Computational Geometry

Lecture 1:
Convex Hull or Mixing Things

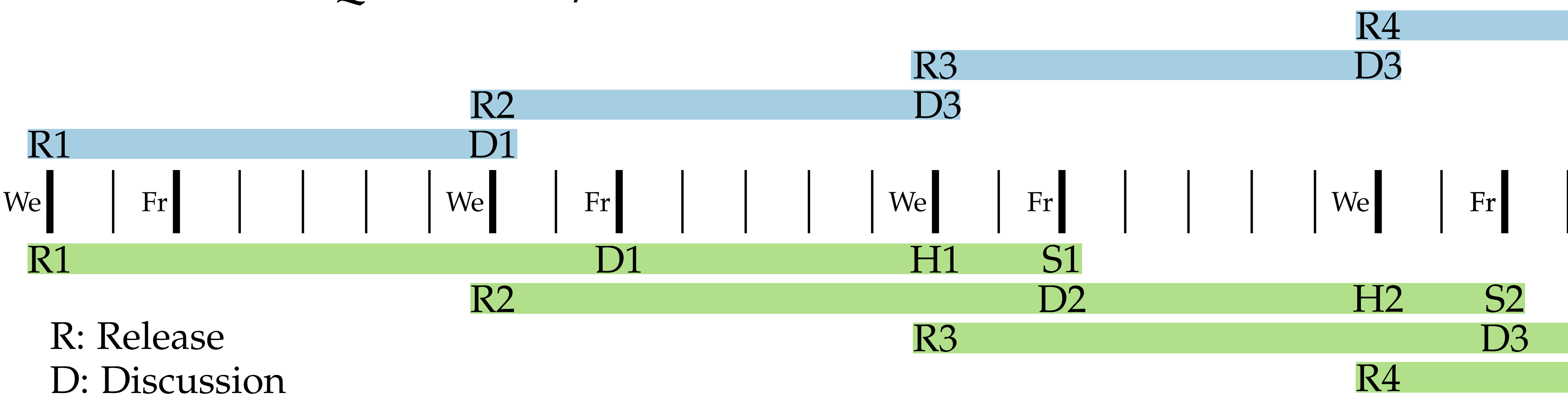
Part I:
Organizational & Overview

Philipp Kindermann

Winter Semester 2020

Organizational

Lectures: Pre-recorded videos (as you see here)
Release date: One week before the lecture
Wed 10:15 – 11:45: Questions/Discussion in Zoom
Questions/Tasks in the Videos



R: Release
D: Discussion
H: Hand In
S: Solutions

Tutorials: One sheet per lecture
$\geq$ 50%: bonus on exam grade
Fri 14:15 – 15:45: Solutions/Discussion in Zoom

Our Lectures and Seminars

Algorithms and Data Structures

Algorithmic Graph Theory

WS	Advanced Algorithms	Computational Geometry	Approximation Algorithms
SS	Exact Algorithms	Graph Visualization	Algorithms for Geographic Information Systems

Seminar Graph Visualization

Seminar Algorithms for Programming Contests

Master Project

Master Thesis

Computational Geometry

Learning goals: At the end of this lecture you will be able to

- decide which algorithms help to solve a number of fundamental *geometric* problems,
- analyze new problems and find *efficient* solutions with the concepts of the lecture.

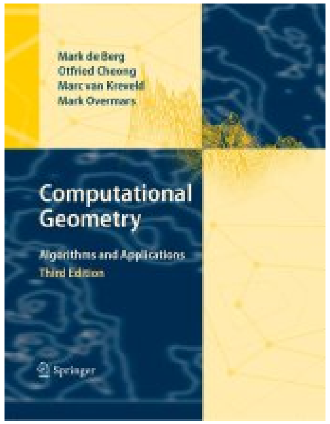
Requirements:

- Big-Oh notation (Landau); e.g., $O(n \log n)$
- Some basic *Algorithms & Data Structures*
  (Balanced) binary search tree, priority queue
- Some basic *Algorithmic Graph Theory*
  Breadth-first search, Dijkstra's algorithm

Content (Prelim.)

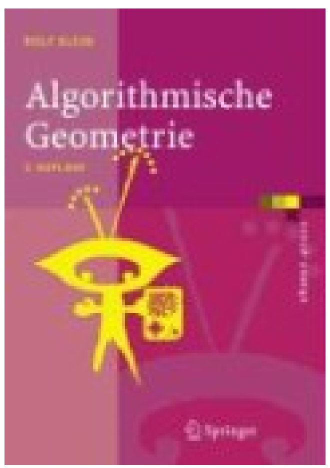
1. Convex Hull in 2D
2. Segment Intersection
3. Polygon Triangulation
4. Linear Programming
5. Orthogonal Range Queries
6. Point Location
7. Voronoi Diagram
8. Delaunay Triangulation
9. Convex Hull in 3D
10. Motion Planning
11. Simplex Range Searching
12. Visibility Graph & Shortest Path

