Computational Geometry

Lecture 2:
Line-Segment Intersection
or
Map Overlay

Part I:
Map Overlay

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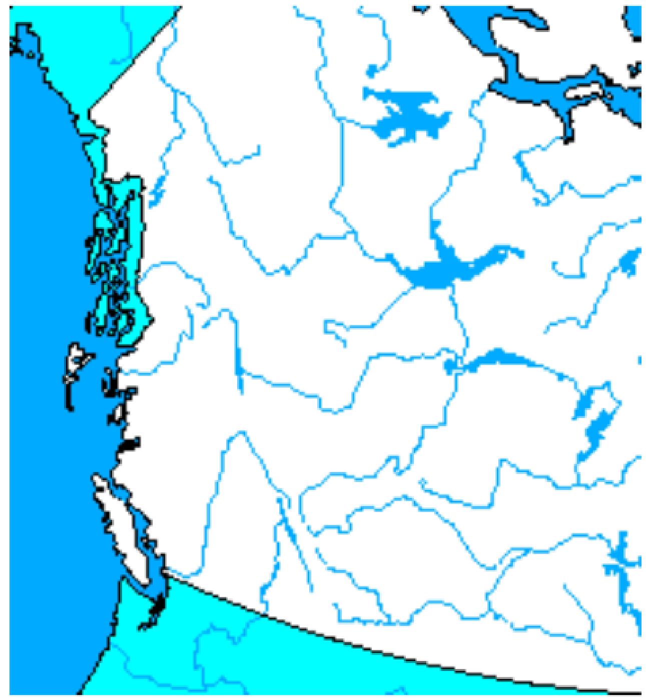
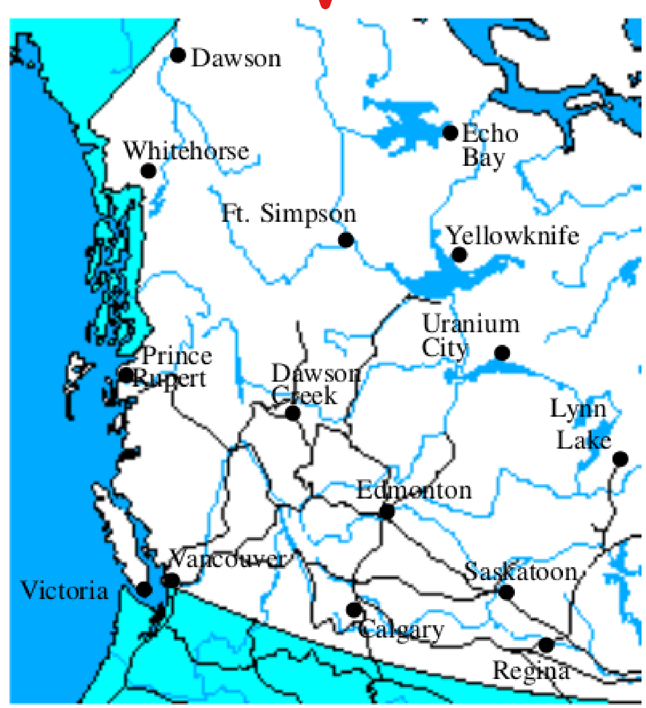
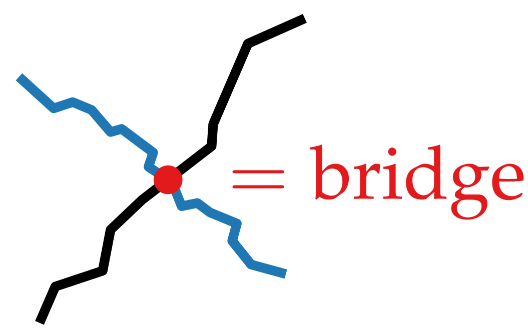
Winter Semester 2020



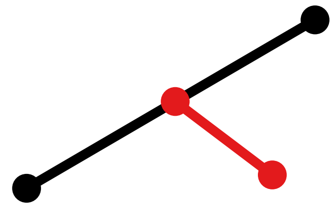
Map Overlay in Geographic Information Systems (GIS)



Here:



Line-Segment Intersection

Definition: Is  an intersection?

