Julius-Maximilians-
UNIVERSITÄT
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Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exakte Algorithmen

Lecture 9.2 Reductions and the $W[t]$ Hierarchy

Based on: [Parameterized Algorithms: §13]

(slides by Thomas Bläsius)

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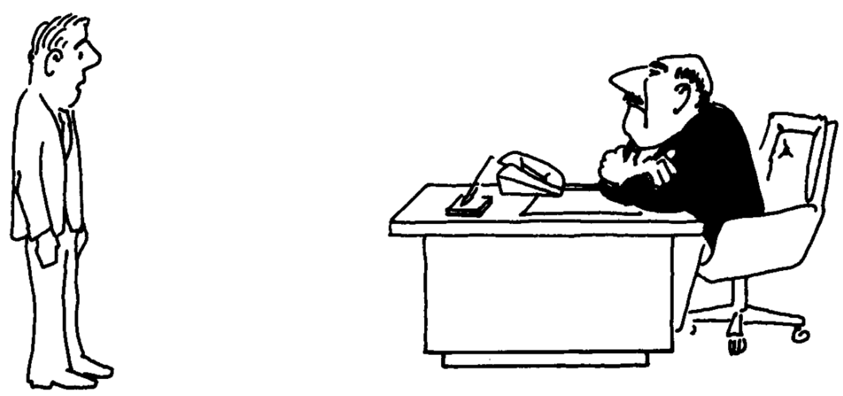
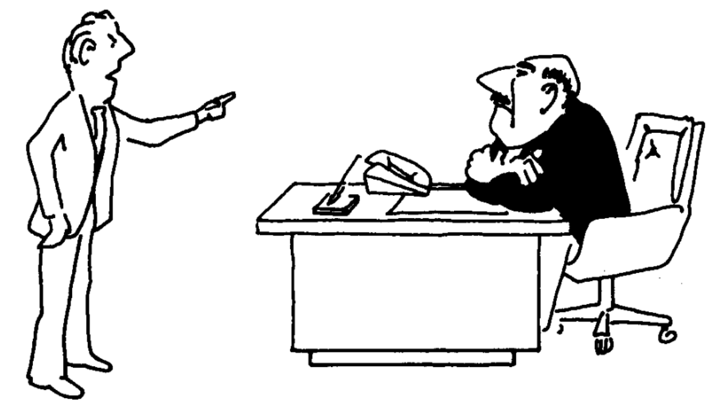
Lehrstuhl für Informatik I

How to obtain lower bounds?



"I can't find an efficient algorithm, I guess I'm just too dumb."



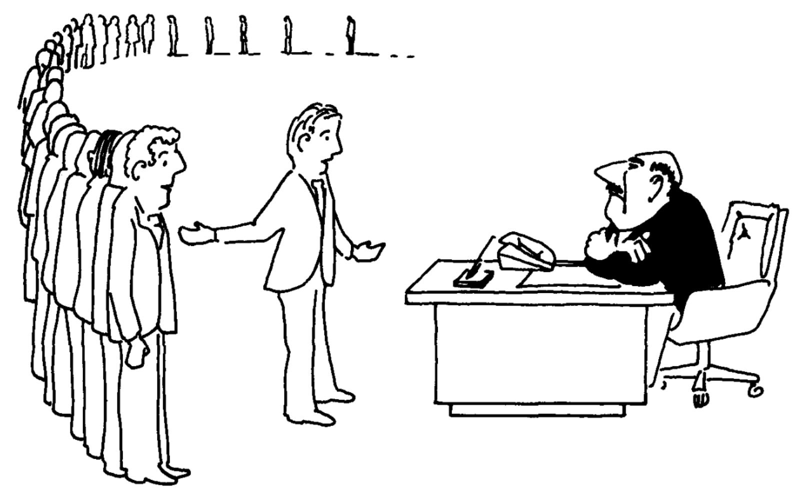
"I can't find an efficient algorithm, because no such algorithm is possible!"

Problem

- difficult to show absence of "nice" algorithms

Solution

- argue that having an efficient algorithm$\Rightarrow$ some well studied diffcult problem can be solved efficiently
- Tool: reduction between problems



"I can't find an efficient algorithm, but neither can all these famous people."

