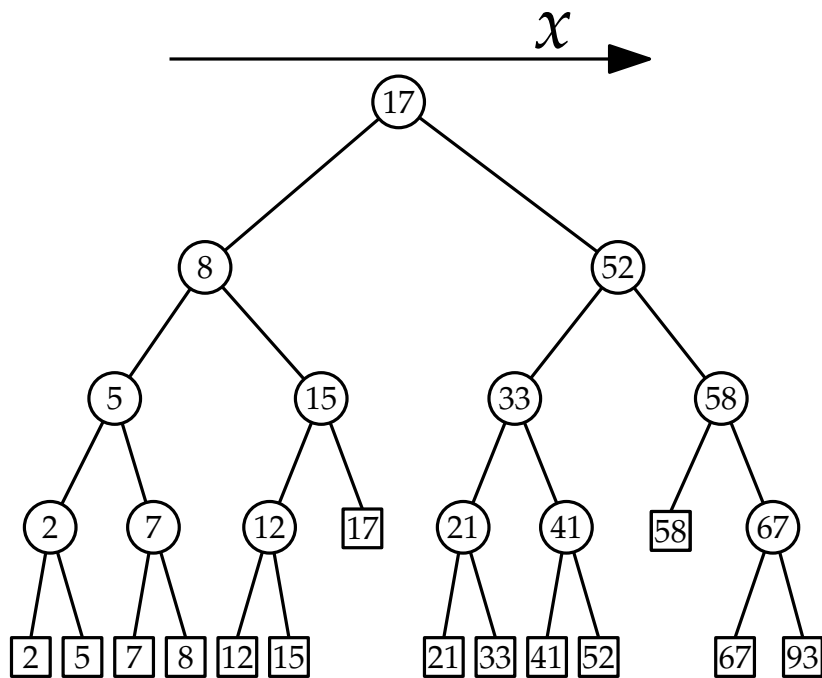
Layered Range Trees



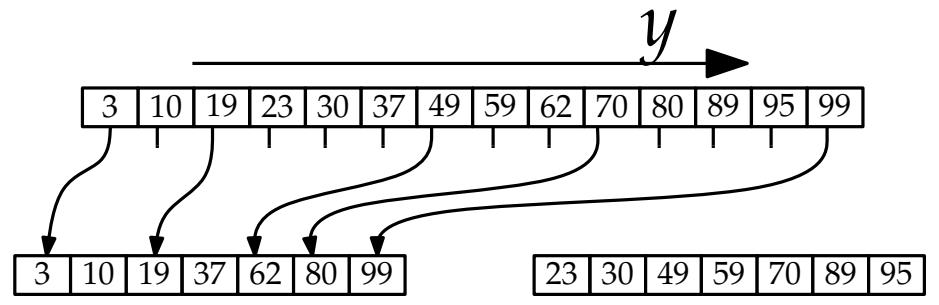
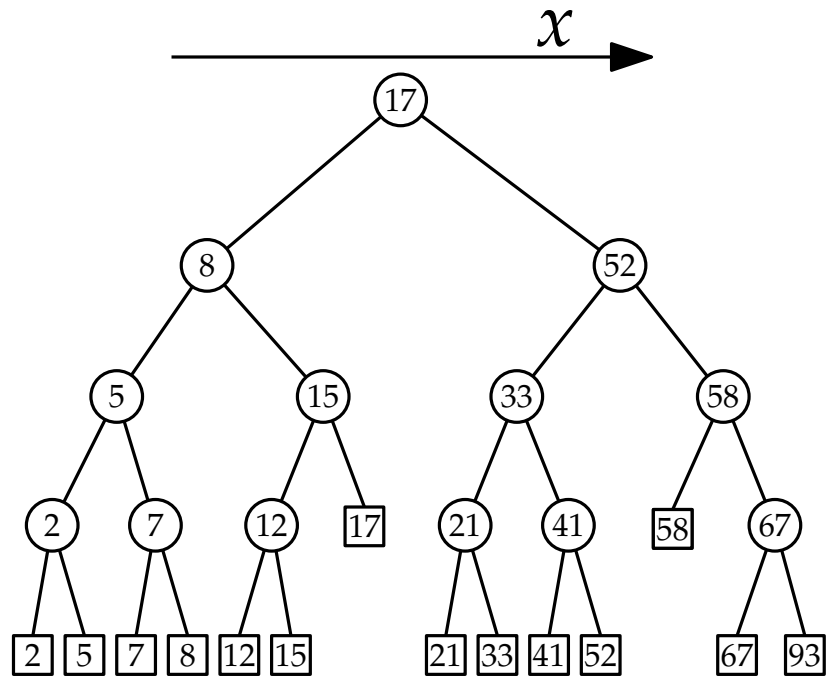
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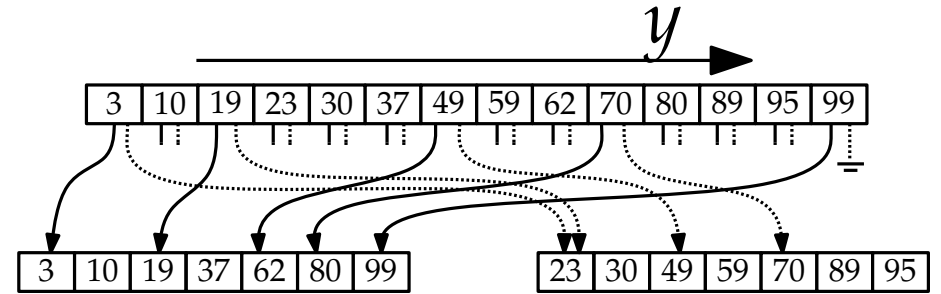
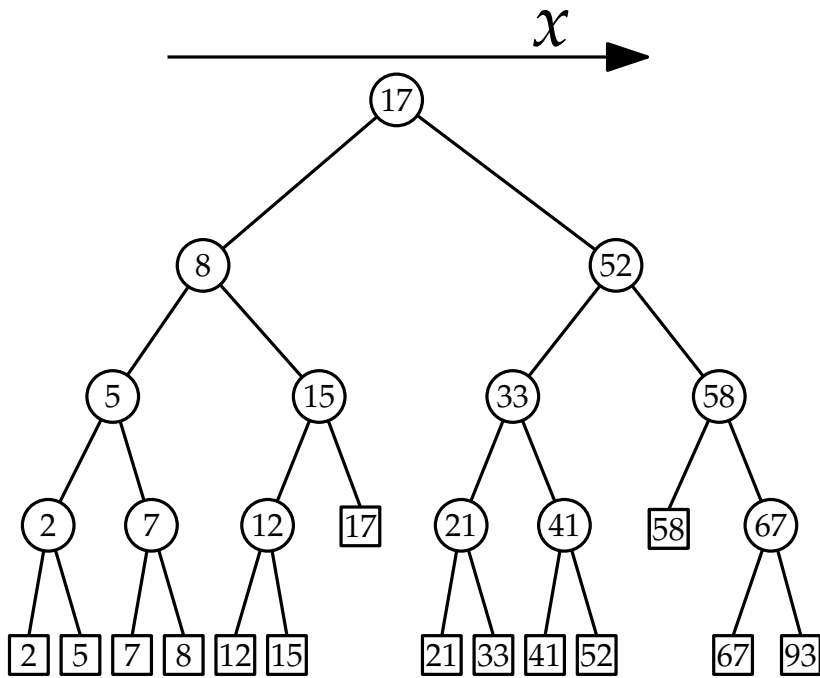
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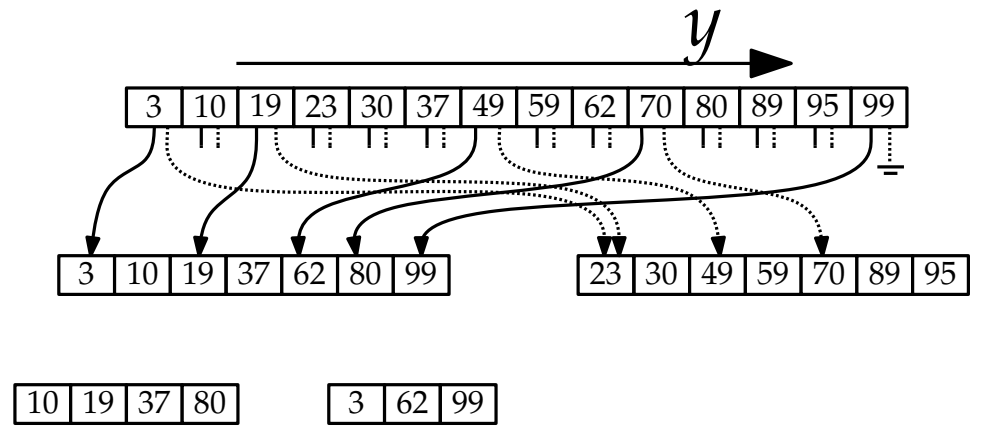
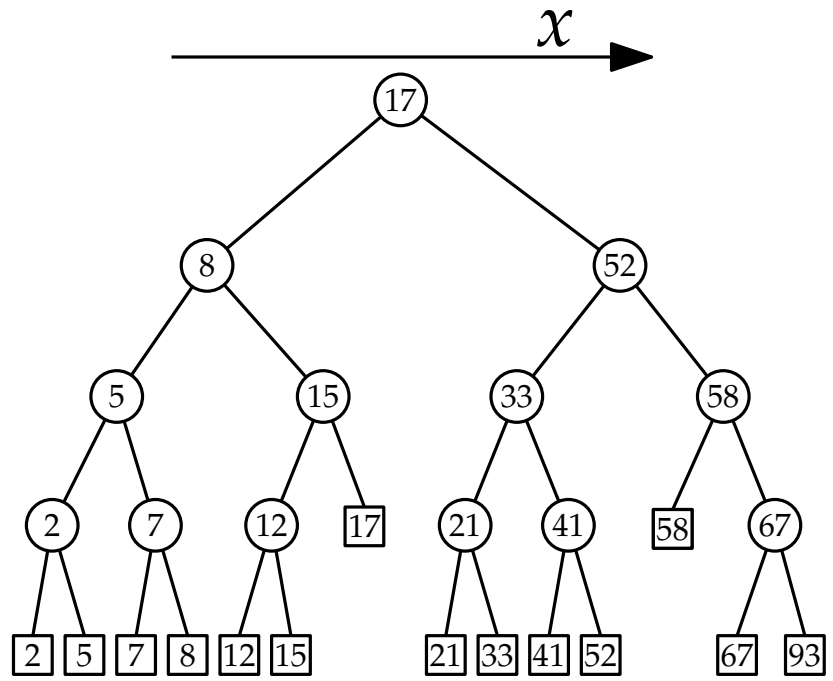
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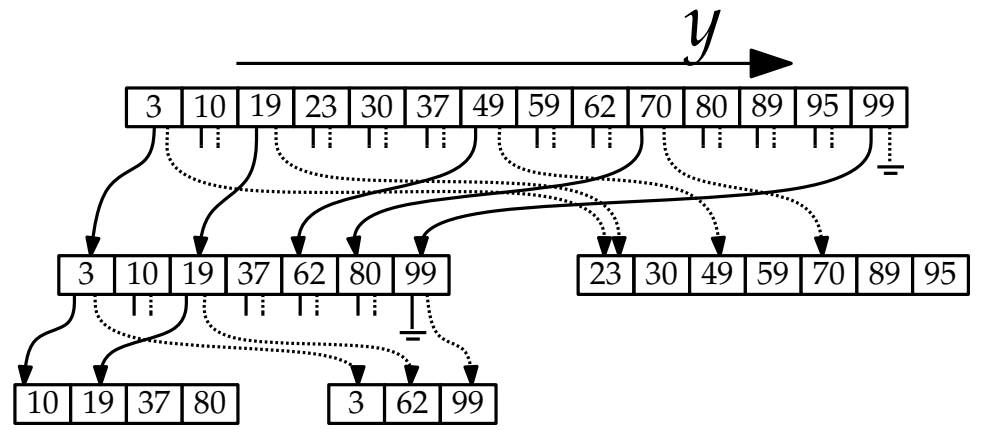
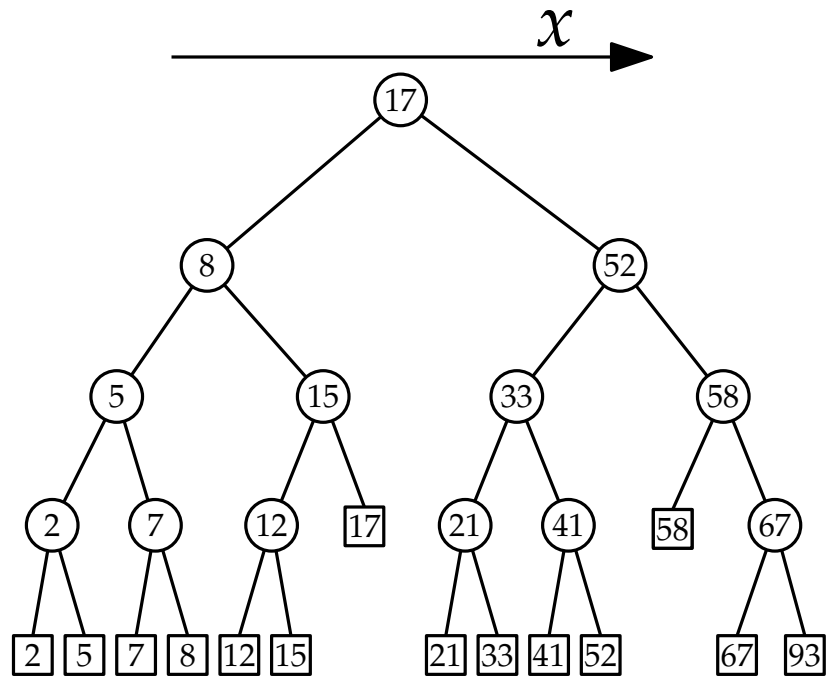
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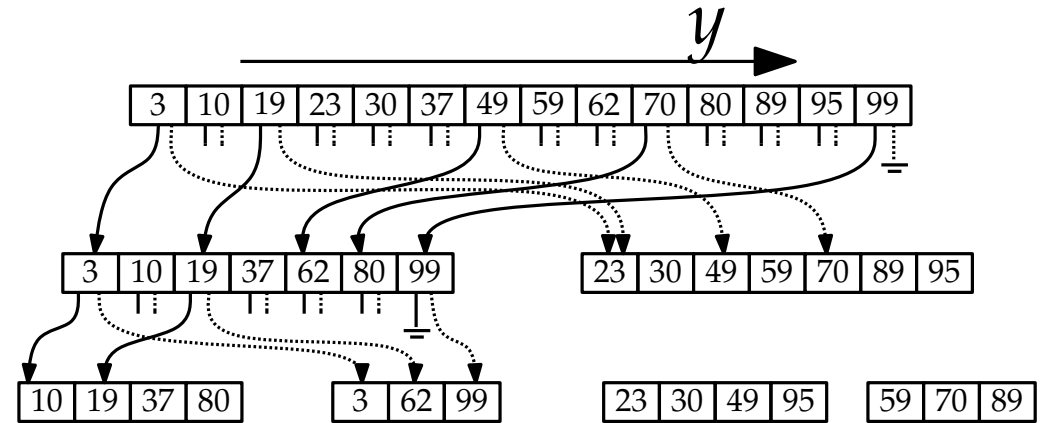
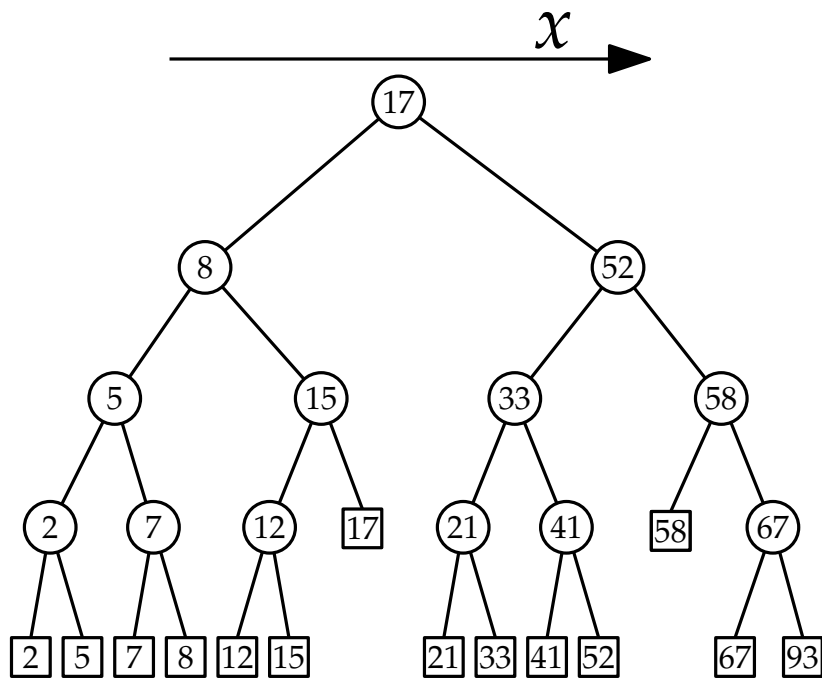
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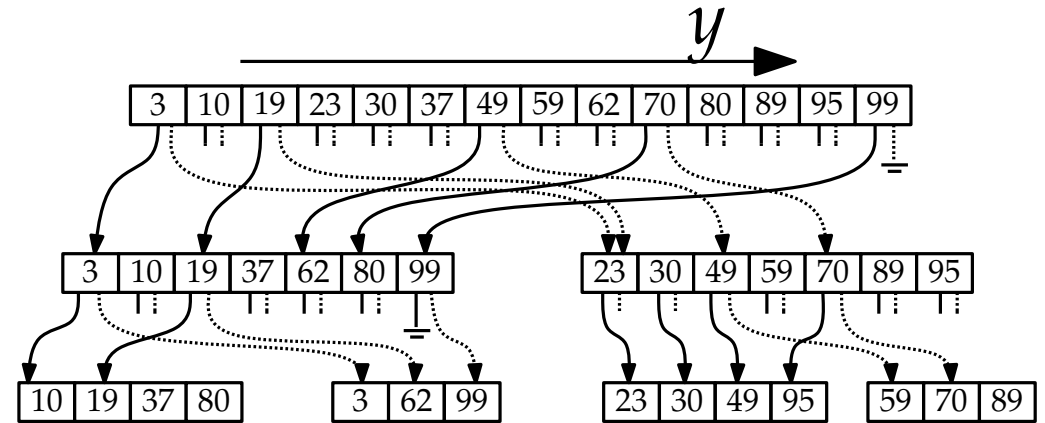
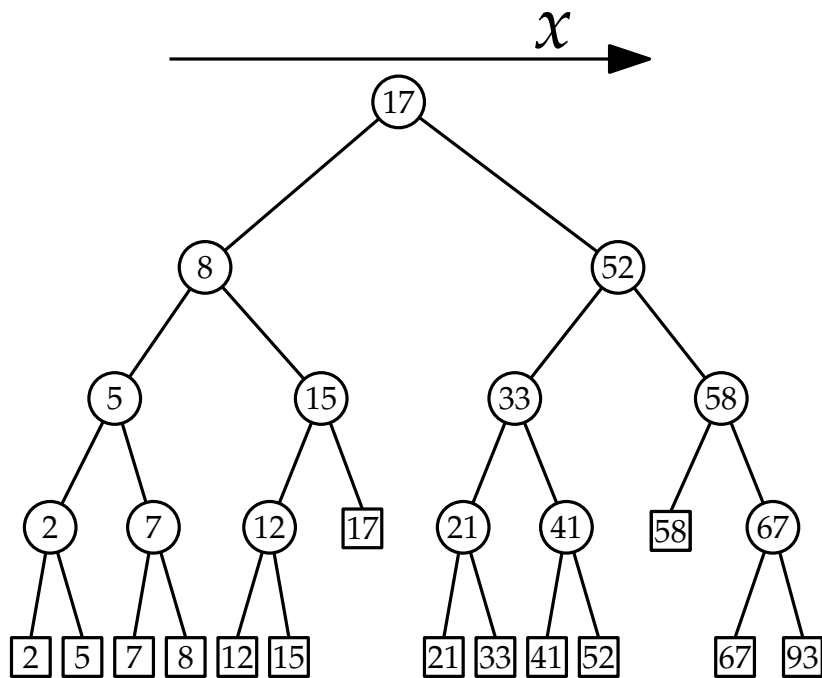
Layered Range Trees



