Computational Geometry

Lecture 5: Orthogonal Range Queries

or

Fast Access to Data Bases

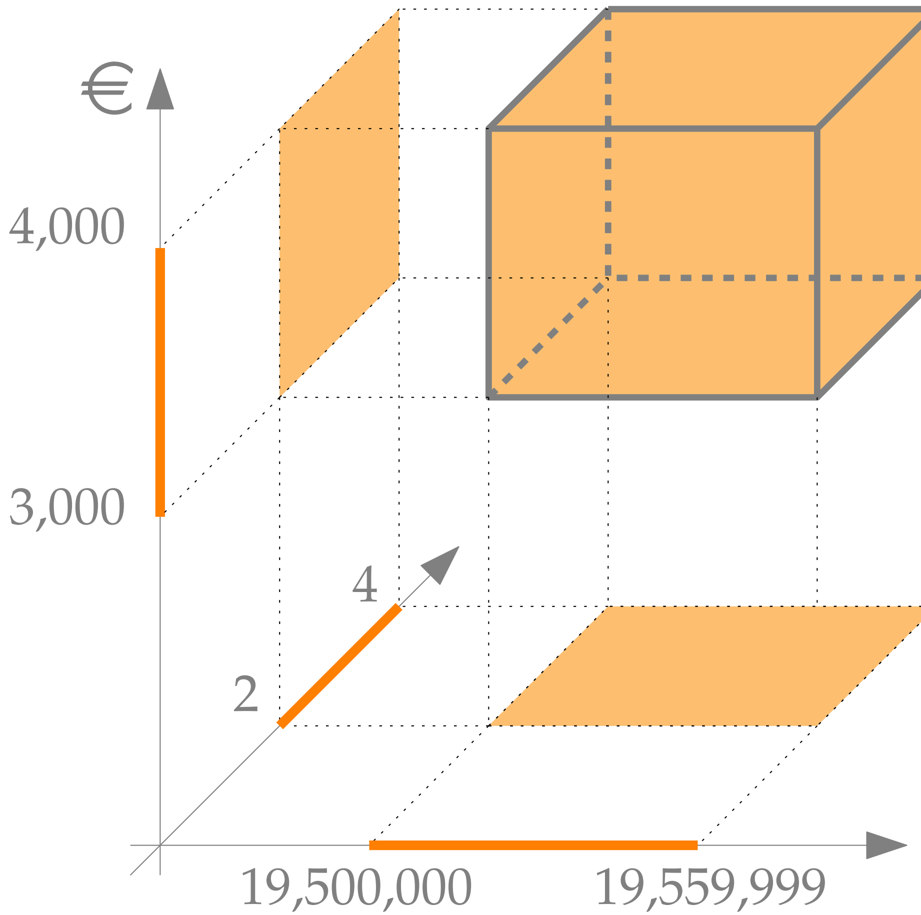
Part I: 1D Range Searching

Philipp Kindermann

Winter Semester 2020

Orthogonal Range Queries

Example: Personnel management in a company



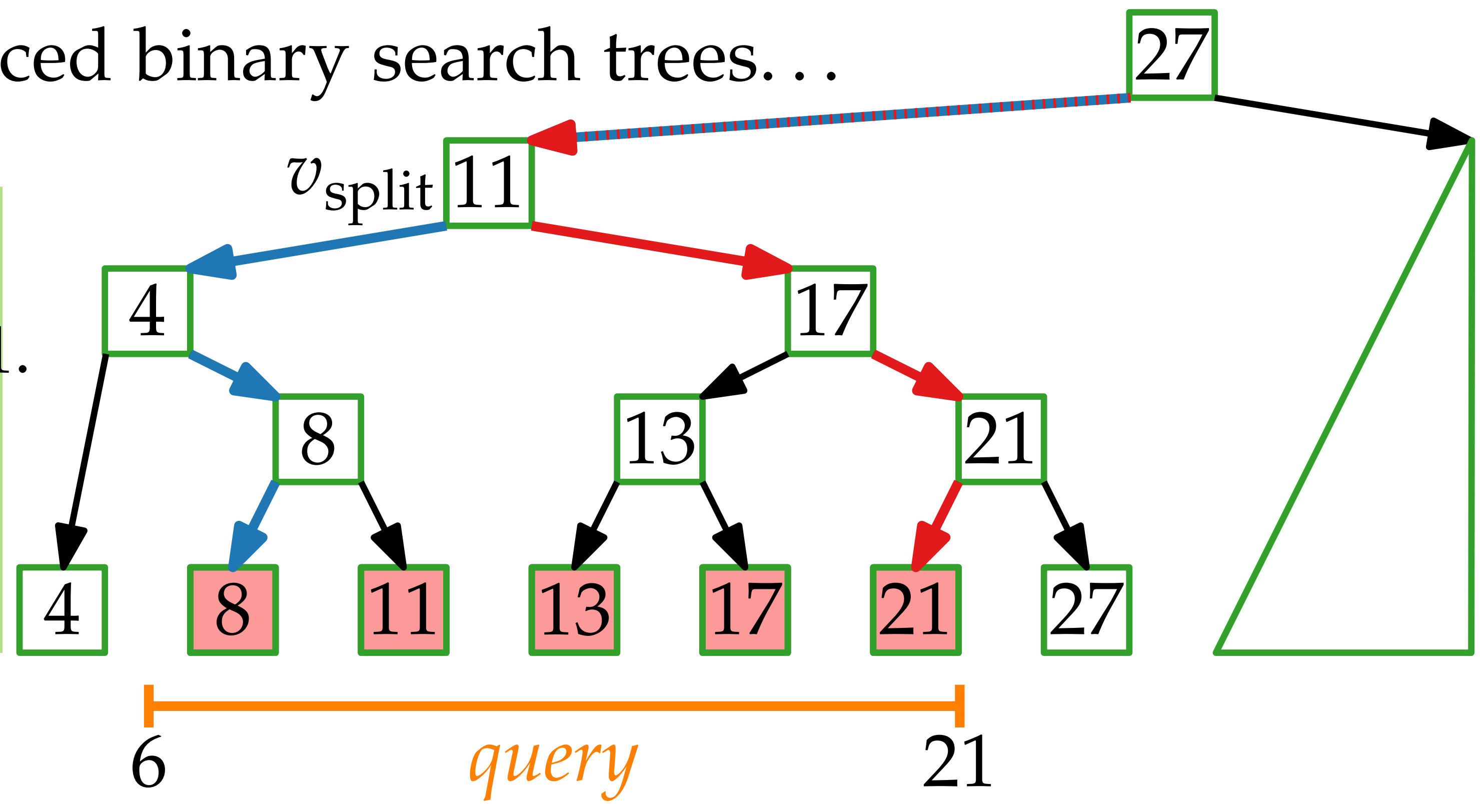
Typical queries for data bases!

