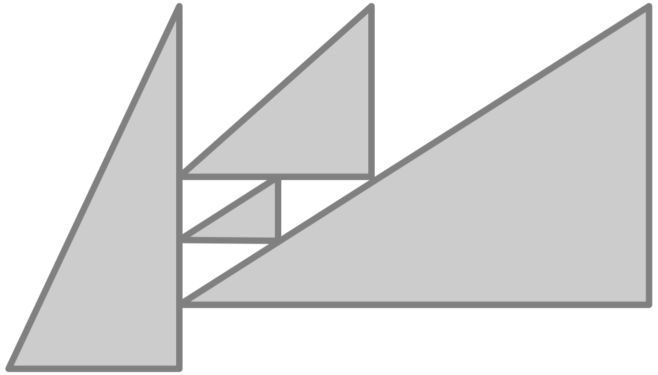
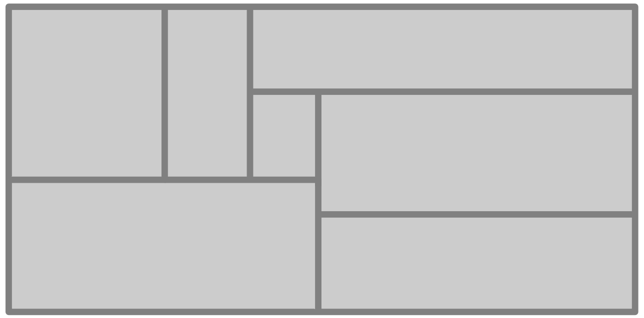
Visualization of graphs

Contact representations of planar graphs

Triangle contacts and rectangular duals

Jonathan Klawitter · Summer semester 2020

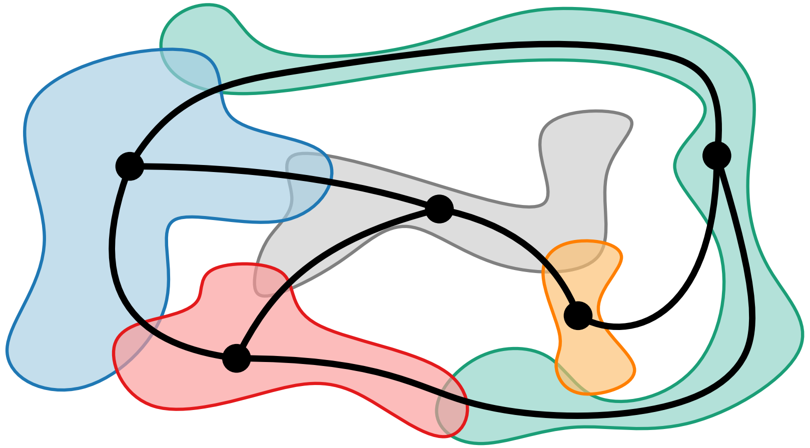




Intersection representation of graphs

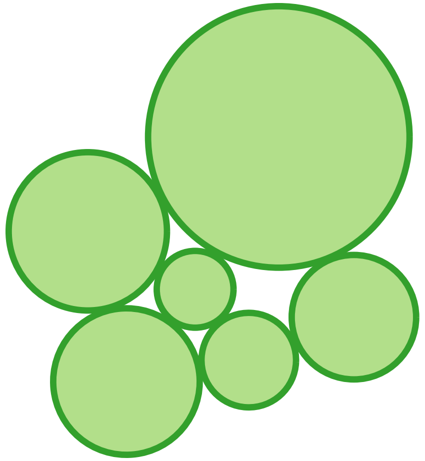
Definitions.
In an **intersection representation** of a graph each vertex is represented as a set such that two sets intersect if and only if the corresponding vertices are adjacent.

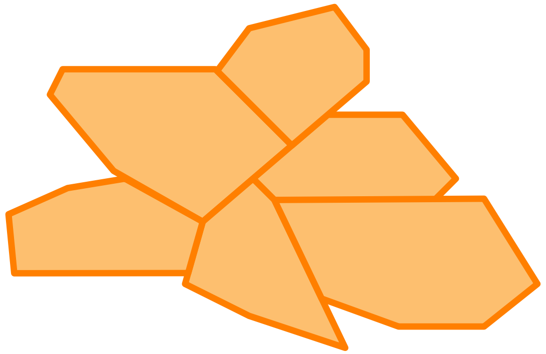
For a collection $\mathcal{S}$ of sets $S_1, \ldots, S_n$,
the **intersection graph** $G(\mathcal{S})$ of $\mathcal{S}$ has vertex set $\mathcal{S}$ and edge set
$\{S_iS_j \colon i, j \in \{1, \ldots, n\}, i \neq j, \text{ and } S_i \cap S_j \neq \emptyset\}$.