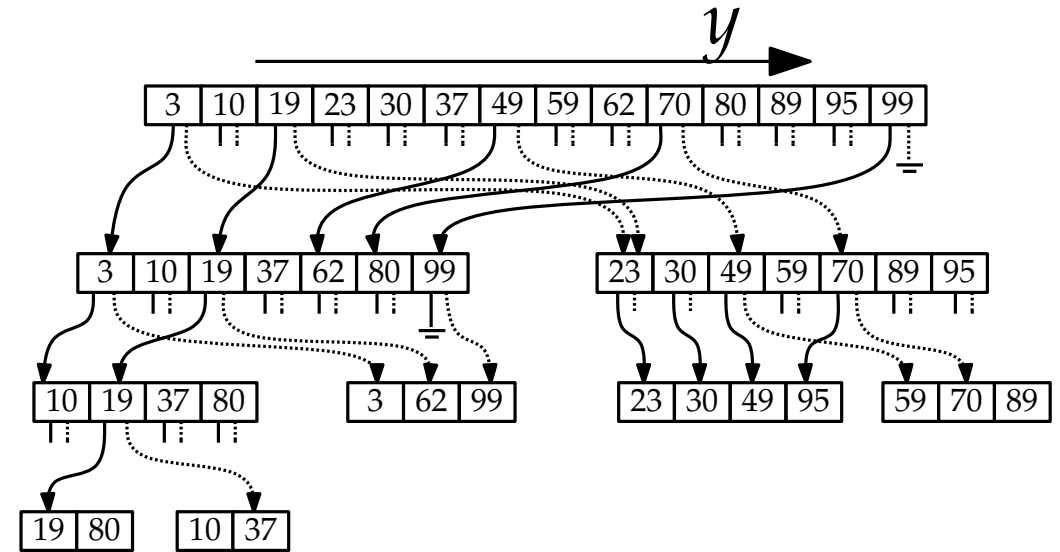
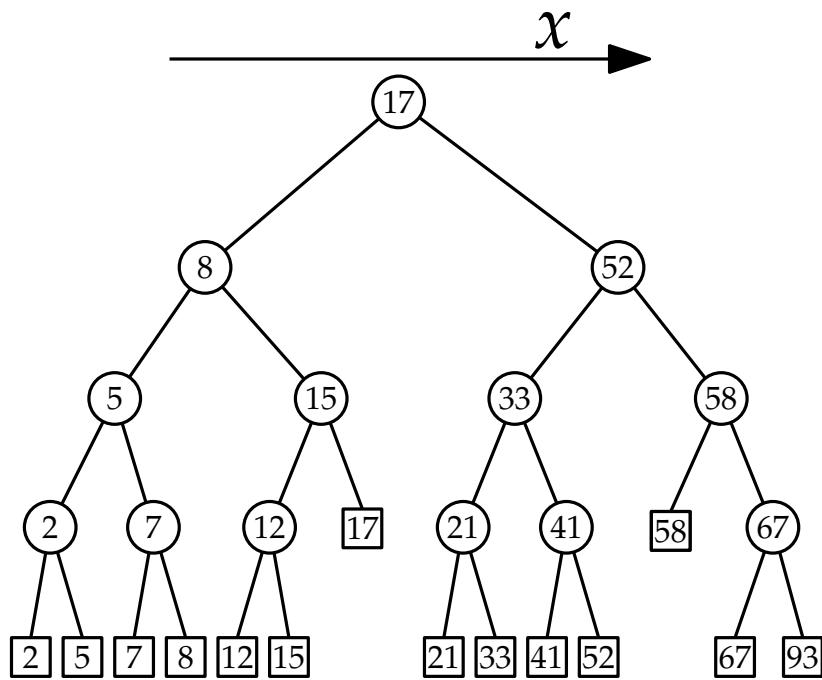
Layered Range Trees



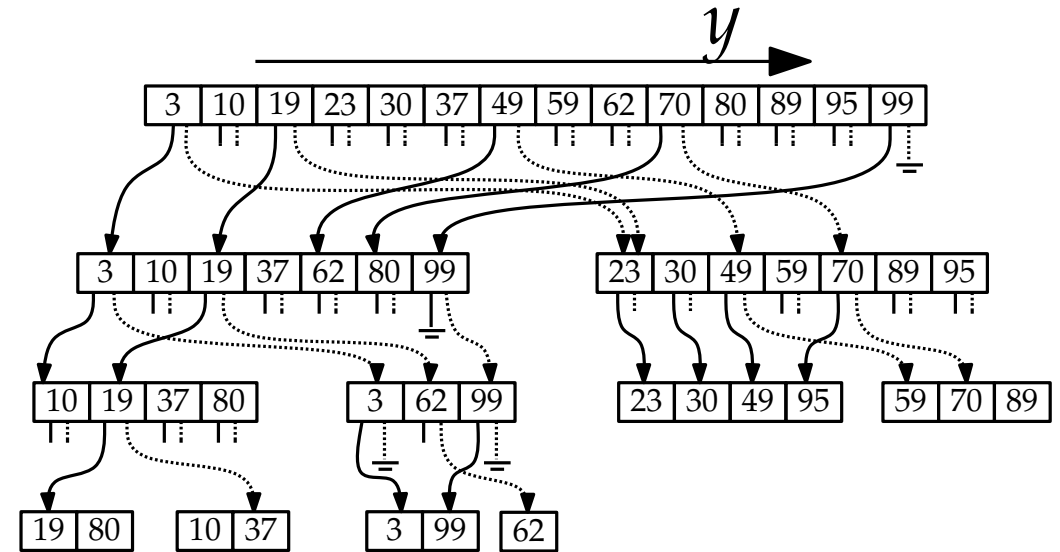
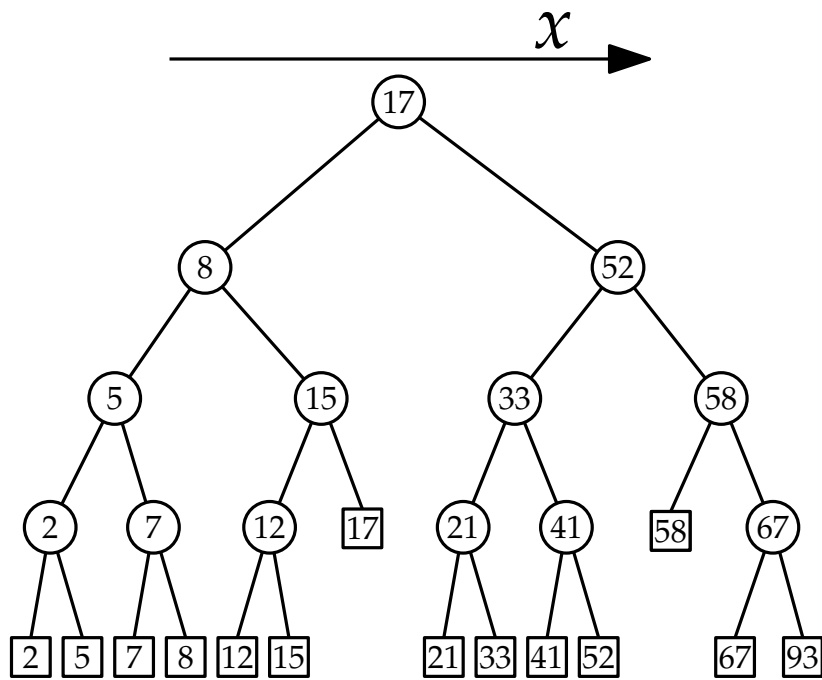
Layered Range Trees



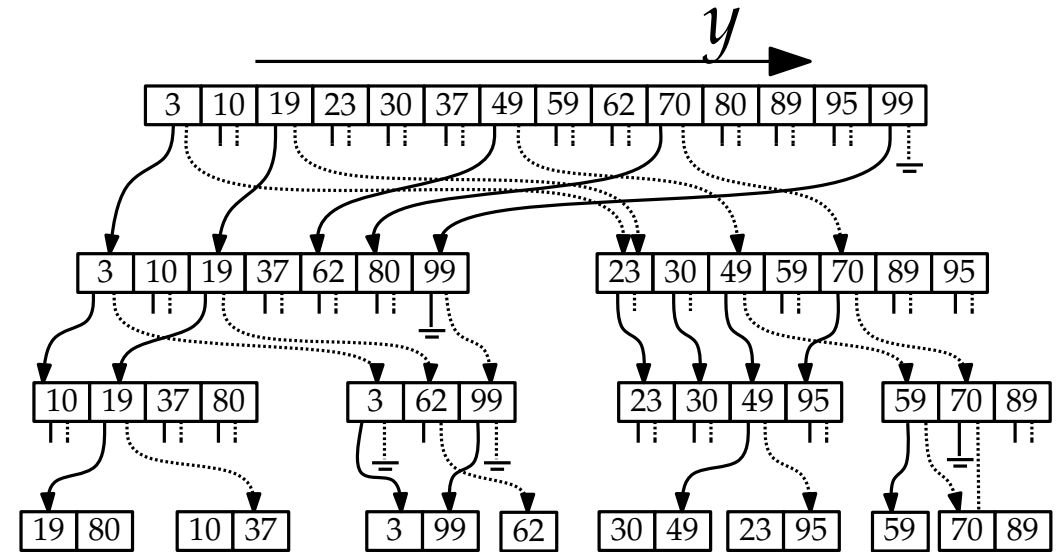
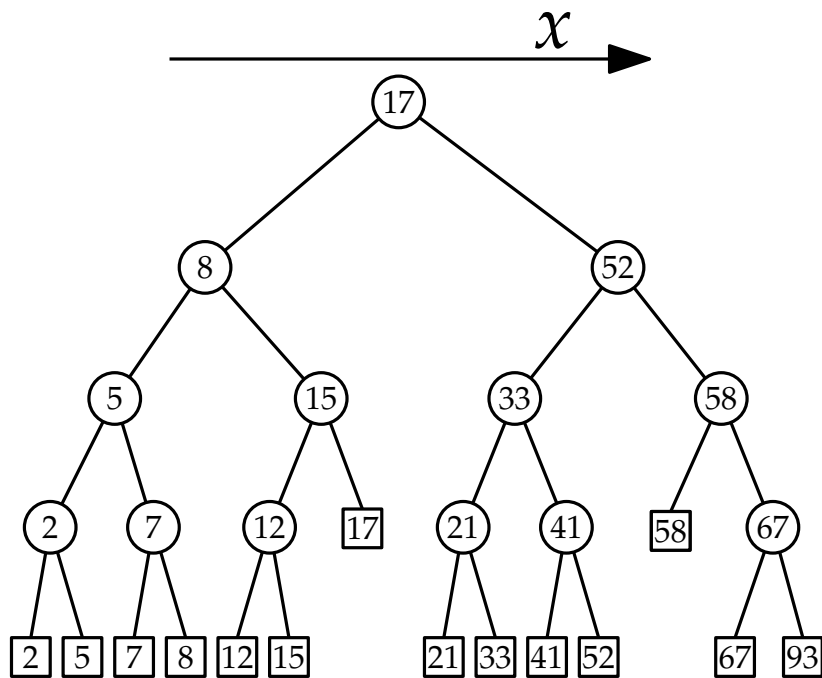
Layered Range Trees