1D Range Searching

Task: Preprocess a finite set $P \subset \mathbb{R}$ such that for any interval $[x, x']$ the set $P \cap [x, x']$ can be reported quickly.

Solution: balanced binary search trees. . .

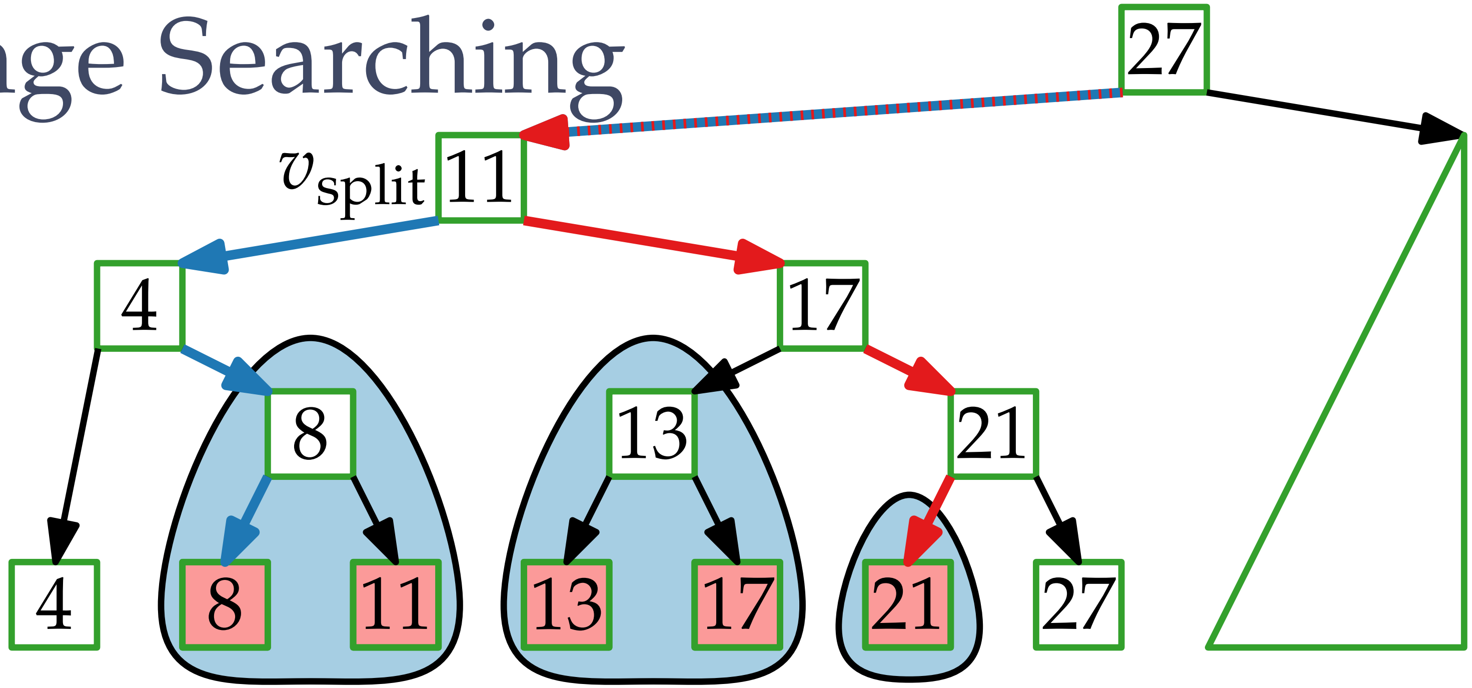
1. Search $x = 6$.
2. Search $x' = 21$.
3. Return all leaves 'inbetween".



Small changes: – keys only in leaves

– inner nodes store max. of their left subtrees

1D Range Searching



Observe: The result of a query is the disjoint union of at most $2h$ *canonical subsets*, where
– $h \in O(\log n)$ is the tree height,
– a canonical subset is an interval that contains all points stored in a subtree.

Theorem. A set of n real numbers can be preprocessed in $O(n \log n)$ time and $O(n)$ space such that 1D range queries take $O(k + \log n)$ time, where $k = |\text{output}|$.

output sensitive!

Extensions to 2D

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

Solutions:

- *one* tree; query path alternates between x- and y-coord. } *kd-tree*
- first-level tree for x-coordinates; many second-level trees for y-coord. } *range tree*

Assume: *General position!*

Here: no two points have the same x- or y-coordinate.