Polynomial Reduction

Reduction from Problem $\mathcal{L}$ to Problem $\mathcal{L}'$

- map each instance I of $\mathcal{L}$ to an instance I' of $\mathcal{L}'$ so that
- I is a YES-instance $\Leftrightarrow$ I' is a YES-instance
- the mapping can be computed in polynomial time

Implications

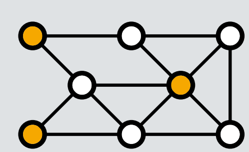
- polynomial time algorithm for $\mathcal{L}'$ $\Rightarrow$ polynomial time algorithm for $\mathcal{L}$
- solving my problem $\mathcal{L}'$ efficiently $\Rightarrow$ solving "difficult" problem $\mathcal{L}$ efficiently

(here, we use "efficient" $\sim$ "polynomial")

Easy Example

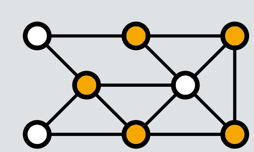
- reduce INDEPENDENT SET to VERTEX COVER
- trivial reduction: G has IS of size k $\Leftrightarrow$ G has VC of size $n-k$
- also: VC $\in$ P $\Rightarrow$ IS $\in$ P
- what about VC $\in$ FPT $\Rightarrow$ IS $\in$ FPT?
  $\rightarrow$ NO, the parameter depends on n
- for "efficient" $\sim$ "FPT" we need a different type of reduction

INDEPENDENT SET



$V' \subseteq V$, so that
$\forall e \in E\colon |e \cap V'| \leq 1$

VERTEX COVER



$V' \subseteq V$, so that
$\forall e \in E\colon |e \cap V'| \geq 1$

Parameterized Reductions

Reduction from Problem $\mathcal{L}$ to Problem $\mathcal{L}'$

- map each instance (I, k) of $\mathcal{L}$ to an instance (I', k') of $\mathcal{L}'$ so that
- (I, k) is a YES-instance $\Leftrightarrow$ (I', k') is a YES-instance where $k' \leq g(k)$
- the map must be computable in FPT-time $(f(k) \cdot |I|^{O(1)})$

(f and g must also be computable functions)

Implications

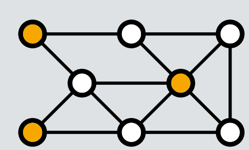
- FPT-algorithm for $\mathcal{L}'$ $\Rightarrow$ FPT algorithm for $\mathcal{L}$
- solving my problem $\mathcal{L}'$ efficiently $\Rightarrow$ solving "difficult" problem $\mathcal{L}$ efficiently

(now, "efficient" $\sim$ "FPT")

Easy Example

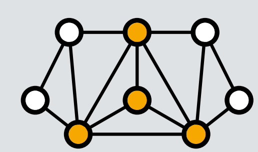
- reduce Independent Set to Clique
- G has a size k IS $\Leftrightarrow$ edge-complement of G has a size k Clique.
- also: Clique $\in$ FPT $\Rightarrow$ IS $\in$ FPT
- reverse reduction also applies:
  Clique $\in$ FPT $\Leftrightarrow$ IS $\in$ FPT
- expectation: Clique, IS $\notin$ FPT

Independent Set



$V' \subseteq V$, so that
$\forall e \in E\colon |e \cap V'| \leq 1$

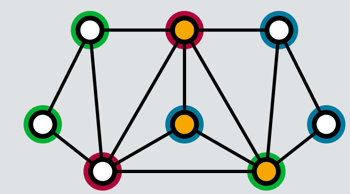
Clique



$V' \subseteq V$, so that
$G[V']$ complete

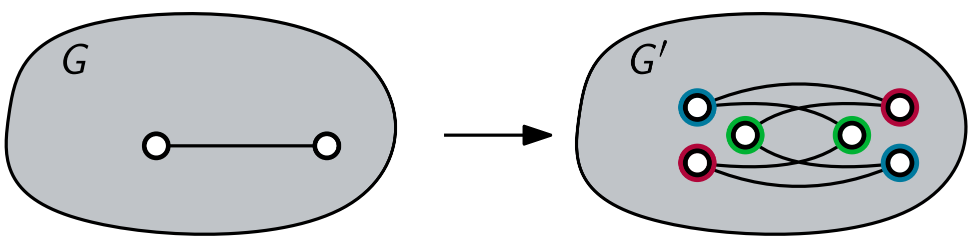
Colored Cliques

Problem: MULTICOLORED CLIQUE
Given: Graph $G = (V, E)$, parameter k,
and partition $(V_1, \ldots, V_k)$ of V.
Find: Clique $V' \subseteq V$ of size k, so that $|V' \cap V_i| = 1$ for each i.