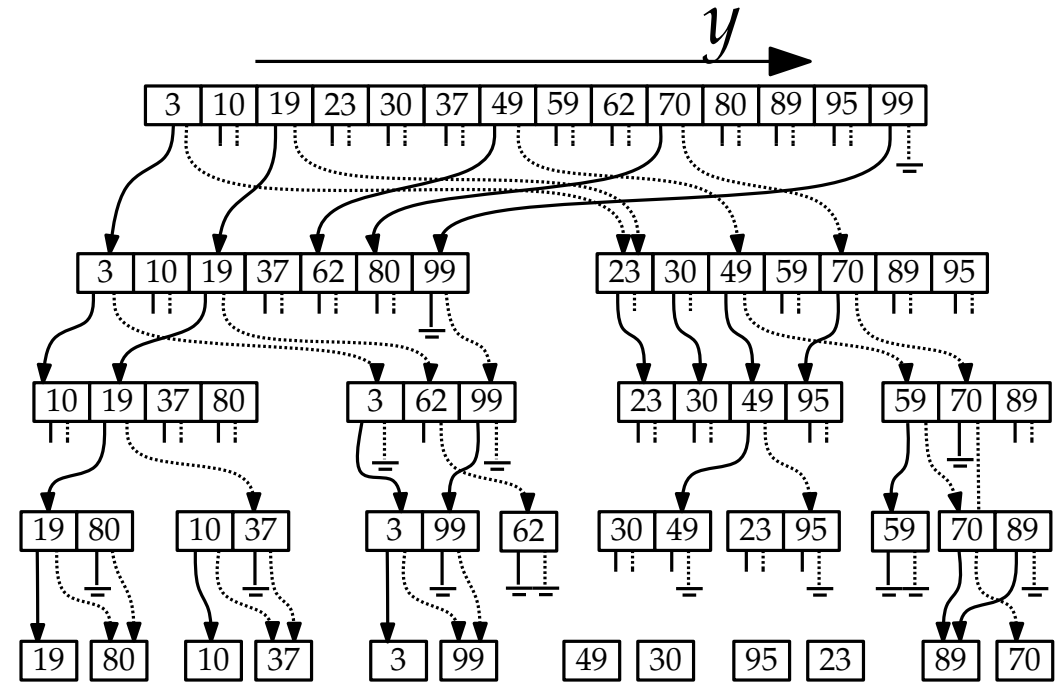
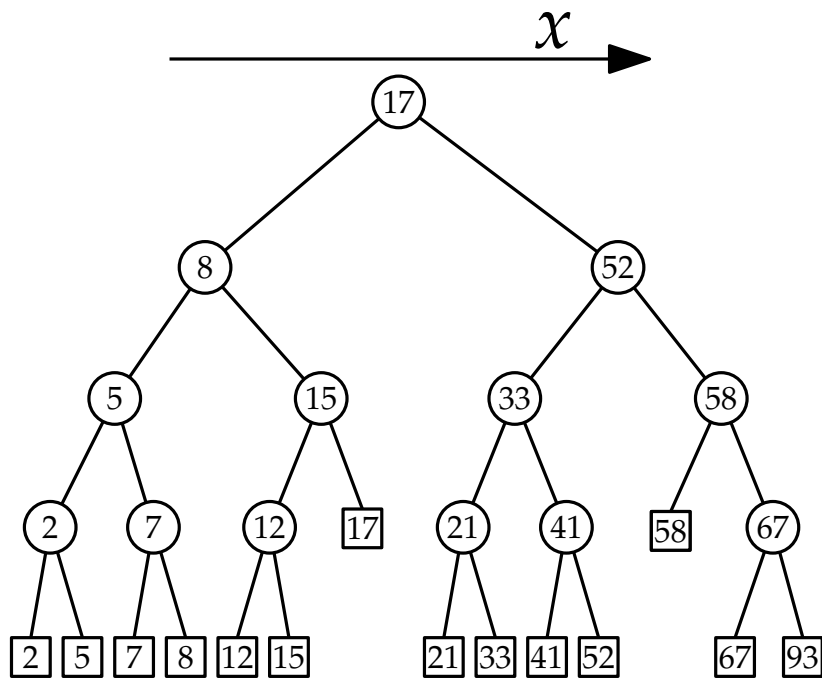
Layered Range Trees



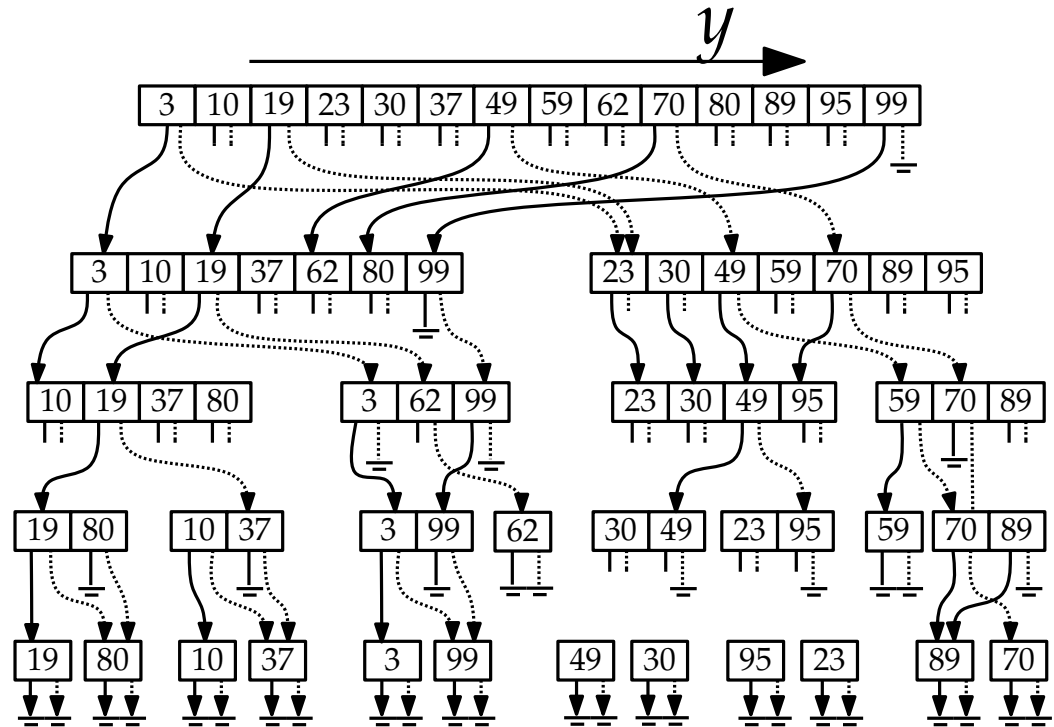
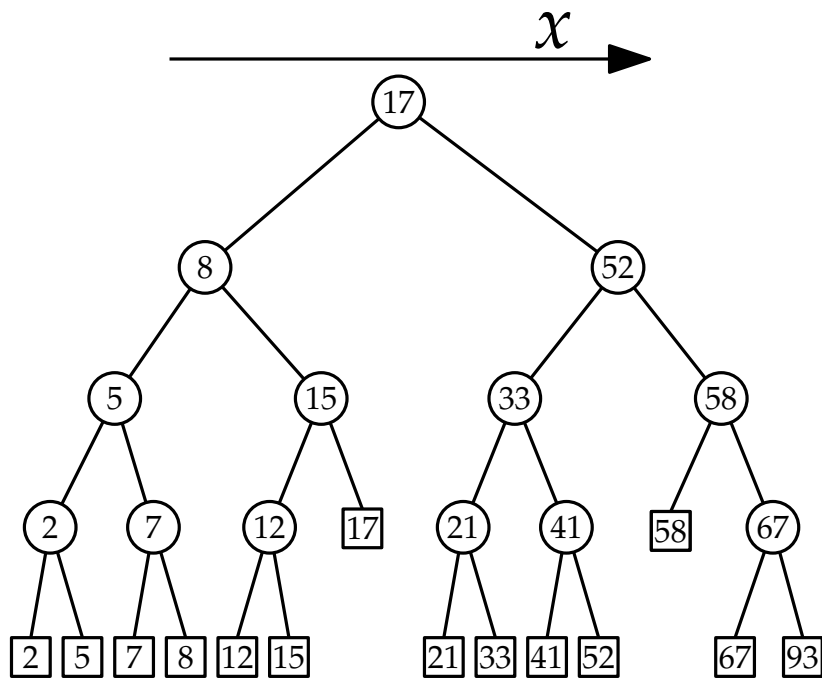
Layered Range Trees



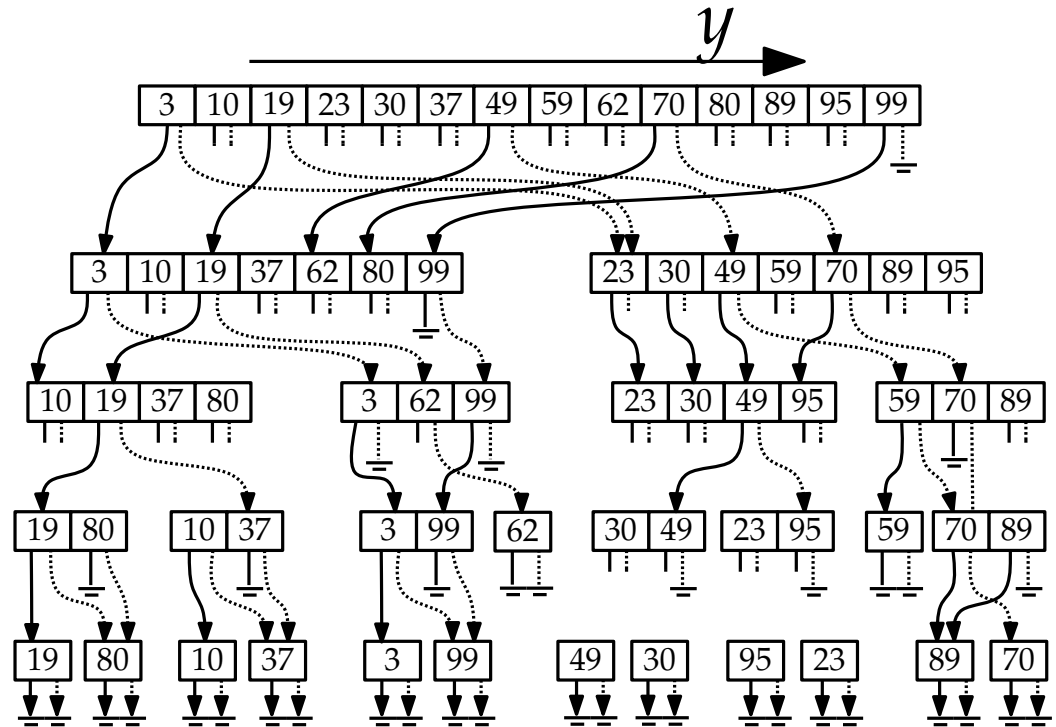
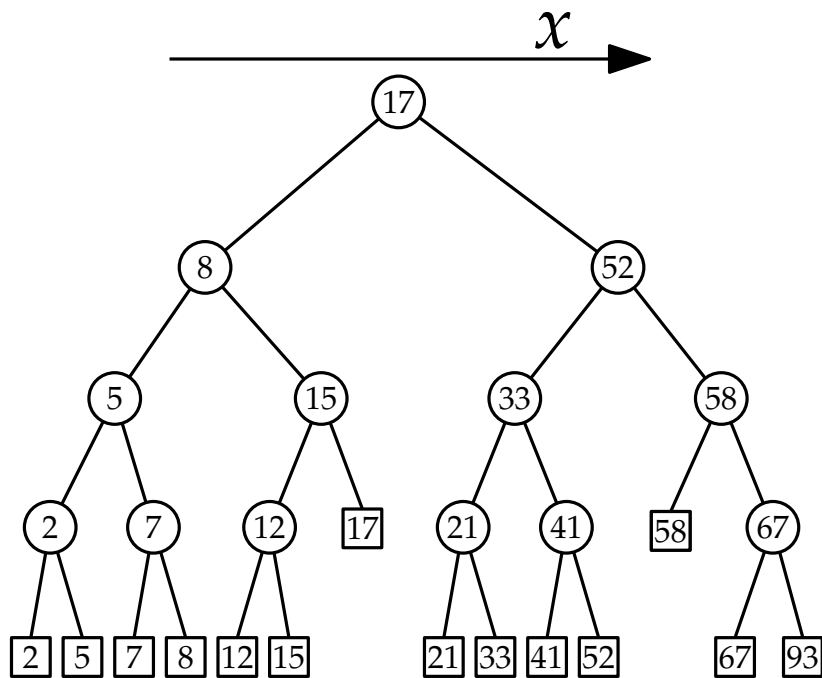
Layered Range Trees



Layered Range Trees

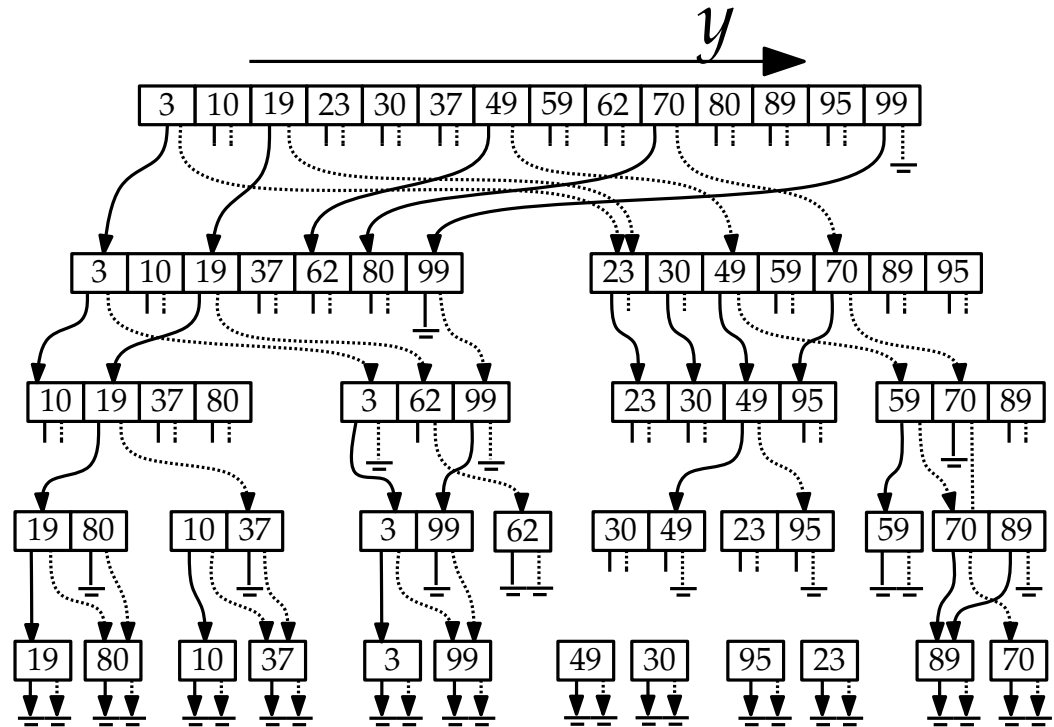
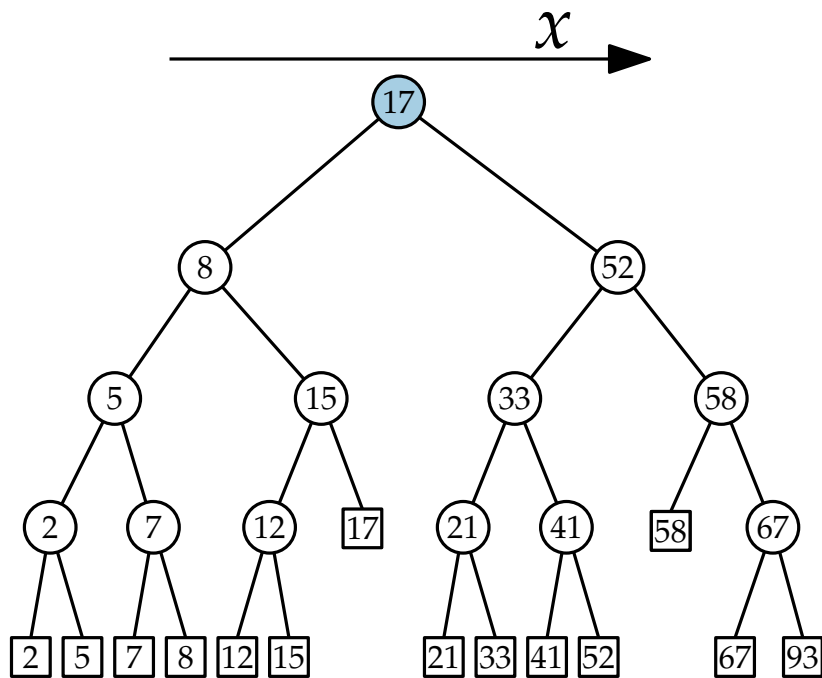