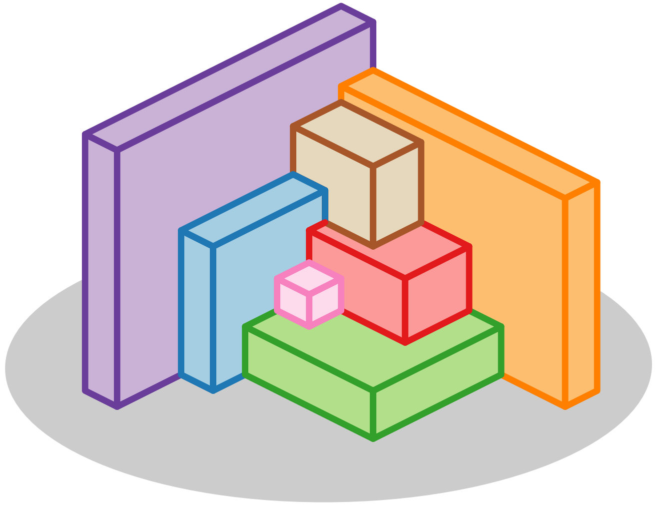


Contact representation of graphs

Definitions.
A collection of interiorly disjoint **objects** $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a called a **contact representation** of its interesction graph $G(\mathcal{S})$.







- Objects could be circles, line segments, triangles, boxes, ...
- Domain could be 2D, 3D, ...

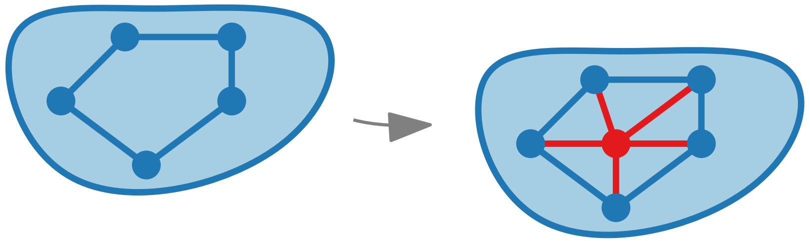
Contact representation of planar graphs

- Is the intersection graph of a contact representation always planar?
  - No, not even for planar object types.
- Which object types can be used to represent all planar graphs?
  - Contact of disks [Koebe '36]
  - Corner contact of triangles and T-shapes [de Fraysseix et al. '94]
  - Side contacts of 3D Boxes [Thomassen '86]
  - ...
- Some object types are used to represent special classes of planar graphs:
  - Line segment contact on grids for bipartite planar graphs [Hartman et al. '91, de Fraysseix et al. '94]
  - Rectangle dissections for so-called properly triangulated planar graphs [Kant, He '97]
  - L-shapes, k-bend path, ...

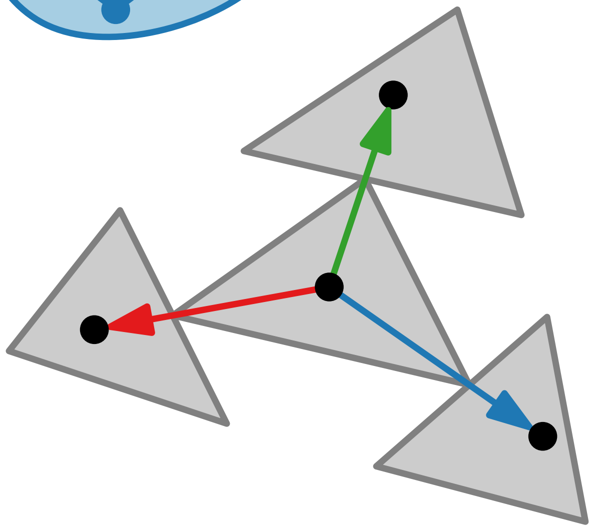
General approach

How to compute a contact representation of a given graph G?