Answer: Depends…

yes!

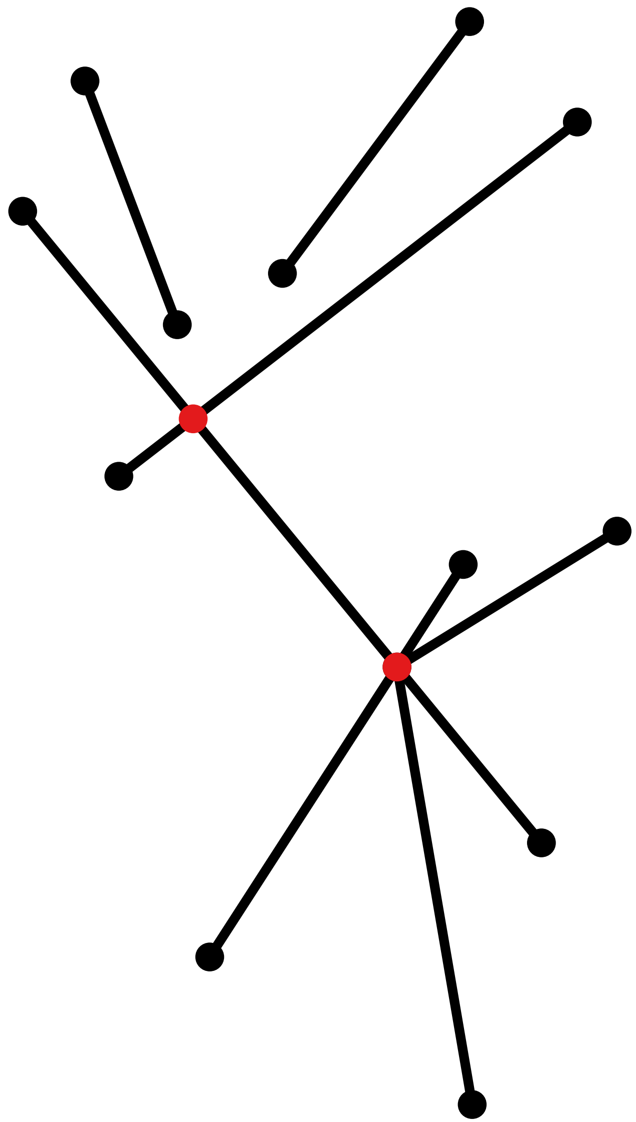
Problem: Given a set S of n *closed* non-overlapping line segments in the plane, compute…

– all points where at least two segments intersect and

– for each such point report all segments that contain it.

Task: How would *you* do it?

Example



Brute Force?

$O(n^2)$ ... can we do better?

Idea:
Process segments top-to-bottom using a "sweep line".

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Part II:
Sweep-Line Algorithm