Layered Range Trees



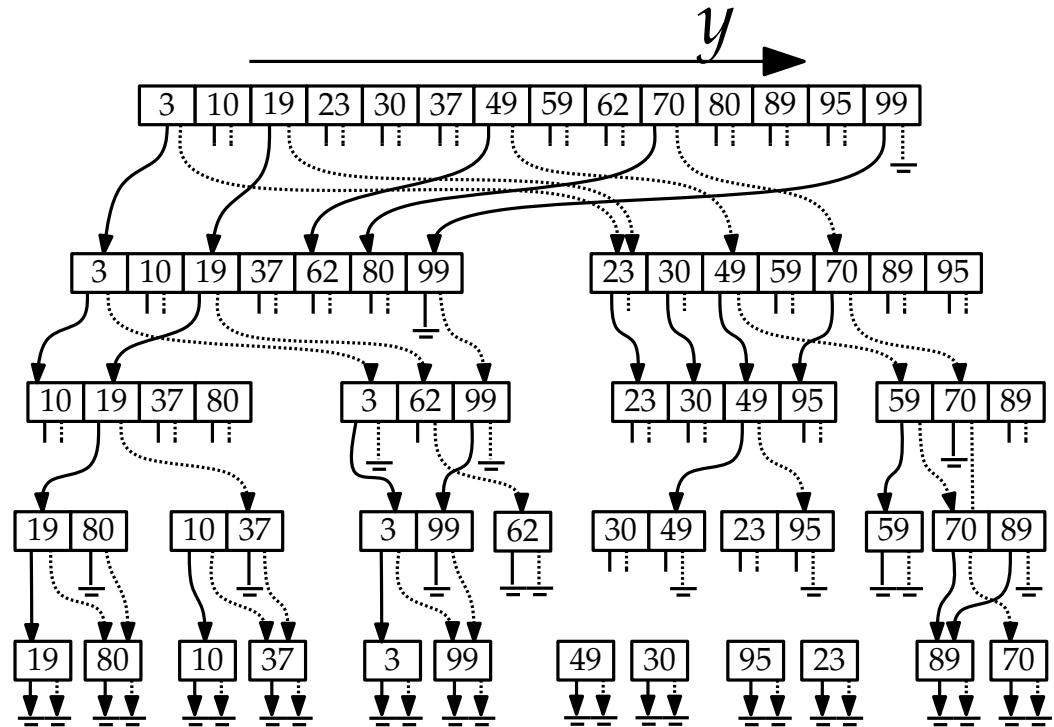
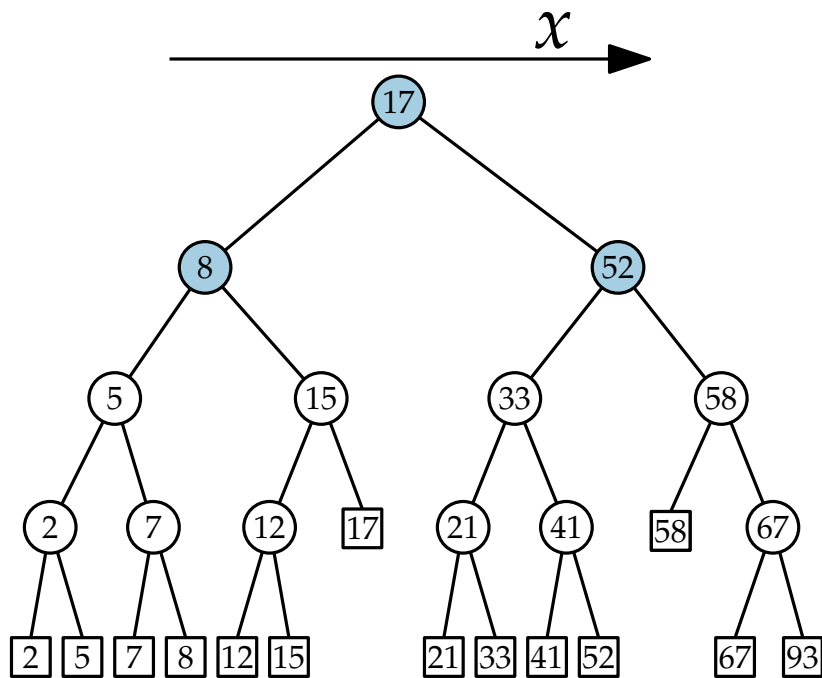
$$[16, 53] \times [18, 60]$$

Layered Range Trees



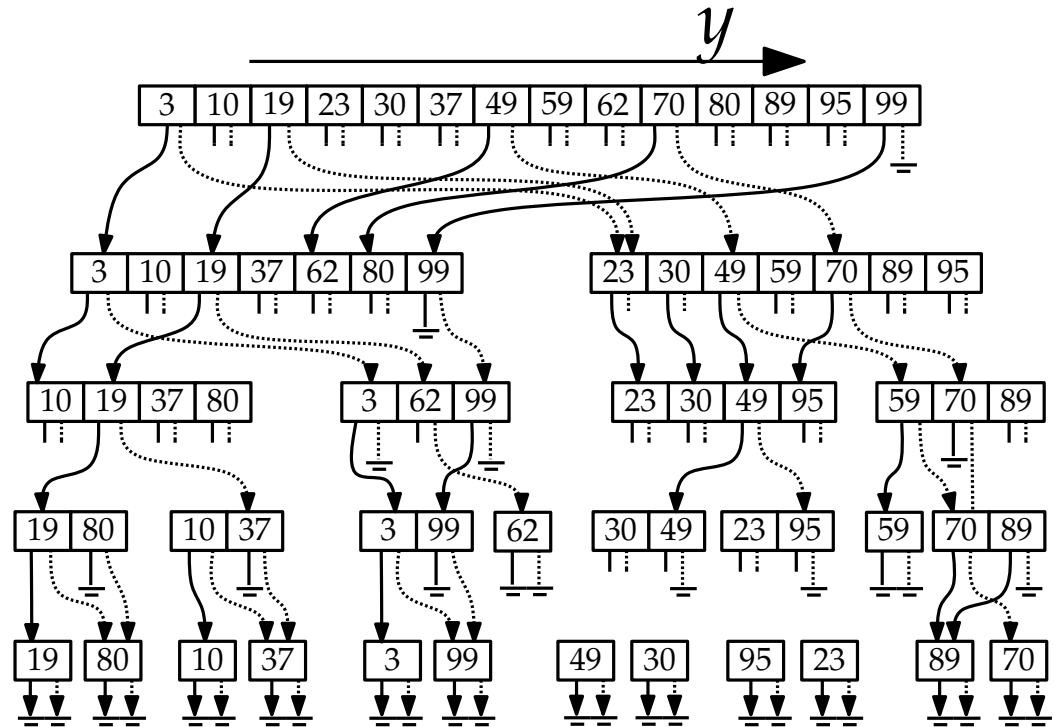
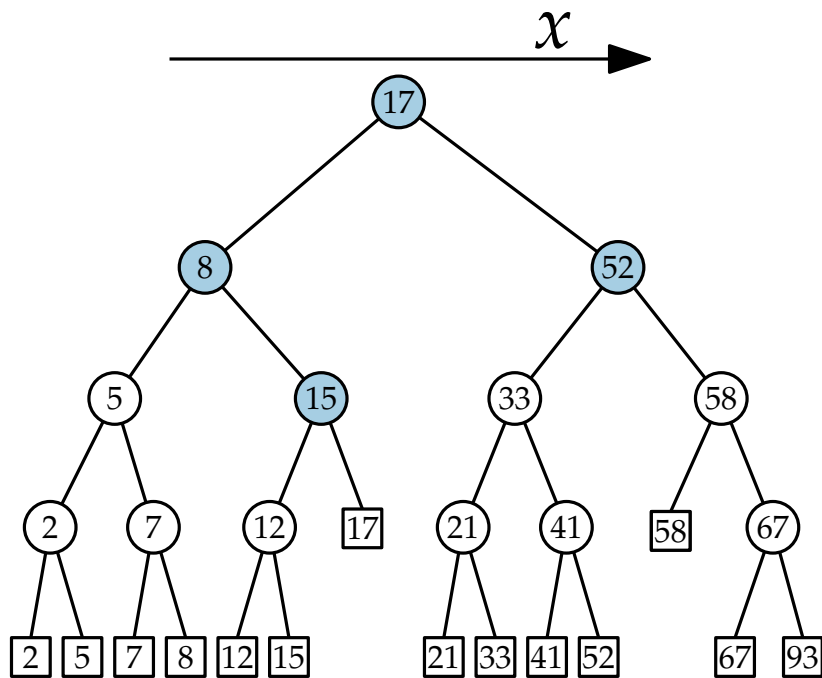
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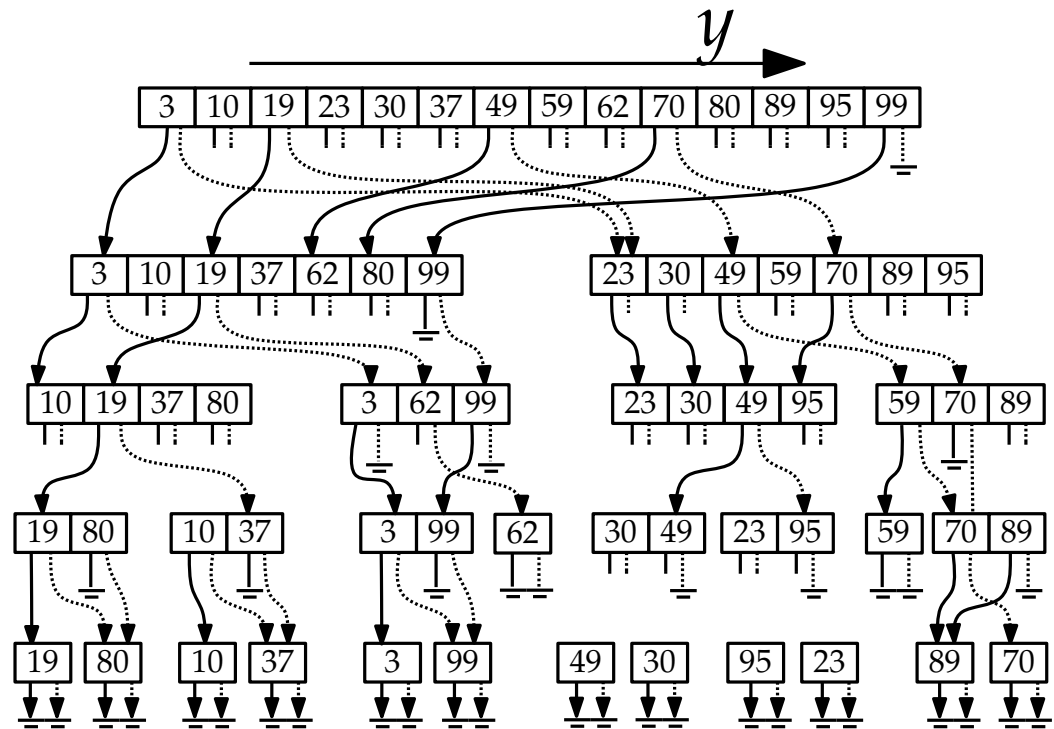
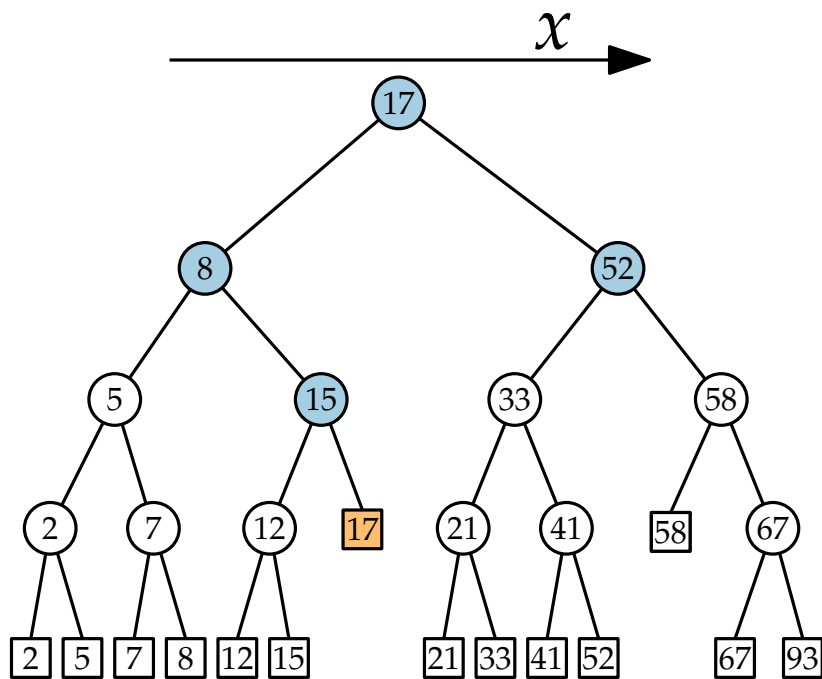
$$[16, 53] \times [18, 60]$$

Layered Range Trees



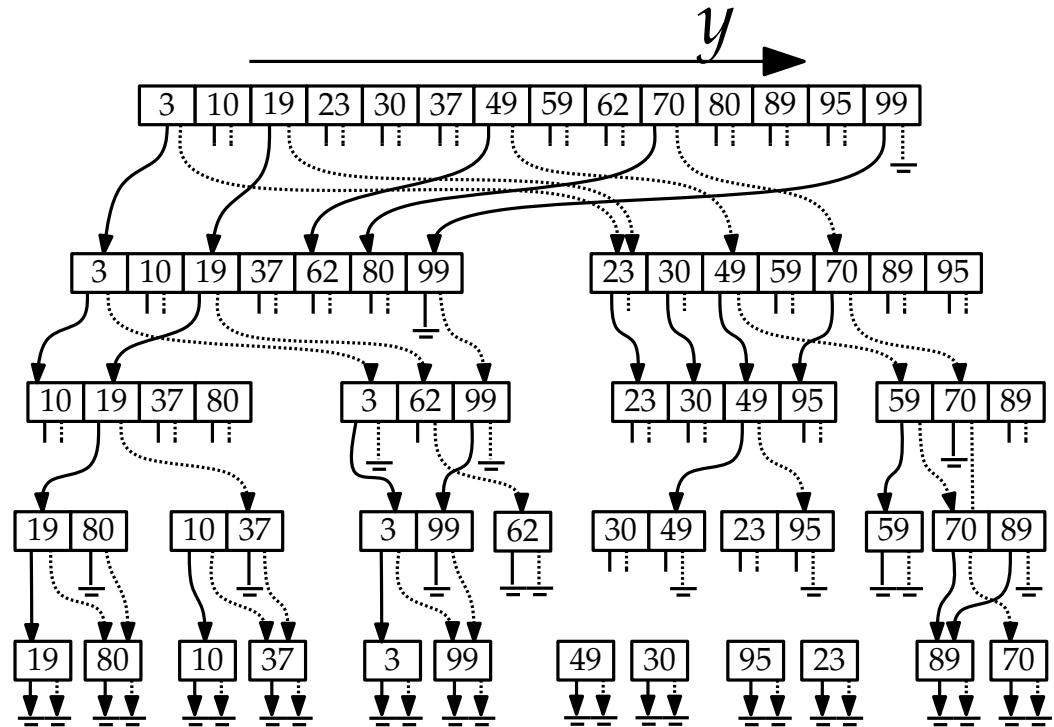
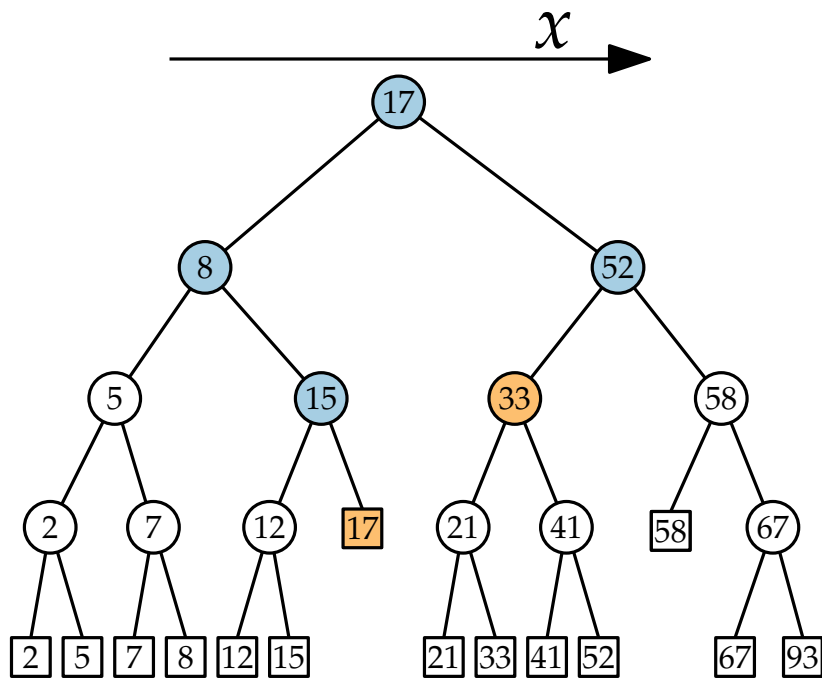
$$[16, 53] \times [18, 60]$$