Literature

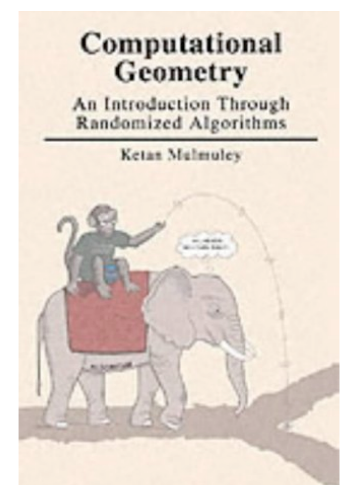


M. de Berg, O. Cheong, M. van Kreveld, M. Overmars:
Computational Geometry: Algorithms & Applications.
Springer, 3rd edition, 2008

Main resource for this course!
Abbreviated as: Comp. Geom A&A



Rolf Klein:
Algorithmische Geometrie: Grundlagen, Methoden, Anwendungen.
Springer, 2nd edition, 2005



Ketan Mulmuley:
Computational Geometry: An Introduction Through Randomized Algorithms. Prentice Hall, 1st edition, 1993

Computational Geometry

Lecture 1:
Convex Hull or Mixing Things

Part II:
Mixing Things

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Mixing Things

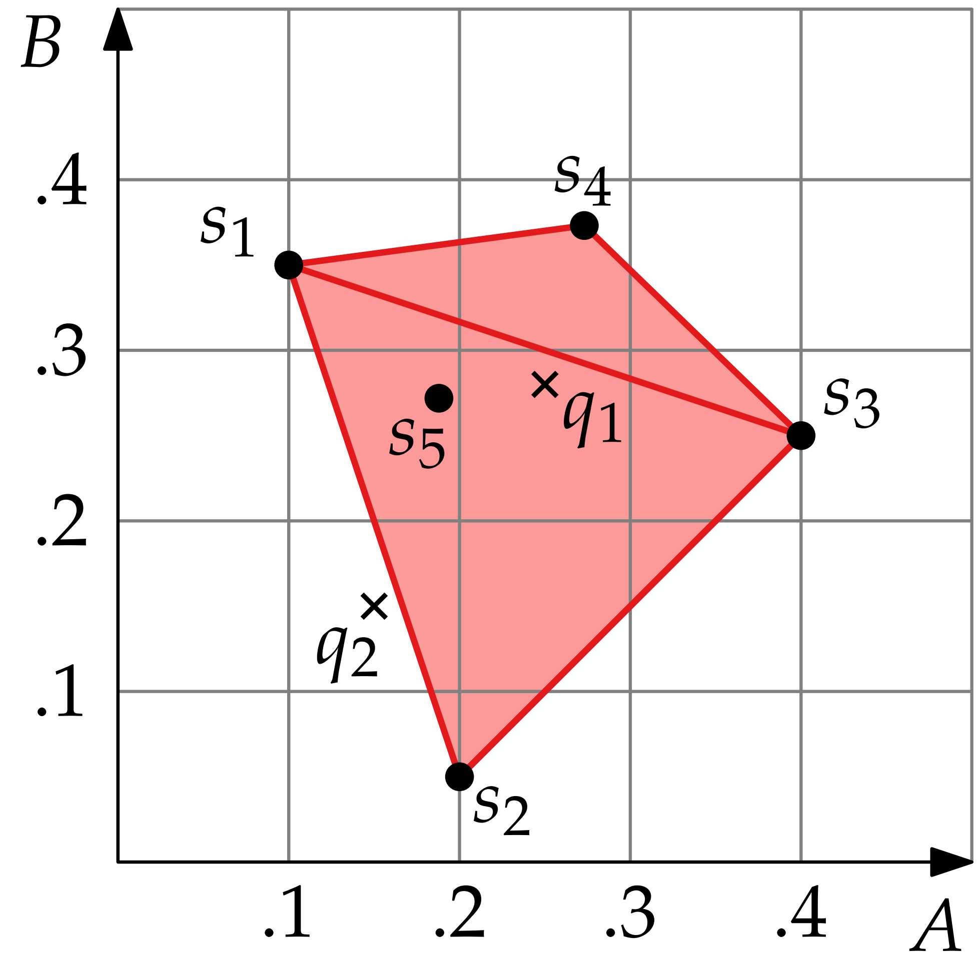
Given...

