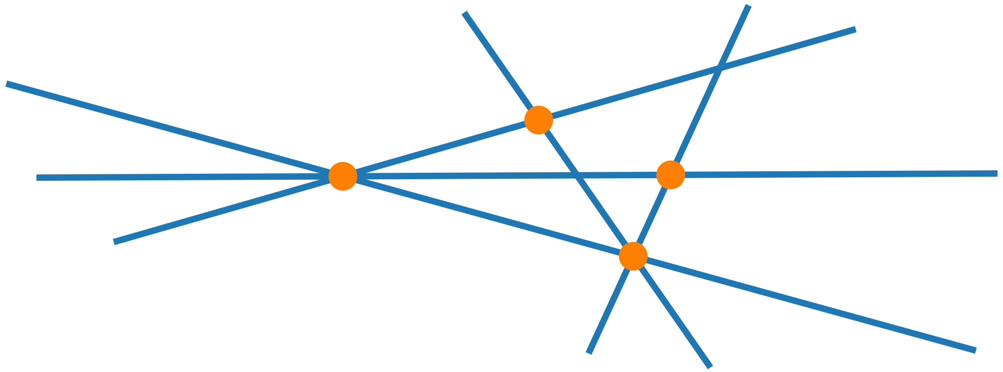
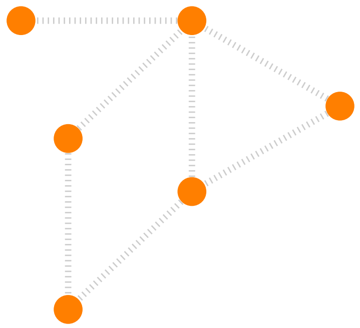
Visualization of graphs

The Crossing Lemma

And its applications

Jonathan Klawitter · Summer semester 2020

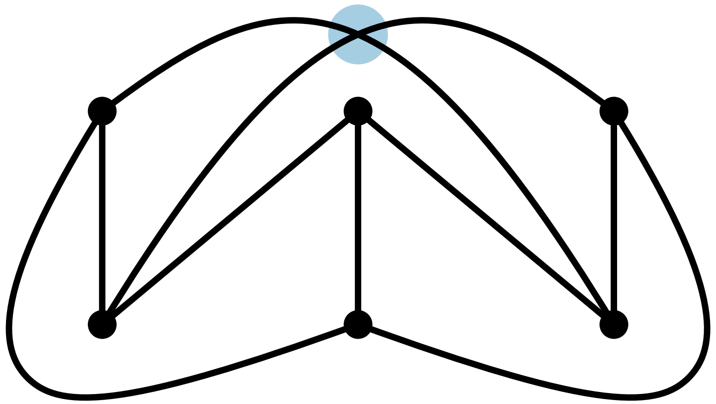




Crossing number and topological graphs

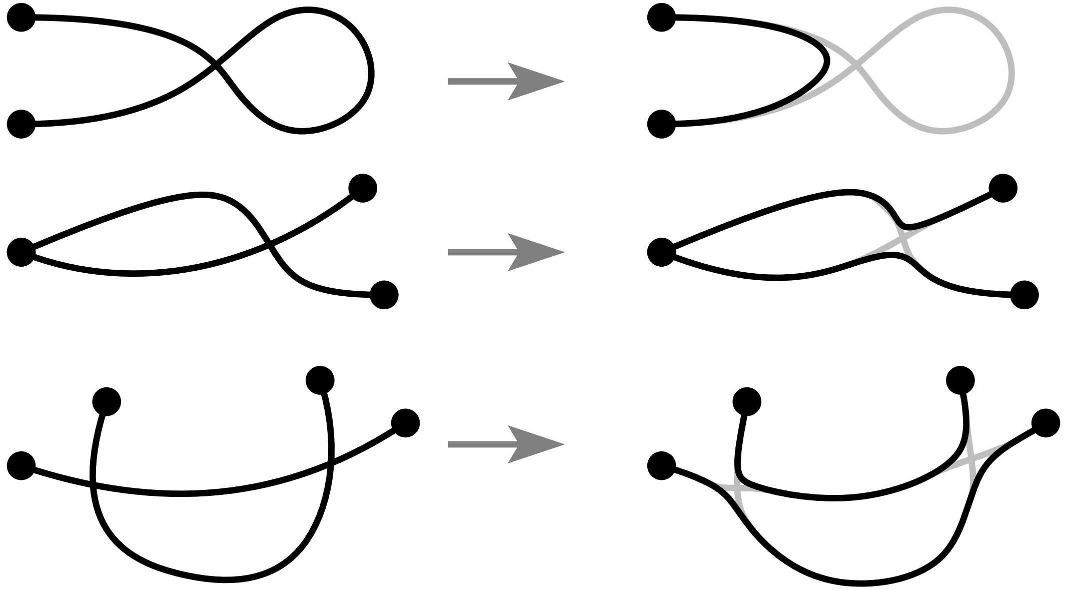
Definition.
For a graph G, the **crossing number** $\text{cr}(G)$ is the smallest number of crossings in a drawing of G (in the plane).