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Part II:
Kd-Trees: Contruction

Philipp Kindermann

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Extensions to 2D

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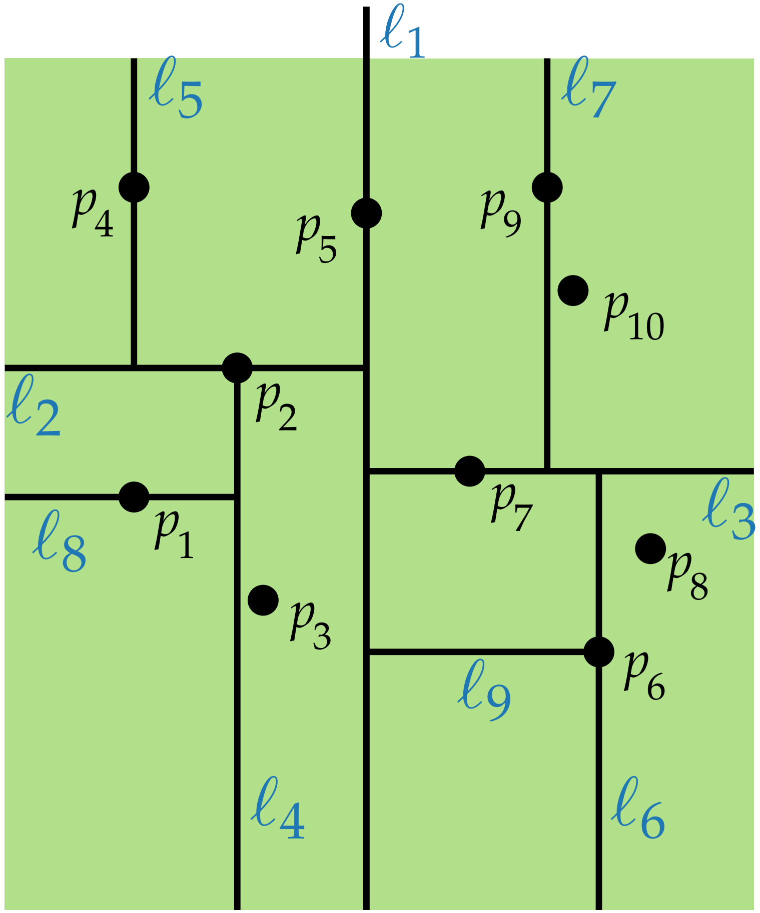
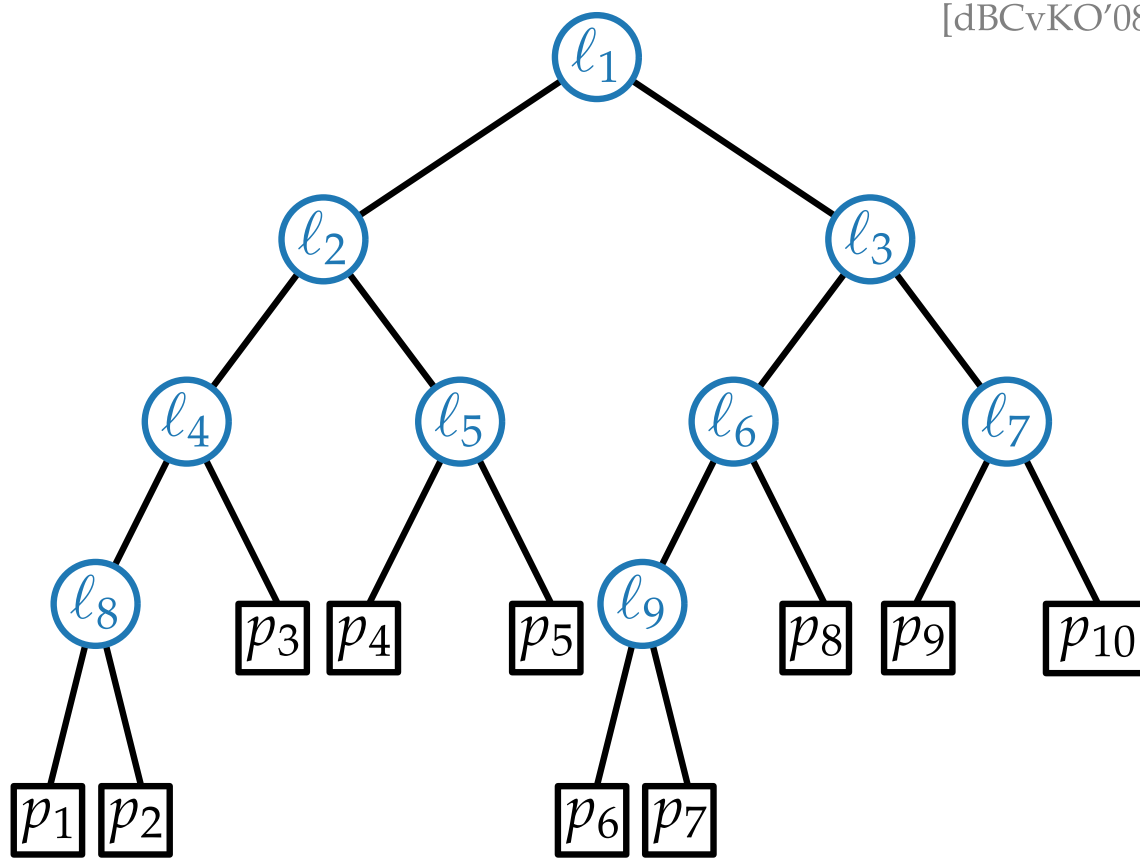
Kd-Trees: Example