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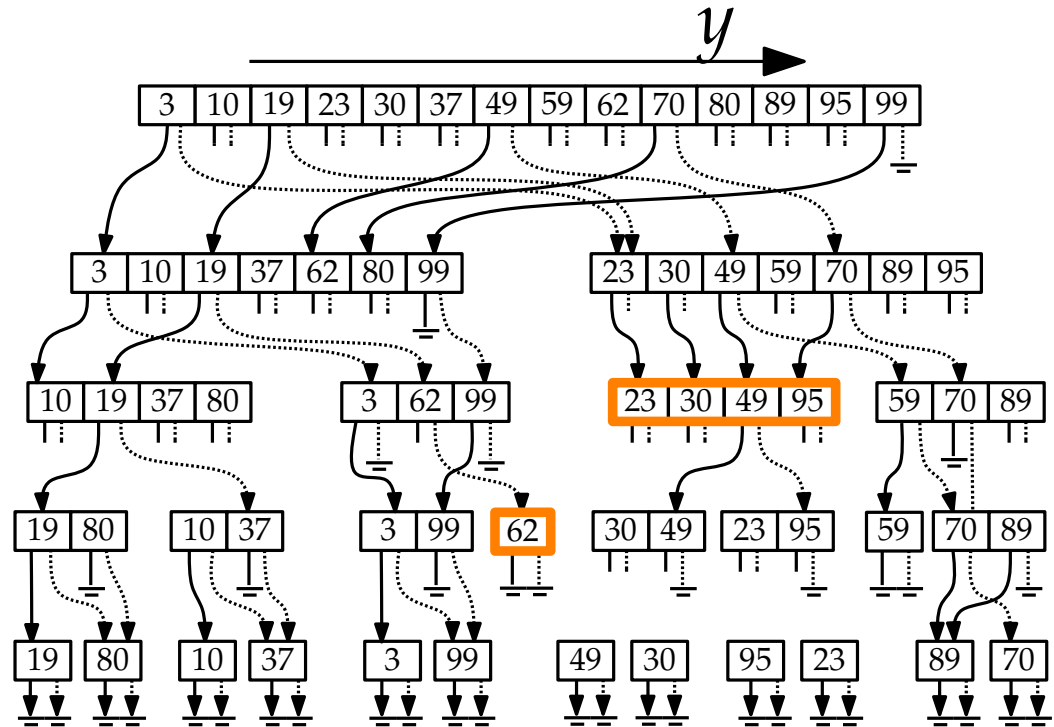
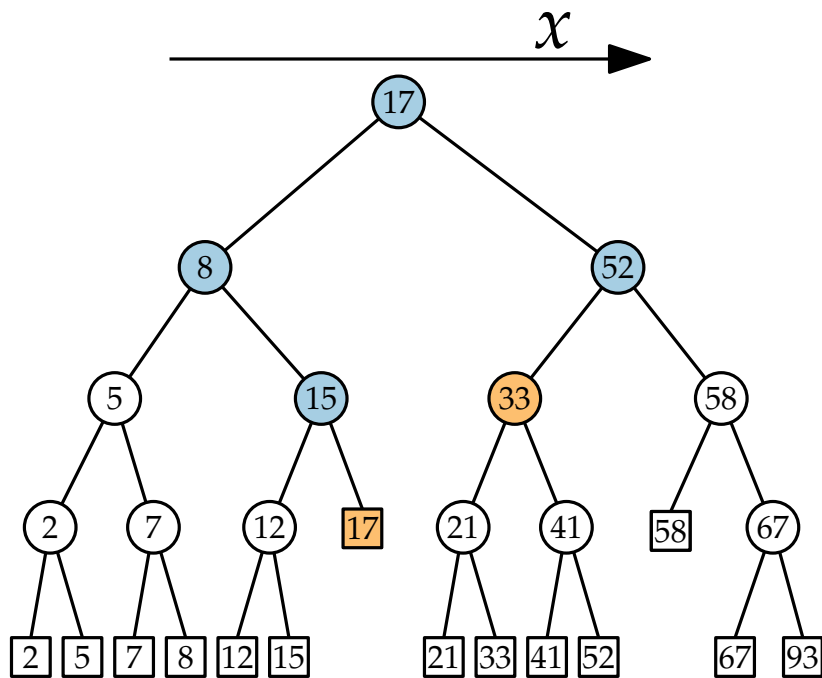
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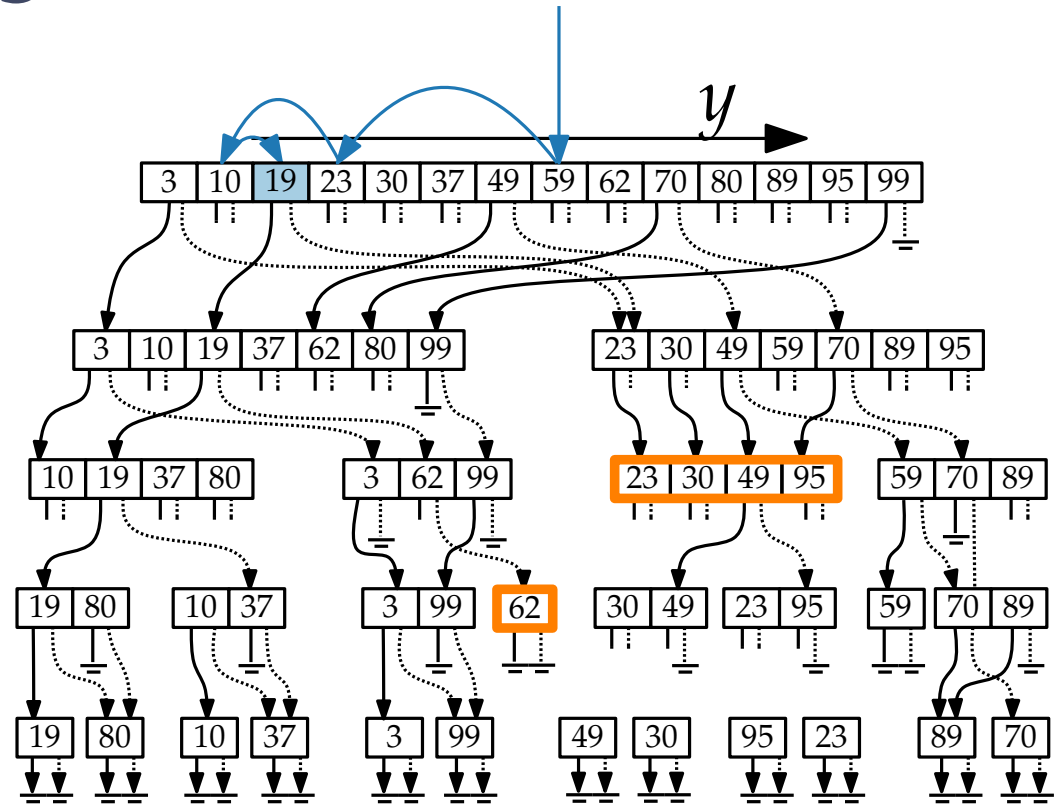
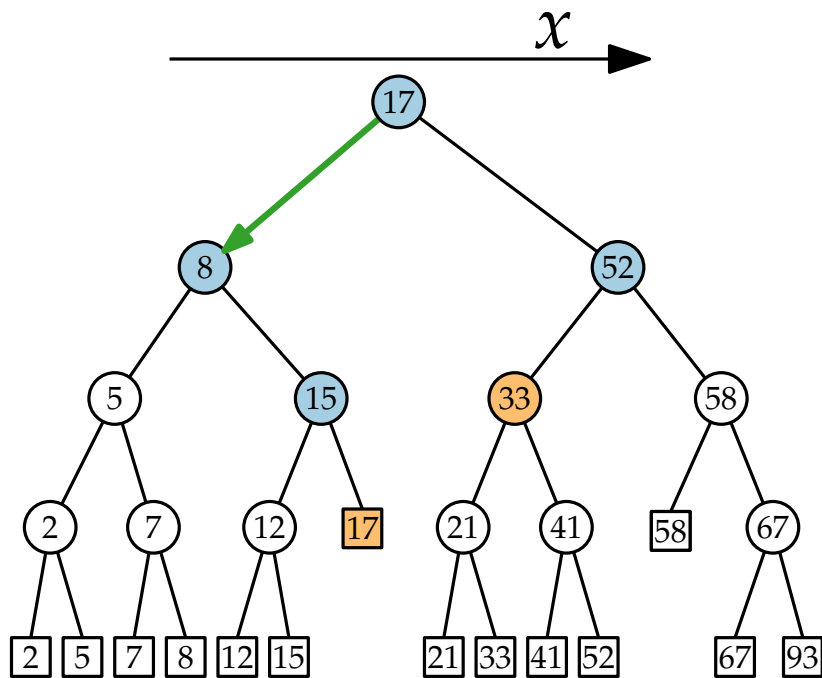
$$[16, 53] \times [18, 60]$$

Layered Range Trees



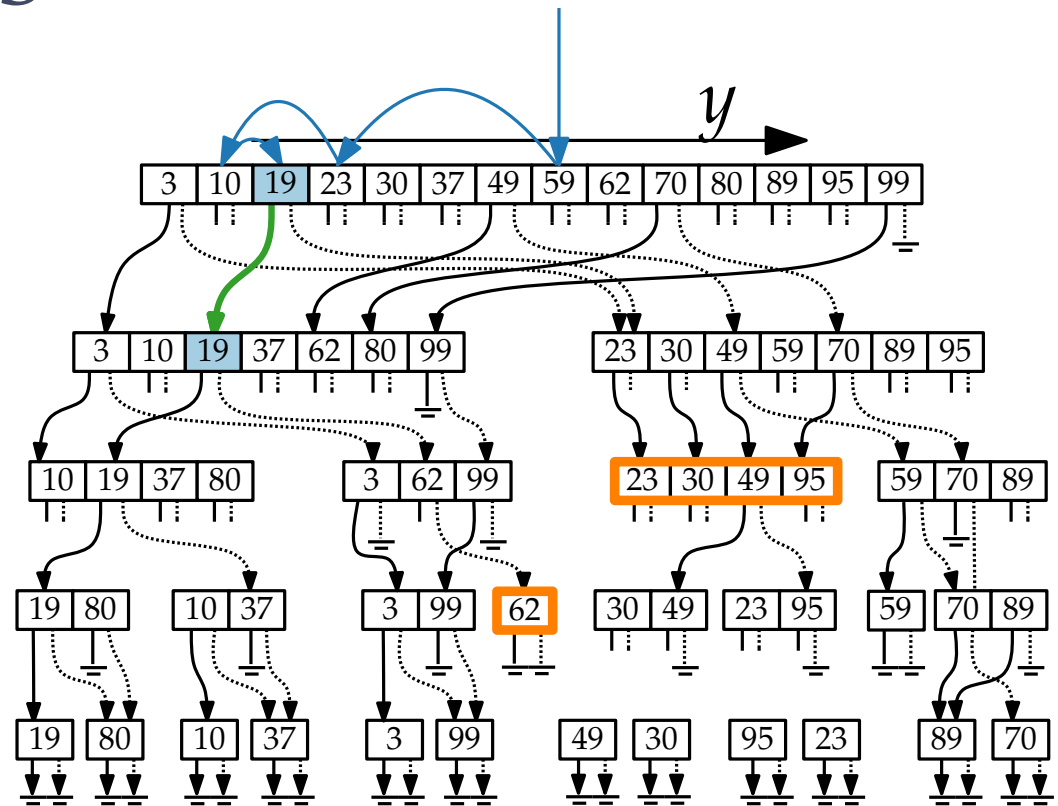
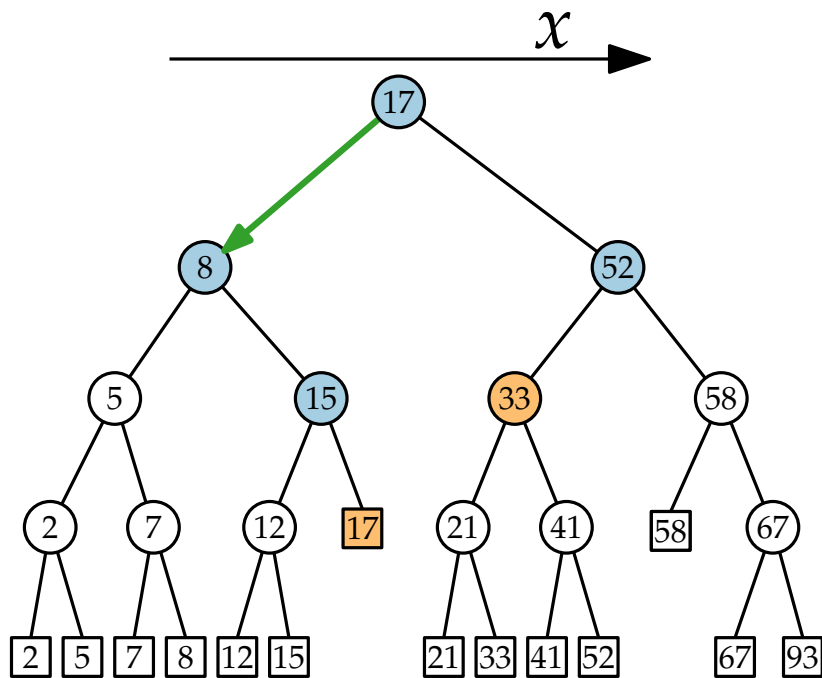
$$[16, 53] \times [18, 60]$$

