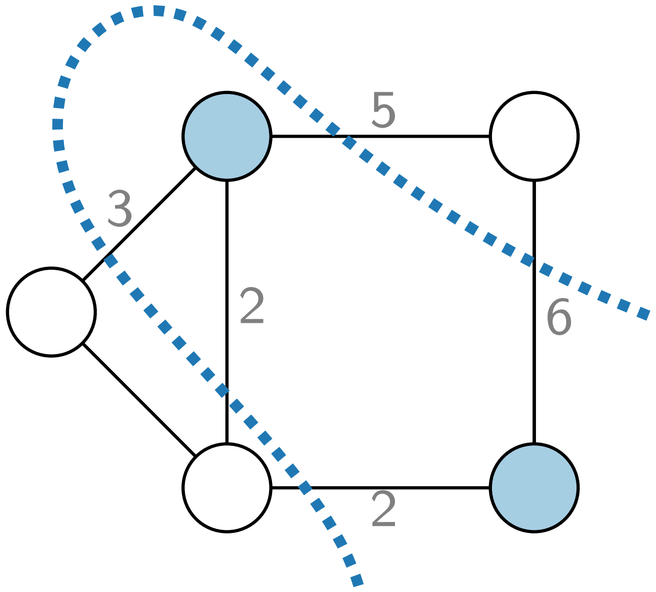
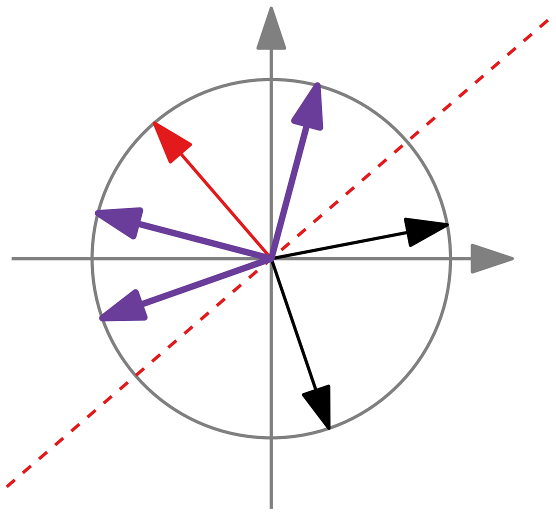
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

QP-Relaxation
for Max Cut

Jonathan Klawitter · WS20

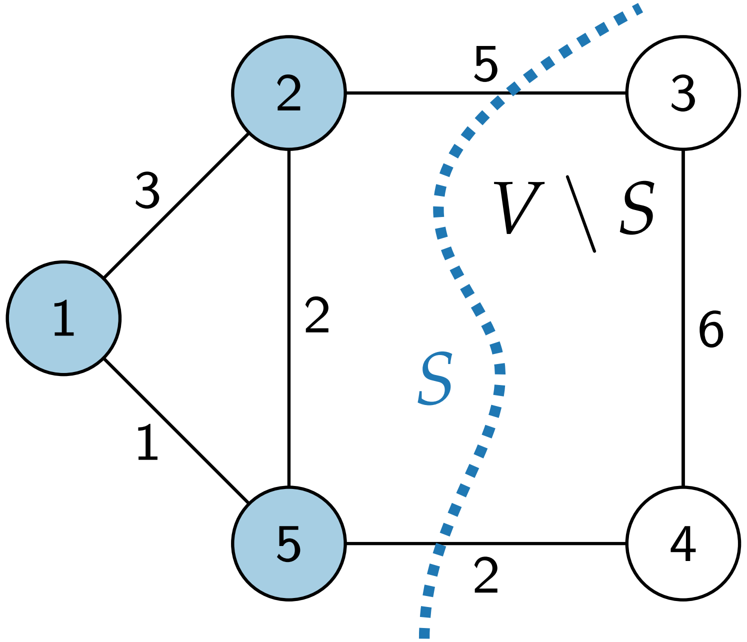




Cut

- Let $G = (V, E)$ be a graph with edge weights $c\colon E \to \mathbb{N}$.
- A **cut** of G is a partition $(S, V \setminus S)$ of V.
- The **weight** of a cut $(S, V \setminus S)$ is

$$c(S, V \setminus S) = \sum_{\substack{uv \in E, \\ u \in S, v \in V \setminus S}} c(uv)$$

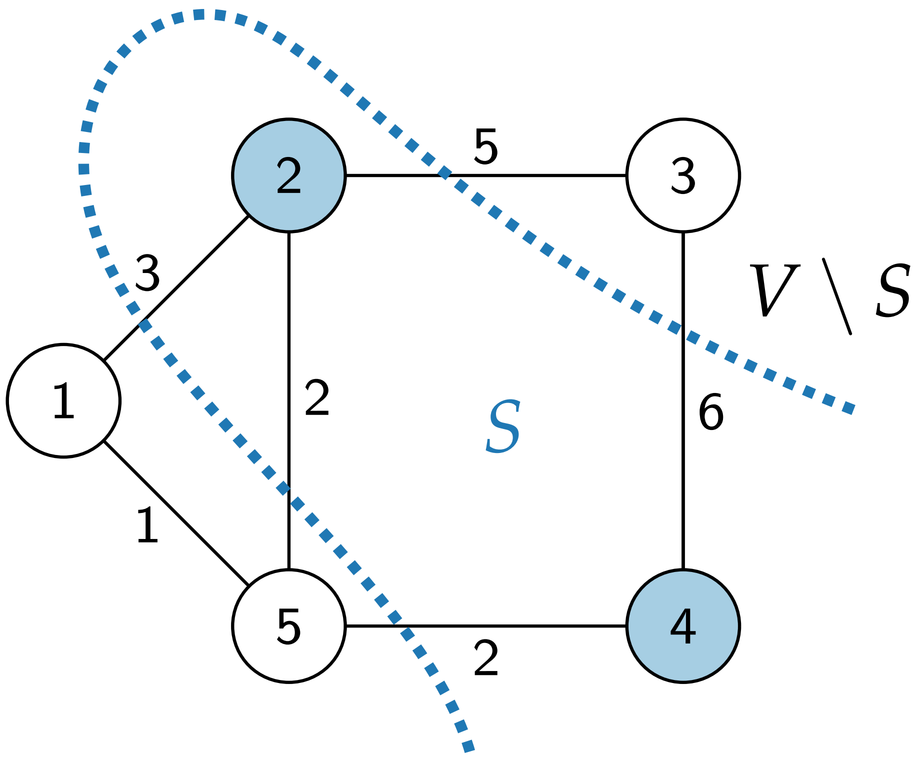


$$c(\{1, 2, 5\}, \{3, 4\}) = 7$$

The **MaxCut** Problem

Input. Graph $G = (V, E)$, edge weights $c\colon E \to \mathbb{N}$.

Output. Cut $(S, V \setminus S)$ of G with **maximum** weight.



$c(S, V \setminus V) = 18$

The **MaxCut** Problem

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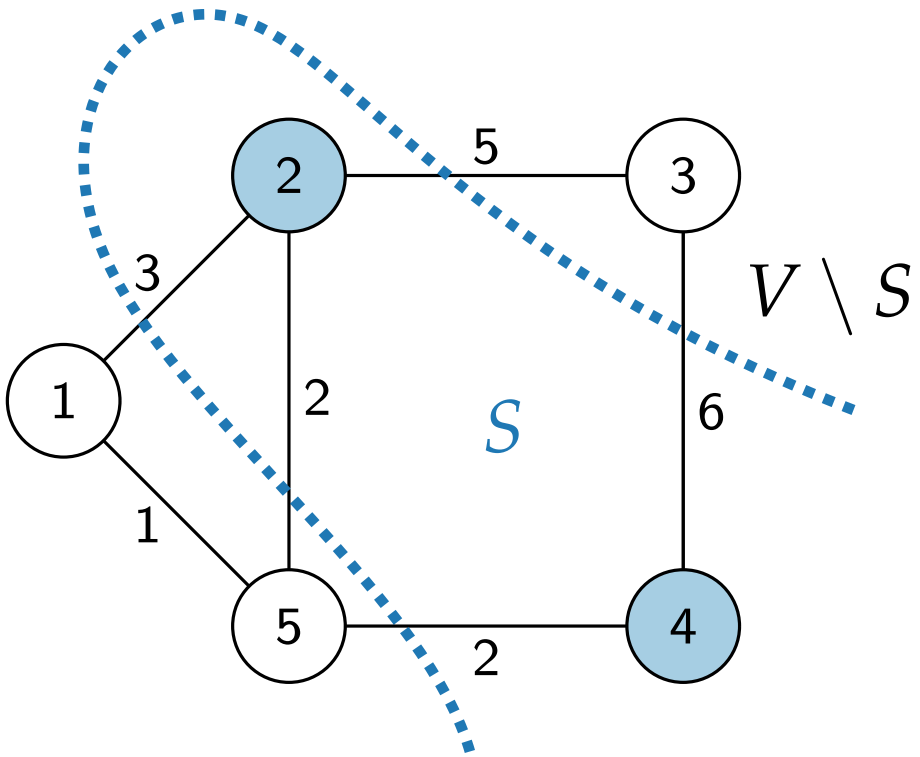
- MaxCut is NP-hard.



$c(S, V \setminus V) = 18$

Randomized 0.5-approximation for (unweighted) MaxCut

$\textsc{CoinFlipMaxCut}(G, c\colon E \to 1)$

$S \leftarrow \emptyset$
foreach $v \in V$ **do**
  if coin flip shows $\textsc{Heads}$ **then**
    $S \leftarrow S \cup \{v\}$
return $c(S, V \setminus S), S$

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Theorem 1.
CoinFlipMaxCut is a randomized 0.5-approximation algorithm for MaxCut.

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- Runs in $O(n+m)$.

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$$\mathrm{E}[c(\textsc{CoinFlipMaxCut}(G))]$$

$$\geq \frac{1}{2}\mathrm{OPT}(G)$$

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- Can be "derandomized". Exercise.

LP-Relaxation

Integer Linear Program

$$
\begin{array}{lrcr}
\textbf{maximize} & c^\intercal x & & \\
\textbf{subject to} & Ax & \leq & b \\
 & x & \geq & 0 \\
 & x & \in & \mathbb{Z}
\end{array}
$$

LP-Relaxation

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Solve in polynomial time

Solution for LP

$x^\star$

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← e.g. rounding

Assignment for ILP

$x^\star$

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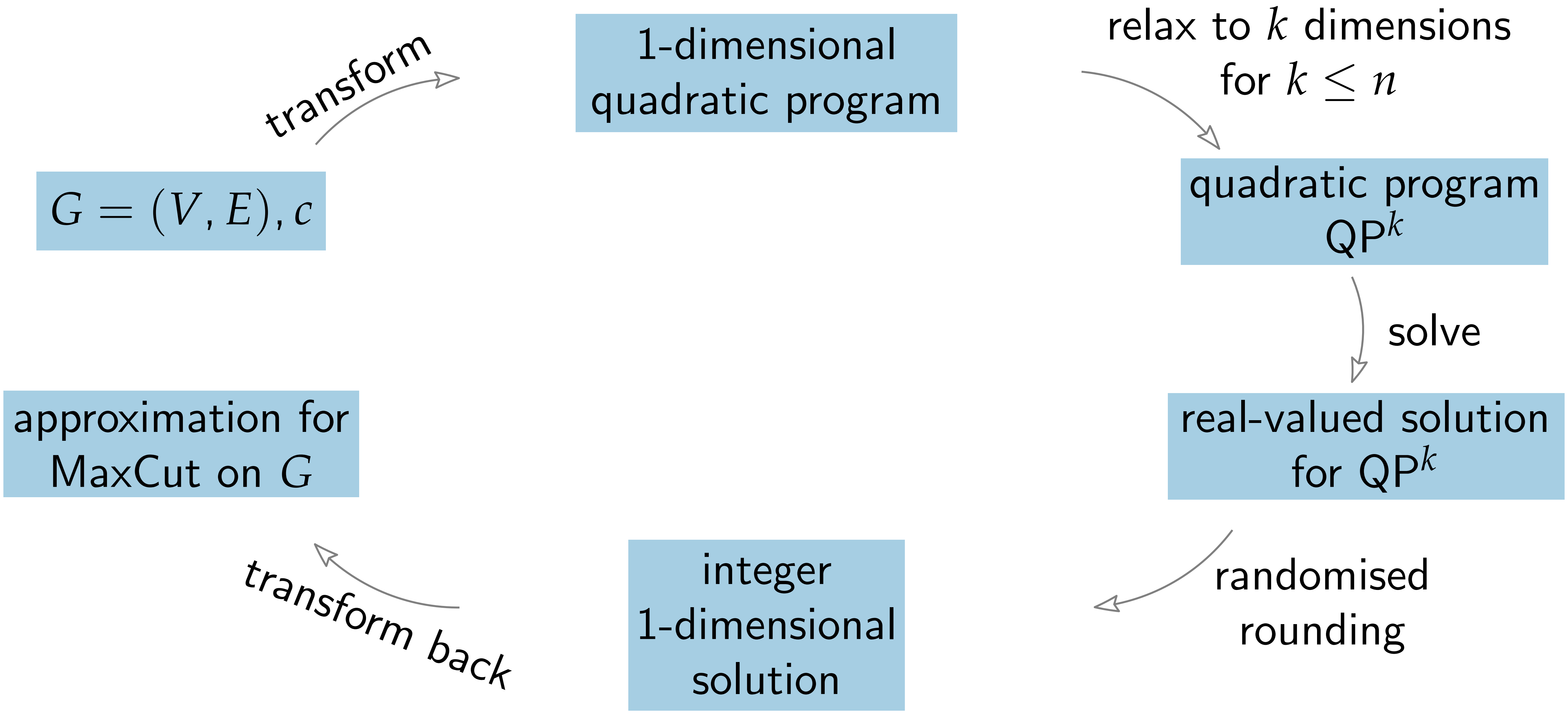
Solution, approximation, or bound