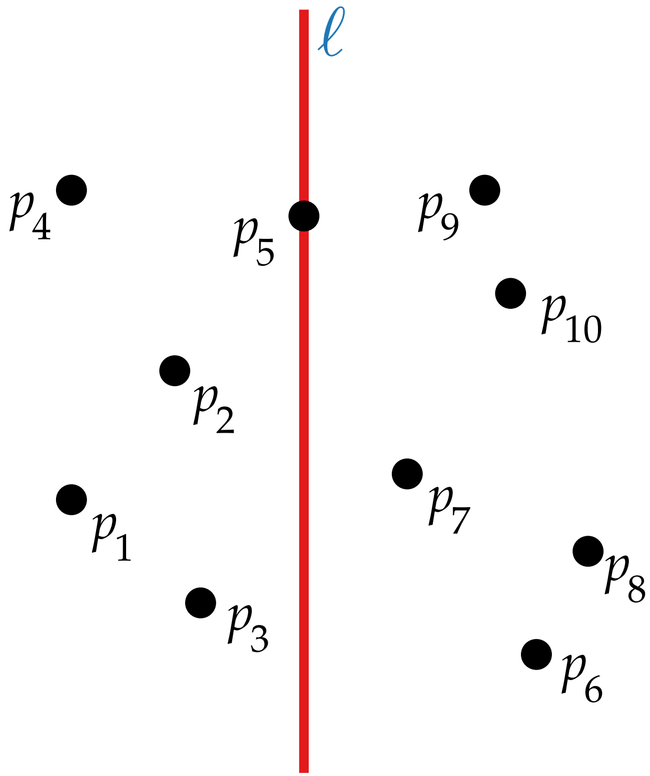
[dBCvKO'08]

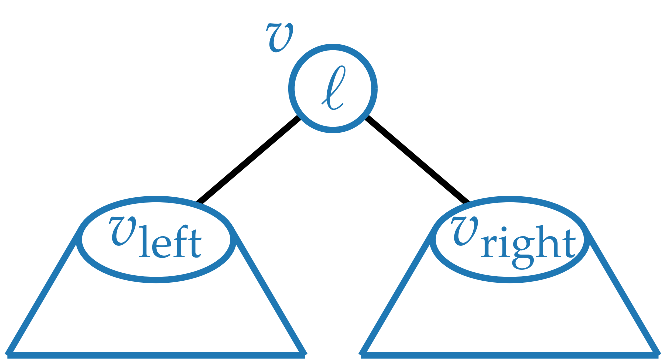




- Split any region that contains more than one point.
- Horizontal split lines/segm. belong to the region below.
  Vertical left.

Kd-Trees: Construction





Pseudo-code:

BuildKdTree(points P, int *depth*)
- **if** $|P| = 1$ **then**
  - **return** (leaf storing the pt in P)
- **else**
  - **if** *depth* is even **then**
    - split P with the vertical line $\ell\colon x = x_{\text{median}(P)}$ into P_1 (pts left of or on ℓ) and $P_2 = P \setminus P_1$
  - **else**
    - split P horizontally...
  - $v_{\text{left}} \leftarrow \text{BuildKdTree}(P_1, \textit{depth} + 1)$
  - $v_{\text{right}} \leftarrow \text{BuildKdTree}(P_2, \textit{depth} + 1)$
  - create a node v storing ℓ
  - make v_{left} and v_{right} the children of v
  - **return** (v)

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Part III:
Kd-Trees: Analysis & Querying

Philipp Kindermann

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Kd-Trees: Analysis

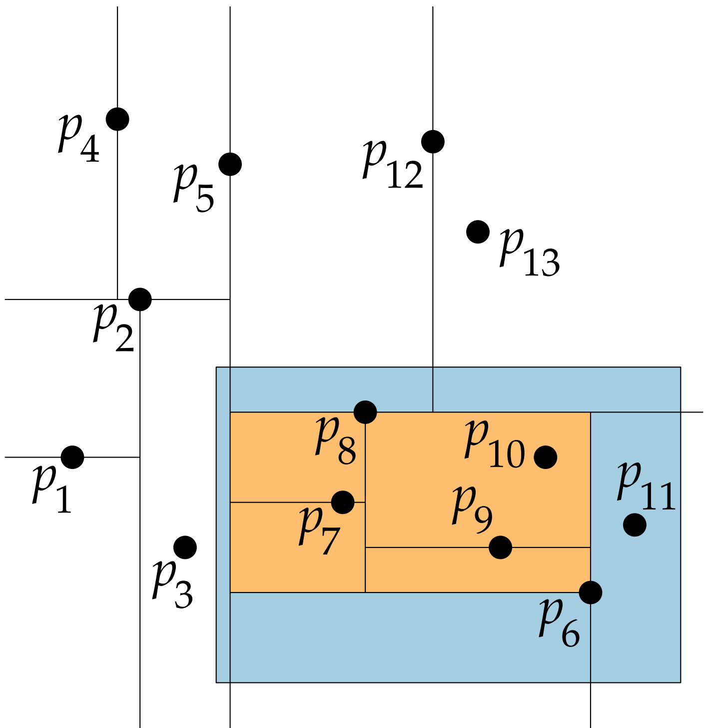
Construction time?