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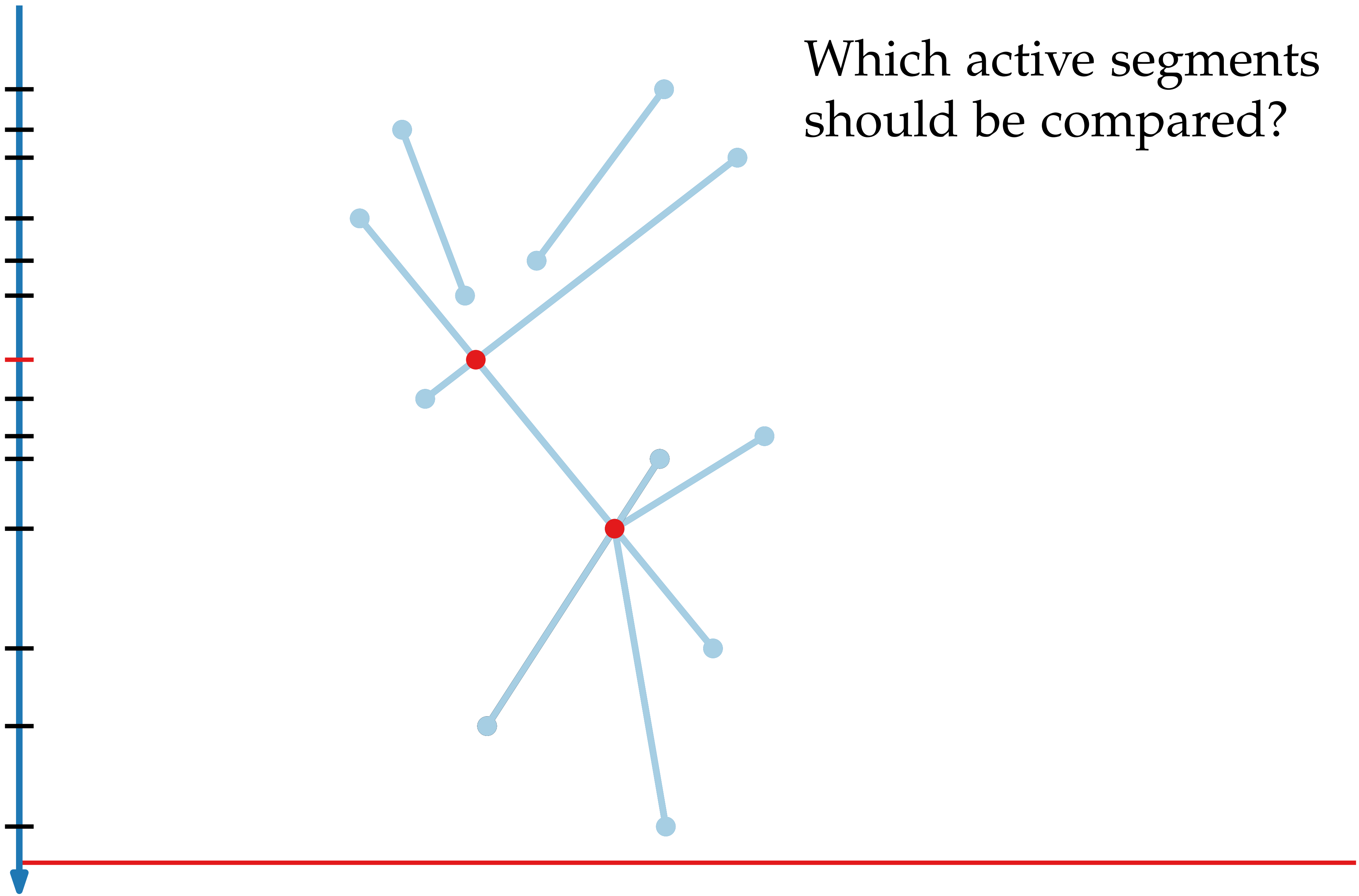
Sweep-Line Algorithm

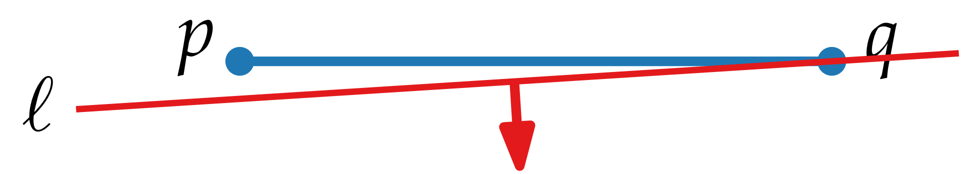


Which active segments should be compared?

Data Structures

1) event (-point) queue $\mathcal{Q}$

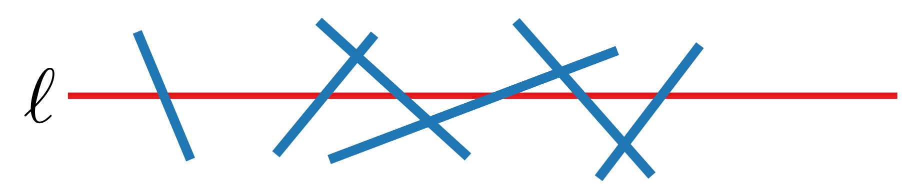
$$p \prec q \quad \Leftrightarrow_{\text{def.}} \quad y_p > y_q \quad \text{or} \quad (y_p = y_q \text{ and } x_p < x_q)$$



Store event pts in *balanced binary search tree* acc. to $\prec$

$\Rightarrow$ nextEvent() and del/insEvent() take $O(\log |\mathcal{Q}|)$ time

2) (sweep-line) status $\mathcal{T}$



Store the segments intersected by ℓ in left-to-right order.