subst.	fract. A	fract. B
s_1	10 %	35 %
s_2	20 %	5 %
s_3	40 %	25 %

can we mix

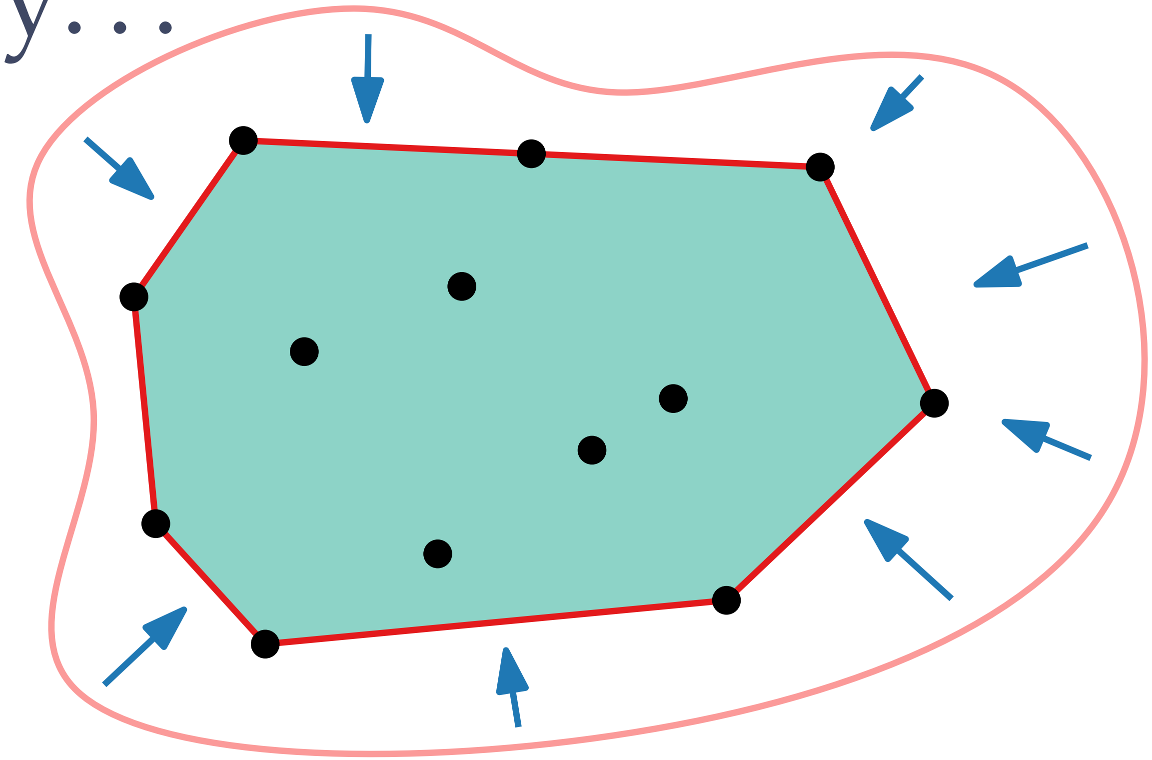
q_1	25 %	28 %
q_2	15 %	15 %

using s_1, s_2, s_3 **?**



Observation. Given a set $S \subset \mathbb{R}^{\cancel{2}d}$ of substances, we can mix a substance $q \in \mathbb{R}^{\cancel{2}d}$ using the substances in $S \quad \Leftrightarrow \quad q \in \mathrm{CH}(S)$.

Formally...



Given $S \subset \mathbb{R}^2$, how do we define the *convex hull* $\mathrm{CH}(S)$?

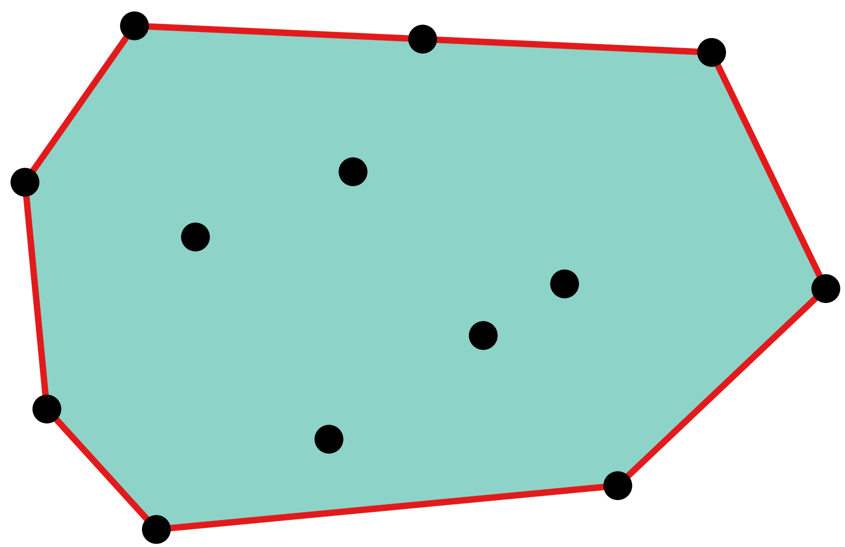
Physics approach: – take (large enough) elastic rope
– stretch and let go
– take area inside (and on) the rope

Maths approach: – define *convex*
– define $\mathrm{CH}(S) = \bigcap_{C \supseteq S:\ C \text{ convex}} C$

Towards Computation

$$\mathrm{CH}(S) \stackrel{\text{def}}{=} \bigcap_{C \supseteq S\colon C \text{ convex}} C$$

Problem with maths approach: *This set is* HUGE!