V_1, V_2, V_3

Reduction: from CLIQUE to MULTICOLORED CLIQUE



Instance $(G, 3)$ of CLIQUE

Instance of MC CLIQUE: $(G', 3, (V_1, V_2, V_3))$

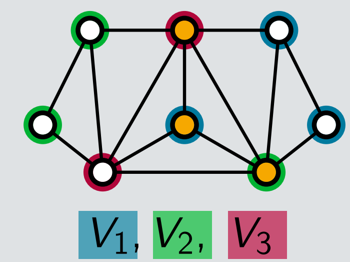
- copy each $v \in V$ to $v^1, \ldots, v^k$, and set $v^i \in V_i$
- for each $uv \in E$ connect u^i with v^j for each $i \neq j$
- $k = k'$

G has a size k clique $\Rightarrow$ G' has a size k colored clique

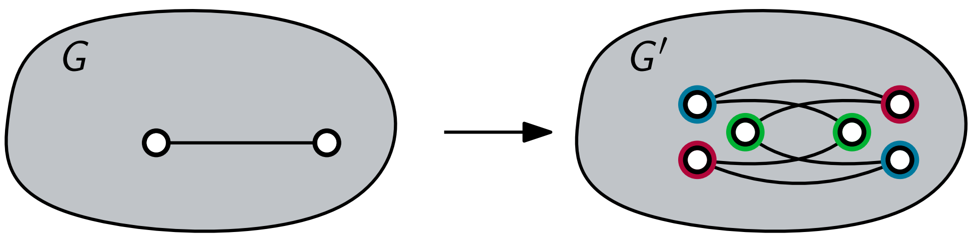
- Let $v_1, \ldots, v_k$ be a clique in G
- then $v_1^1, \ldots, v_k^k$ is a clique in G'
- all these vertices have distinct colors

Colored Cliques

Problem: **MULTICOLORED CLIQUE**
Given: Graph $G = (V, E)$, parameter k,
and partition $(V_1, \ldots, V_k)$ of V.
Find: Clique $V' \subseteq V$ of size k, so that $|V' \cap V_i| = 1$ for each i.



Reduction: from CLIQUE to MULTICOLORED CLIQUE



Instance $(G, 3)$ of CLIQUE

Instance of MC CLIQUE: $(G', 3, (V_1, V_2, V_3))$

- copy each $v \in V$ to $v^1, \ldots, v^k$, and set $v^i \in V_i$
- for each $uv \in E$ connect u^i with v^j for each $i \neq j$
- $k = k'$

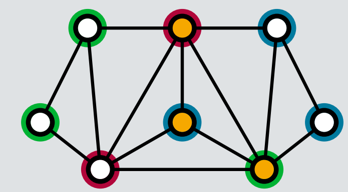
***G'* has a size *k* colored clique $\Rightarrow$ *G* has a size *k* clique**

- Let $v^1_{\pi(1)}, \ldots, v^k_{\pi(k)}$ be a colored clique in G with $\pi\colon \{1, \ldots, k\} \to \{1, \ldots n\}$
- π is injective (the colored clique does not contain two copies of the same vertex)
- thus, $v_{\pi(1)}, \ldots, v_{\pi(k)}$ are k distinct nodes that form a clique in G

$\square$

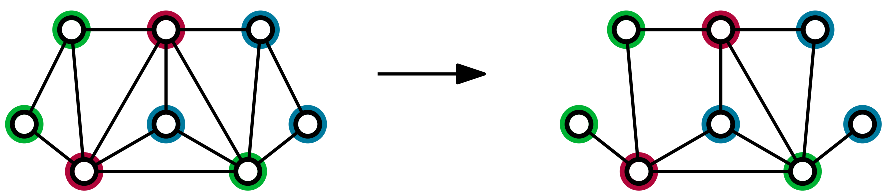
Colored Cliques

Problem: MULTICOLORED CLIQUE
Given: Graph $G = (V, E)$, parameter k,
and partition $(V_1, \ldots, V_k)$ of V.
Find: Clique $V' \subseteq V$ of size k, so that $|V' \cap V_i| = 1$ for each i.