Layered Range Trees



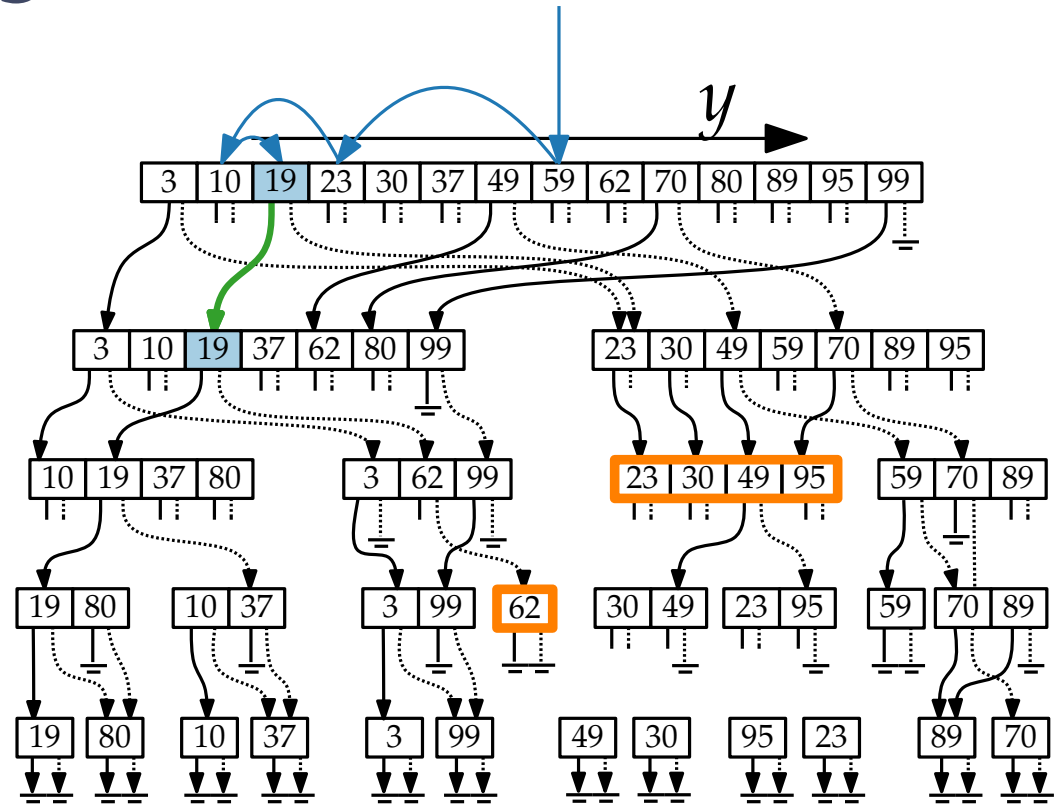
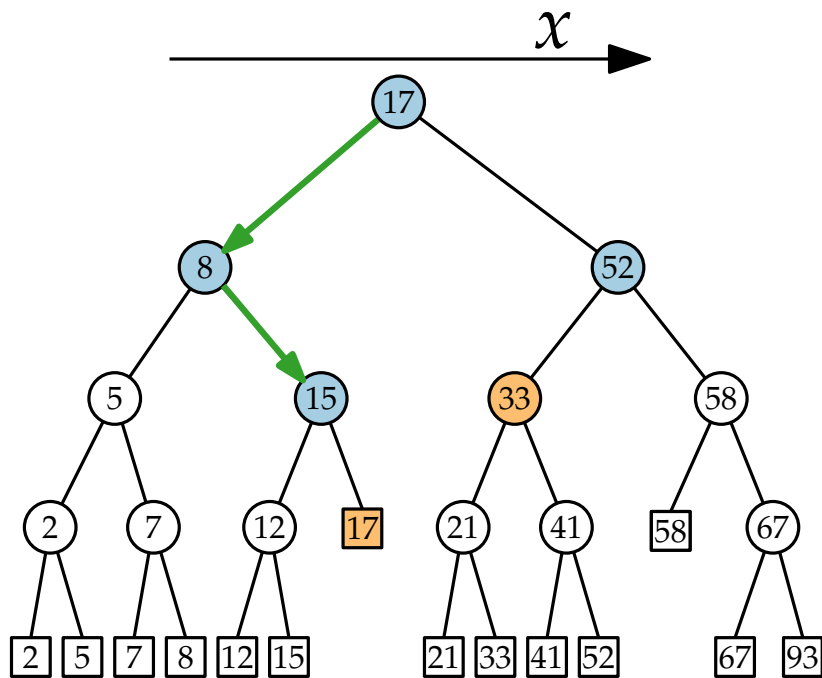
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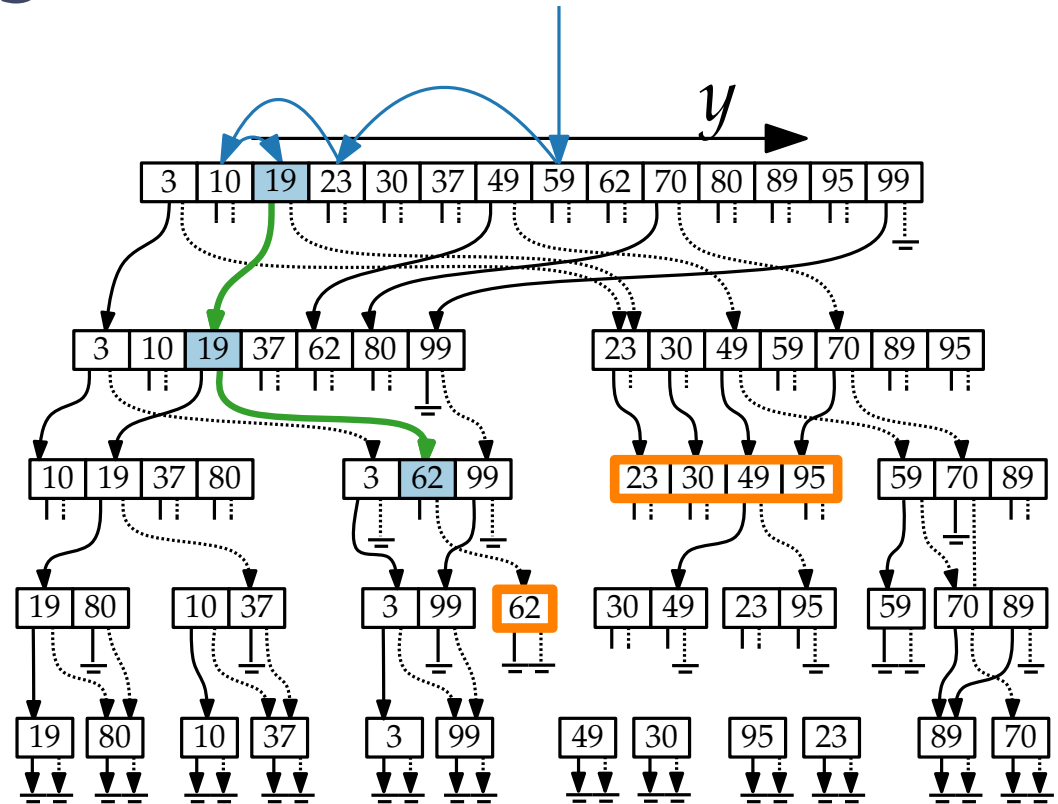
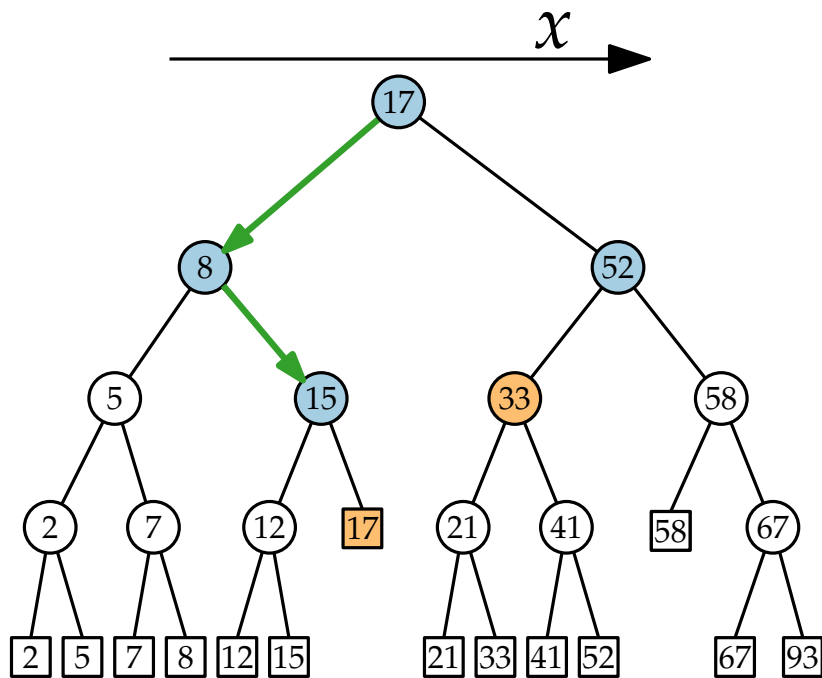
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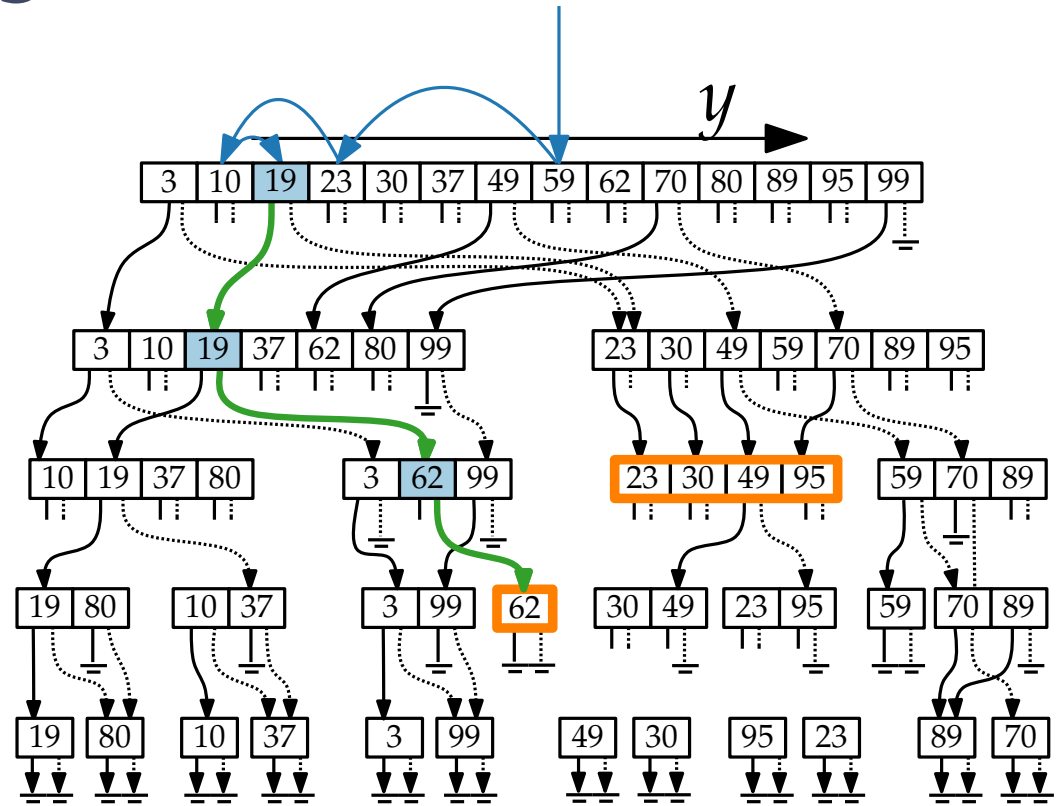
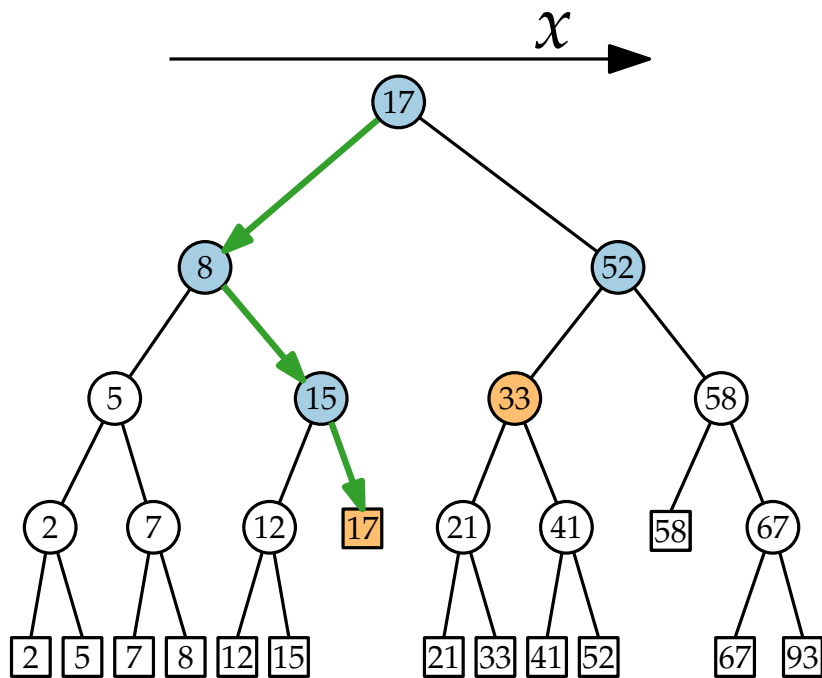
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Layered Range Trees



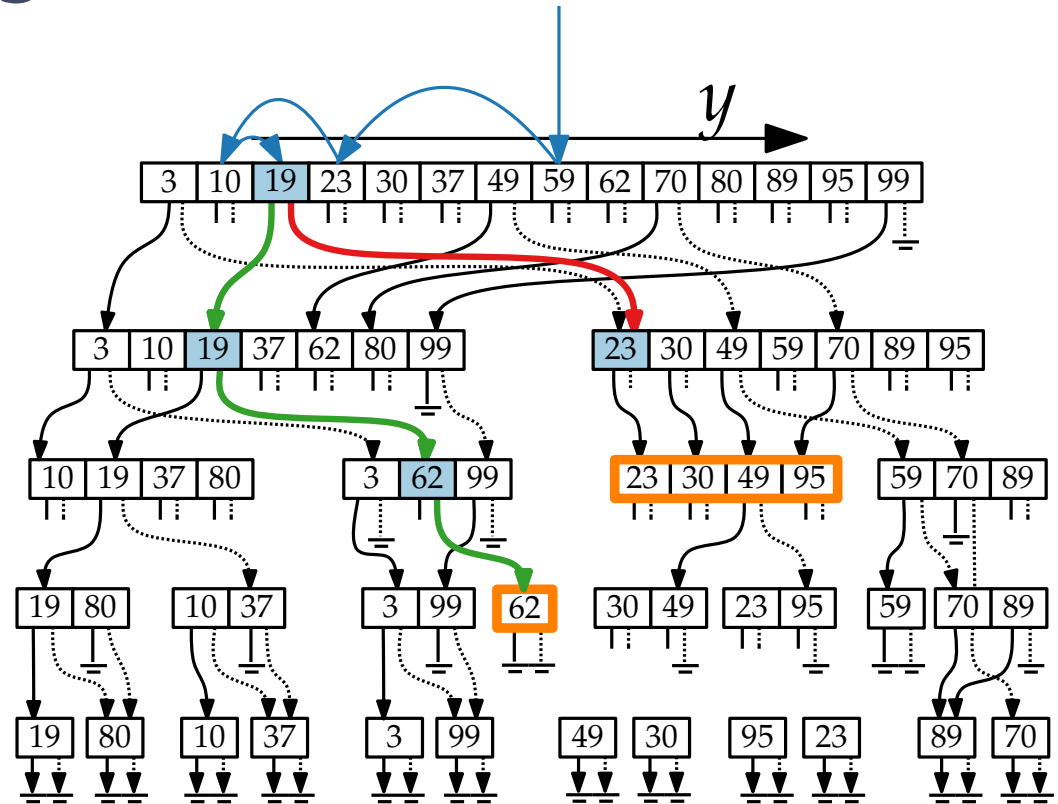
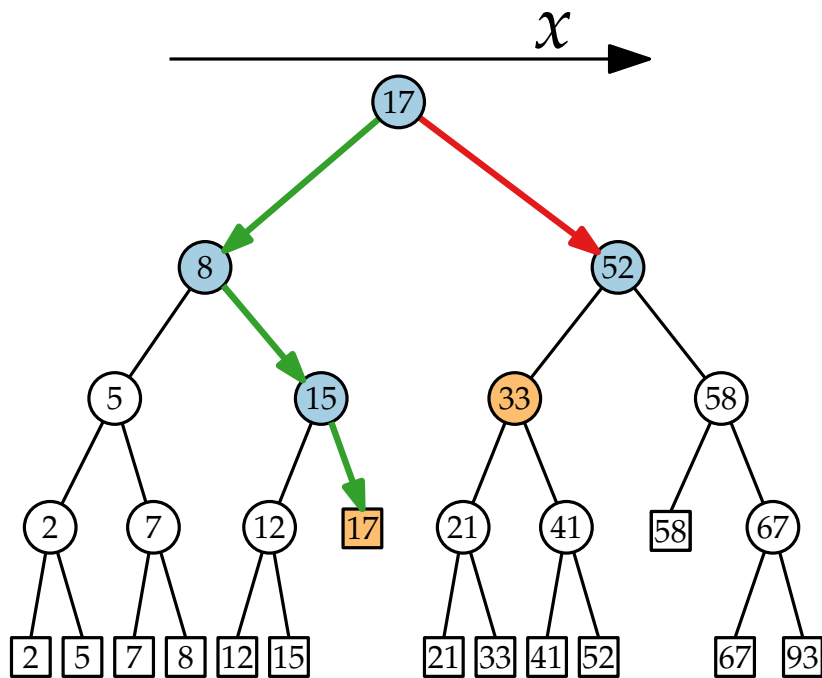
$$[16, 53] \times [18, 60]$$