Maybe we can do with a little less?

Claim. $$\mathrm{CH}(S) = \bigcap_{\substack{H \supseteq S\colon \\ H \text{ closed halfplane}}} H = \bigcap_{\substack{H \supseteq S\colon H \text{ cl. halfplane}, \\ |\partial H \cap S| \geq 2}} H$$

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Part III:
Algorithmic Approach

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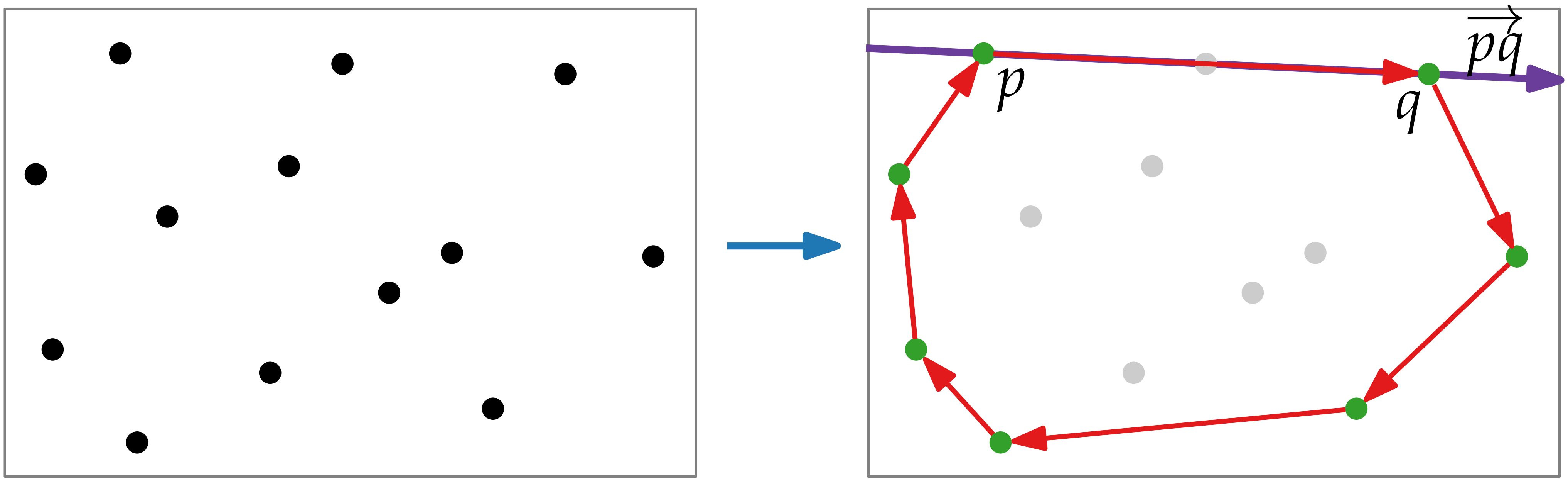
Algorithmic Approach

Input: set S of n points in the plane, that is, $S \subset \mathbb{R}^2$



Output: list of vertices of $\mathrm{CH}(S)$ in clockwise order

Observation. (p, q) is an edge of $\mathrm{CH}(S)$ $\Leftrightarrow$
each point in S lies
– strictly to the right of the directed line $\overrightarrow{pq}$ or
– on the line segment $\overline{pq}$

Finally, an Algorithm

FirstConvexHull(S)

$E \leftarrow \emptyset$

foreach $(p, q) \in S \times S$ with $p \neq q$ **do**

 $valid \leftarrow true$

 foreach $r \in S$ **do**

  if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**

   $valid \leftarrow false$

 if $valid$ **then**

  $E \leftarrow E \cup \{(p, q)\}$

from E construct sorted list L of vertices of CH(S)

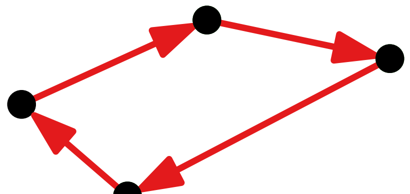
return L

r strictly right of $\overrightarrow{pq}$

$$\Updownarrow$$

$$\begin{vmatrix} x_r & y_r & 1 \\ x_p & y_p & 1 \\ x_q & y_q & 1 \end{vmatrix} < 0$$

Important:
Test takes $O(1)$ time!



