Computational Geometry

Lecture 4:
Linear Programming
or
Profit Maximization

Part I:
Introduction to Linear Programming

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Winter Semester 2020

Maximizing Profits

You're the boss of a small company that produces two products P_1 and P_2. For the production of x_1 units of P_1 and x_2 units of P_2, you're profit in € is:

$$G(x_1, x_2) = 30x_1 + 50x_2$$

Three machines M_A, M_B and M_C produce the required components A, B and C for the products. The components are used in different quantities for the products, and each machine requires some time for the production.

$$\begin{aligned} M_A: &\quad 4x_1 + 11x_2 \leq 880 \\ M_B: &\quad x_1 + x_2 \leq 150 \\ M_C: &\quad x_2 \leq 60 \end{aligned}$$

Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

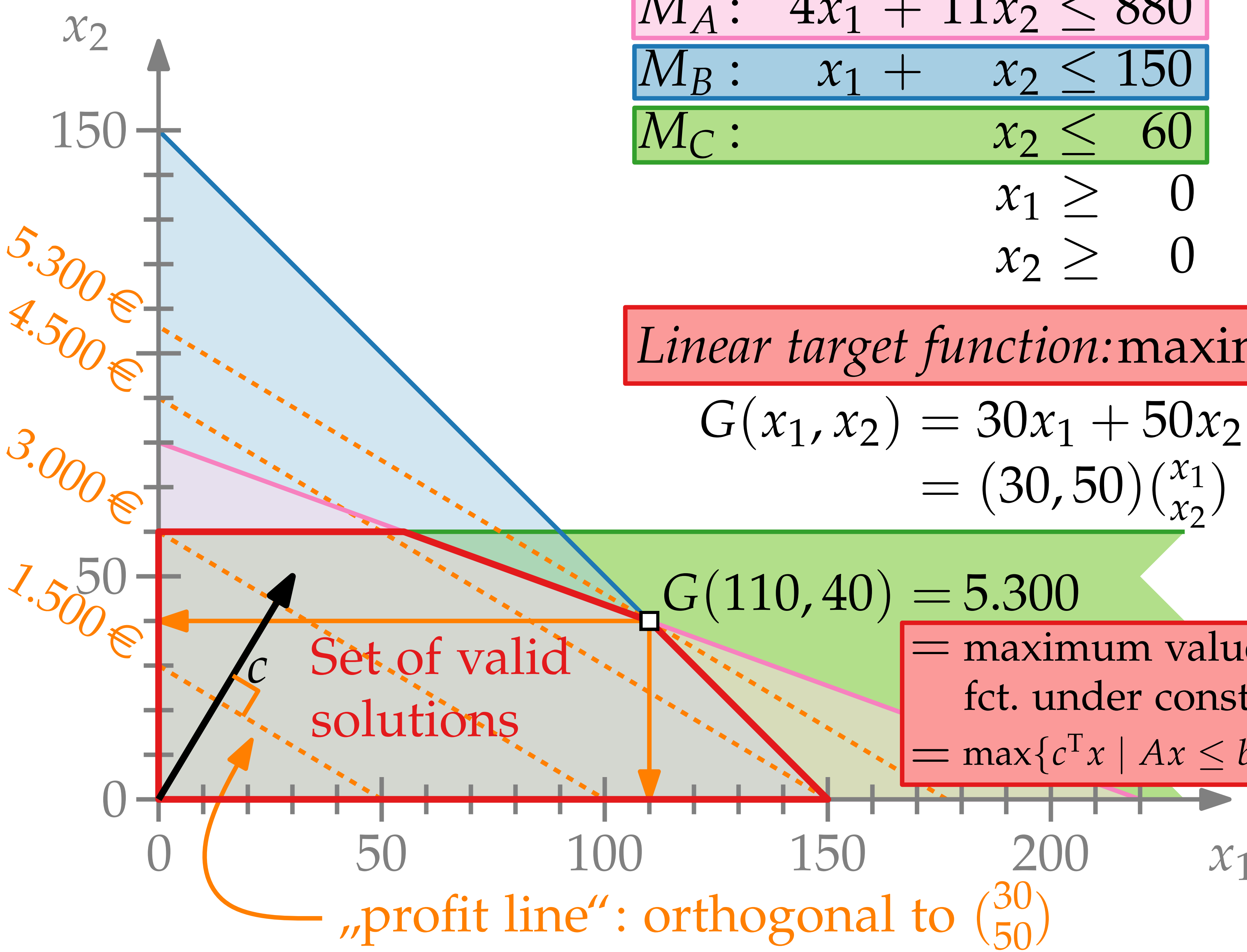
$$\begin{aligned} M_A: \quad 4x_1 + 11x_2 &\leq 880 \\ M_B: \quad x_1 + x_2 &\leq 150 \\ M_C: \quad x_2 &\leq 60 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