- Consider only inner triangulations (or maximally bipartite graphs, etc)
  - Triangulate by adding vertices, not by adding edges



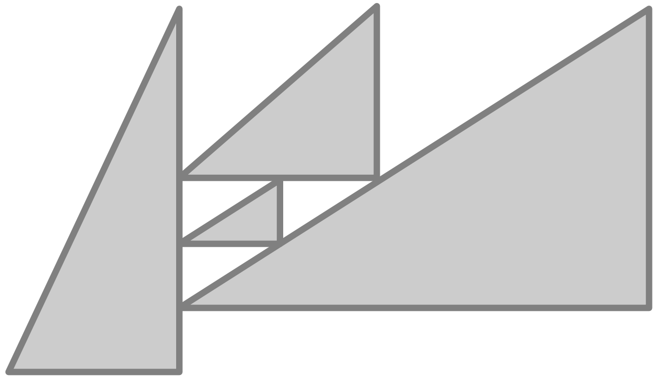
- Describe contact representation combinatorically.
  - Which objects contact each other in which way?



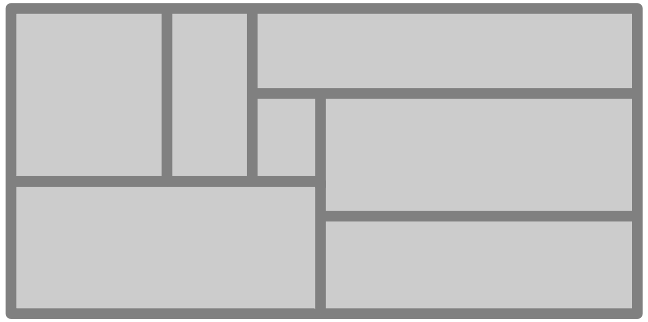
- Compute combinatorical description.
- Show that combinatorical description can be used to construct drawing.

In this lecture

- Representations with right-triangles and corner contact
  - Use Schnyder realizer to describe contacts between triangles
  - Use canonical order to calculate drawing

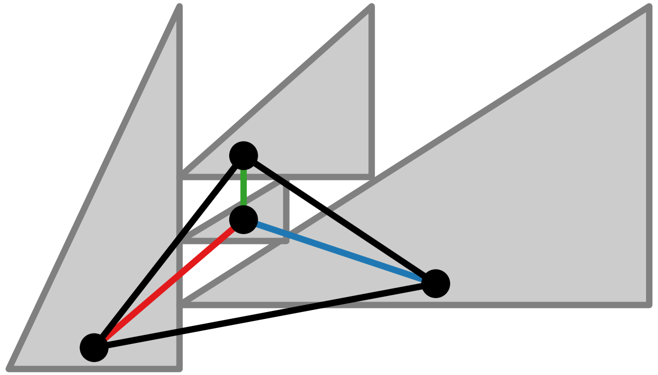


- Representation with dissection of a rectangle, called **rectangular dual**
  - Find similar description like Schnyder realizer for rectangles
  - Construct drawing via st-digraphs, duals, and topological sorting.