Running Time Analysis

FirstConvexHull(S)

$E \leftarrow \emptyset$

foreach $(p, q) \in S \times S$ with $p \neq q$ **do** $(n^2 - n) \cdot$

  valid $\leftarrow$ *true*

  foreach $r \in S$ **do** $n \cdot$

    if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then** $\Theta(1)$

      valid $\leftarrow$ *false*

  $\Theta(n)$

  if *valid* **then**

    $E \leftarrow E \cup \{(p, q)\}$

$\Theta(n^3)$

from E construct sorted list L of vertices of CH(S)

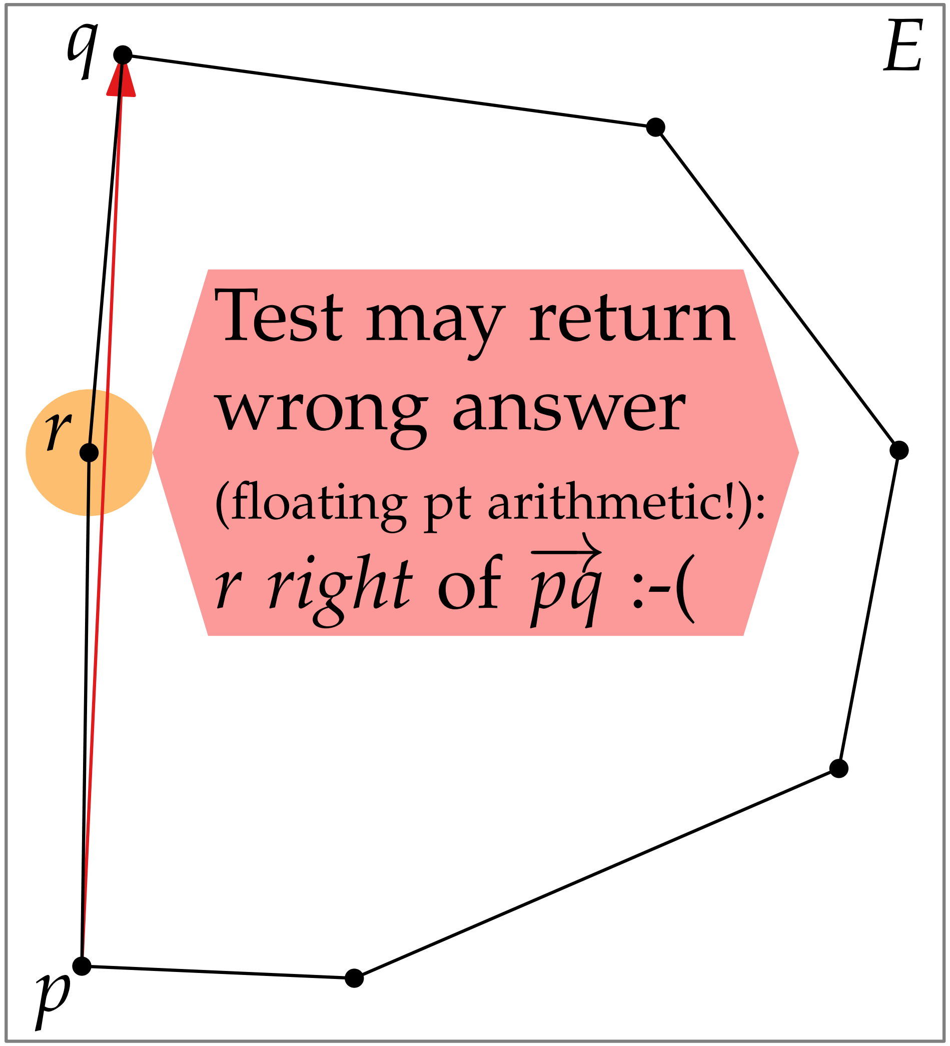
return L

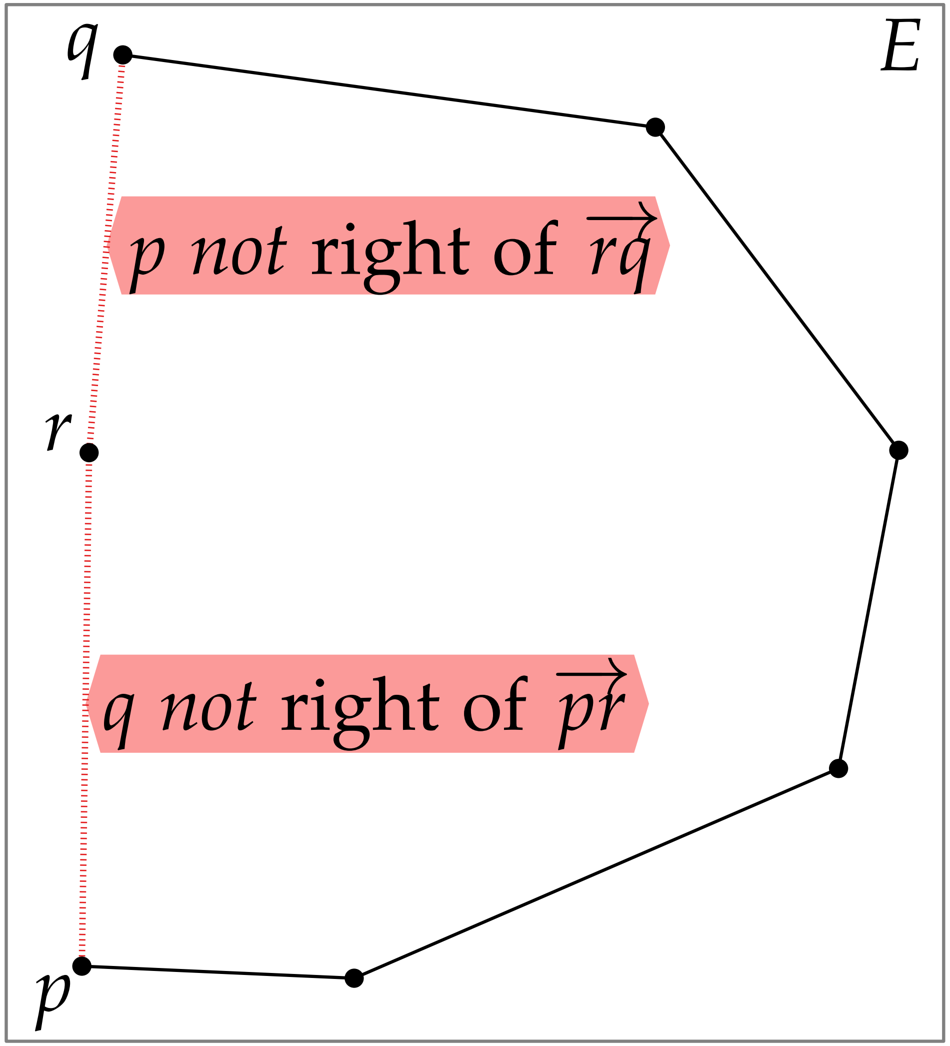
$O(n^2)$

Lemma. We can compute the convex hull of n pts in the plane in $\Theta(n^3)$ time.

Discussion

if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**
  $\lfloor$ *valid* $\leftarrow$ *false*





Observation. Algorithm FirstConvexHull is not *robust*.

Computational Geometry

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Part IV:
Graham Scan

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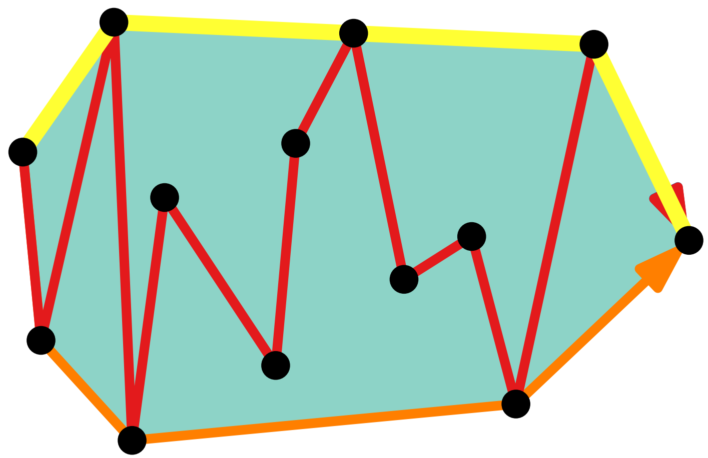
Winter Semester 2020

New Ideas (Graham Scan)



- split computation in two
- bring pts in lexicographic order
- proceed incrementally

UpperConvexHull(S: set of pts in the plane)
  $\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort S lexicographically
  $L \leftarrow \langle p_1, p_2 \rangle$
  for $i \leftarrow 3$ **to** n **do** // compute upper convex hull of $\{p_1, p_2, \ldots, p$
    L.append(p_i)
    while $|L| > 2$ **and** last 3 pts in L make a left turn **do**
      remove second last pt from L
  return L

Running Time Analysis

UpperConvexHull(S: set of pts in the plane)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort S lexicographically $\quad O(n \log n)$

$L \leftarrow \langle p_1, p_2 \rangle$

for $i \leftarrow 3$ **to** n **do** $\quad \cancel{(n-2)\cdot}$

L.append(p_i) $\quad O(n)$

while $|L| > 2$ **and** last 3 pts in L make a left turn **do**

remove second last pt from L $\quad \cancel{?}$

return L

Amortized analysis:

– each pt $p_2, \ldots, p_{n-1}$ pays 1 € for its potential removal later on

– this pays for the total effort of all executions of the while loop

Theorem. We can compute the convex hull of n pts in the plane in $O(n \log n)$ time – in a robust way.