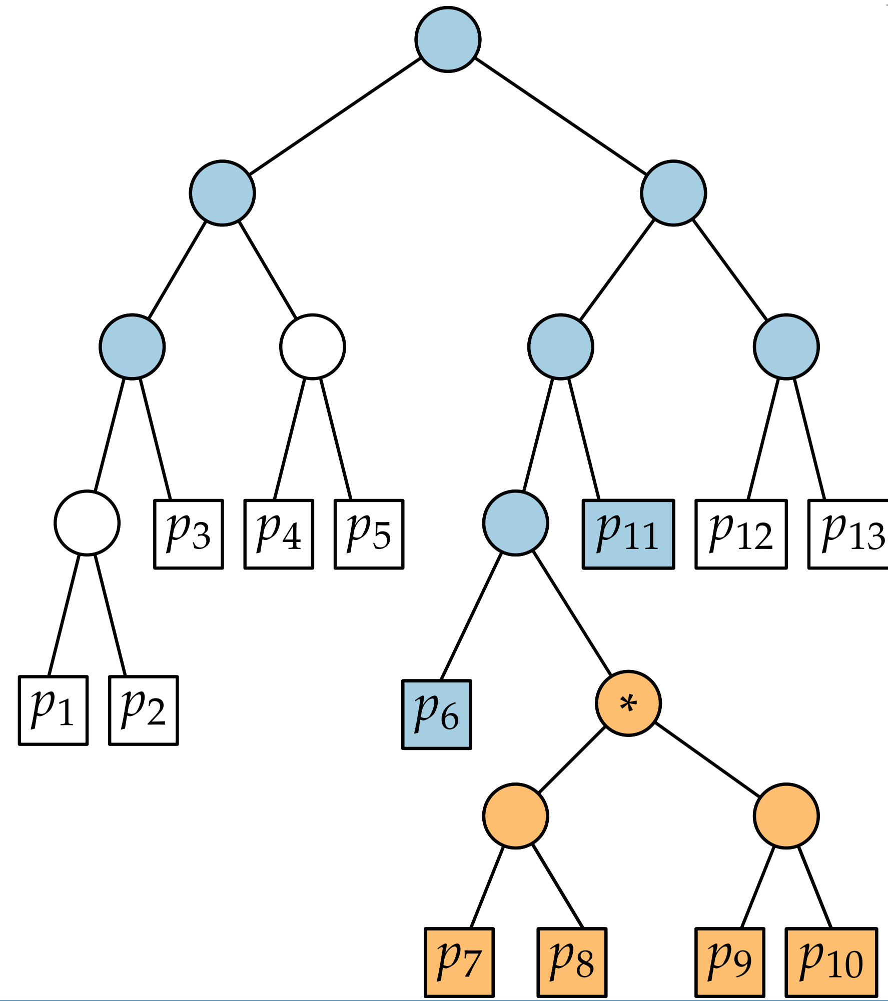
see Mergesort!

$$T(n) = \left.\begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\lceil n/2 \rceil) & \text{else.} \end{cases}\right\} = O(n \log n)$$

Lemma. A kd-tree for a set of n pts in the plane takes $O(n \log n)$ time to construct and uses $O(n)$ storage.

Kd-Trees: Querying





Lemma. Querying a kd-tree for n pts in the plane with an axis-parallel rectangle R takes $O(k + \sqrt{n}\,)$ time, where $k = |\text{output}|$.

regions intersected by query boundary:

$$Q(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2 + 2Q(\lceil n/4 \rceil) & \text{else.} \end{cases} = O(\sqrt{n})$$

in $\mathbb{R}^d$

$O(k + n^{1-1/d})$

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Part IV:
Range Trees: Querying & Construction

Philipp Kindermann

Winter Semester 2020

Extensions to 2D

Task: Preprocess a finite set $P \subset \mathbb{R}^2$ such that for any range query $R = [x, x'] \times [y, y']$ the set $P \cap R$ can be reported quickly.

Solutions:

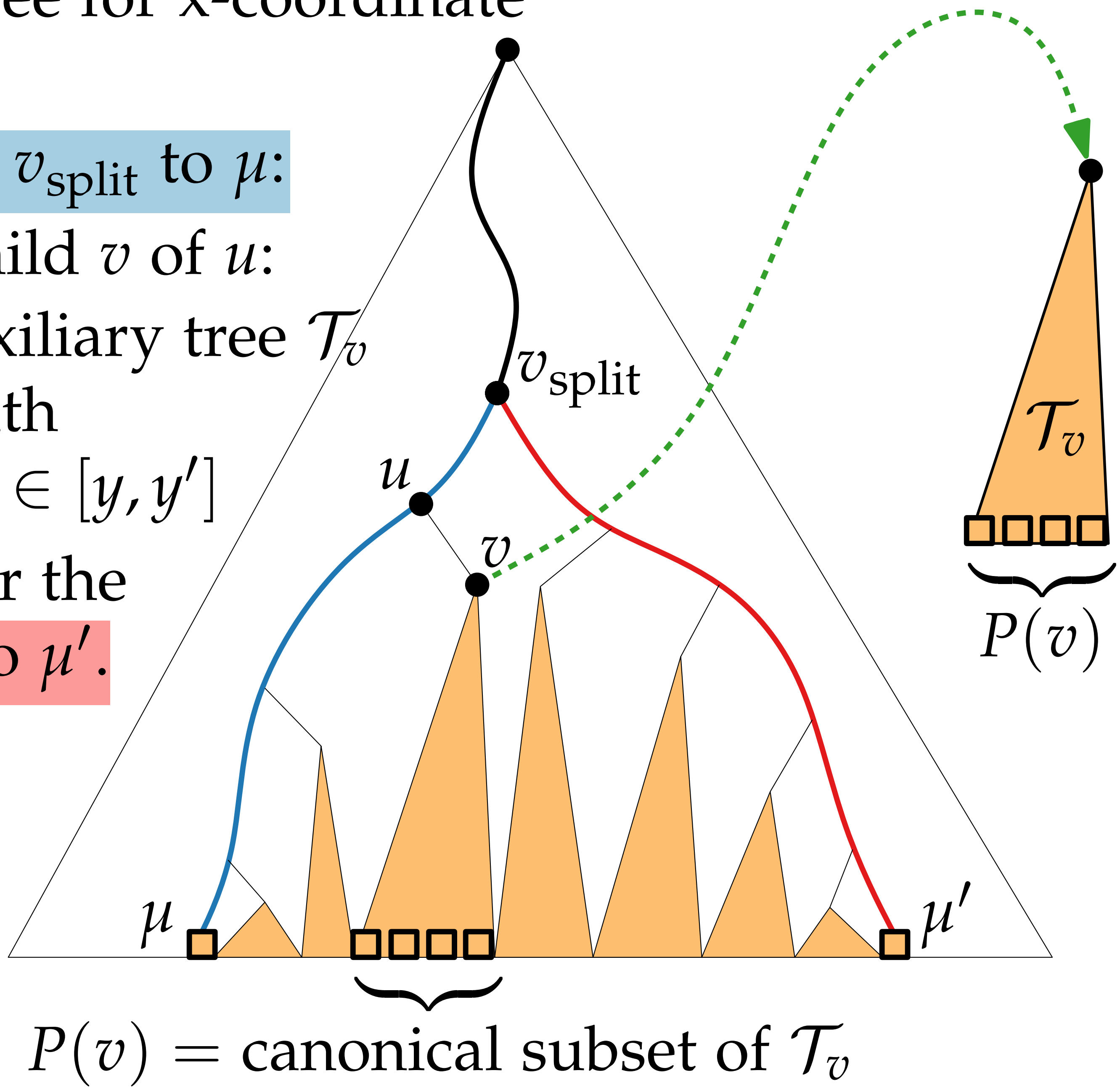
- *one* tree; query path alternates between x- and y-coord. } *kd-tree*
- first-level tree for x-coordinates; many second-level trees for y-coord. } *range tree*