Example.
$\text{cr}(K_{3,3}) = 1$



In a crossing minimal drawing of G

- no edge is self-intersecting,
- edges with common endpoints do not intersect,
- two edges intersect at most once,
- and wlog, at most two edges intersect at the same point.



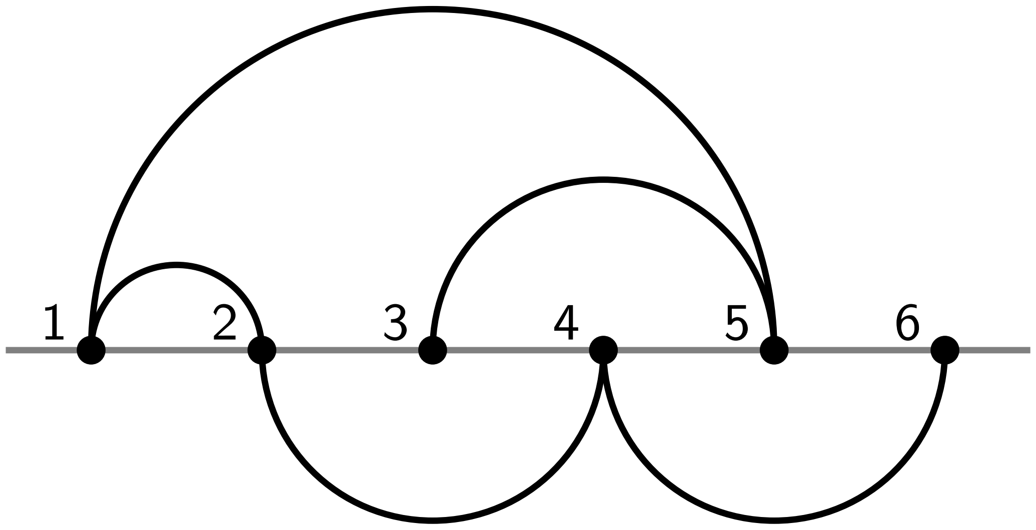
Such a drawing is called a **topological drawing** of G.

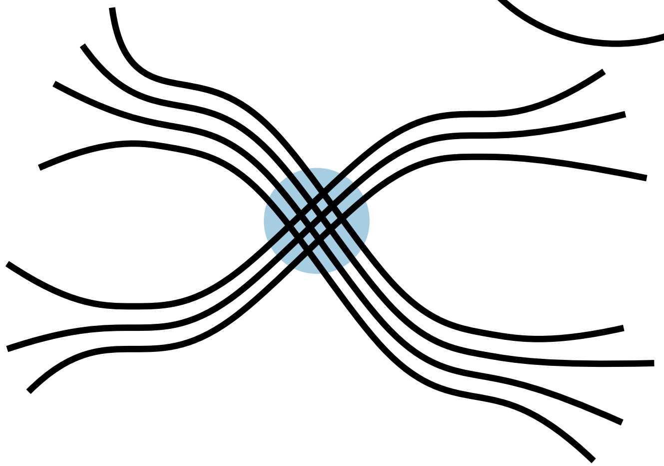
Computing the crossing number

- Computing cr(G) is NP-hard. [Garey, Johnson '83]
- cr(G) often unknown, only conjectures exist
  - for K_n it is only known for up to $\sim$ 12 vertices
- In pratice, cr(G) is often not computed directly but rather drawings of G are optimised with
  - force-based methods (next lecture),
  - multidimensional scaling,
  - heuristics, ...
- cr(G) is a measure of how far G is from being planar
- Planarization, where we replace crossings with dummy vertices, also uses only heuristics

Other crossing numbers

- Schaefer [Schae20] offers a huge survey on different crossings numbers (and more precise definitions)
- One-sided crossing minimization . . .
- Fixed Linear Crossing Number
- In book embeddings
- Crossings of edge bundles
- On other surfaces, like on donuts
- Weighted crossings
- Crossing minimization is NP-hard for most of the variants





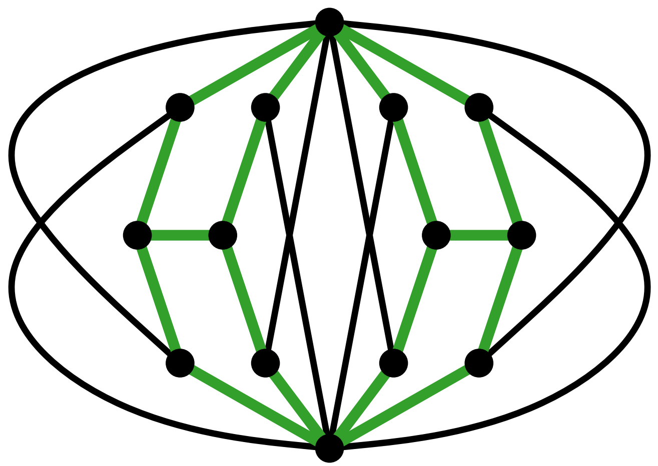
Rectilinear crossing number

Definition.
For a graph G, the **rectilinear (straight-line) crossing number** $\bar{\text{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of G.