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Part V:
Output-Sensitive Algorithms

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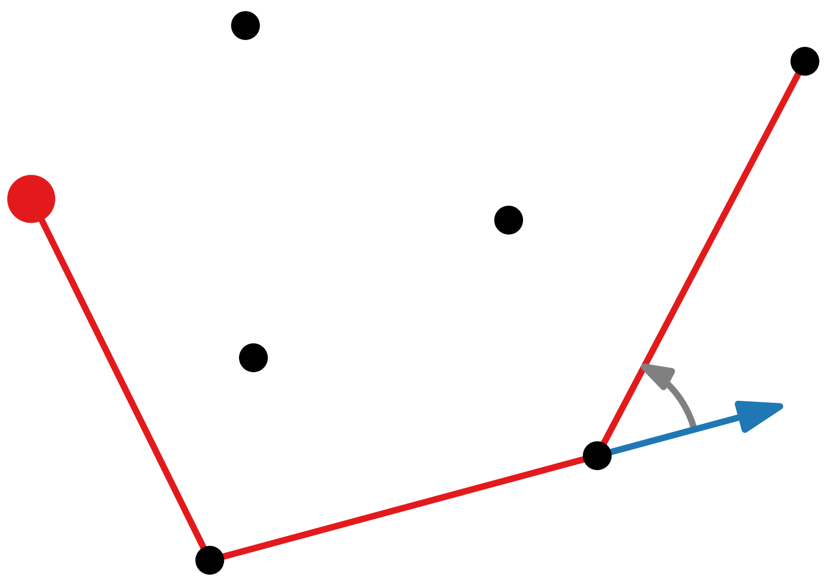
Output-Sensitive Algorithms

- Jarvis' gift-wrapping algorithm (aka Jarvis' march)

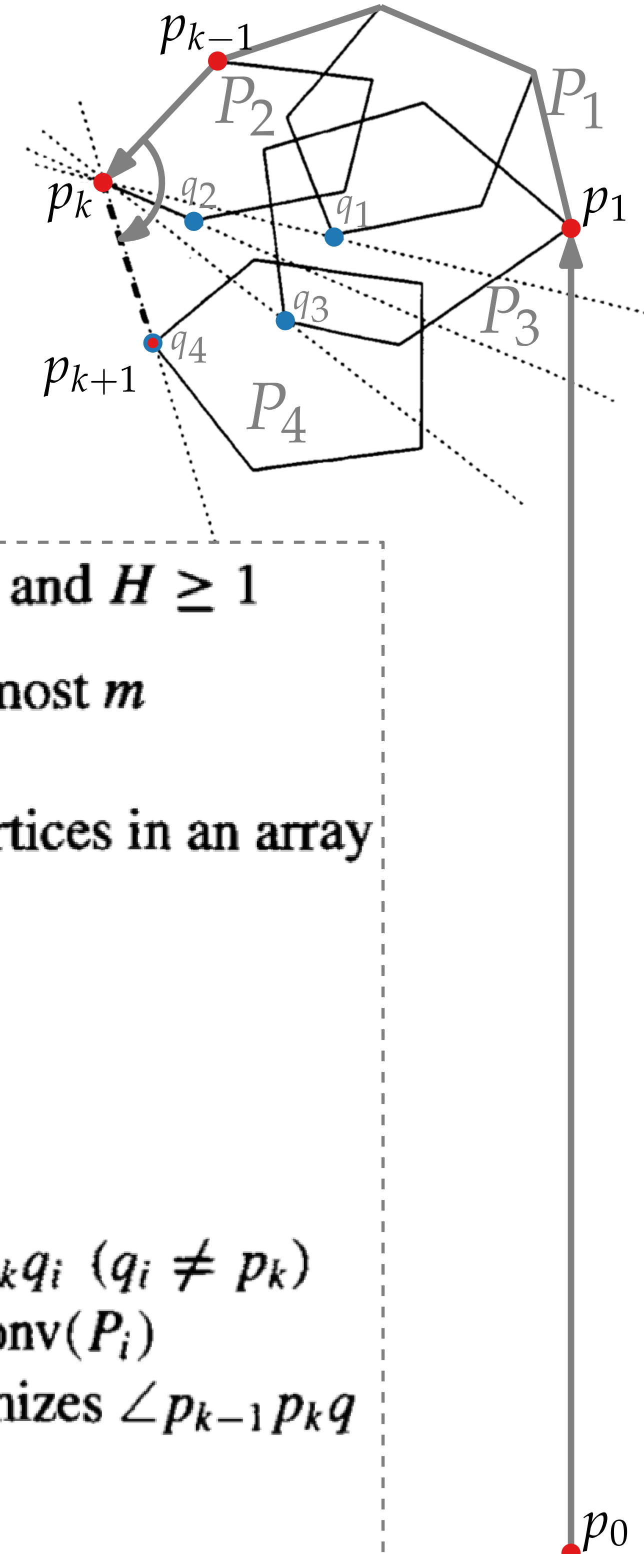
  Runtime? $O(n \cdot h)$



- Chan's exponential-search algorithm $O(n \log h)$

…where $h = |\mathrm{CH}(S)|$ = size of the output

Chan's Algorithm



Algorithm Hull2D(P), where $P \subset E^2$

for $t = 1, 2, \ldots$ do
  $L \leftarrow$ Hull2D(P, m, H), where $m = H = \min\{2^{2^t}, n\}$
  if $L \neq$ *incomplete* then return L