V_1, V_2, V_3

Reduction: from MULTICOLORED CLIQUE to CLIQUE



Instance of MC CLIQUE: $(G, 3, (V_1, V_2, V_3))$

Instance $(G', 3)$ of CLIQUE

- delete edges within each color class
- $k' = k$

G has a colored size k clique $\Rightarrow$ G' has a size k clique

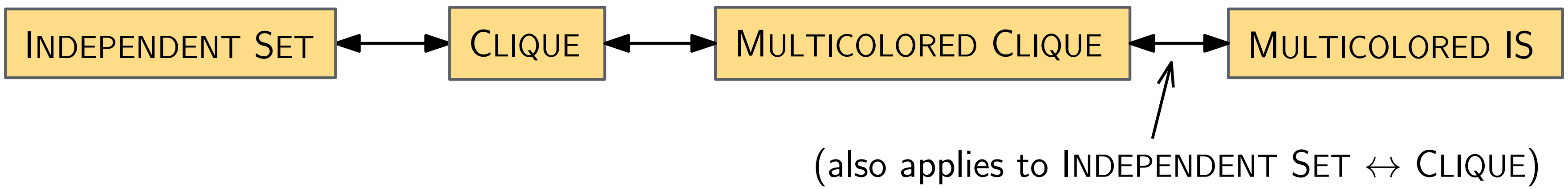
- the colored clique does not use any edges inside a color class
- thus, G' contains a size k clique

G' has a size k clique $\Rightarrow$ G has a colored size k clique

- in G', no monochromatic vertices are adjacent
- thus, each clique must be a colored clique

$\square$

FPT-Reductions so far



Dominating Set

Reduce Multicolored Independent Set to Dominating Set

- enforce the following properties on the Dominating Set instance
  - select exactly one element from each color class
  - avoid simultaneous selection of designated node pairs

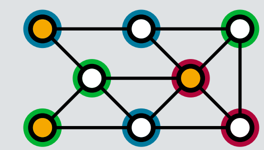
One element in each color class

- vertex x_i adjacent to all vertices in V_i and no others

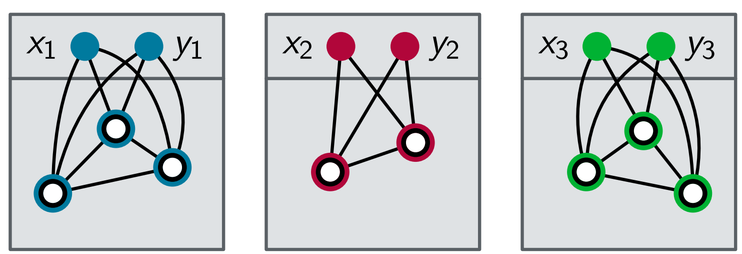
(forces the selection of at least one of $V_i \cup \{x_i\}$)

- Problem: can still choose x_i
- Solution: create a copy y_i of x_i
  (picking both x_i and y_i is too costly to form a size k DS)
- make V_i into a clique as to allow any vertex from V_i to dominate it and both x_i and y_i.

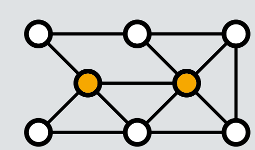
MC Independent Set



$V' \subseteq V$ colored, so that
$\forall e \in E : |e \cap V'| \leq 1$



Dominating Set



$V' \subseteq V$, so that
$\forall v \in V : |N[v] \cap V'| \geq 1$

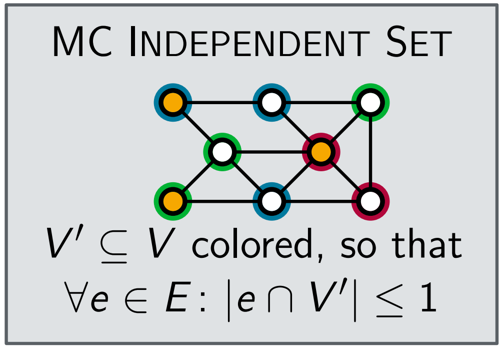
Dominating Set

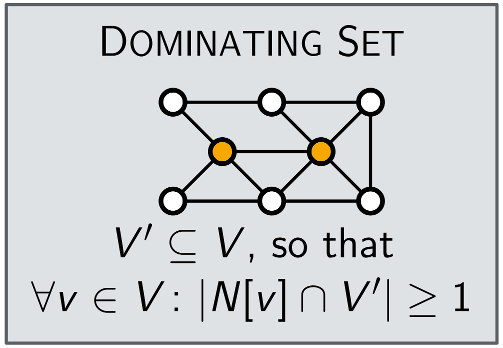
Reduce Multicolored Independent Set to Dominating Set