Goemans-Williamson algorithm for MaxCut



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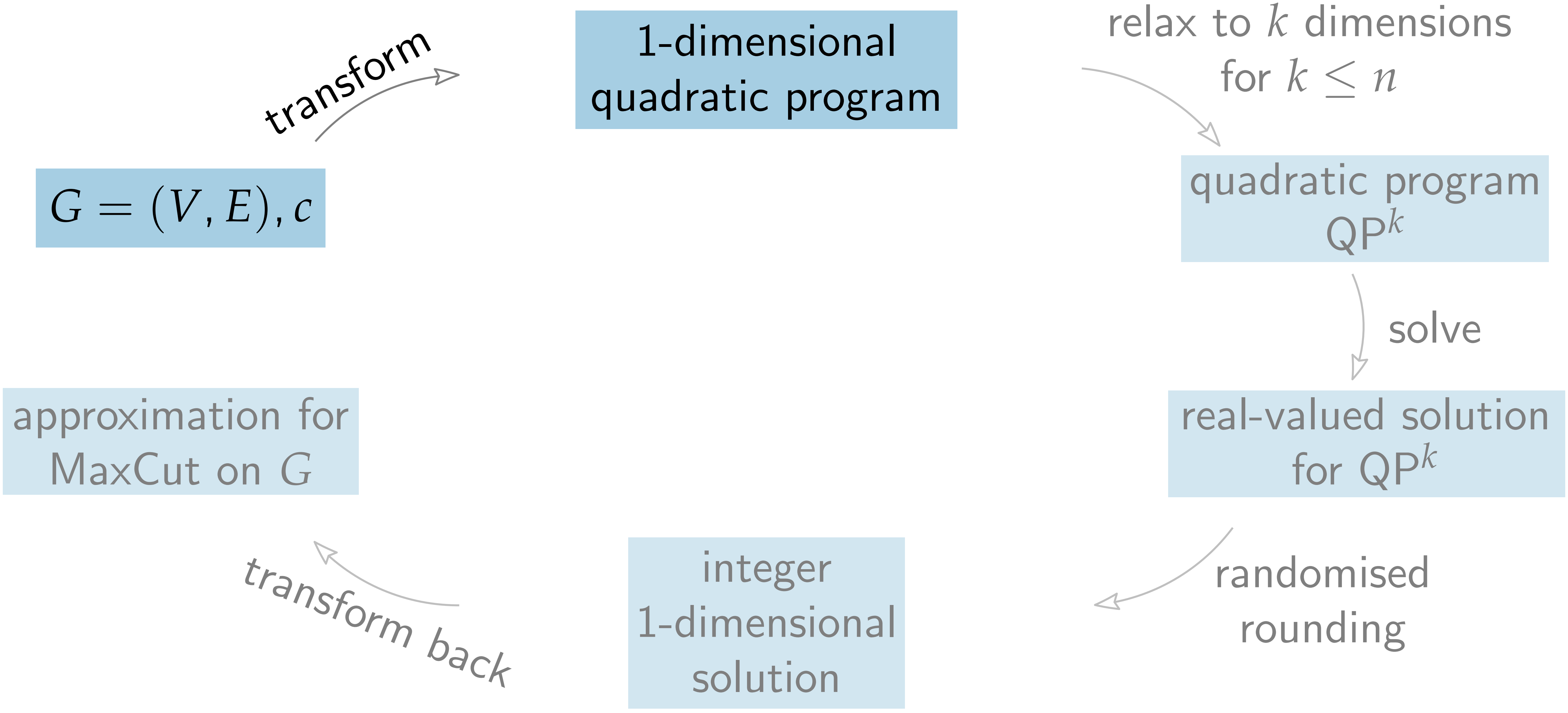

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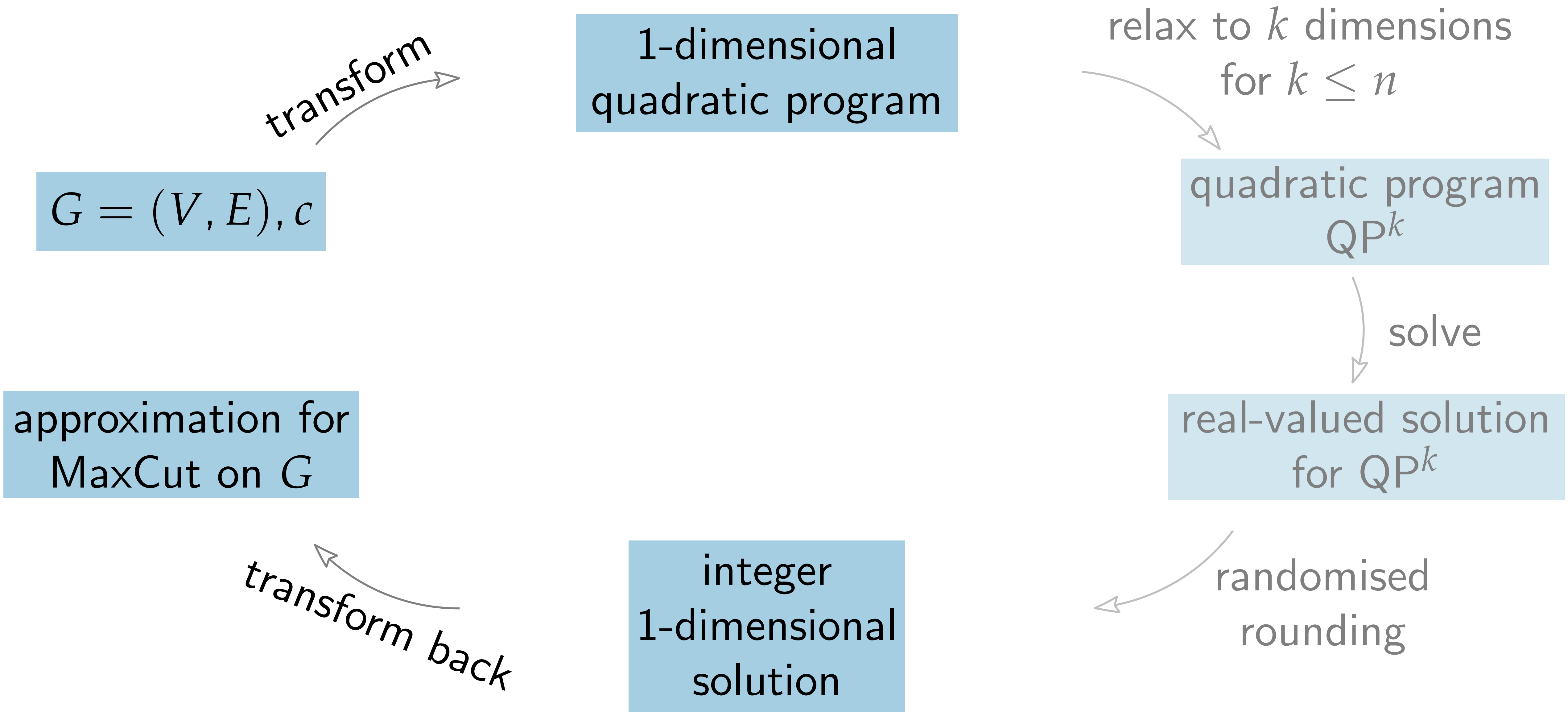
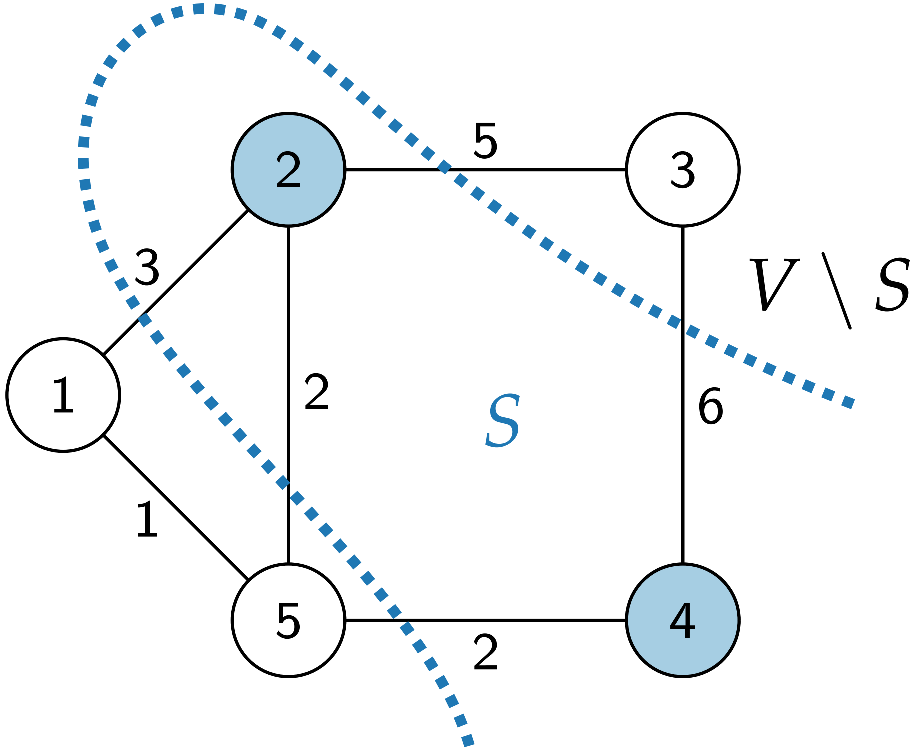


QP(G, c)

Idea.



QP(G, c)

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subject to

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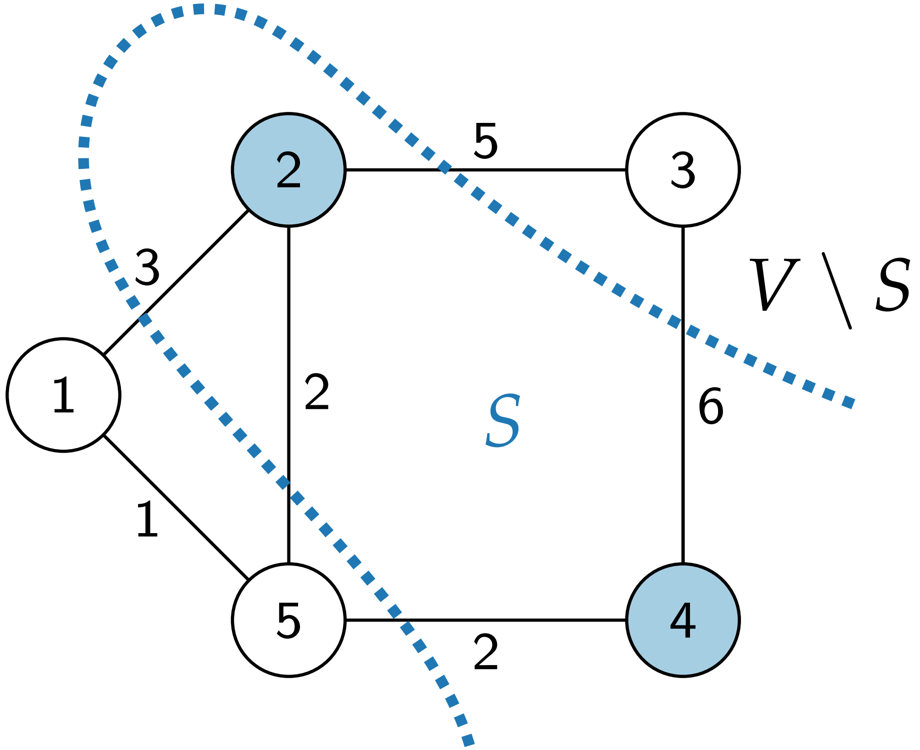
Idea.