Assume: *General position!*

Here: no two points have the same x- or y-coordinate.

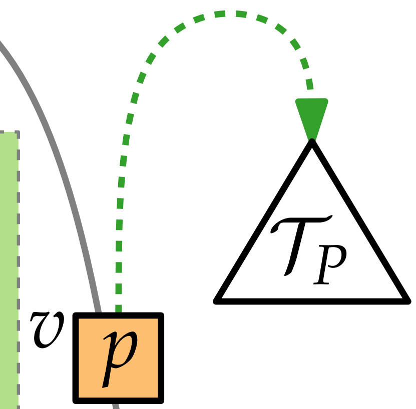
Range Trees: Query Algorithm

1. Search in main tree for x-coordinate
2. For each node u
   on the path from v_{split} to μ:
   For the right child v of u:
   Search in auxiliary tree $\mathcal{T}_v$
   for points with
   y-coordinate $\in [y, y']$
3. Symmetrically for the
   path from v_{split} to μ'.

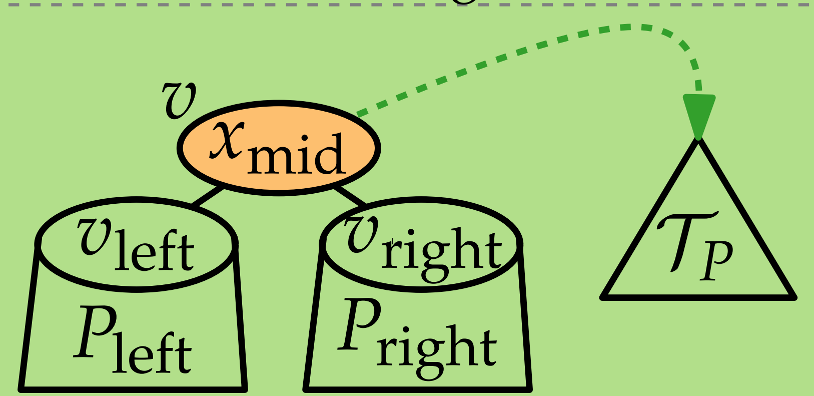


Range Trees: Construction

Build2DRangeTree(point[ ] P)
construct 2nd-level tree $\mathcal{T}_P$ on P (y-order)
if $P = \{p\}$ **then**
create leaf v:



else
x_{mid} = median x-coordinate of P
P_{left} = pts in P with x-coordinate $\leq x_{\text{mid}}$
P_{right} = $>$
v_{left} = Build2DRangeTree(P_{left})
v_{right} = Build2DRangeTree(P_{right})
create node v:



return v

Running time?
$O(n \log n)$:-(

Better:
Pre-sort once, then build tree bottom-up in linear time.
$\Downarrow$
Total construction time $O(n \log n)$

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Part V:
Range Trees: Analysis