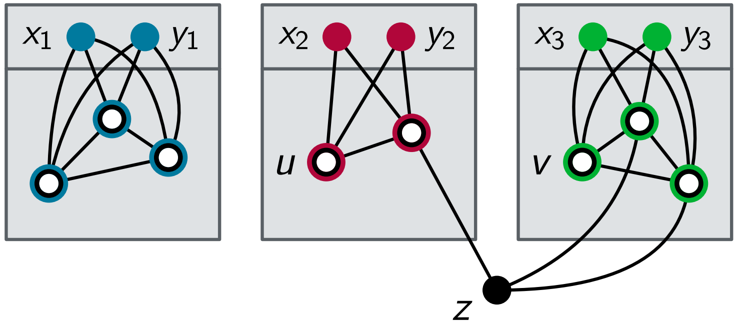
- enforce the following properties on the Dominating Set instance
  - select exactly one element from each color class
  - avoid simultaneous selection of designated node pairs

Avoid selecting both *u* and *v*

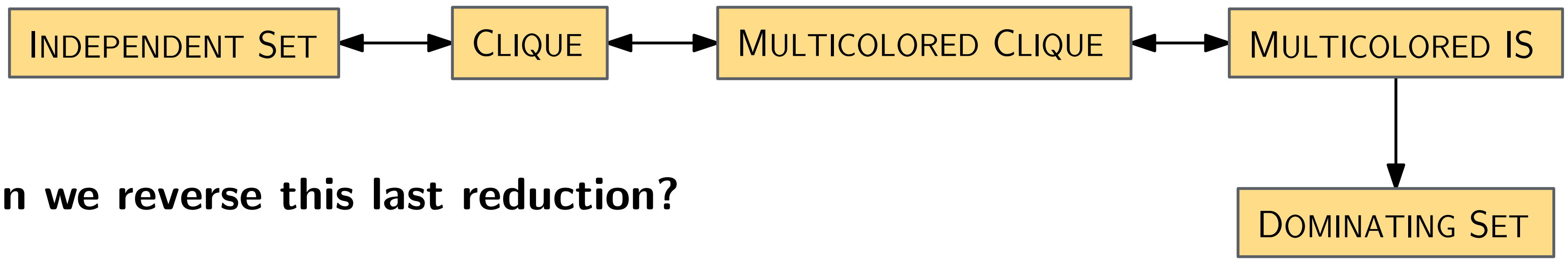
- Idea: insert a vertex z that is dominated unless both u and v are picked
  - z is not adjacent to u and not adjacent to v but is adjacent to all other vertices in the color classes of u and v







FPT-Reductions so far



Can we reverse this last reduction?

- it is not known ... but seems very unlikely
- DOMINATING SET is (probably) more difficult than CLIQUE oder INDEPENDENT SET
  (DS $\in$ FPT $\Rightarrow$ IS $\in$ FPT, but not the other way around)
- we need a refined notion of "hardness"

Similar to the complexity class P

- usually, one focuses on: NP-hardness
- but there are also **intermediate** problems (assuming P $\neq$ NP)
- however, no natural hierarchy of NP-intermediate problems
  (candidates: prime factorization, graph isomorphism)

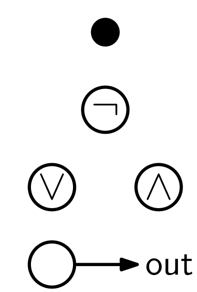
And now? What about FPT?