- Indicator variables $x_i \in \{1, -1\}$



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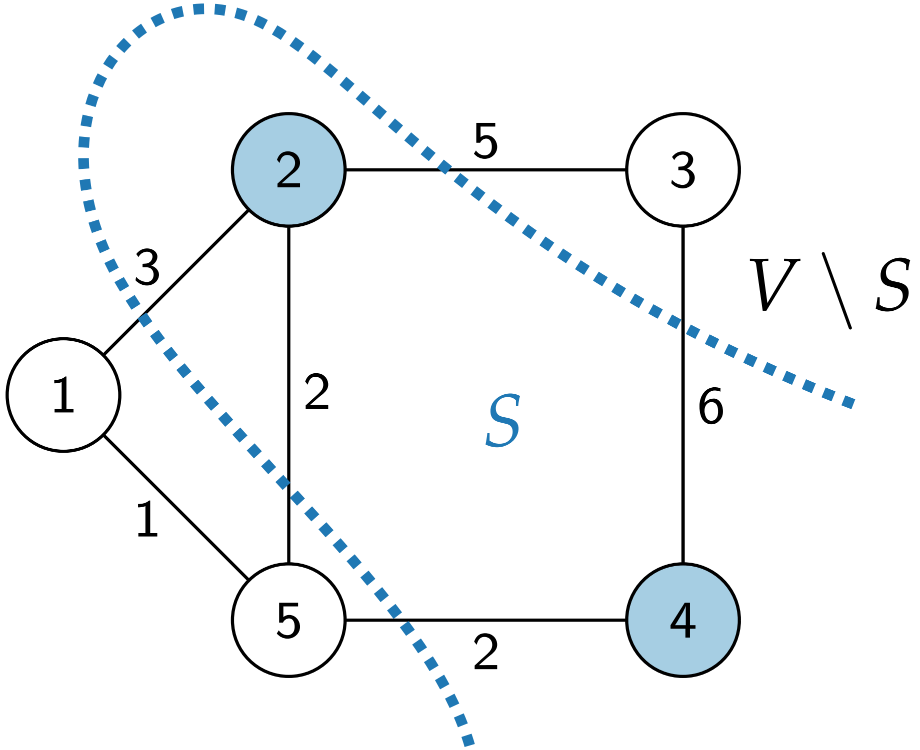


QP(G, c)

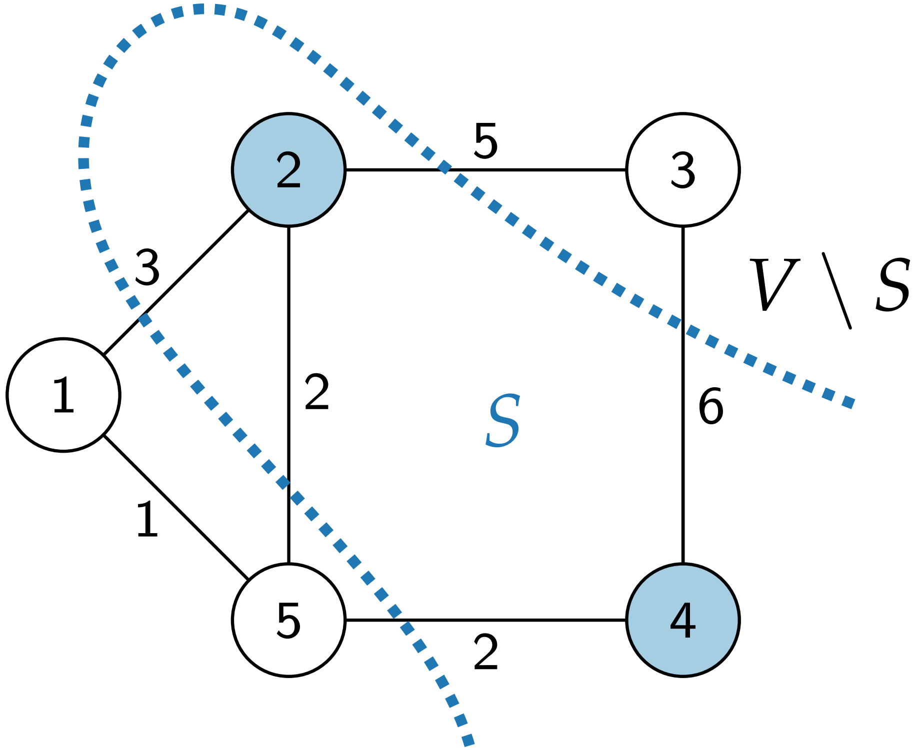
maximize

subject to $x_i^2 = 1$

QP(G, c)

Idea.

- Indicator variables $x_i \in \{1, -1\}$
- $x_i x_j = \begin{cases} 1 & \text{if } i, j \text{ in same partition} \\ -1 & \text{otherwise} \end{cases}$



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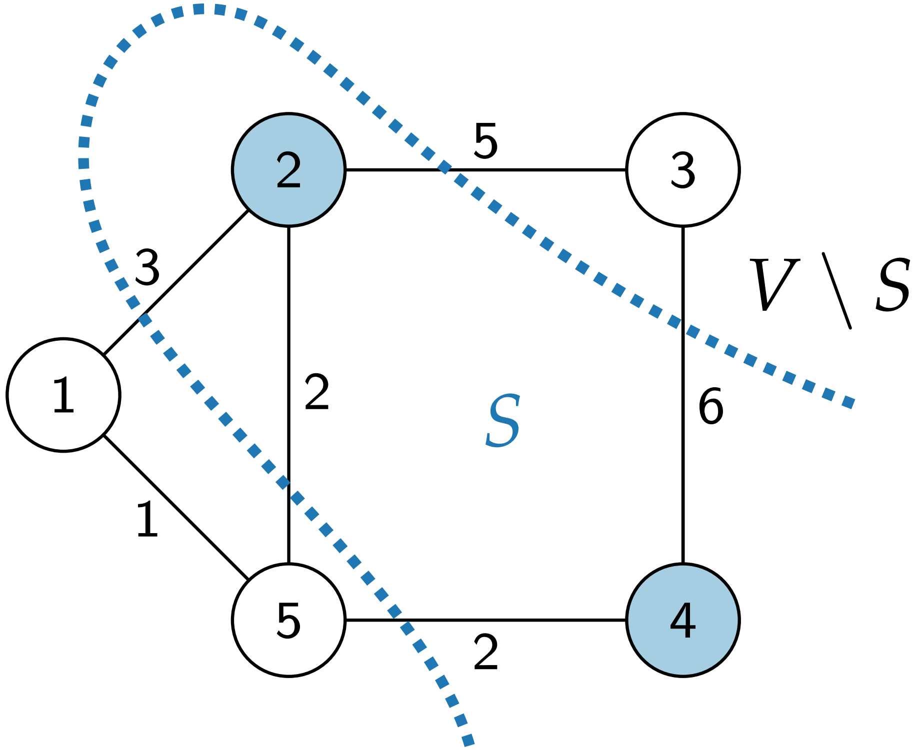

QP(G, c)

maximize $(1 - x_i x_j)$

subject to $x_i^2 = 1$

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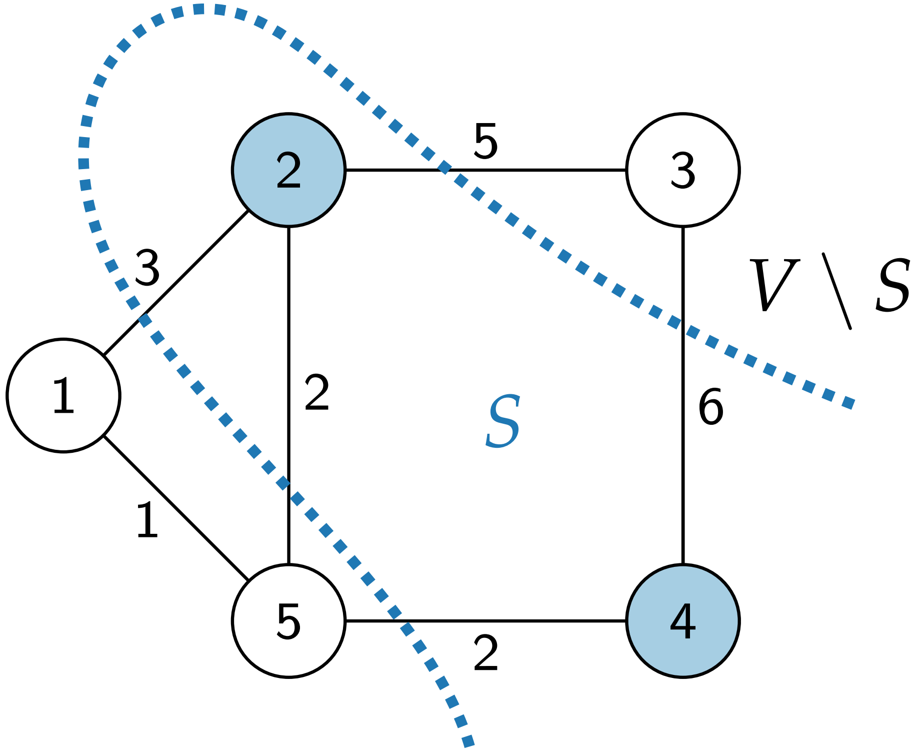

- Weight matrix c_{ij}

	1	2	3	4	5
1		3			1
2	3		5		2
3		5		6	
4			6		2
5	1	2		2	

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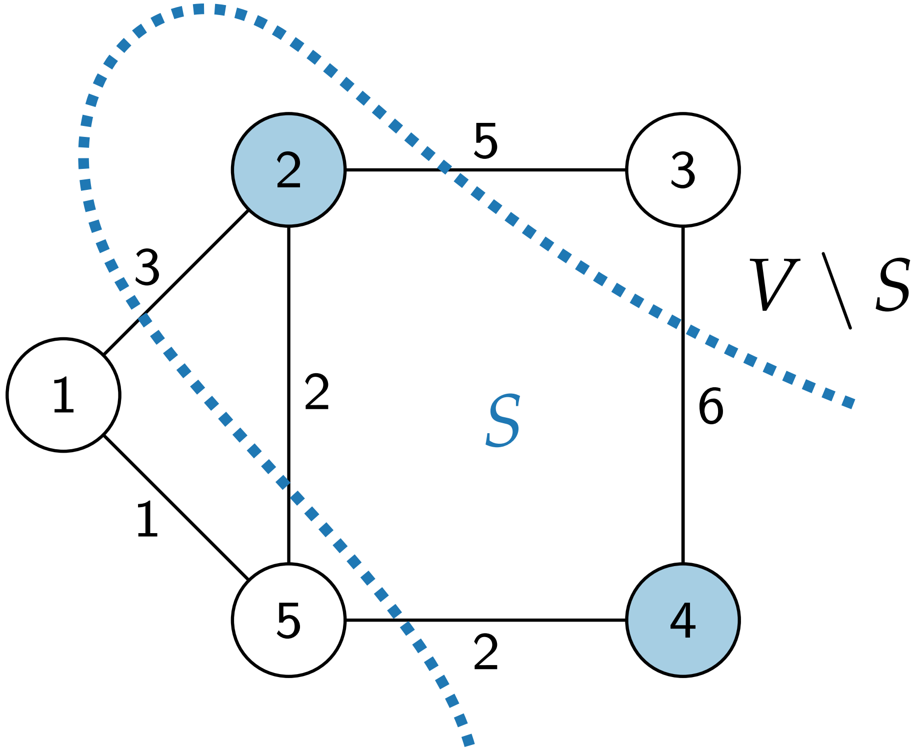
maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$