Triangle corner contact representation

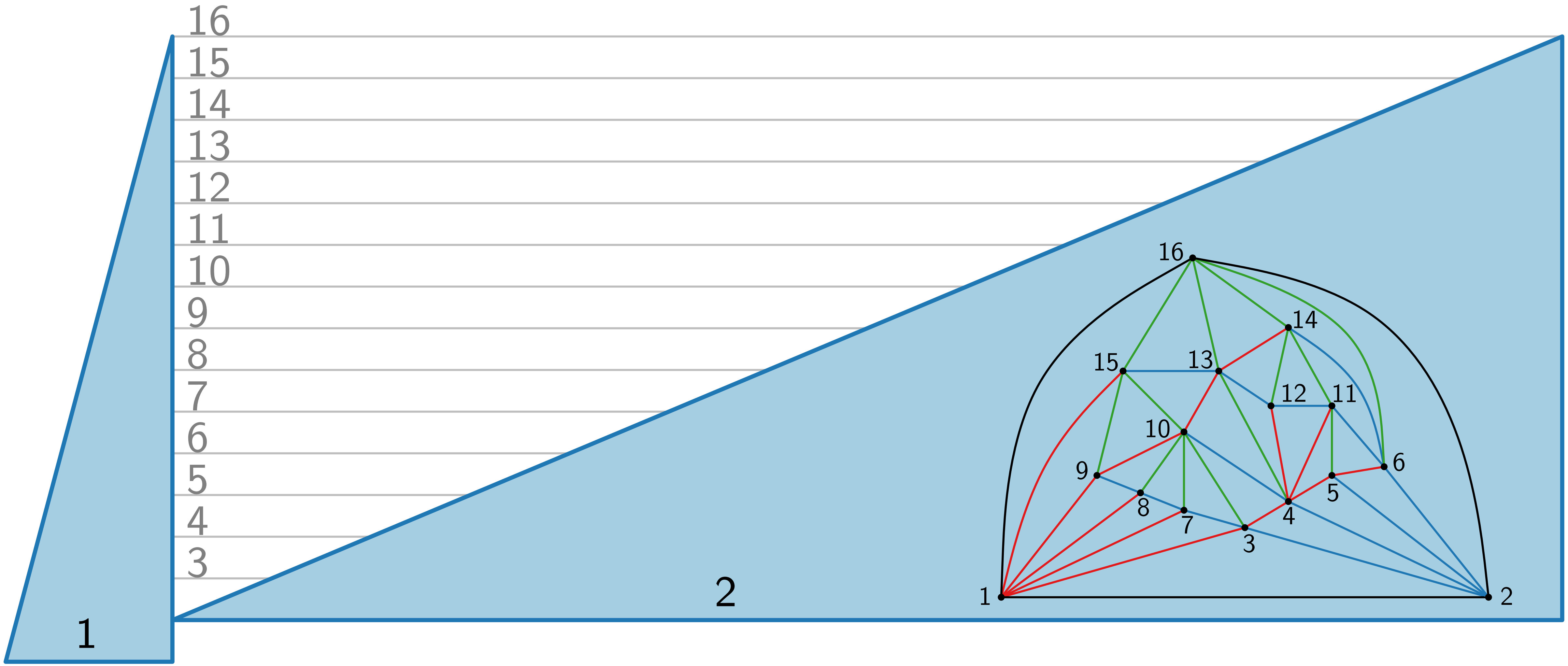
Idea.
Use canonical order and Schnyder forest to find coordinates for triangles.



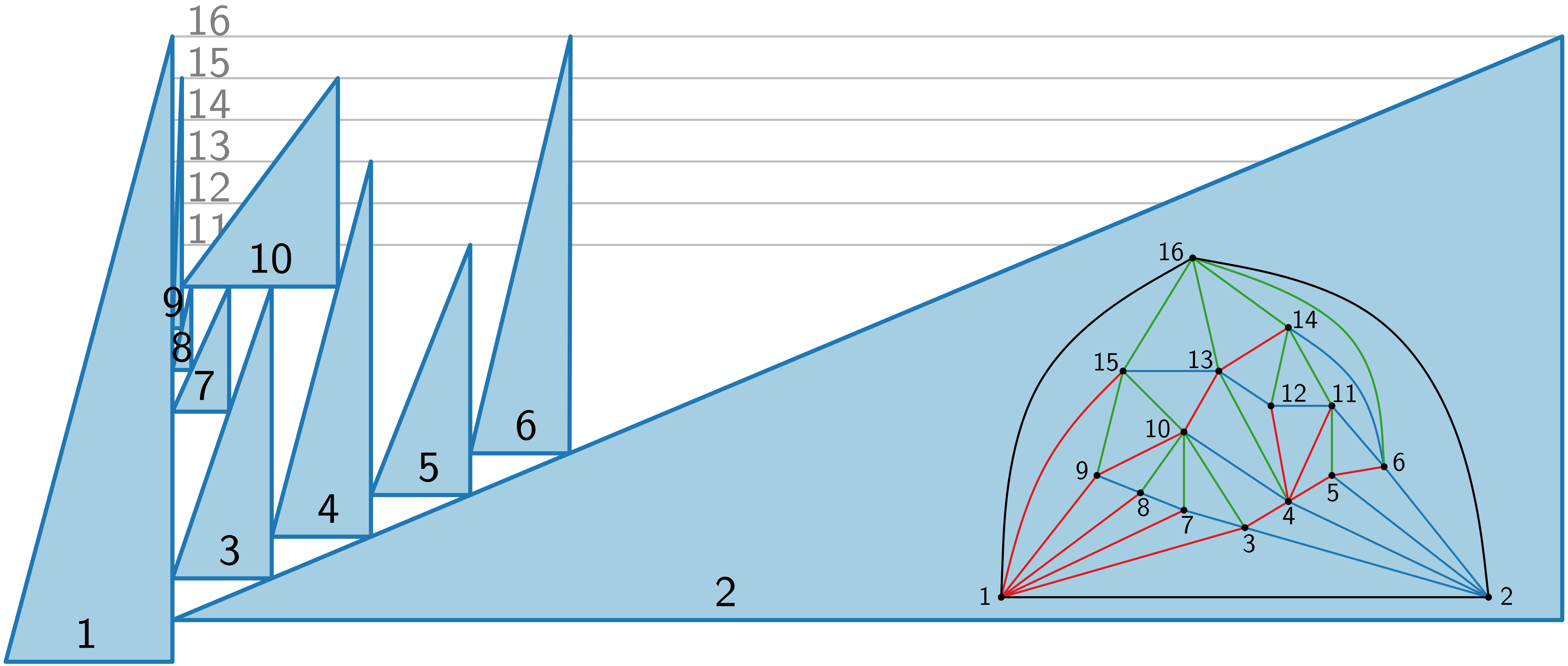
Observation.