- define natural hierarchy of complexity classes
- establish prototypical problem for each level

Weighted Circuit Satisfiability

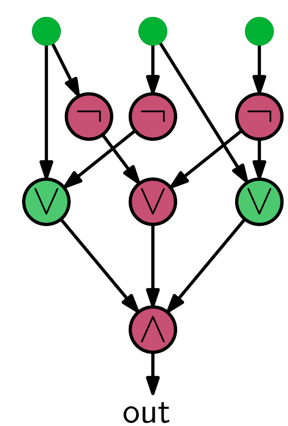
Boolean Circuits

- directed acyclic graph (DAG) with the following node types:
  - sources (in-degree 0) are **input nodes**
  - **NEGATION** nodes have in-degree 1
  - **AND**- resp. **OR**-nodes have in-degree ≥ 2
  - one sink (out-degree 0) is the **output node**



- an assignment of 0 or 1 to each input node propogates through the other nodes, providing an output value. (in the natural way)
- an assignment **statisfying** when the output value is 1
- the **weight** of an assignment is the number of 1s used

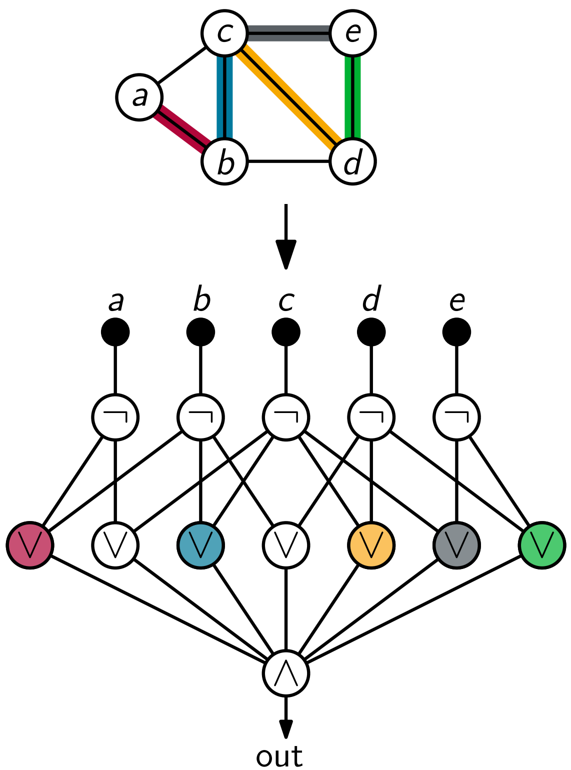
example



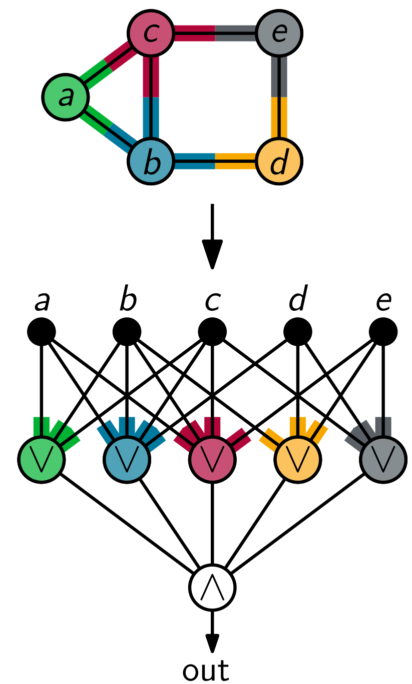
Problem: Weighted Circuit Satisfiability (WCS)
Given a boolean circuit and a parameter k
Find: A weight k satisfying assignment