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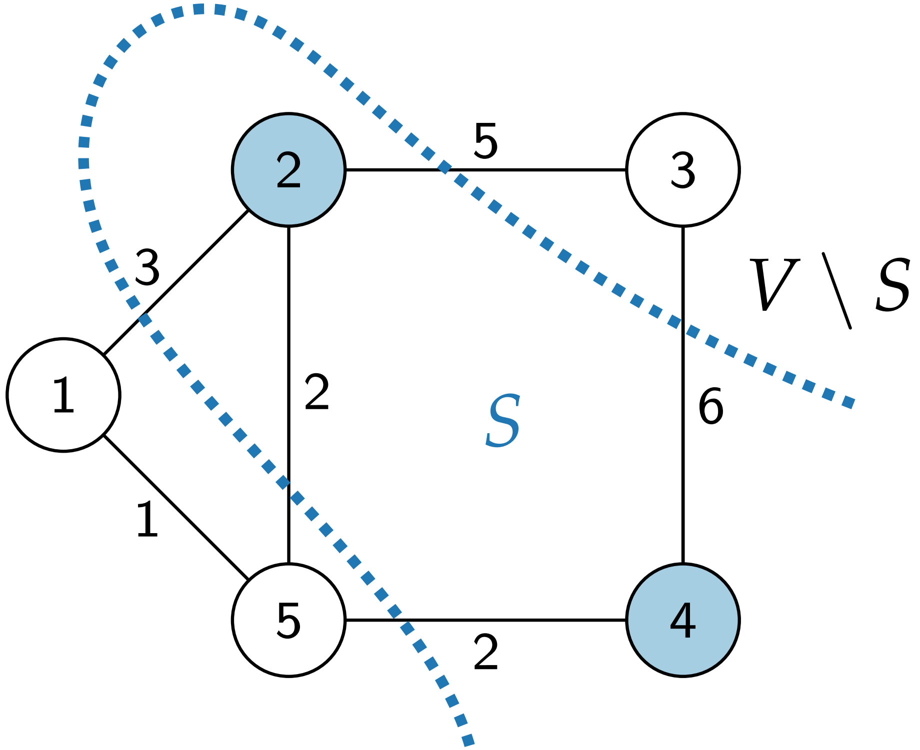

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- Solution

$x_2 = x_4 = 1$

$x_1 = x_3 = x_5 = -1$

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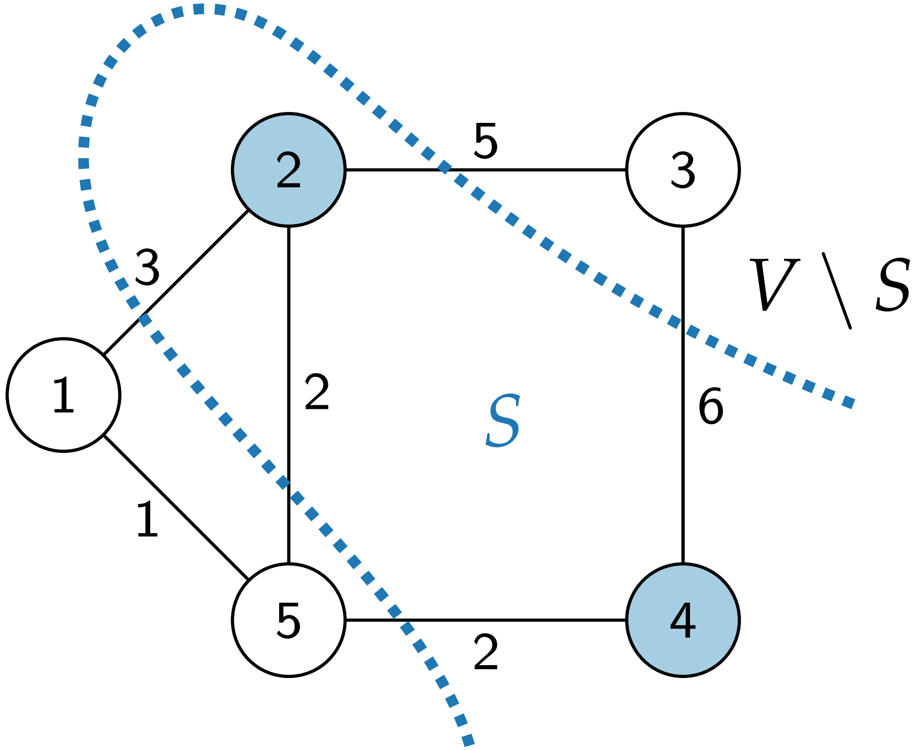

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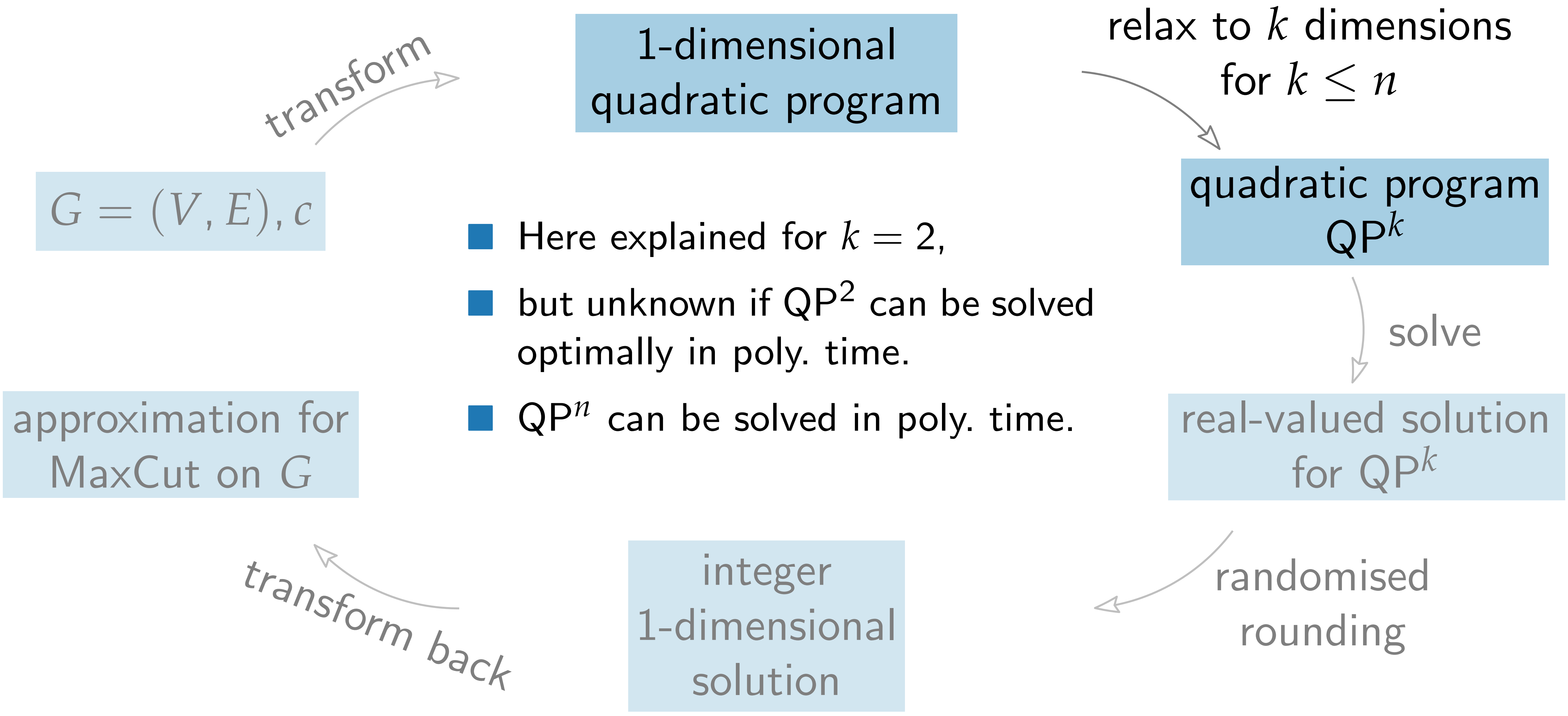
Note.

- Solving QP(G) is NP-hard.
- Otherwise MaxCut wouldn't be NP-hard.

Goemans-Williamson algorithm for MaxCut



- Here explained for $k = 2$,
- but unknown if QP^2 can be solved optimally in poly. time.
- QP^n can be solved in poly. time.

Relaxation of QP(G, c)

QP$^2(G, c)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - x^i \cdot x^j)$

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$x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$

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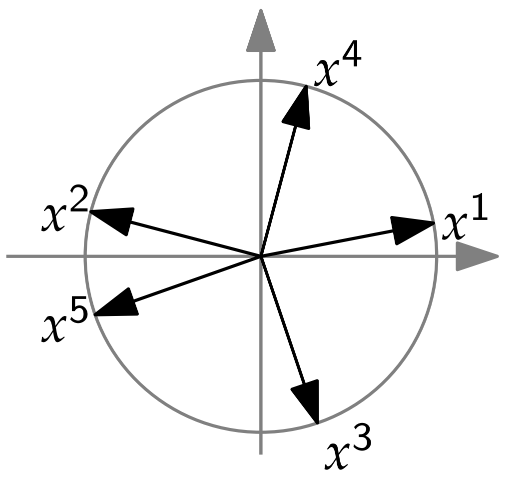
- x^i lies on unit circle.



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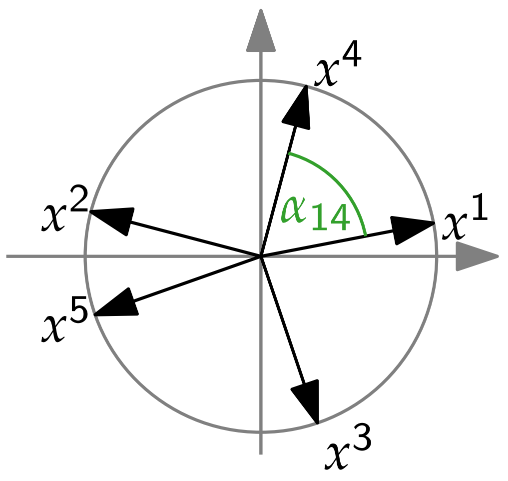
- x^i lies on unit circle.
- $x^i x^j = x_1^i x_1^j + x_2^i x_2^j = \cos(\alpha_{ij})$ with $0 \leq \alpha_{ij} \leq \pi$.



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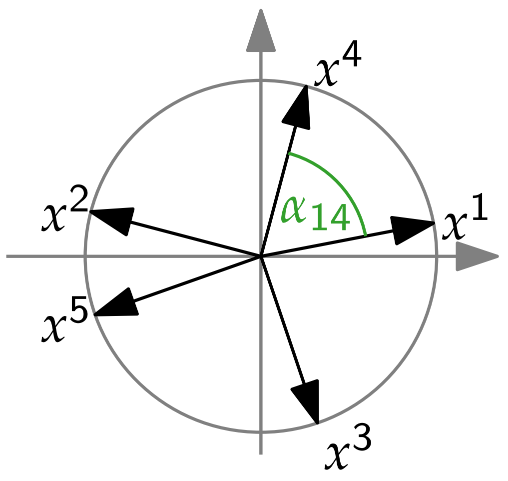

- We maximize angles α_{ij}: $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - \cos(\alpha_{ij}))$