How? In a balanced binary search tree!

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Part III:
Algorithmic Details

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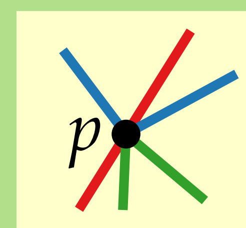
Pseudo-code

findIntersections(S)

Input: set S of n non-overlapping closed line segments

Output: – set I of intersection pts
– for each $p \in I$ every $s \in S$ with $p \in s$

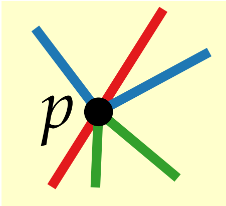
$\mathcal{Q} \leftarrow \emptyset$; $\mathcal{T} \leftarrow \langle$ vertical lines at $x = -\infty$ and $x = +\infty \rangle$ // sentinels
foreach $s \in S$ **do** // initialize event queue $\mathcal{Q}$
  foreach endpoint p of s **do**
    if $p \notin \mathcal{Q}$ **then** $\mathcal{Q}$.insert(p); $L(p) = U(p) = C(p) = \emptyset$
    if p lower endpt of s **then** $L(p)$.append(s)
    if p upper endpt of s **then** $U(p)$.append(s)



while $\mathcal{Q} \neq \emptyset$ **do**
  $p \leftarrow \mathcal{Q}$.nextEvent()
  $\mathcal{Q}$.deleteEvent(p)
  handleEvent(p)

This subroutine does the real work.
How would you implement it?

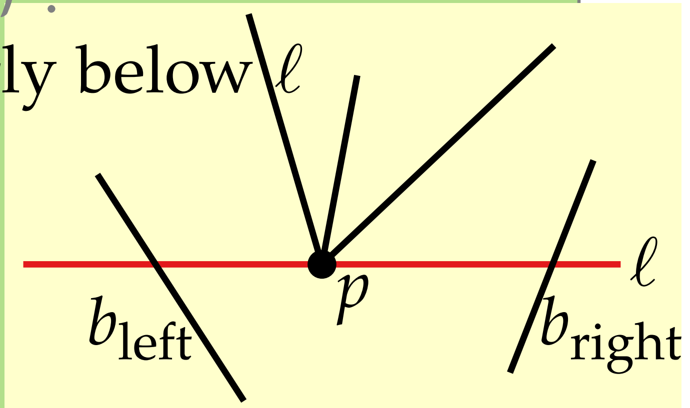
Handling an Event

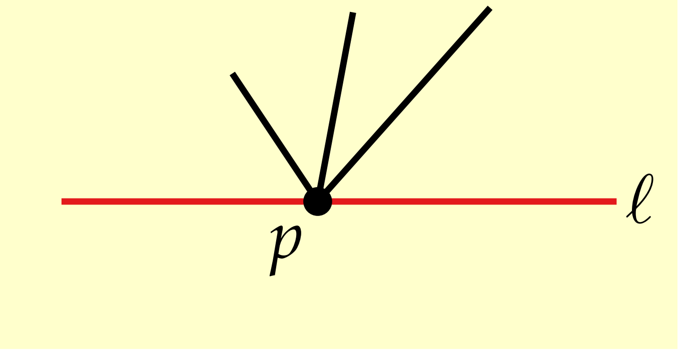


$C(p)$, $L(p)$, $U(p)$

findNewEvent(s, s', p)
if $s \cap s' = \emptyset$ **then return**
$\{x\} = s \cap s'$
if x below ℓ or on ℓ to the right of p **then**
  if $x \notin \mathcal{Q}$ **then** $\mathcal{Q}$.add(x)
  if $x \in$ rel-int(s) **then** $C(x) \leftarrow C(x) \cup \{s\}$
  if $x \in$ rel-int(s') **then** $C(x) \leftarrow C(x) \cup \{s'\}$

handleEvent(event p)
if $|U(p) \cup L(p) \cup C(p)| > 1$ **then**
  report intersection in p, report segments in $U(p) \cup L(p) \cup C(p)$
delete $L(p) \cup C(p)$ from $\mathcal{T}$ // consecutive in $\mathcal{T}$!
insert $U(p) \cup C(p)$ into $\mathcal{T}$ in their order slightly below ℓ
if $U(p) \cup C(p) = \emptyset$ **then**
  $b_{\text{left}} / b_{\text{right}}$ = left/right neighbor of p in $\mathcal{T}$
  findNewEvent$(b_{\text{left}}, b_{\text{right}}, p)$
else