Philipp Kindermann

Winter Semester 2020

Range Trees: Space Consumption

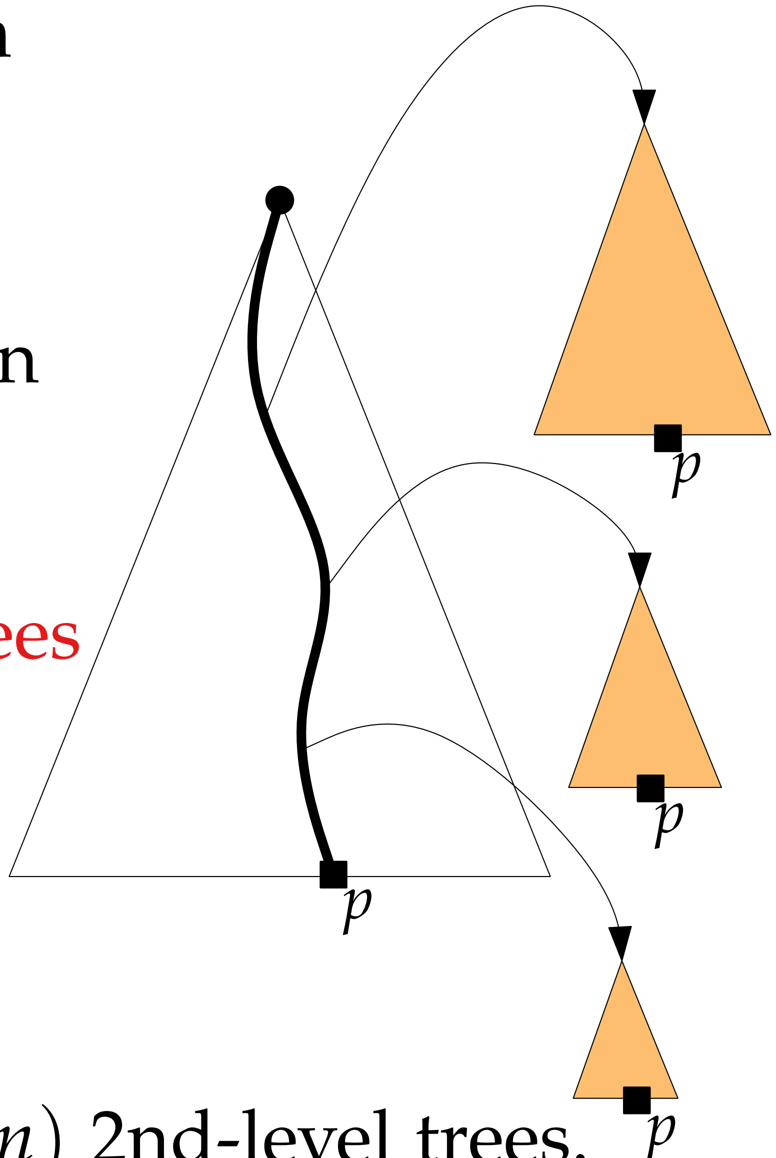
Each node v of the 1st-level tree has a pointer to a 2nd-level tree $\mathcal{T}_v$ with $|\mathcal{T}_v| = \Theta(|P(v)|)$.

Q: What's the *total* space consumption of all 2nd-level trees?

What's your guess:

- $\Theta(n^2)$,
- $\Theta(n \log n)$,
- $\Theta(n \log^2 n)$, or
- $\Theta(n)$?

How many trees will contain a given point?



A: Each $p \in P$ is stored in $h = \Theta(\log n)$ 2nd-level trees.

$\Rightarrow$ $\Theta(n \log n)$ space

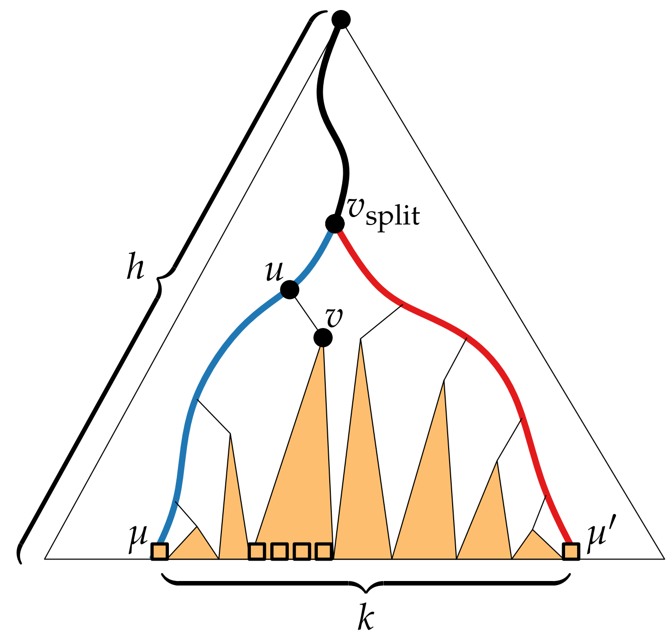
Range Trees: Query time

$$T(n,k) = \sum_{u \in \text{paths to } \mu \text{ and } \mu'} O(k_u + \log n)$$

$$= O(\textstyle\sum_u k_u) + O(\textstyle\sum_u \log n)$$

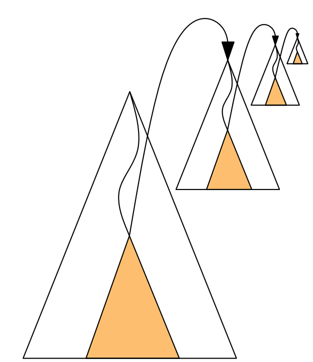
$$= O(k) \qquad + 2h \cdot O(\log n)$$

$$= O(k + \log^2 n)$$



$\mathbb{R}^d$? $O(n \log^{d-1} n)$ storage and construction time

$O(k + \log^d n)$ query time



See Chapter 5.4 in Comp. Geom A&A

Comparison

	kd-tree	range tree
construction time	$O(n \log n)$	$O(n \log n)$
storage	$O(n)$	$O(n \log n)$
query time	$O(k + \sqrt{n})$	$O(k + \log^2 n)$

Note: *trade-off* between space and query time

General Sets of Points

Idea: use *composite numbers* $(a|b)$ with lex order

$p = (x, y)$ ➞ $\hat{p} = ((x|y),\ (y|x))$ ➞ *unique* coordin.

range $R = [x, x'] \times [y, y']$

↓

$$\hat{R} = [(x|-\infty), (x'|+\infty)] \times [(y|-\infty), (y'|+\infty)]$$

Show: $p \in R \;\Leftrightarrow\; \hat{p} \in \hat{R}$

This removes our assumption about the input points being in general position.

We can use kd-trees and range trees for *any* set of points; no matter how many points have the same x- or y-coord.