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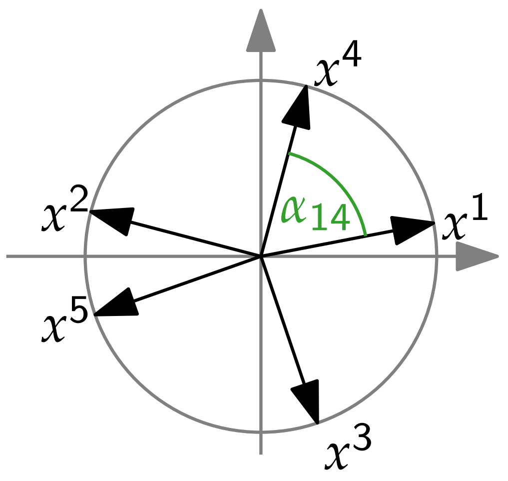
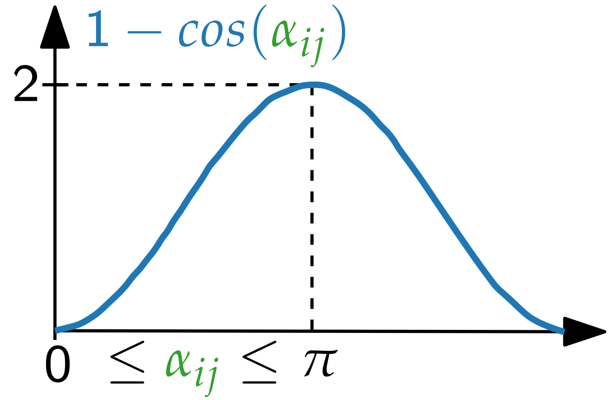
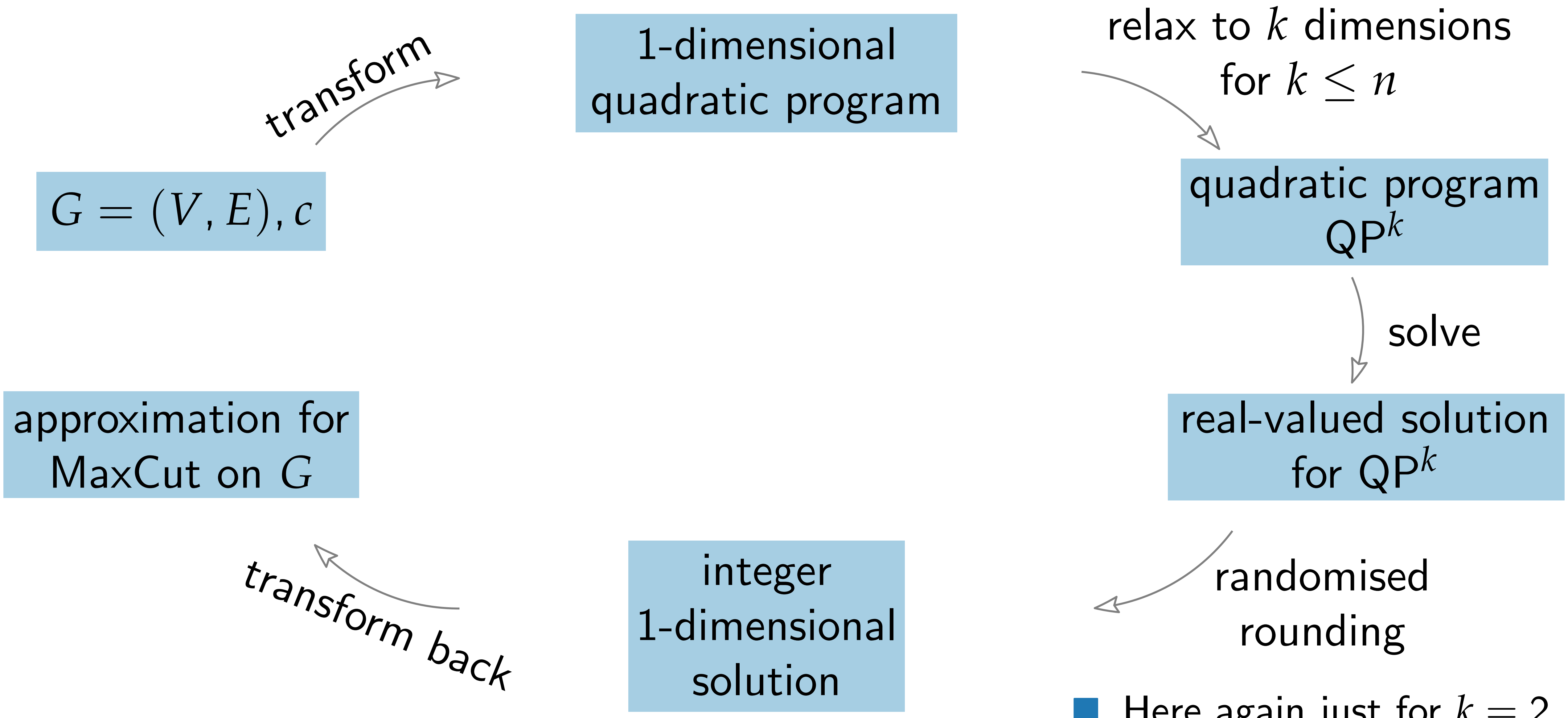


- We maximize angles α_{ij}: $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - \cos(\alpha_{ij}))$
- since larger α_{ij}, increases contribution of c_{ij}.

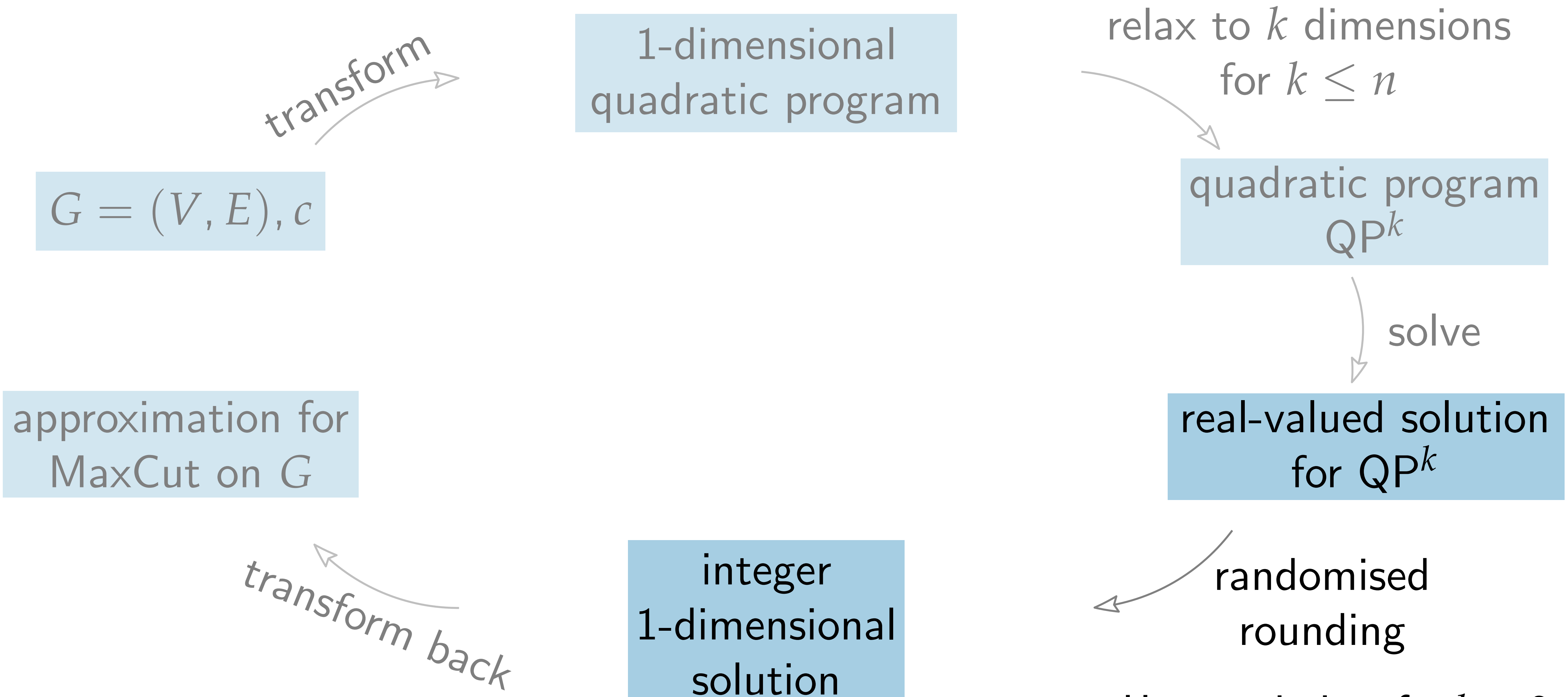


Goemans-Williamson algorithm for MaxCut



- Here again just for $k = 2$.

Goemans-Williamson algorithm for MaxCut



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Algorithm $\textsc{RandomizedMaxCut}$

$\textsc{RandomizedMaxCut}(G, c)$

Compute optimal solution $(\tilde{x}^1, \ldots, \tilde{x}^n)$ for $\mathrm{QP}^2(G, c)$
Pick random vector $r \in \mathbb{R}^2$
$S \leftarrow \{i \in V : \tilde{x}^i \cdot r \geq 0\}$
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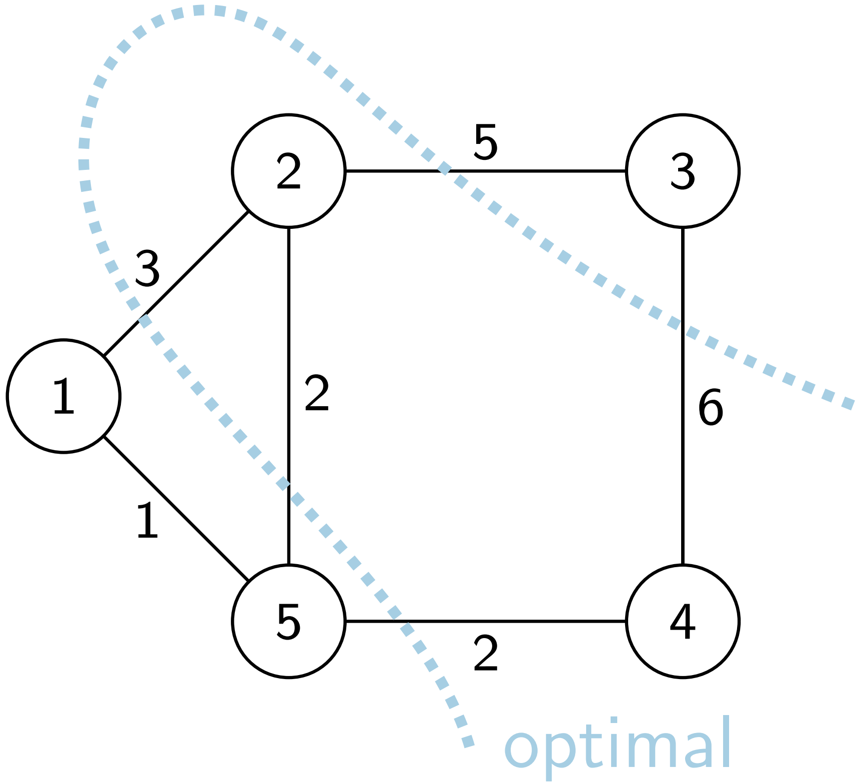

Algorithm RandomizedMaxCut

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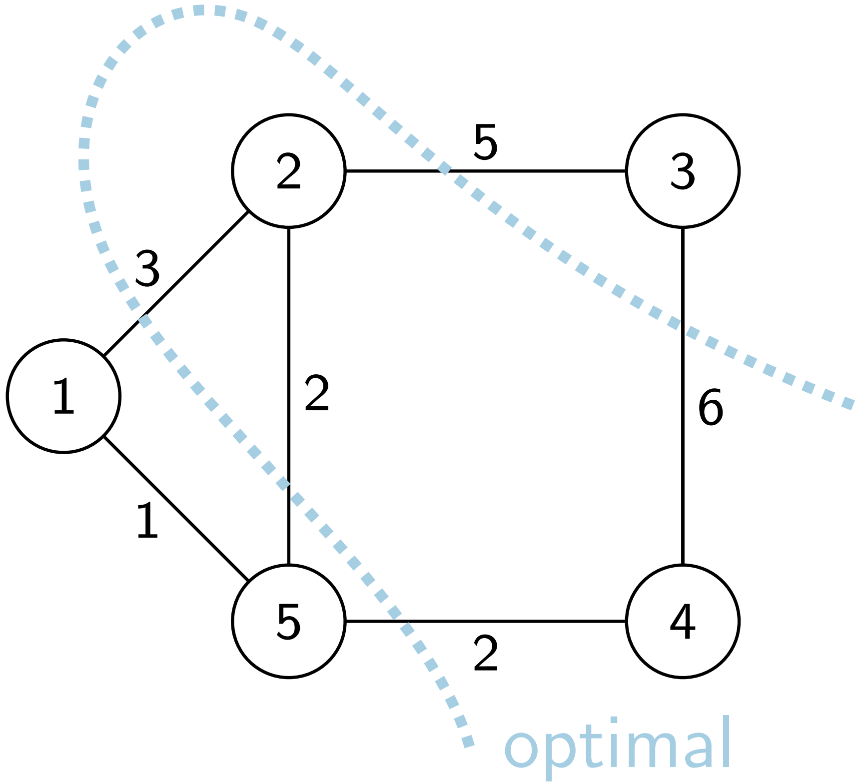
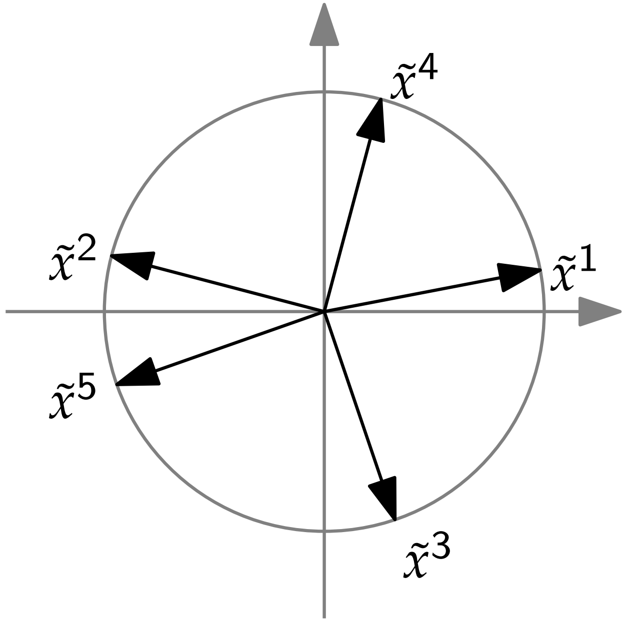
Compute optimal solution $(\tilde{x}^1, \ldots, \tilde{x}^n)$ for QP$^2(G, c)$