Reductions visualized

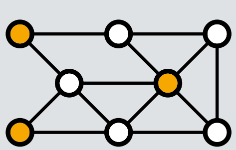
Independent Set → WCS



Dominating Set → WCS

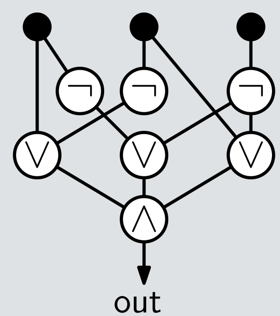


Independent Set

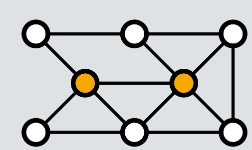


$V' \subseteq V$, so that
$\forall e \in E \colon |e \cap V'| \leq 1$

WCS



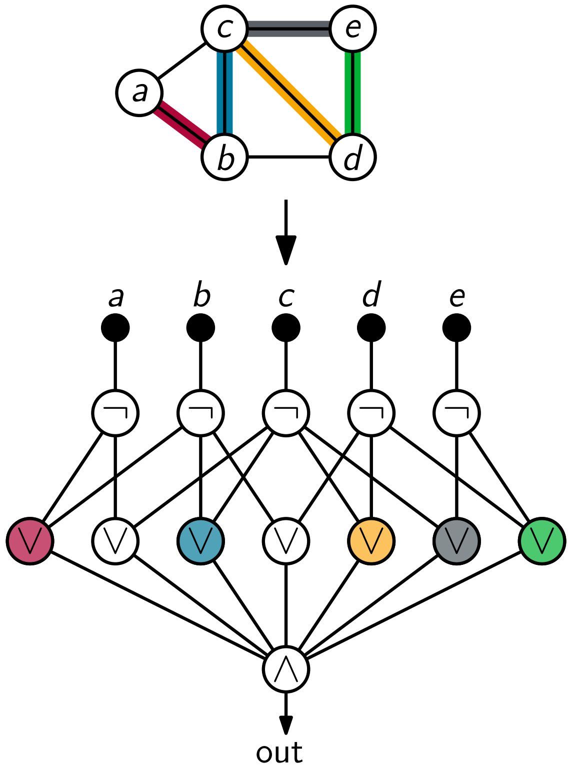
Dominating Set



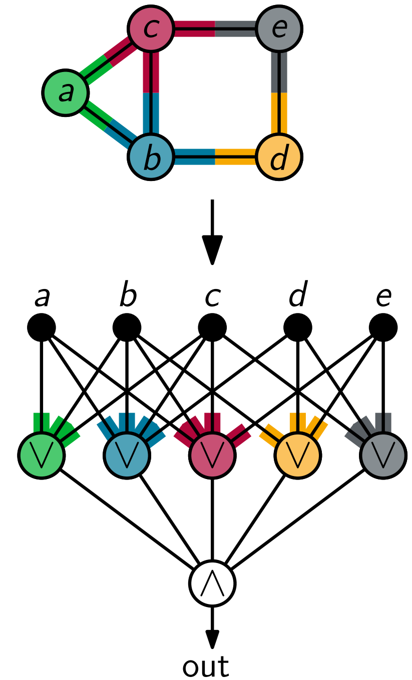
$V' \subseteq V$, so that
$\forall v \in V \colon |N[v] \cap V'| \geq 1$

Reductions visualized

Independent Set → WCS



Dominating Set → WCS



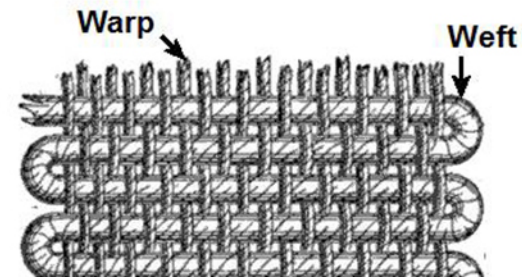
Observations

- the circuits have constant depth
- the circuit for DS contains more nodes with in-degree > 2
- is DS harder (in terms of FPT) than IS?

Weft

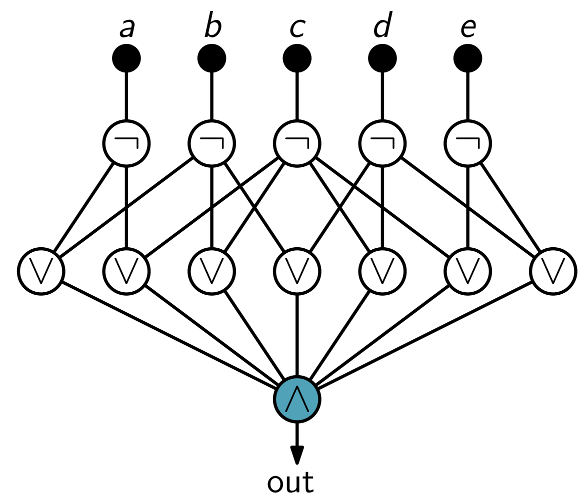
Definition
The **Weft** of a boolean circuit is the maximum number of nodes with in-degree > 2 on a directed path.

Problem
WCS[t] is WCS limited to circuits with constant depth and weft at most t.



Definition
The class $\boldsymbol{W[t]}$ contains all problems with a paramterized reduction to WCS[t].