$$Ax \leq b$$

$$x \geq 0$$

Linear target function: maximize $c^{\mathrm{T}}x$

$$G(x_1, x_2) = 30x_1 + 50x_2 = (30, 50)\binom{x_1}{x_2}$$



$G(110, 40) = 5.300$

$=$ maximum value of target fct. under constraints.

$= \max\{c^{\mathrm{T}}x \mid Ax \leq b, x \geq 0\}$

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Part II:
A First Approach

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Definition and Known Algorithms

Given a set H of n halfspaces in $\mathbb{R}^d$ and a direction c, find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

Many algorithms known, e.g.:

- Simplex [Dantzig '47]
- Ellipsoid method [Khatchiyan '79]
- Inner-point method [Karmakar' 84]

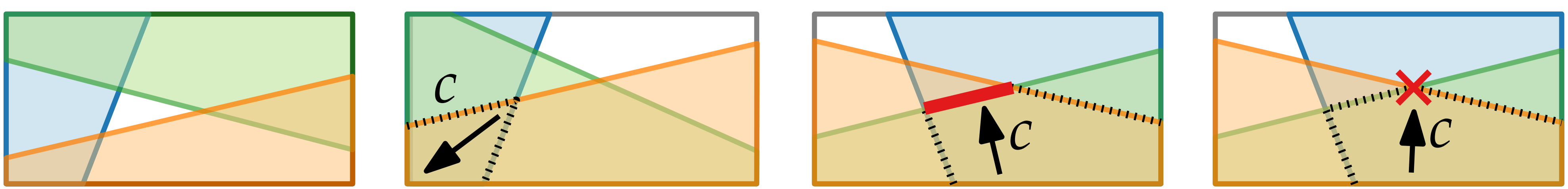
Good for instances where n *and* d are large.

We consider $d = 2$.

VERY important problem, e.g., in Operations Research.

["Book" application: casting]

$\bigcap H$ bounded.



$\bigcap H = \emptyset$ | $\bigcap H$ unbnd. in dir. c | set of optima: segment vs. point

First Approach

- compute $\bigcap H$ explicitly
- walk along $\partial\left(\bigcap H\right)$ to find a vertex x with cx maximum

```
IntersectHalfplanes(H)
  if |H| = 1 then
  |   C ← h, where {h} = H
  else
  |   split H into sets H_1 and H_2 with |H_1|, |H_2| ≈ |H|/2
  |   C_1 ← IntersectHalfplanes(H_1)
  |   C_2 ← IntersectHalfplanes(H_2)
  |   C ← IntersectConvexRegions(C_1, C_2)
  return C
```

How??

Running time: $T_{\mathrm{IH}}(n) = 2T_{\mathrm{IH}}(n/2) + T_{\mathrm{ICR}}(n)$

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Part III:
Intersecting Convex Regions

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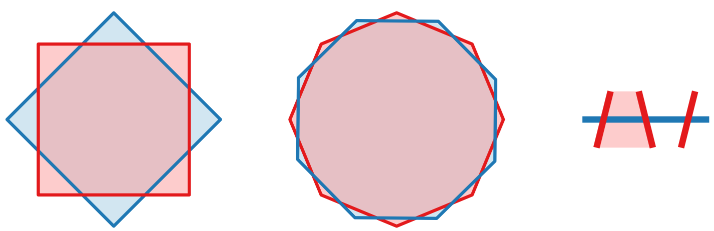
Intersecting Convex Regions

Any ideas?

Use sweep-line alg. for map overlay (line-segment intersections)!