Pick random vector $r \in \mathbb{R}^2$

$S \leftarrow \{i \in V : \tilde{x}^i \cdot r \geq 0\}$

return $c(S, V \setminus S)$





(Sketch)

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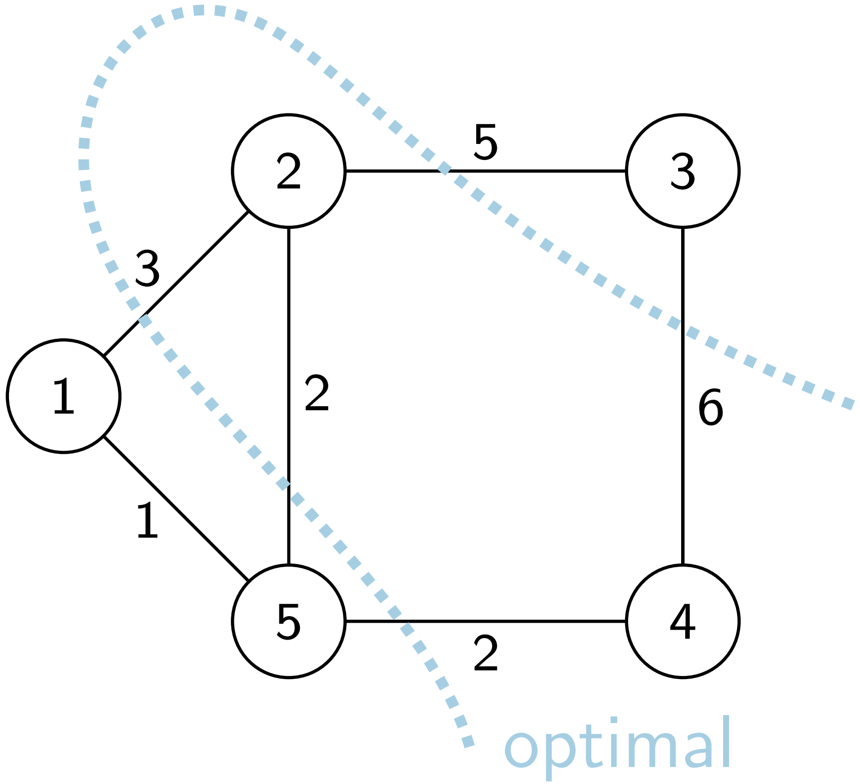

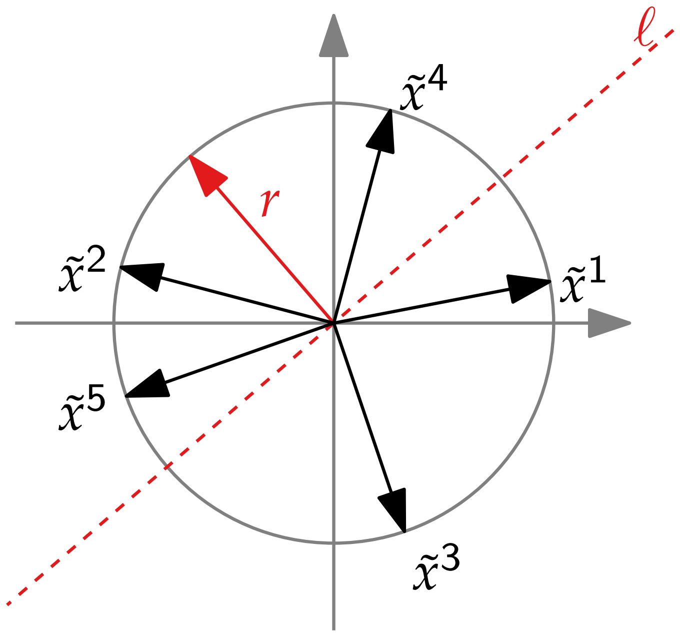

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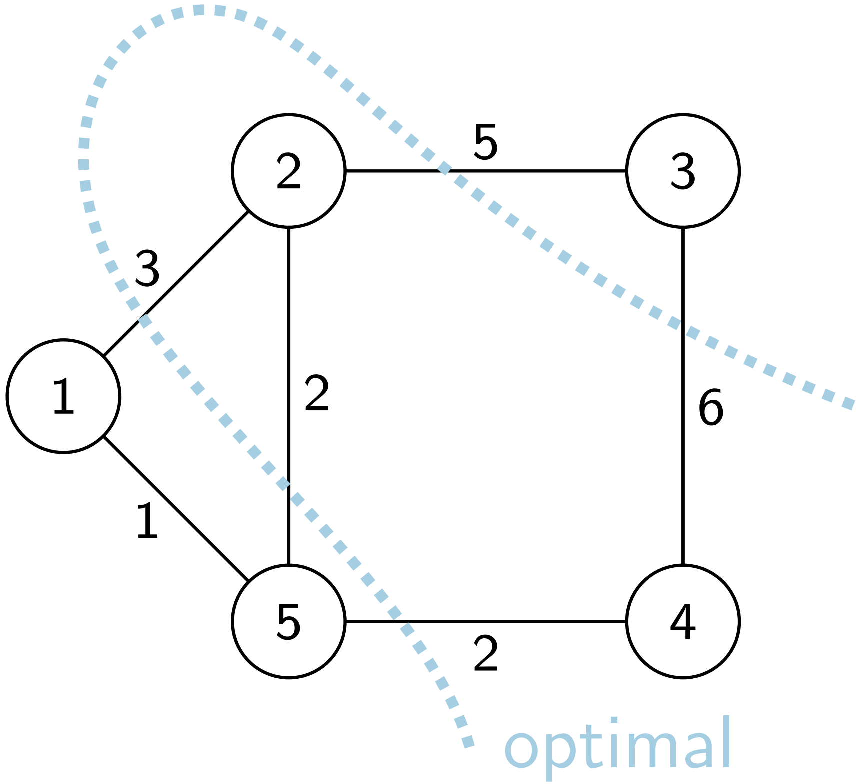
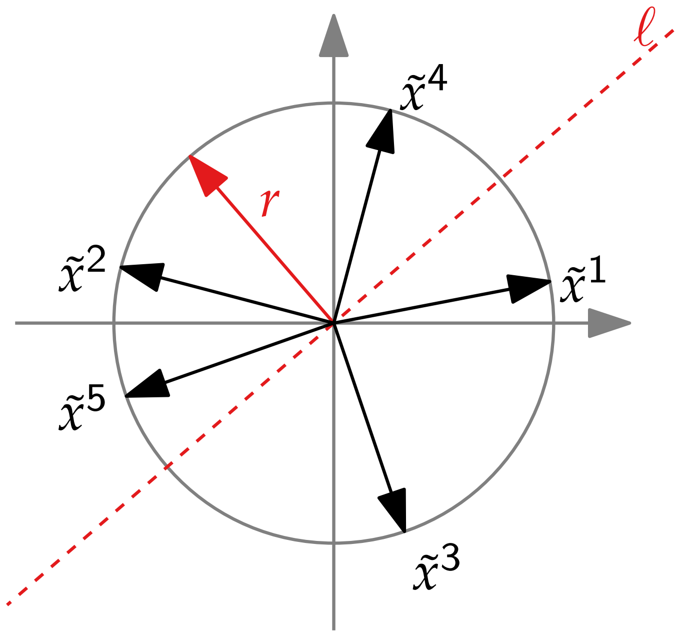
- $\tilde{x}^i$ lies above line ℓ orthogonal to r





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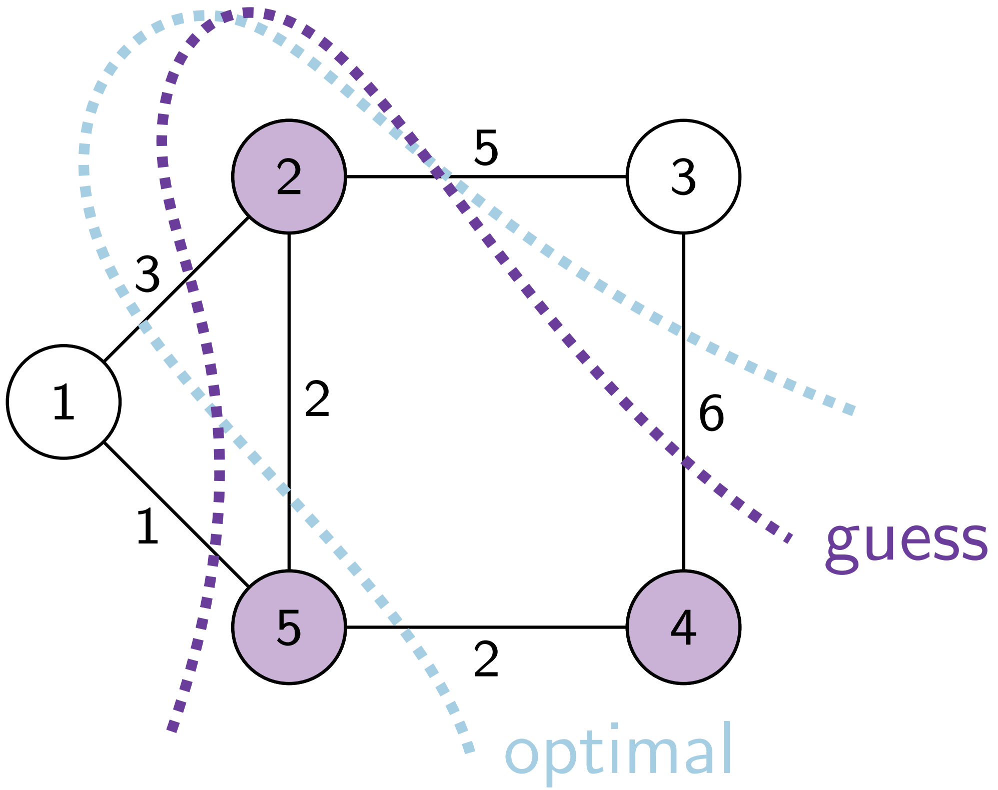

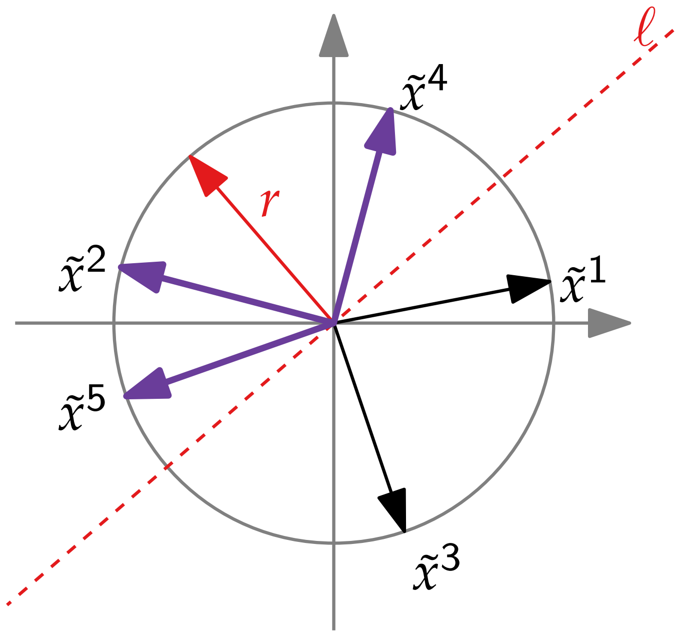

(Sketch)

RandomMaxCut – expected value

Lemma 2.
Let X be the solution of RandomizedMaxCut(G, c).
If r is picked uniformally at random, then

$$\mathrm{E}[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij} \frac{\alpha_{ij}}{\pi}.$$

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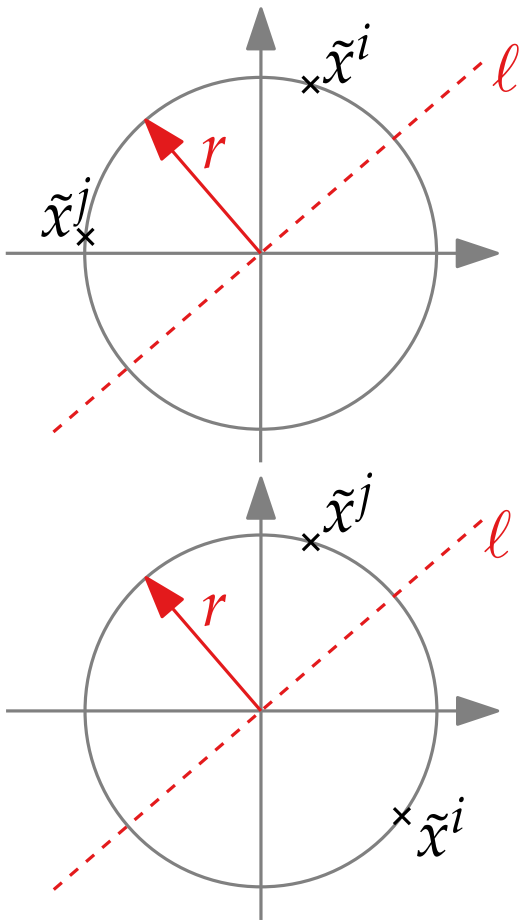