Separation.
$\text{cr}(K_8) = 18$, but $\bar{\text{cr}}(K_8) = 19$.

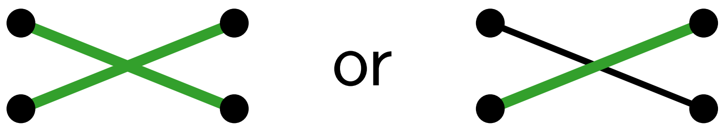
Even more . . .

Lemma 1. [Bienstock, Dean '93]
For $k \geq 4$, there exists a graph G_k with $\text{cr}(G_k) = 4$ and $\bar{\text{cr}}(G_k) \geq k$.

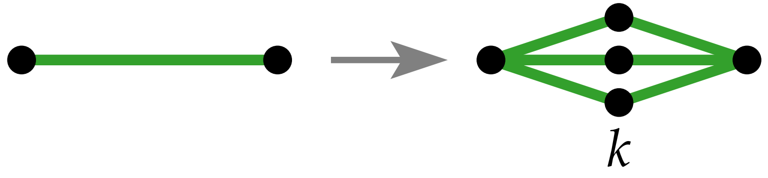


G_1

- Each straight-line drawing of G_1 has at least one crossing of the following types:



- From G_1 to G_k do



First lower bounds on $\text{cr}(G)$

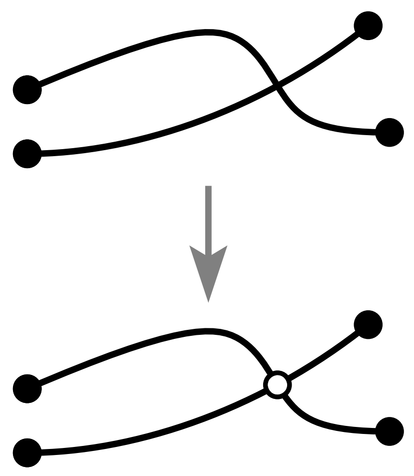
Lemma 2.
For a graph G with n vertices and m edges,

$$\text{cr}(G) \geq m - 3n + 6.$$

Proof.

- Consider a drawing of G with $\text{cr}(G)$ crossings.
- Obtain a graph H by turning crossings into dummy vertices.
- H has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- H is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6.$$



The Crossing Lemma

- 1973 Erdös and Guy conjectured that $\text{cr}(G) \in \Omega(\frac{m^3}{n^2})$.
- In 1982 Leighton and, indepedently, Ajtai, Chávtal, Newborn and Szemerédi showed that

$$\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

- Bound is asymptotically sharp.
- Result stayed hardly known until Székely in 1997 demonstrated its usefulness.
- We look at a proof "from THE BOOK" by Chazelle, Sharir and Welz.
- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.

The Crossing Lemma

Crossing Lemma.
For a graph G with n vertices and m edges, $m \geq 4n$,

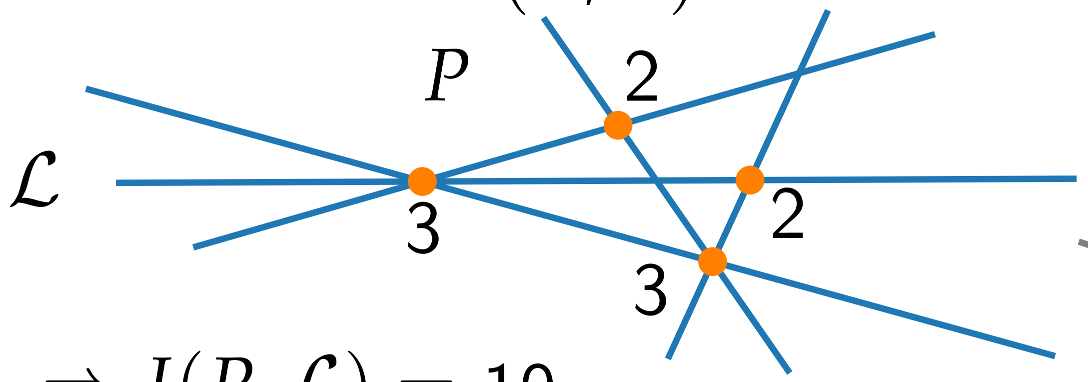
$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Proof.