Algorithm Hull2D(P, m, H), where $P \subset E^2$, $3 \leq m \leq n$, and $H \geq 1$

partition P into subsets $P_1, \ldots, P_{\lceil n/m \rceil}$ each of size at most m
for $i = 1, \ldots, \lceil n/m \rceil$ do
  compute conv(P_i) by Graham's scan and store its vertices in an array in ccw order [in $O(m \log m)$ time]
$p_0 \leftarrow (0, -\infty)$
$p_1 \leftarrow$ the rightmost point of P
for $k = 1, \ldots, H$ do
  for $i = 1, \ldots, \lceil n/m \rceil$ do
    compute the point $q_i \in P_i$ that maximizes $\angle p_{k-1} p_k q_i$ ($q_i \neq p_k$) by performing a binary search on the vertices of conv(P_i)
  $p_{k+1} \leftarrow$ the point q from $\{q_1, \ldots, q_{\lceil n/m \rceil}\}$ that maximizes $\angle p_{k-1} p_k q$
  if $p_{k+1} = p_1$ then return the list $\langle p_1, \ldots, p_k \rangle$
return *incomplete*

Chan's Algorithm

[Text copied on October 17, 2017 from: https://en.wikipedia.org/wiki/Chan's_algorithm]

Initially, we assume that the value of h is known and make a parameter $m = h$. This assumption is not realistic, but we remove it later. The algorithm starts by arbitrarily partitioning P into at most $1 + \frac{n}{m}$ subsets Q with at most m points each. Then, it computes the convex hull of each subset Q using an $O(n \log n)$ algorithm – **Graham's scan**. Note that, as there are $O(n/m)$ subsets of $O(m)$ points each, this phase takes $O(n/m) \cdot O(m \log m) = O(n \log m)$ time.

The second phase consists of executing the **Jarvis' march** algorithm algorithm and using the precomputed convex hulls to speed up the execution. At each step in Jarvis's march, we have a point p_i in the convex hull, and need to find a point $p_{i+1} = f(p_i, P)$ such that all other points of P are to the right of the line $p_i p_{i+1}$. If we know the convex hull of a set Q of m points, then we can compute $f(p_i, Q)$ in $O(\log m)$ time, by using binary search. We can compute $f(p_i, Q)$ for all the $O(n/m)$ subsets Q in $O(n/m \log m)$ time. Then, we can determine $f(p_i, P)$ using the same technique as normally used in Jarvis's march, but only considering the points that are $f(p_i, Q)$ for some subset Q. As Jarvis's march repeats this process $O(h)$ times, the second phase also takes $O(n \log m)$ time, and therefore $O(n \log h)$ time if $m = h$.
By running the two phases described above, we can compute the convex hull of n points in $O(n \log h)$ time, assuming that we know the value of h. If we make $m < h$, we can abort the execution after $m + 1$ steps, therefore spending only $O(n \log m)$ time (but not computing the convex hull). We can initially set m as a small constant (we use 2 for our analysis, but in practice numbers around 5 may work better), and increase the value of m until $m > h$, in which case we obtain the convex hull as a result.
If we increase the value of m too slowly, we may need to repeat the steps mentioned before too many times, and the execution time will be large. On the other hand, if we increase the value of m too quickly, we risk making m much larger than h, also increasing the execution time. Similar to strategy used by Chazelle and Matoušek's algorithm, Chan's algorithm squares the value of m at each iteration, and makes sure that m is never larger than n. In other words, at iteration t (starting at 1), we have $m = \min\left(n, 2^{2^t}\right)$. The total running time of the algorithm is

$$\sum_{t=1}^{\lceil \log\log h \rceil} O\left(n \log(2^{2^t})\right) = O(n) \sum_{t=1}^{\lceil \log\log h \rceil} O(2^t) = O\left(n \cdot 2^{1+\lceil \log\log h \rceil}\right) = O(n \log h).$$

To generalize this construction for the 3-dimensional case, an $O(n \log n)$ algorithm to compute the 3-dimensional convex hull should be used instead of Graham scan, and a 3-dimensional version of Jarvis's march needs to be used. The time complexity remains $O(n \log h)$.