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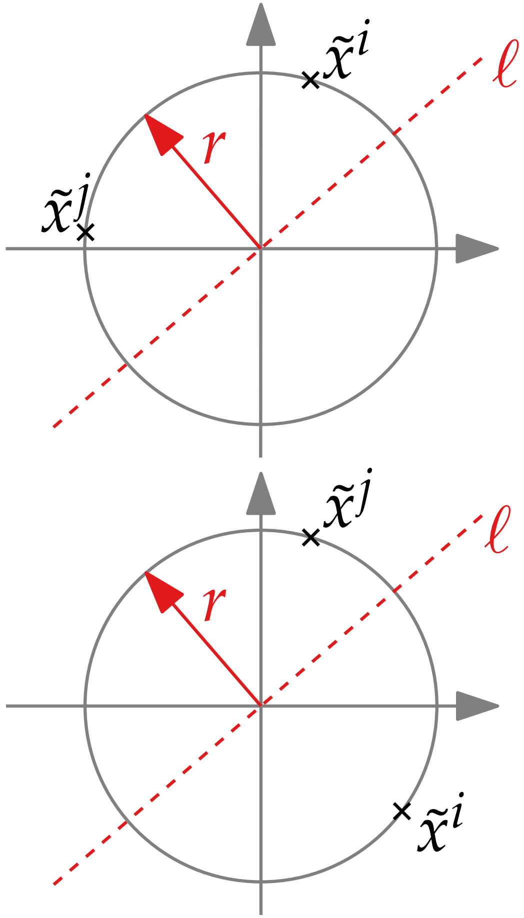

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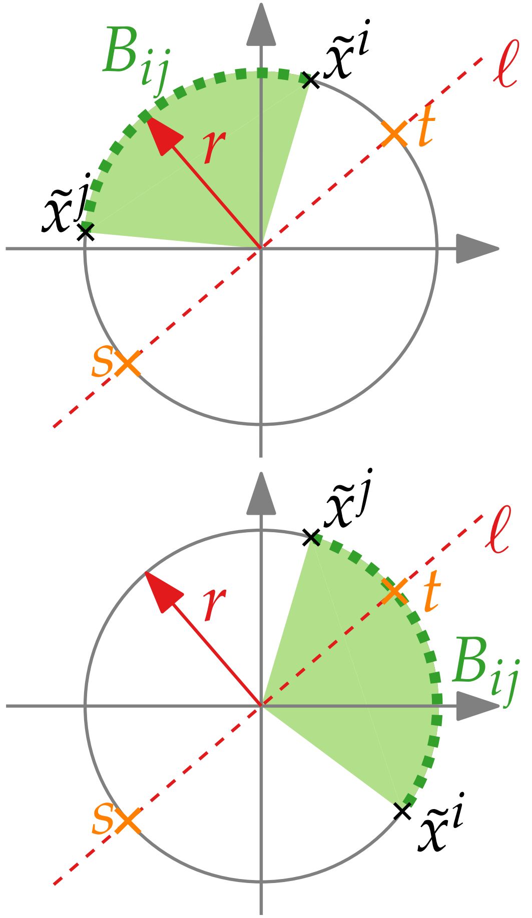

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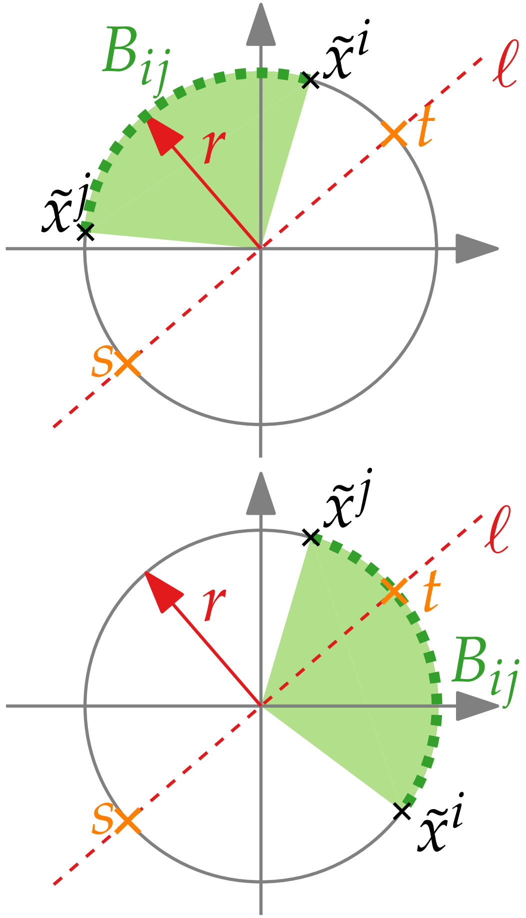

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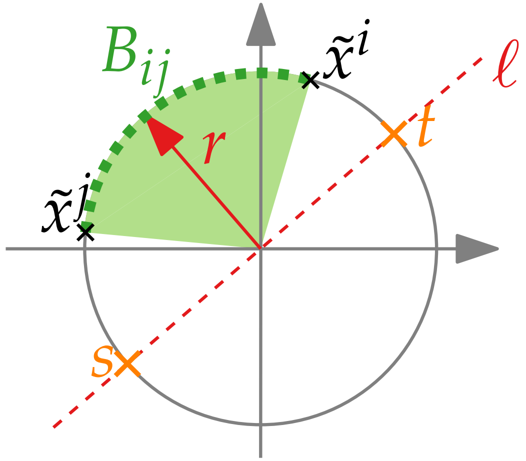

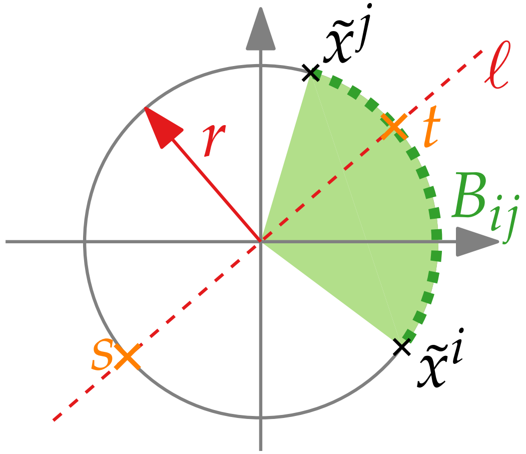

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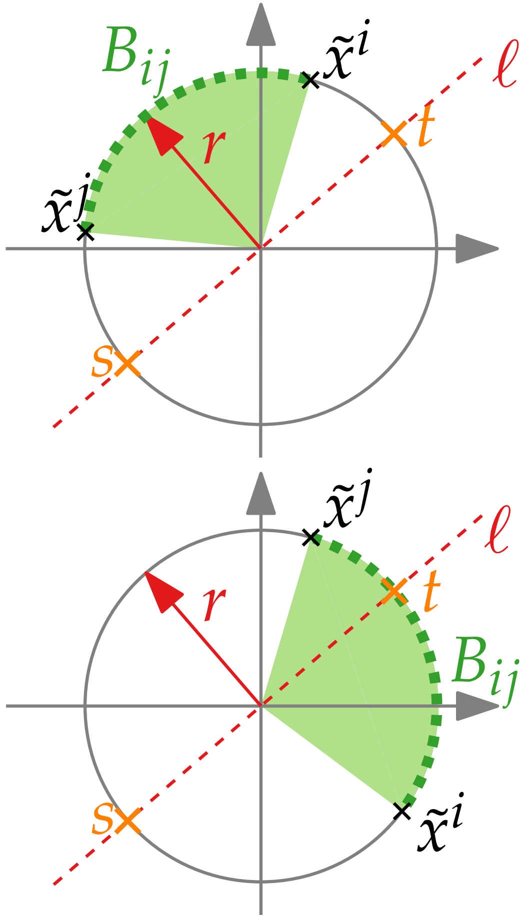

RandomMaxCut – quality

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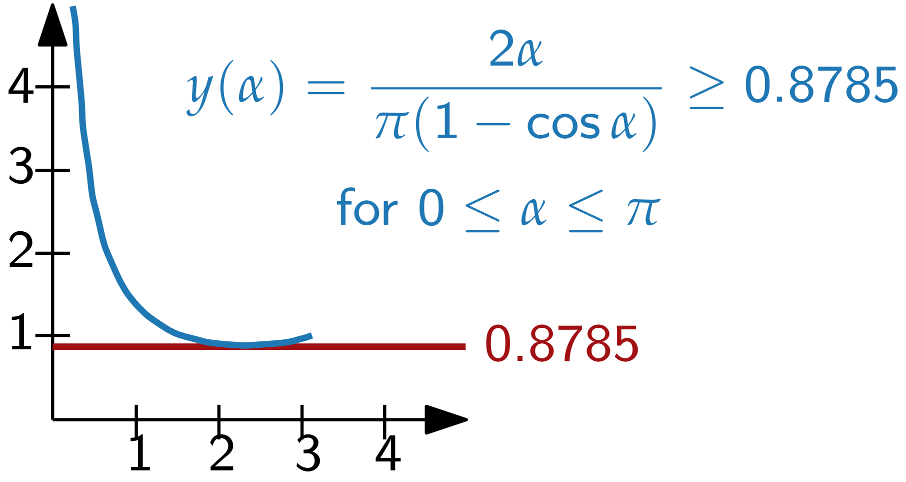
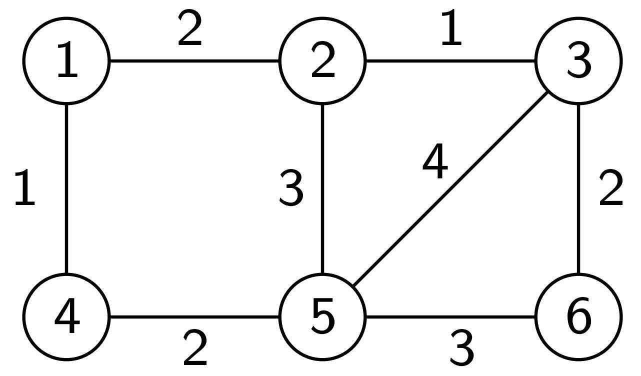


Example



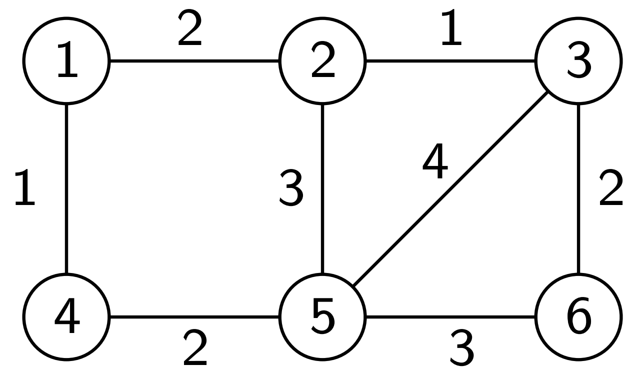
Example

1. Step: Build QP

$$\begin{aligned} &\textbf{maximize} && \tfrac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} c_{ij}(1 - x_i x_j) \\ &\textbf{subject to} && x_i^2 = 1 \end{aligned}$$

Weight matrix c_{ij}

	1	2	3	4	5	6
1		2		1		
2	2		1		3	
3		1			4	2
4	1				2	
5		3	4	2		3
6			2		3	



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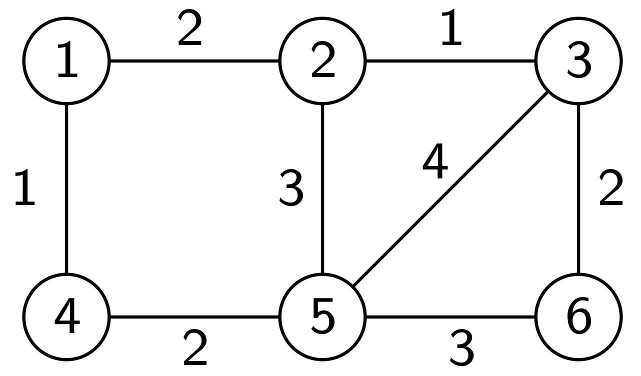