Running time $T_{\mathrm{ICR}}(n) = O((n + I)\log n)$,



where $I = \#$ intersection points.

here: $I \le n$

Running time $T_{\mathrm{IH}}(n) = 2T_{\mathrm{IH}}(n/2) + T_{\mathrm{ICR}}(n)$

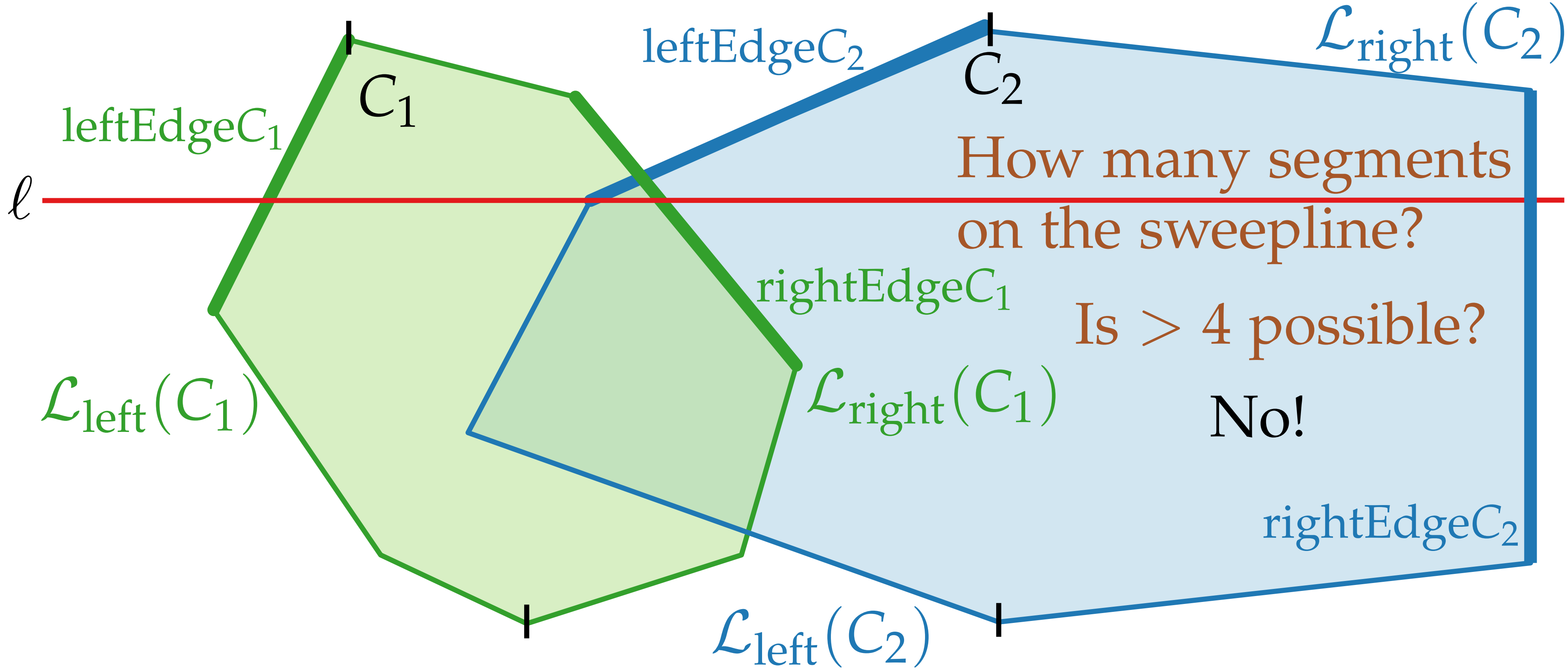
$$\le 2T_{\mathrm{IH}}(n/2) + O(n \log n)$$

$$\in O(n \log^2 n)$$

Better ideas?

Better analysis of the sweep-line for *convex* regions/polygons!

Intersecting Convex Regions Faster



Theorem. The intersection of two convex polygonal regions can be computed in linear time.

Corollary. The intersection of n half planes can be computed in $O(n \log n)$ time.

Can we do better?