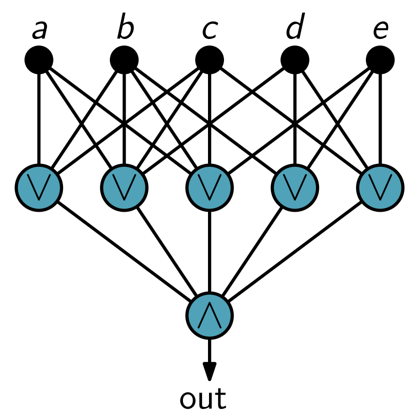
Weft 1



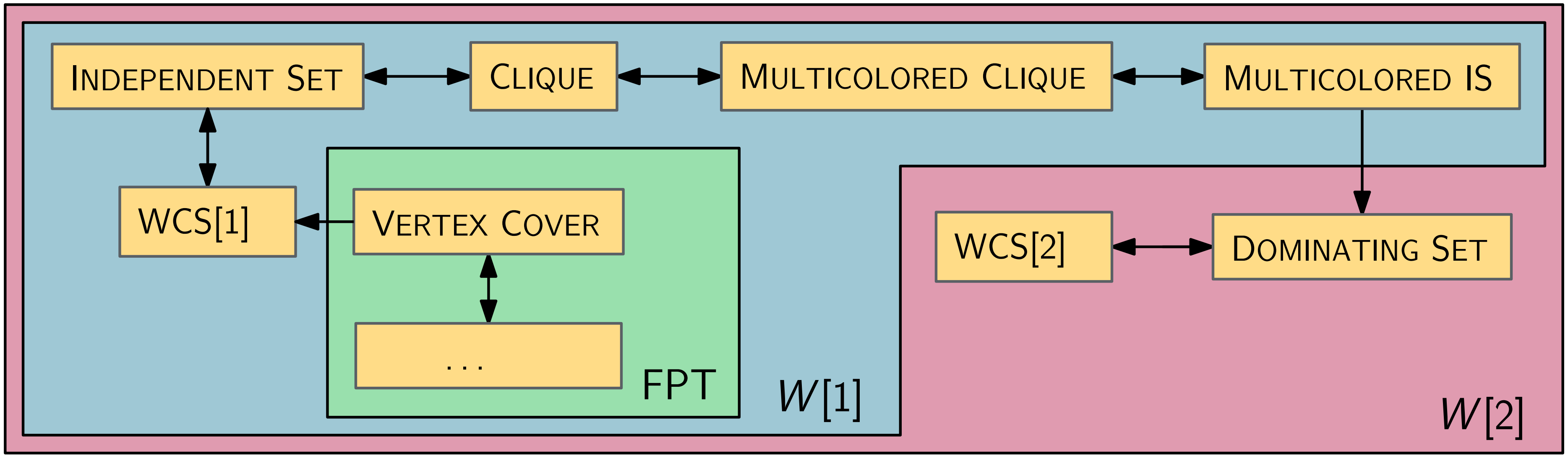
We have seen

- INDEPENDENT SET $\in W[1] \subseteq W[2]$
- DOMINATING SET $\in W[2]$

Weft 2



FPT-Reductions so far



Further reductions

- one can also reduce WCS[1] to Independent Set
- and WCS[2] to Dominating Set
- so, all problems in $W[1]$ reduce to IS
  $\rightsquigarrow$ IS is $W[1]$-complete
- similarly DS is $W[2]$-complete
- note: $W[1] \subseteq W[2]$ why?
- also: FPT $\subseteq W[1] \subseteq W[2]$ why?

Summary

The W-Hierarchy

- Complexity Classes $\mathrm{FPT} \subseteq W[1] \subseteq W[2] \subseteq W[3] \subseteq \cdots$
- $W[t]$ defined via a prototypical complete problem $\mathrm{WCS}[t]$:
  $\mathcal{L} \in W[t] \Leftrightarrow \mathcal{L}$ can be reduced to $\mathrm{WCS}[t]$ (by FPT-reduction)
- Inclusions **expected** to be strict

Is my $W[t]$-completeness proof useless if $W[t] = \mathrm{FPT}$?

- finding an FPT-algorithm for a complete problem would provide an FPT-algorithm for all problems in the class.

How do I show that my problem is hard?

- reduce a known hard problem to your problem
- reducing from MC Independent Set or MC Clique provides $W[1]$-hardness
- reducing from Dominating Set or Set Cover provides $W[2]$-hardness