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Part IV: Correctness

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Correctness

Lemma. findIntersections(S) correctly computes all intersection points & the segments that contain them.

Proof. Let p be an intersection pt. Assume (by induction):

- Every int. pt $q \prec p$ has been computed correctly.
- $\mathcal{T}$ contains all segments intersecting ℓ in left-to-right order.

Case I: p is not an interior pt of a segment.

$\Rightarrow$ p has been inserted in $\mathcal{Q}$ in the beginning.

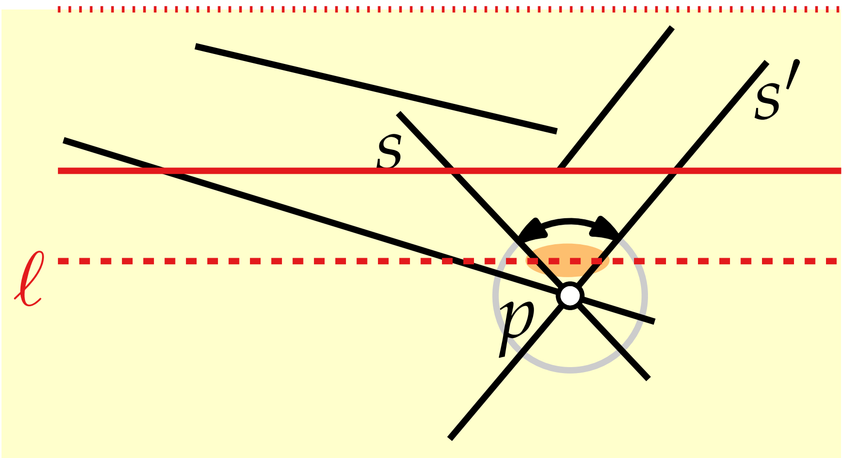
Segm. in $U(p)$ and $L(p)$ are stored with p in the beginning.

When p is processed, we output all segm. in $U(p) \cup L(p)$.

$\Rightarrow$ All segments that contain p are reported.

Correctness (Case II)

Case II: p is an int. point of some segment, i.e., $C(p) \neq \emptyset$.
If p is not an endpt, need that p is inserted into $\mathcal{Q}$ before ℓ reaches p.



We also need that *every* segment with p as an interior point is added to $C(p)$.

Let $s, s' \in C(p)$ be neighbors in the circular ordering of $C(p) \cup \{\ell\}$ around p. Imagine moving ℓ slightly back in time.
Then s, s' were neighbors in the left-to-right order on ℓ (in $\mathcal{T}$).
At the beginning of the alg., they weren't neighbors in $\mathcal{T}$.
$\Rightarrow$ There was some moment when they became neighbors!
This is when $\{p\} = s \cap s'$ was inserted into $\mathcal{Q}$. $\square$

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Part V:
Running Time

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$\mathcal{Q} \leftarrow \emptyset$; $\mathcal{T} \leftarrow \langle$ vertical lines at $x = -\infty$ and $x = +\infty \rangle$ // sentinels
foreach $s \in S$ **do** // initialize event queue $\mathcal{Q}$
  foreach endpoint p of s **do**
    if $p \notin \mathcal{Q}$ **then** $\mathcal{Q}$.insert(p); $L(p) = U(p) = C(p) = \emptyset$
    if p lower endpt of s **then** $L(p)$.append(s)
    if p upper endpt of s **then** $U(p)$.append(s)
while $\mathcal{Q} \neq \emptyset$ **do**
  $p \leftarrow \mathcal{Q}$.nextEvent()
  $\mathcal{Q}$.deleteEvent(p)
  handleEvent(p)