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Part IV:
Incremental Approach

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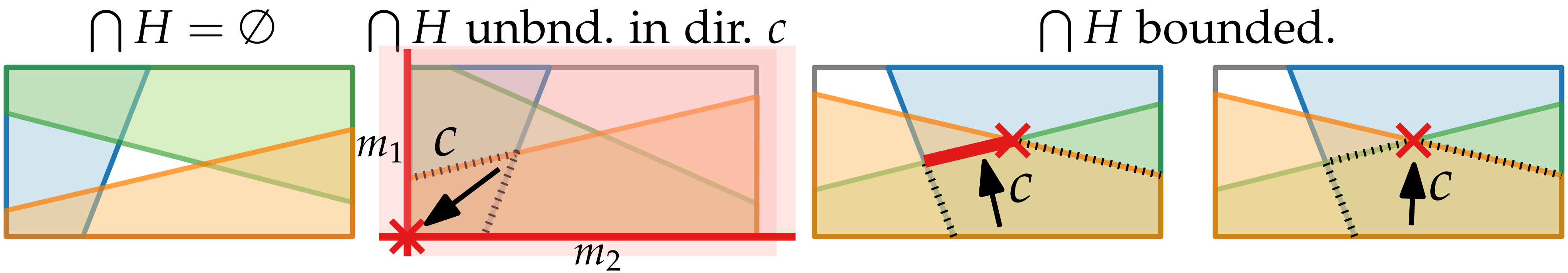
A Small Trick: Make Solution Unique



- Add two bounding halfplanes m_1 and m_2

$$m_1 = \begin{cases} x \leq M & \text{if } c_x > 0, \\ x \geq M & \text{otherwise,} \end{cases} \quad \text{for some sufficiently large } M$$

$$m_2 = \begin{cases} y \leq M & \text{if } c_y > 0, \\ y \geq M & \text{otherwise.} \end{cases}$$

- Take the lexicographically largest solution.

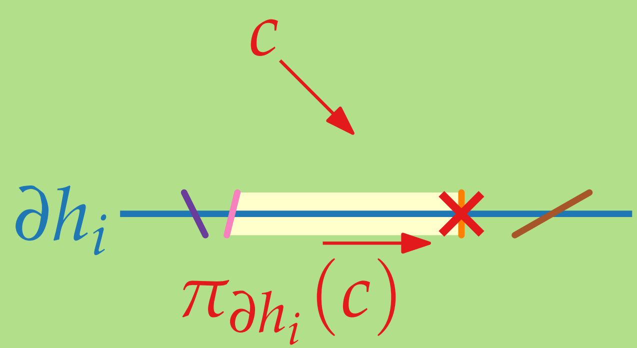
$\Rightarrow$ Set of solutions is either empty or a uniquely defined pt.

Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

Randomized

2DBoundedLP(H, c, m_1, m_2)
  compute random permutation of H
  $H_0 = \{m_1, m_2\}$
  $v_0 \leftarrow$ corner of $m_1 \cap m_2$
  for $i \leftarrow 1$ **to** n **do**
    if $v_{i-1} \in h_i$ **then**
      $v_i \leftarrow v_{i-1}$ $O(1)$
    else
      $v_i \leftarrow$ 1DBoundedLP$(\pi_{\partial h_i}(H_{i-1}), \pi_{\partial h_i}(c))$ $O(i)$
      if $v_i =$ nil **then**
        return nil
    $H_i = H_{i-1} \cup \{h_i\}$ $O(1)$
  return v_n



w-c running time:

$$T(n) = \sum_{i=1}^{n} O(i) = = O(n^2)$$:-(

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Part V:
The Randomized-Incremental Approach

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Result

Theorem. The 2D bounded LP problem can be solved in $O(n)$ expected time.

Proof. Let $X_i = \begin{cases} 1 & \text{if } v_{i-1} \notin h_i, \\ 0 & \text{else.} \end{cases}$ (indicator random var.).

Then the expected running time is

$$\mathbf{E}[T_{2d}(n)] = \mathbf{E}[\textstyle\sum_{i=1}^{n}(1 - X_i) \cdot O(1) + X_i \cdot O(i)]$$
$$= \textstyle\sum \mathbf{E}[1 - X_i] \cdot O(1) + \sum \mathbf{E}[X_i] \cdot O(i)$$
$$\leq O(n) + \textstyle\sum \mathbf{Pr}[X_i = 1] \cdot O(i) = O(n).$$

We fix the i random halfplanes in H_i.

$\mathbf{Pr}[X_i = 1]$ = probability that the optimal solution changes when h_i is added to H_{i-1}.

Proof technique: *Backward analysis!*

= probability that the optimal solution changes when h_i is removed from H_i.

$\leq 2/i$. This is independent of the choice of H_i.