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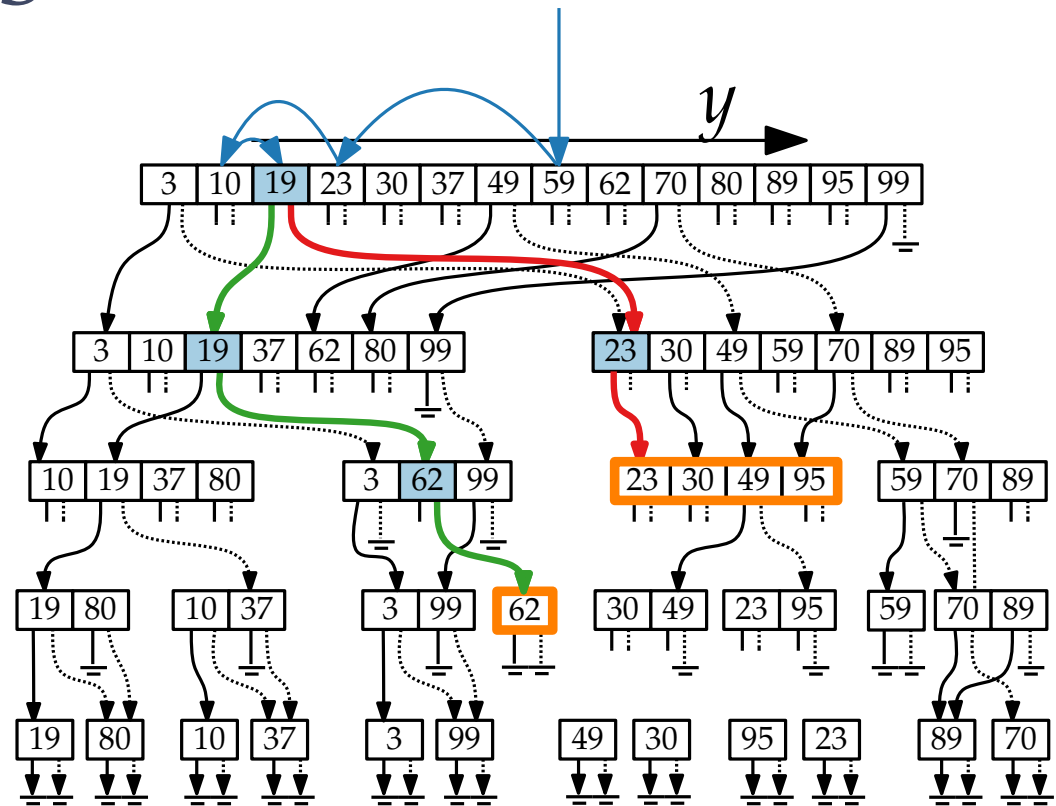
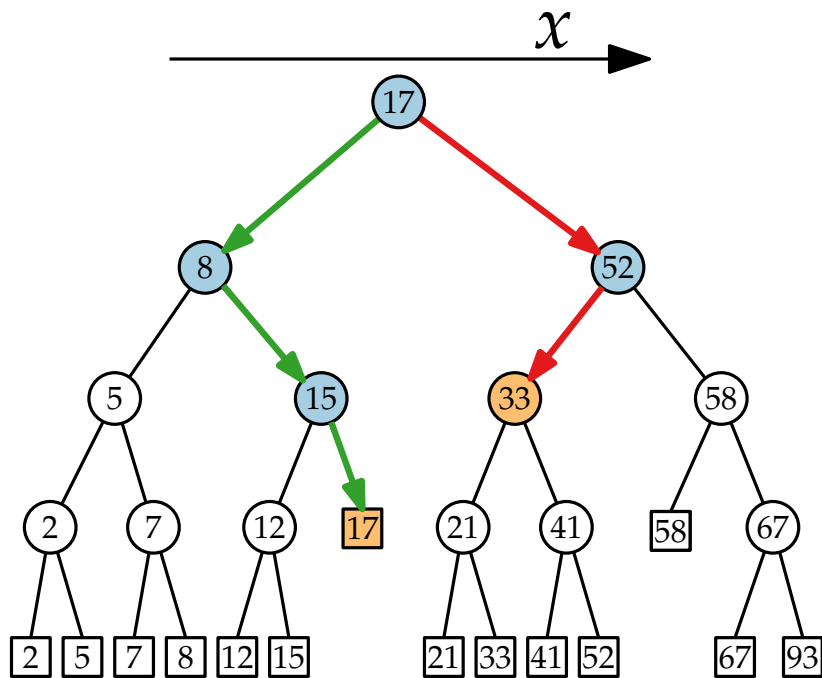
$$[16, 53] \times [18, 60]$$

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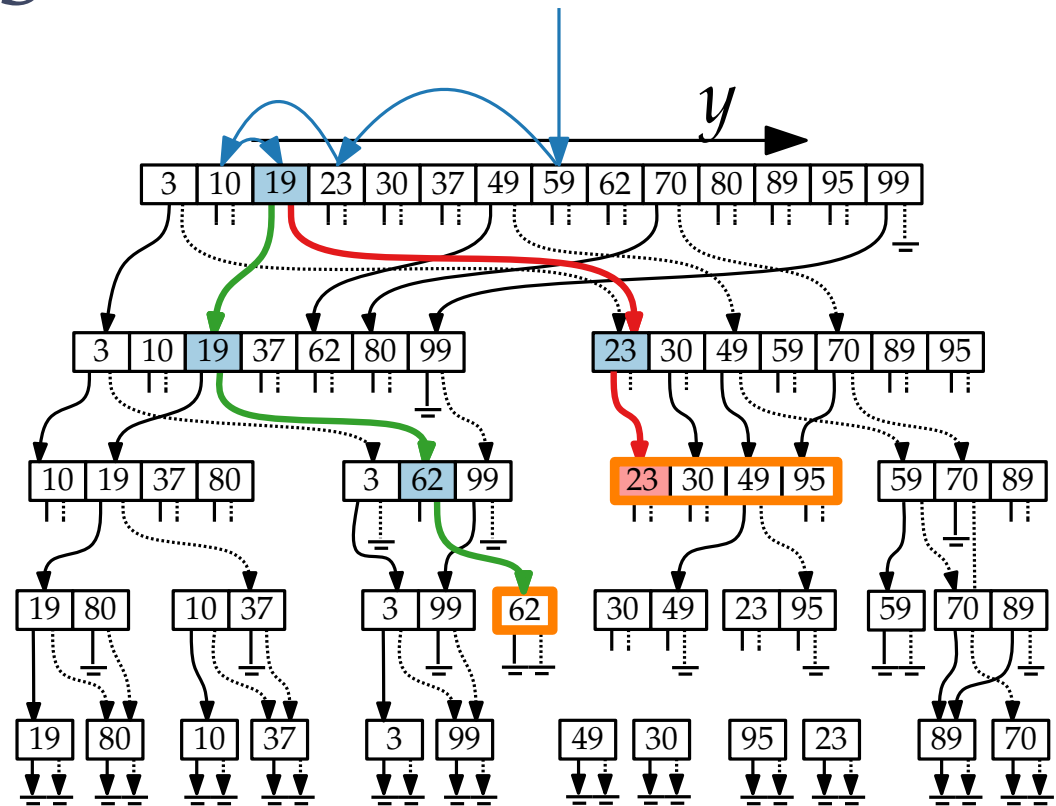
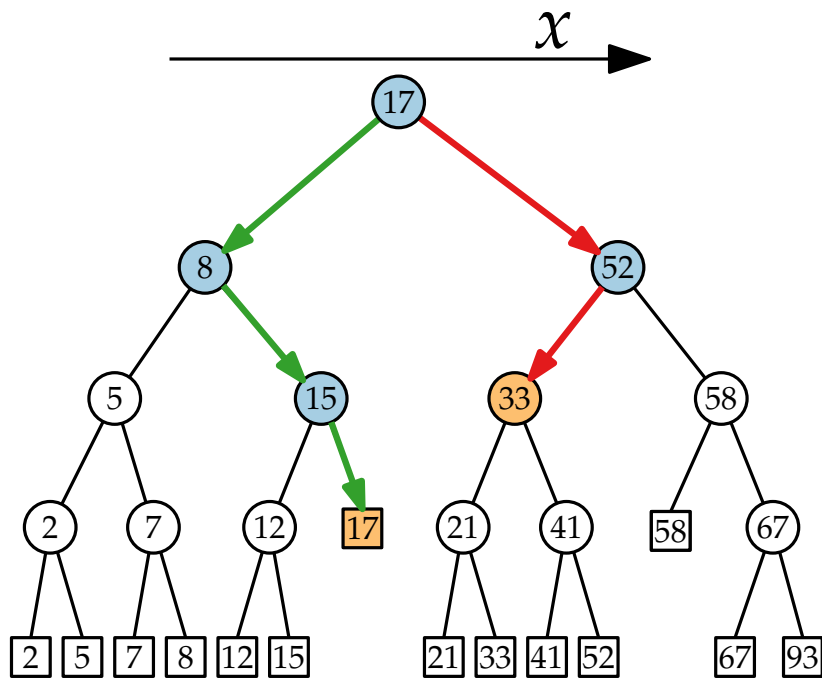
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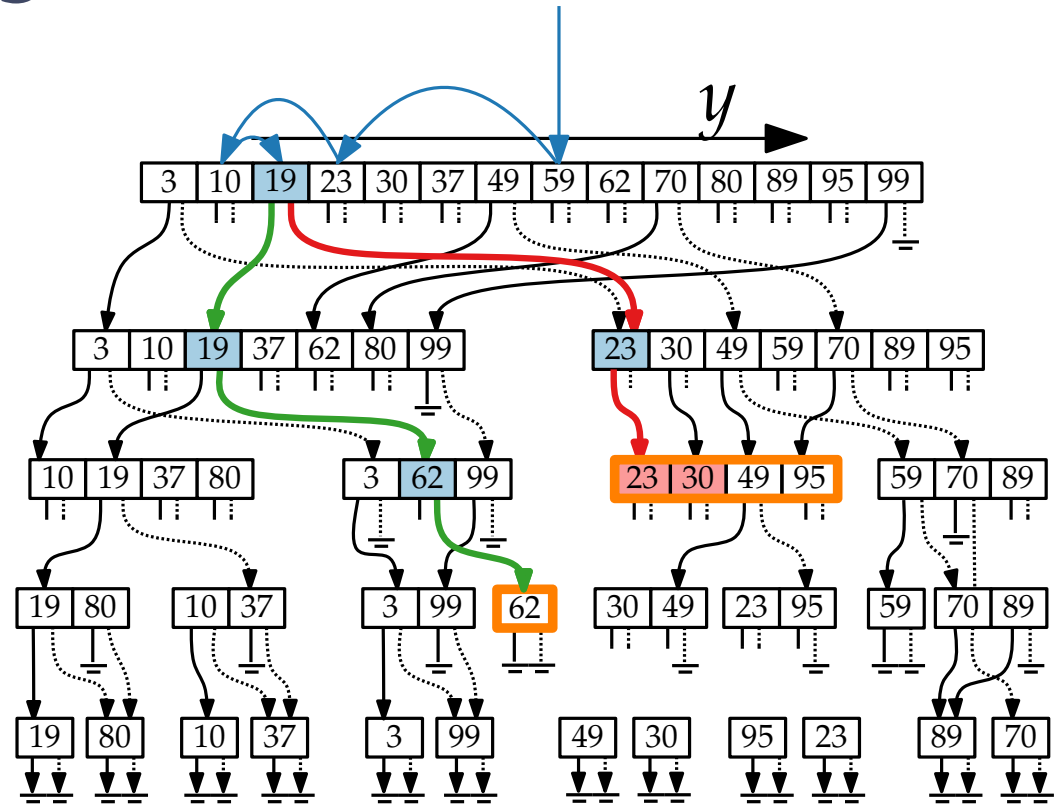
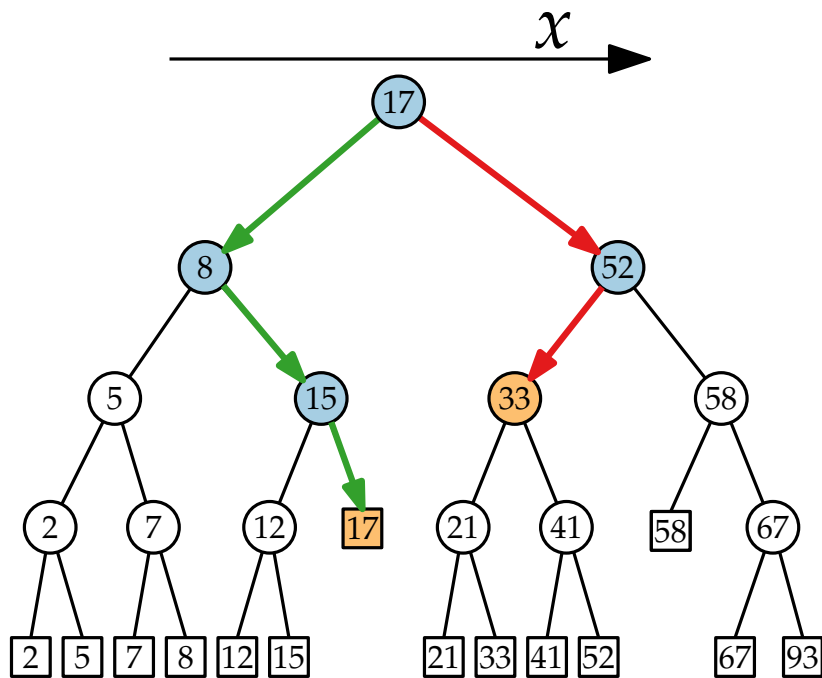
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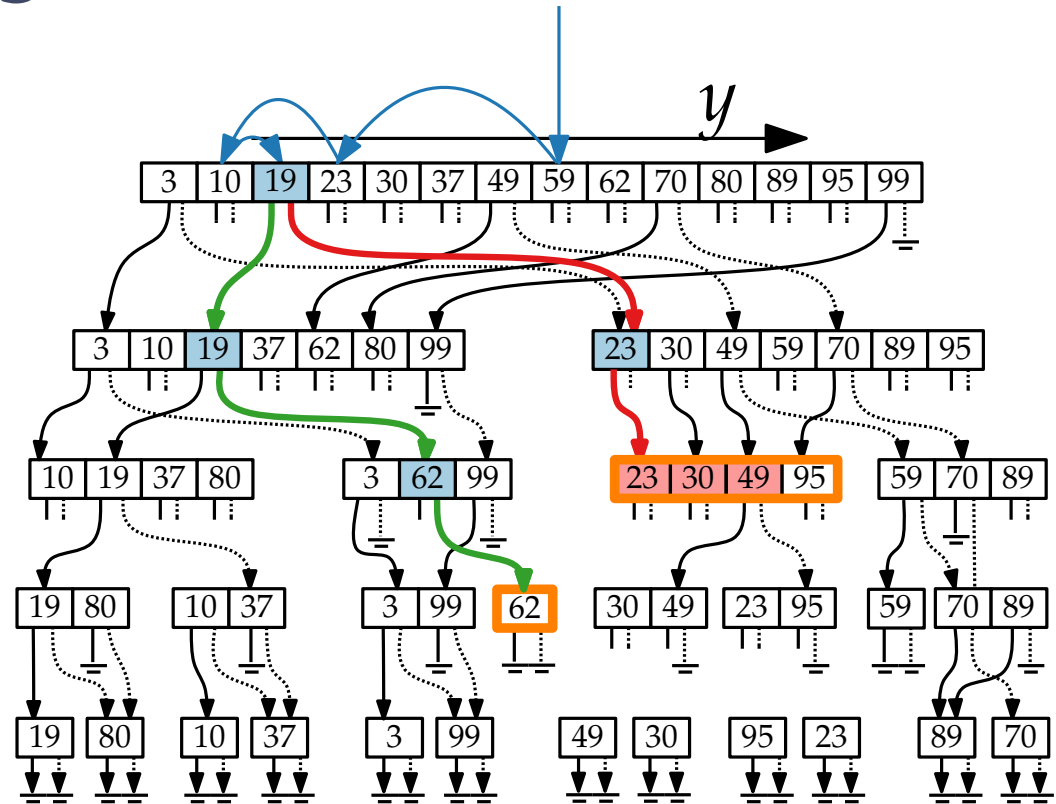
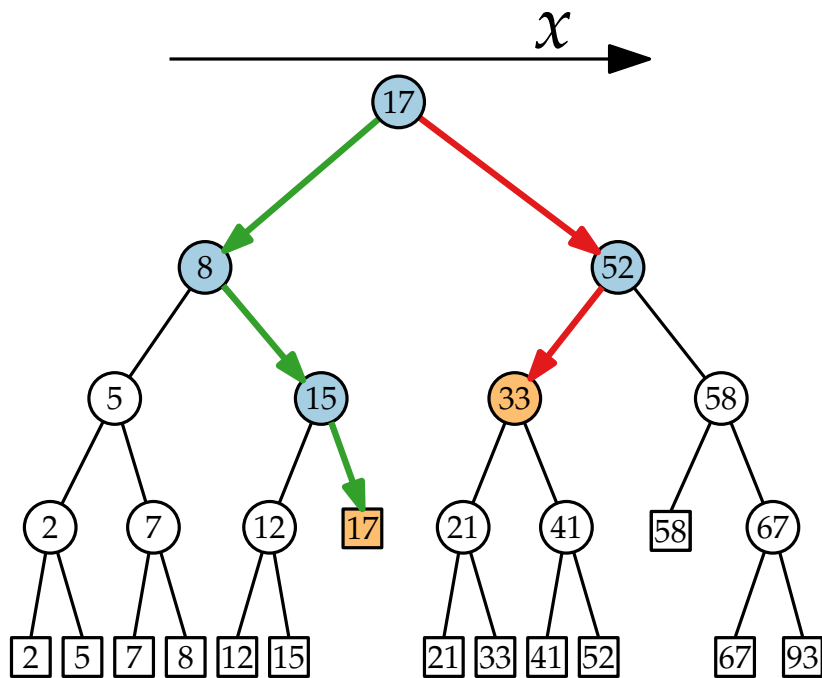
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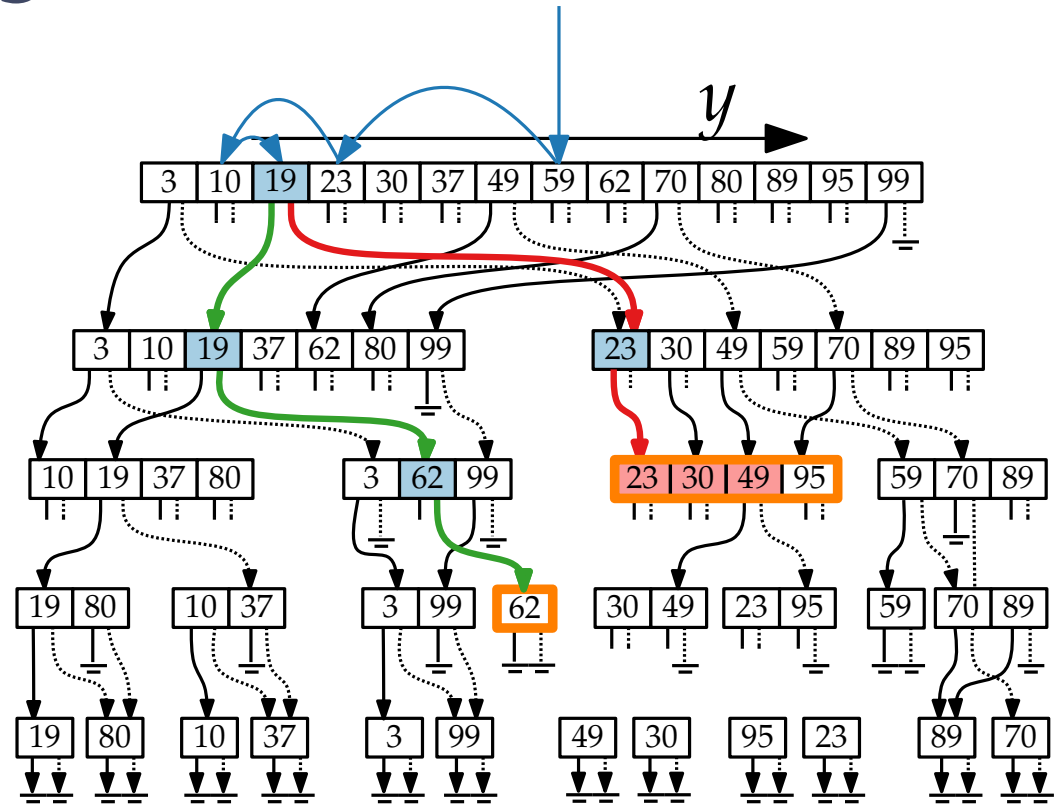
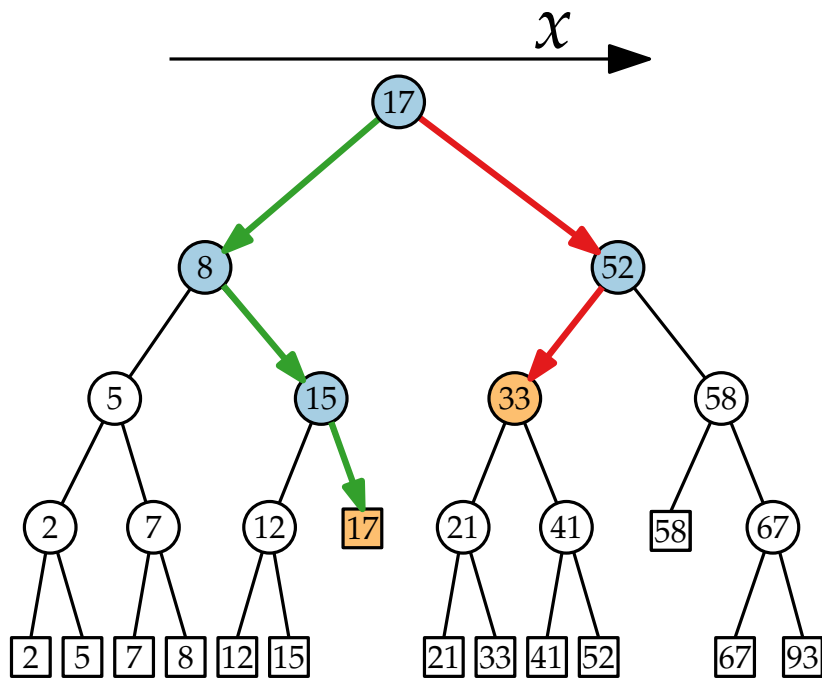
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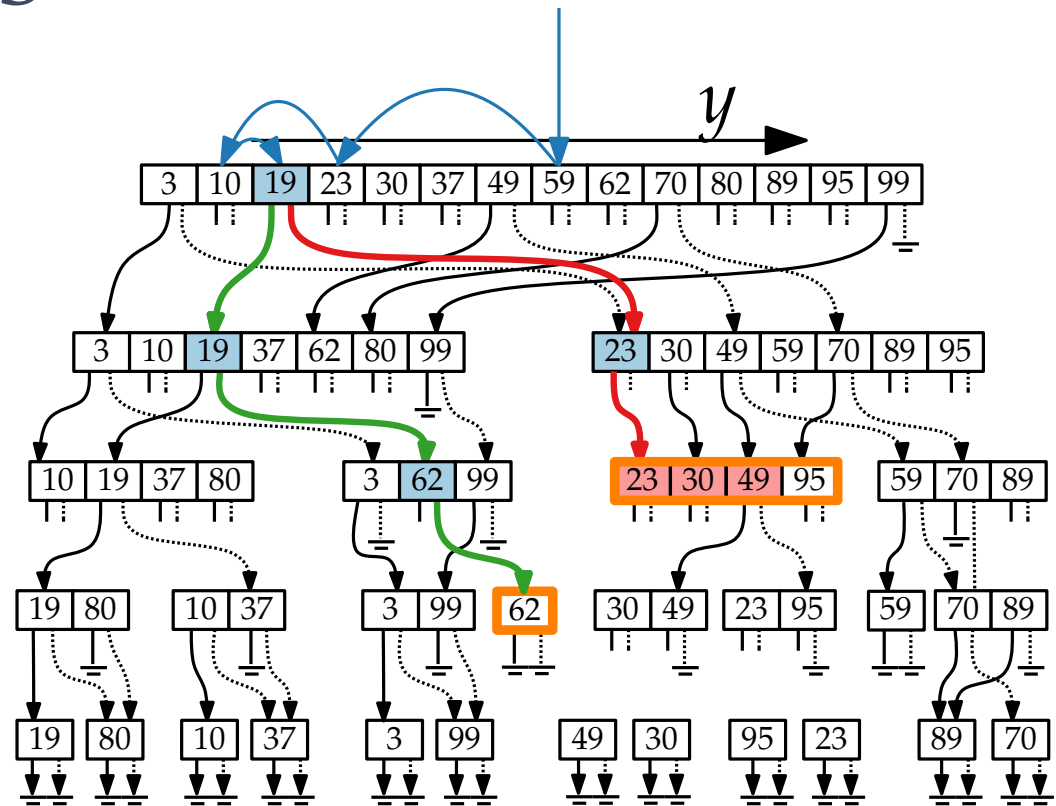
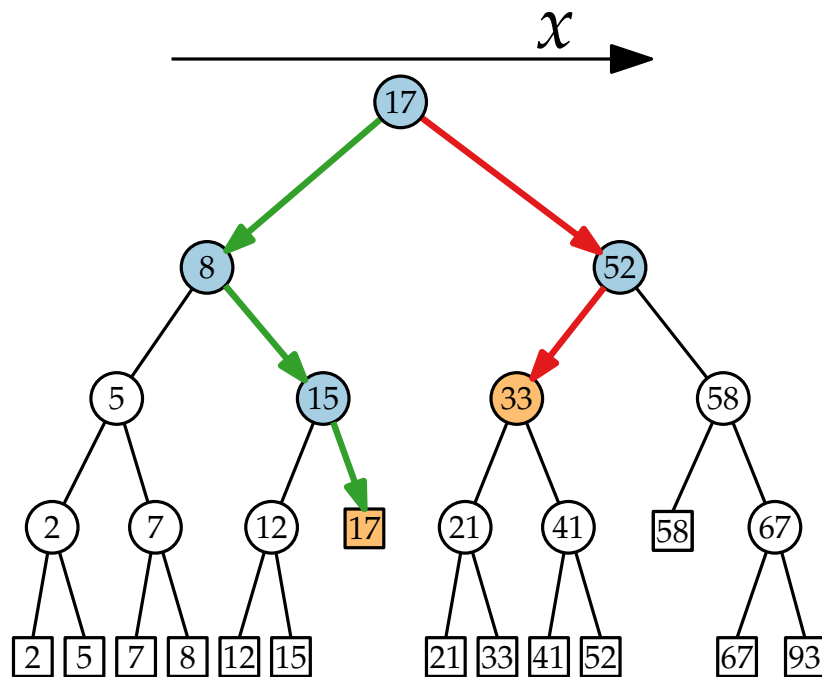
$$[16, 53] \times [18, 60]$$