- Consider a minimal embedding of G.
- Let p be a number in $(0,1)$.
- Keep every vertex of G independently with probability p.
- Let G_p be the remaining graph.
- Let n_p, m_p, X_p be the random variables counting the number of vertices/edges/crossings of G_p.
- By Lem 2, $\mathbb{E}(X_p - m_p + 3n_p) \geq 0$.
- $\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2 m$
- $\mathbb{E}(X_p) = p^4 \mathrm{cr}(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p)$
  $= p^4 \mathrm{cr}(G) - p^2 m + 3pm$
- $\mathrm{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\mathrm{cr}(G) \geq \frac{1}{64}\left[\frac{4m}{(n/m)^2} - \frac{3n}{(n/m)^3}\right] = \frac{1}{64}\frac{m^3}{n^2}$

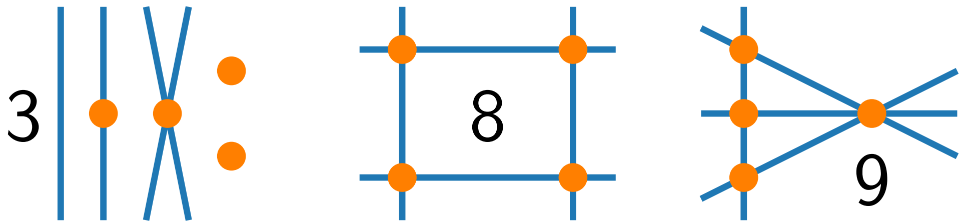
Application 1: Point-line incidences

- For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$, $I(P, \mathcal{L})$ = number of point-line incidences in $(P, \mathcal{L})$.



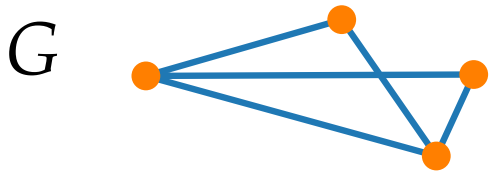
$\Rightarrow I(P, \mathcal{L}) = 10$

- Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.
- For example: $I(4, 4) = 9$



Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
$I(n, k) \leq c(n^{2/3}k^{2/3} + n + k)$.

Proof.



- $\mathrm{cr}(G) \leq k^2$
- $\#$ points on l $= 1 +$ $\#$ edges on l
- $I(n, k) - k \leq m$
- Crossing Lemma: $\frac{1}{64}\frac{m^3}{n^2} \leq \mathrm{cr}(G)$
- $c'(I(n, k) - k)^3/n^2 \leq \mathrm{cr}(G) \leq k^2$
- if $m \ngeq 4n$, then $I(n, k) - k \leq 4n$

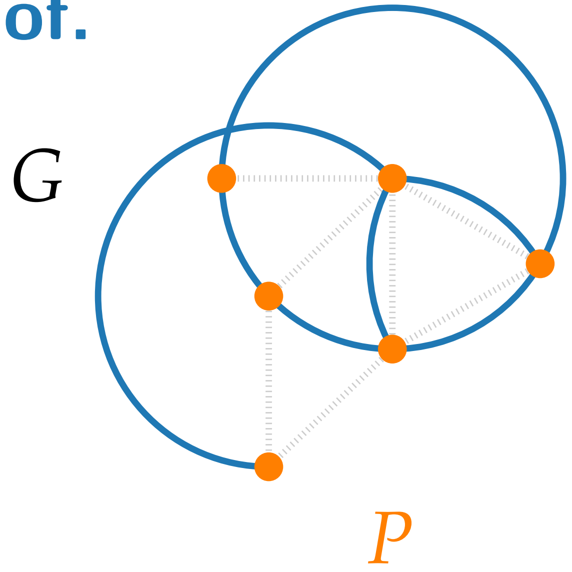
Application 2: Unit distances

For points $P \subset \mathbb{R}^2$ define

- $U(P) =$ number of pairs in P at unit distance and
- $U(n) = \max_{|P|=n} U(P)$.

Theorem 2.
[Spencer, Szemerédi, Trotter '84, Székely '97]
$U(n) < 6.7n^{4/3}$

Proof.



- $U(P) - \mathcal{O}(n) \leq m$
- $\text{cr}(G) \leq 2n^2$
- $c\frac{(U(P)-\mathcal{O}(n))^3}{n^2} \leq \text{cr}(G) \leq 2n^2$

Literature

- [Aigner, Ziegler] Proofs from THE BOOK
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao blog post "The crossing number inequality" from 2007
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Székely '97] Crossing Numbers and Hard Erdös Problems in Discrete Geometry
- Documentary/Biography "N Is a Number: A Portrait of Paul Erdös"