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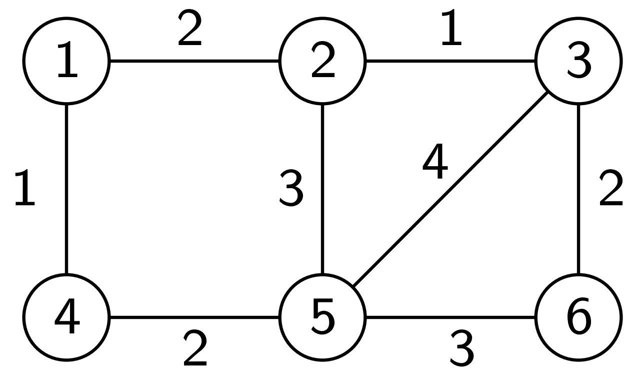
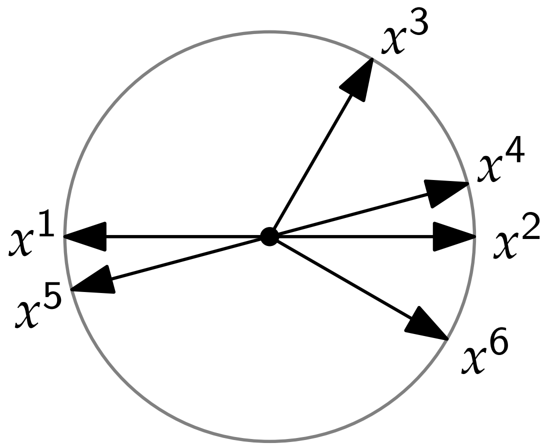
Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0	180	120	165	345	210





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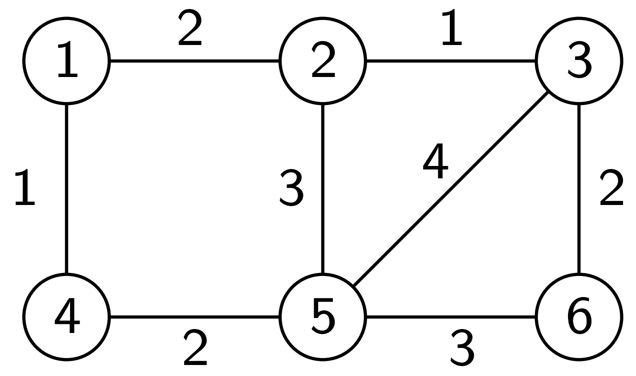

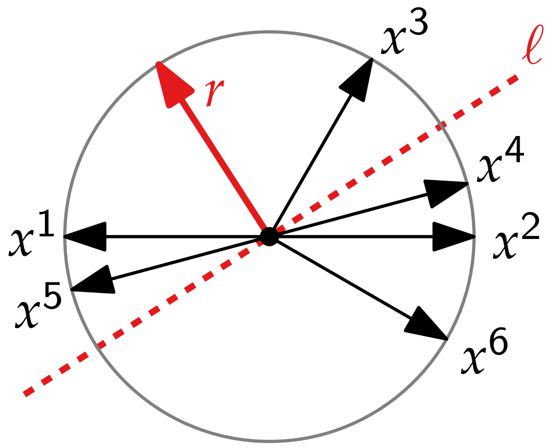

4. Step: Guess r

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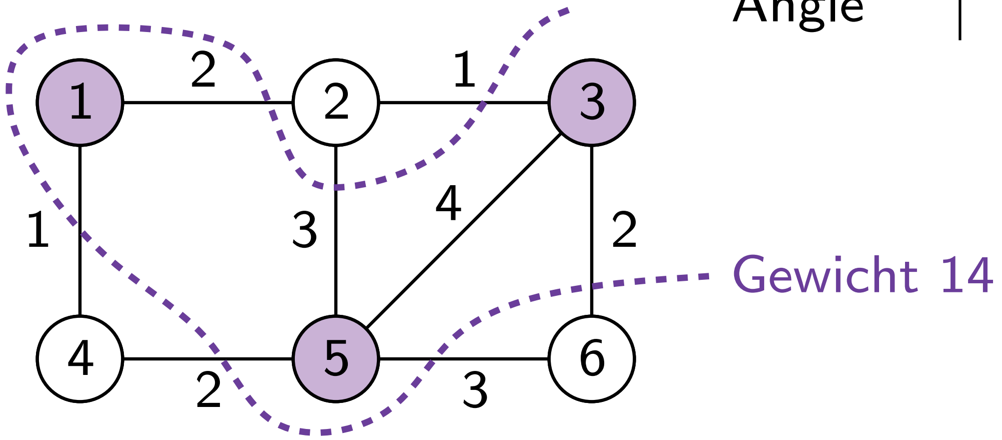

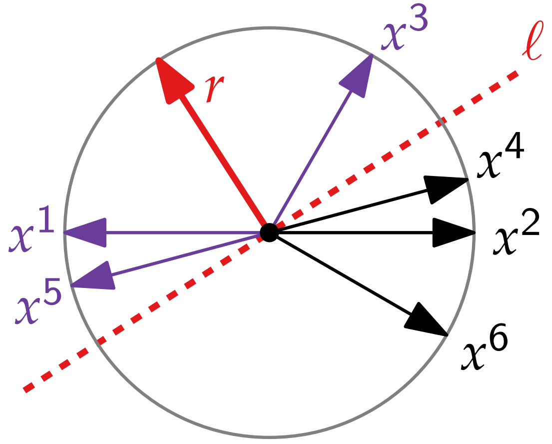

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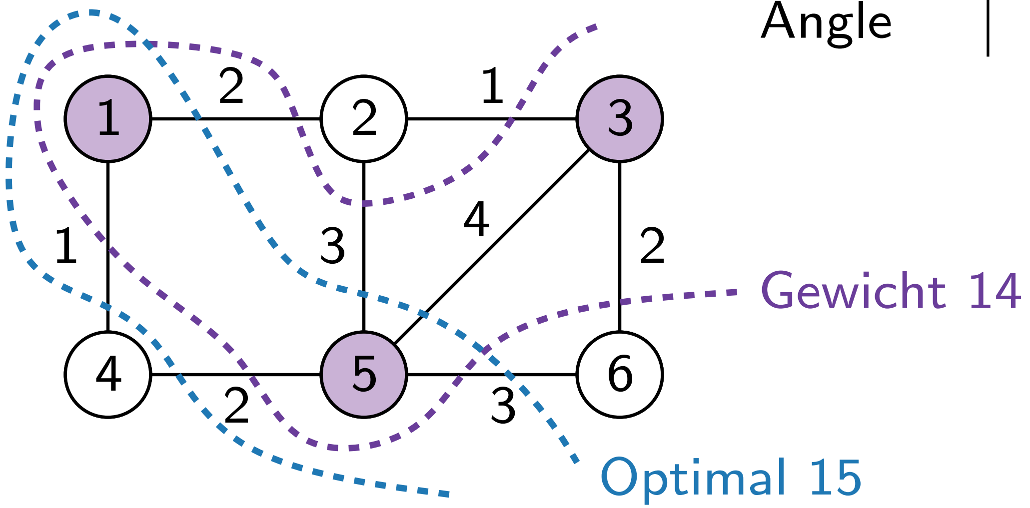

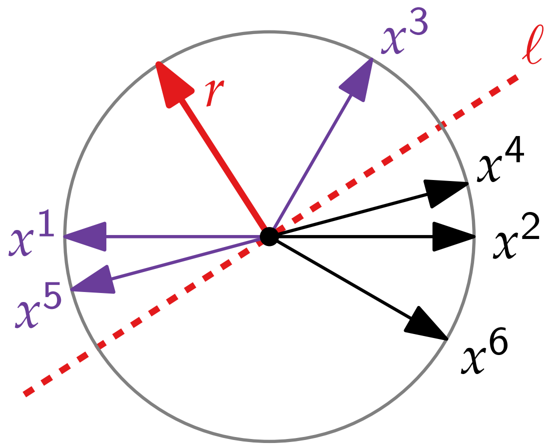

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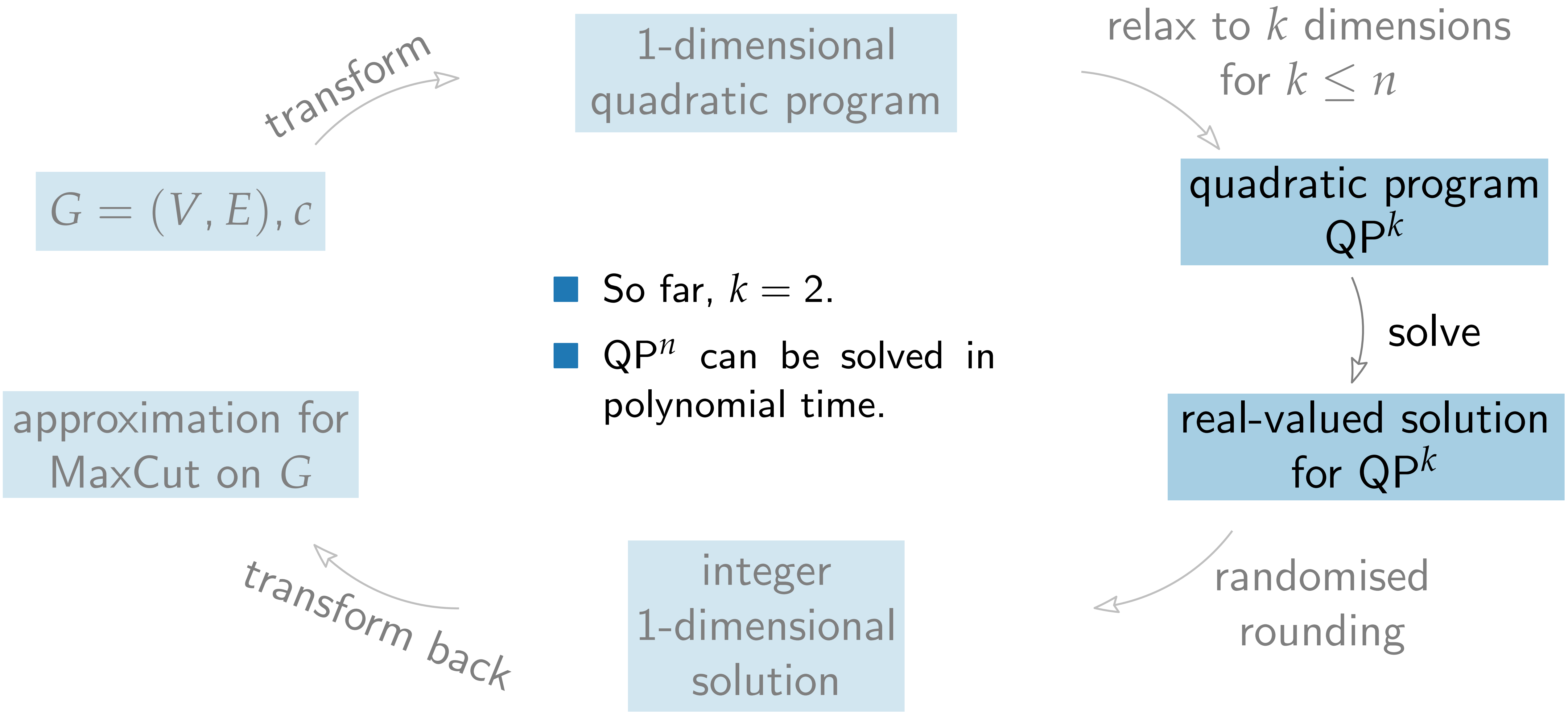
Goemans-Williamson algorithm for MaxCut



- So far, $k = 2$.
- QP^n can be solved in polynomial time.

$\mathrm{QP}^n(G, c)$

$\mathbf{QP}^2(G, c)$

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- $\mathrm{QP}^n(G, c)$ becomes problem $\textsc{SemiDefiniteCut}(G, c)$.
  - Can be approximated in time poly. in (G, c) and $1/\varepsilon$ with additive guarantee ε.
  - For $\varepsilon = 10^{-5}$, approximation guarantee for $\textsc{Random-MaxCut}$ is achieved.

Discussion

- Semidefinite programming is a powerful tool to develop approximation algorithms
- Whole book on this topic:
  - [Gärtner, Matoušek] “Approximation Algorithms and Semidefinite Progamming”

Discussion

- Semidefinite programming is a powerful tool to develop approximation algorithms
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- Using randomness is another tool to design approximation algorithms.
- See future lectures.

Literature

Original paper:

- [GW ’95] “Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming”

Source:

- [Vazirani Ch26] “Approximation Algorithms”

Whole book on this topic:

- [Gärtner, Matoušek] “Approximation Algorithms and Semidefinite Progamming”