Layered Range Trees



$$[16, 53] \times [18, 60] \rightarrow (21, 49), (33, 30), (52, 23)$$

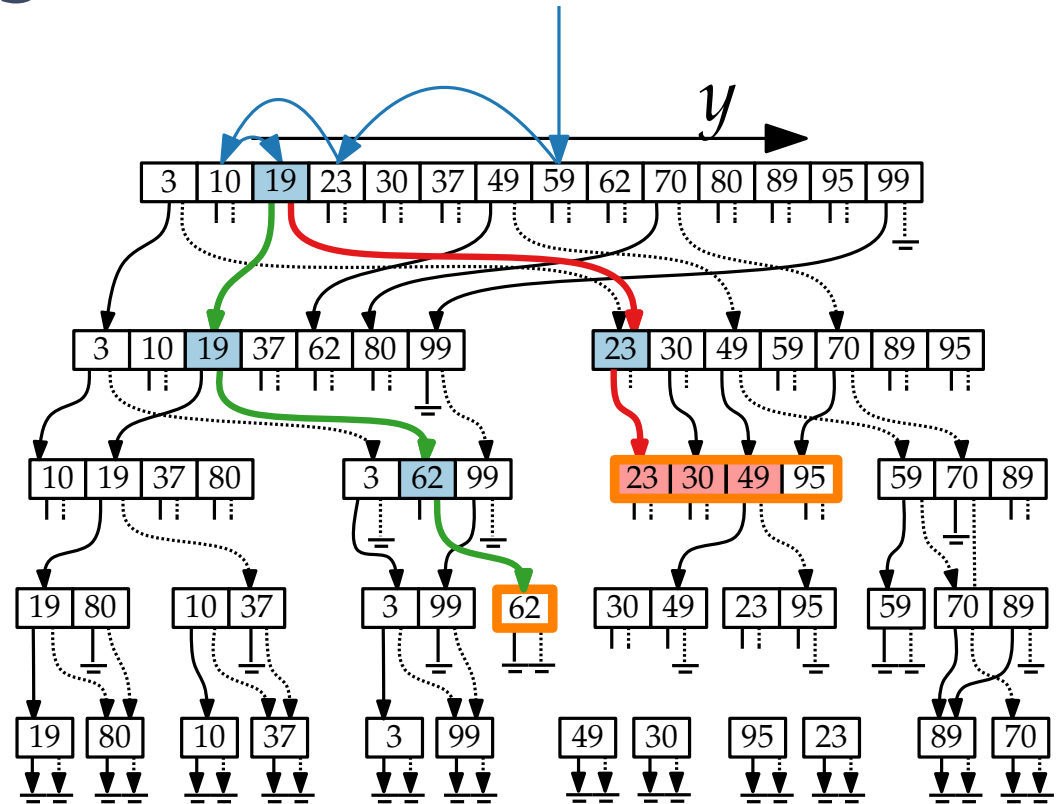
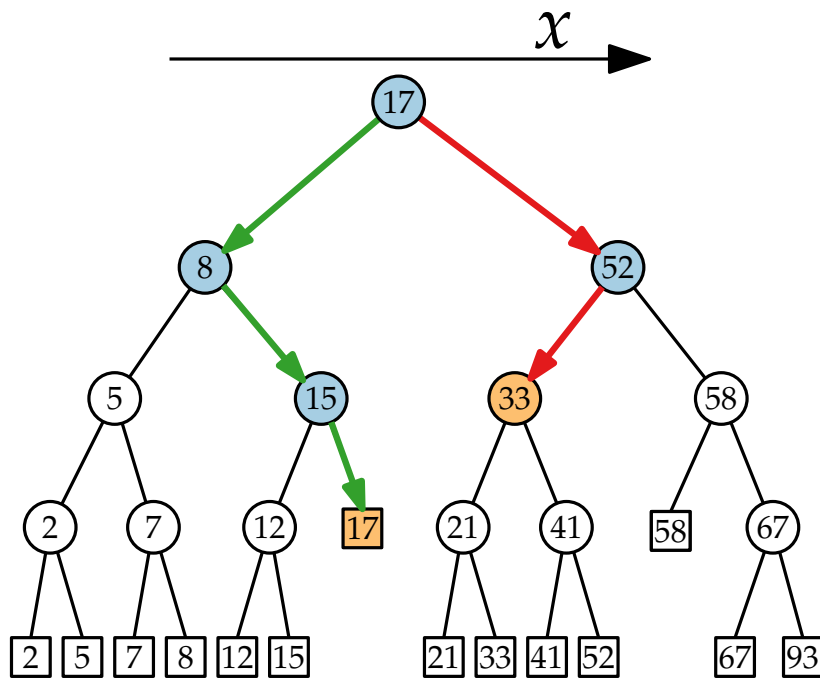
Layered Range Trees



$$[16, 53] \times [18, 60] \rightarrow (21, 49), (33, 30), (52, 23)$$

Theorem: Let $d \geq 2$ and let P be a set of n pts in \mathbb{R}^d . Given $O(n \log^{d-1} n)$ preprocessing time & storage, d -dim range queries on P can be answered in $O(k + \log^{d-1} n)$ time.

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