handleEvent(event p)
if $|U(p) \cup L(p) \cup C(p)| > 1$ **then**
  report int. in p, report segments in $U(p) \cup L(p) \cup C(p)$
delete $L(p) \cup C(p)$ from $\mathcal{T}$ // consecutive in $\mathcal{T}$!
insert $U(p) \cup C(p)$ into $\mathcal{T}$ in their order slightly below ℓ
if $U(p) \cup C(p) = \emptyset$ **then**
  $b_{\text{left}}/b_{\text{right}}$ = left/right neighbor of p in $\mathcal{T}$
  findNewEvent(b_{left}, b_{right}, p) → $\{x\} = s \cap s'$
    if $x \notin \mathcal{Q}$ **then** $\mathcal{Q}$.insert(x)
else
  $s_{\text{left}}/s_{\text{right}}$ = leftmost/rightmost segment in $U(p) \cup C(p)$
  b_{left} = left neighbor of s_{left} in $\mathcal{T}$
  b_{right} = right neighbor of s_{right} in $\mathcal{T}$
  findNewEvent(b_{left}, s_{left}, p)
  findNewEvent(b_{right}, s_{right}, p)

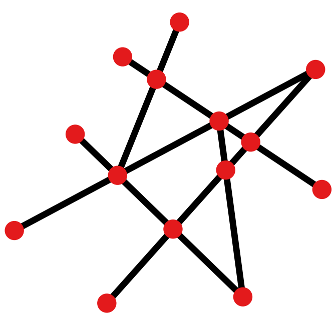
Running time?

Running Time

Check your knowledge about planar graphs!

Lemma. findIntersections() finds I intersection points among n non-overlapping line segments in $O((n+I)\log n)$ time.

Proof. Let p be an event pt,
$m(p) = |L(p) \cup C(p)| + |U(p) \cup C(p)|$
and $m = \sum_p m(p)$.



Then it's clear that the runtime is $O((m+n)\log n)$.
We show that $m \in O(n+I)$. ($\Rightarrow$ lemma)
Define (geometric) graph $G = (V, E)$ with
$V = \{$ endpts, intersection pts $\} \Rightarrow |V| \le 2n + I$.
For any $p \in V$: $m(p) = \deg(p)$.
$\Rightarrow m = \sum_p \deg(p) = 2|E| \le 2 \cdot (3|V| - 6)$
$\in O(n+I)$ $\square$

Euler (G is planar!!)

Today's Main Result

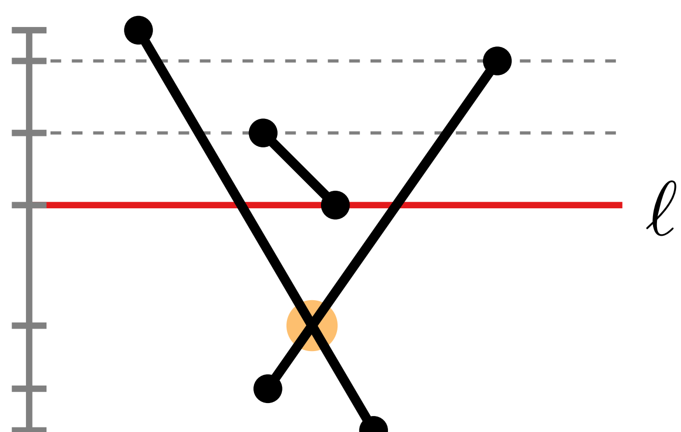
Theorem. We can report all I intersection points among n non-overlapping line segments in the plane and report the segments involved in the intersections in $O((n+I)\log n)$ time and $O(n)$ space.

Sure? The event-point queue $\mathcal{Q}$ contains

- all segment end pts below the sweep line
- all intersection pts below the sweep line

$\Rightarrow$ (worst-case) space consumption $\in \Theta(n+I)$:-(

Can we do better?



– insert $s \cap s'$ into $\mathcal{Q}$
– remove $s \cap s'$ from $\mathcal{Q}$
– re-insert $s \cap s'$ into $\mathcal{Q}$

$\Rightarrow$ need just $O(n)$ space; (asymptotic) running time doesn't change $\square$