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Part VI:
Fractional Cascading

Philipp Kindermann Winter Semester 2020

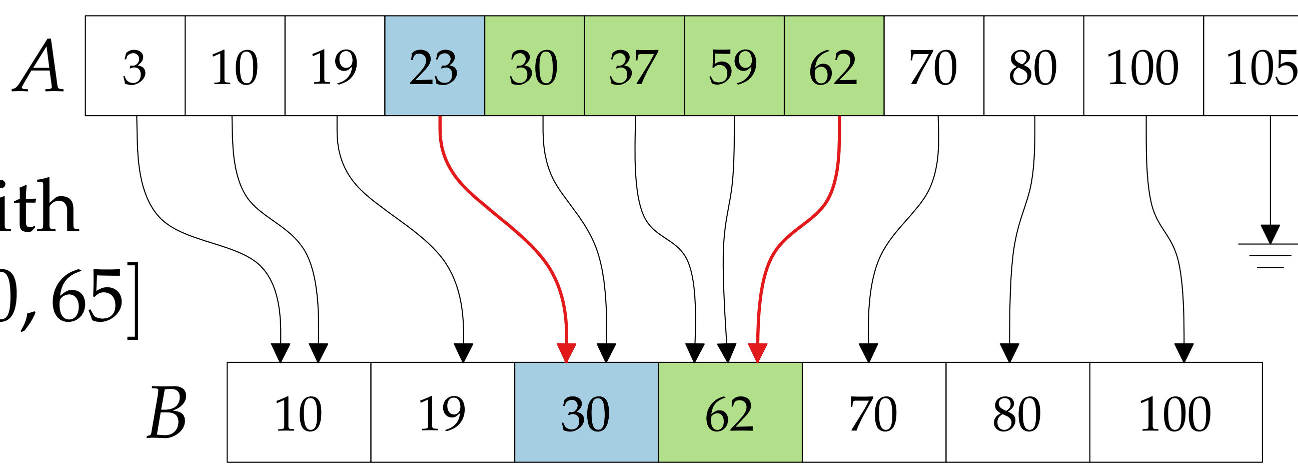
Fractional Cascading

Task 1: Given sets $B \subset A \subset \mathbb{N}$ stored in sorted order in arrays $A[1..n]$ and $B[1..m]$, support 1d range queries in the multiset $A \cup B$ in $k + 1 \cdot \log n$ time!

We allow $n \log m$ bits extra space.

Task 2: Speed up 2d range queries: $O(k + \log^2 n) \to O(k + \log n)$ time!

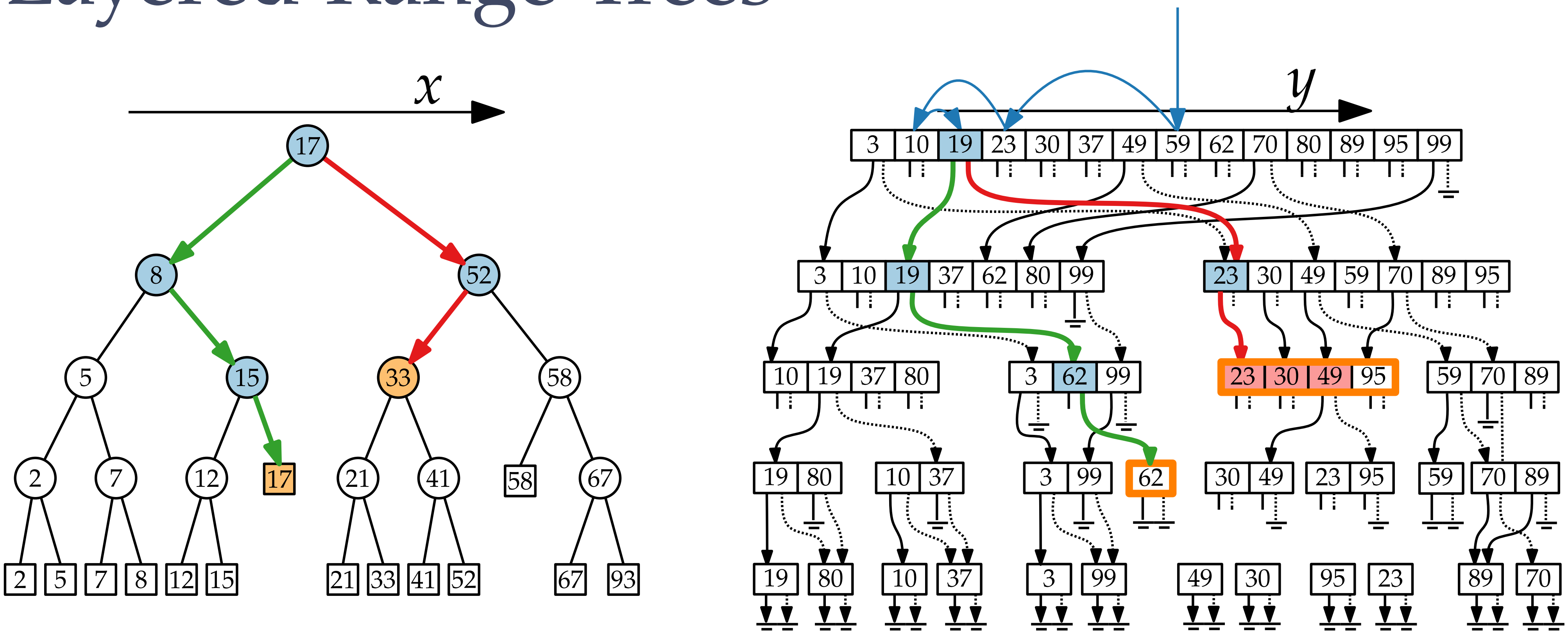
query with range $[20, 65]$



link $a \in A$ with smallest $b \geq a$ in B

[dBCvKO'08]

Layered Range Trees



$[16, 53] \times [18, 60] \rightarrow (21, 49), (33, 30), (52, 23)$

Theorem: Let $d \geq 2$ and let P be a set of n pts in $\mathbb{R}^d$. Given $O(n \log^{d-1} n)$ preprocessing time & storage, d-dim range queries on P can be answered in $O(k + \log^{d-1} n)$ time.