- Can set base of triangle at height equal to position in canonical order.
- Triangle tip is precisely at base of triangle corresponding to cover neighbor.
- Outgoing edges in Schnyder forest indicate corner contacts.

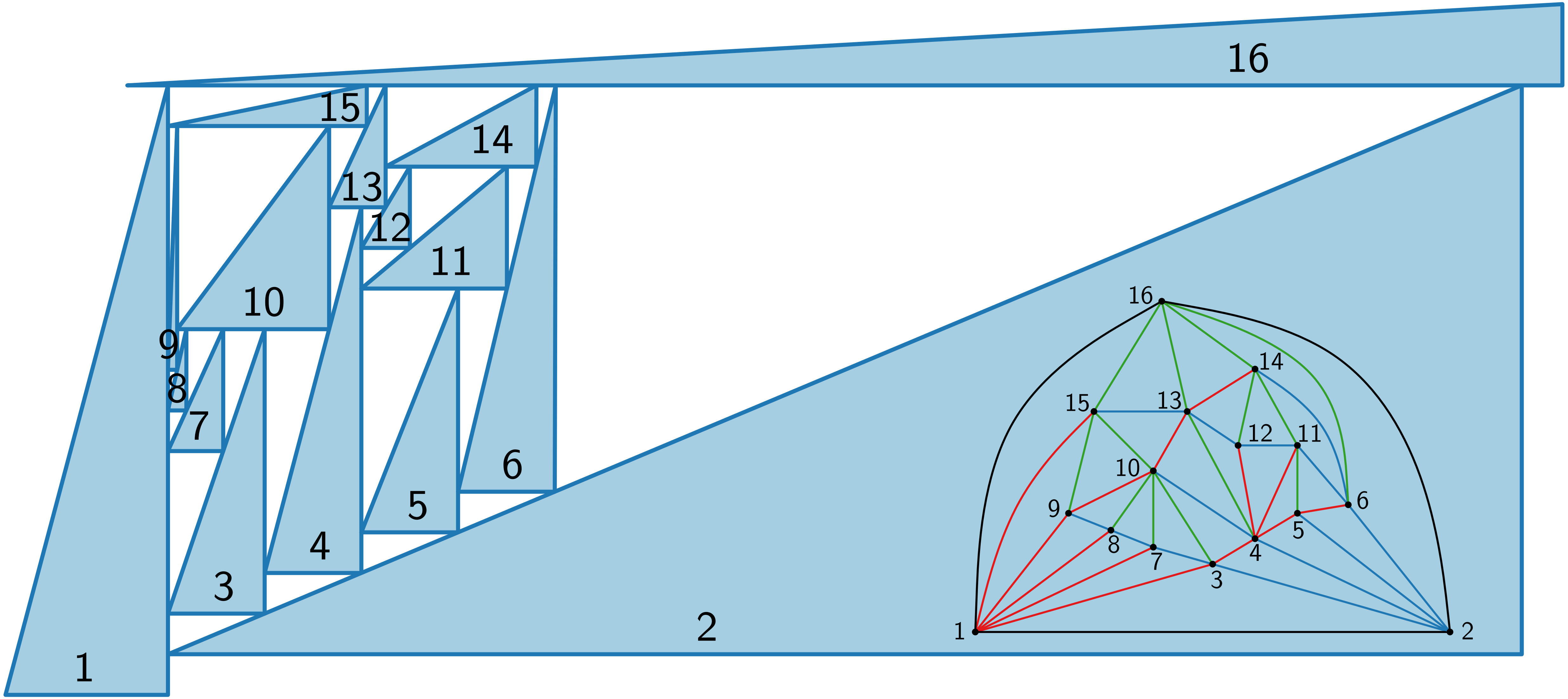
Triangle-contact representation example



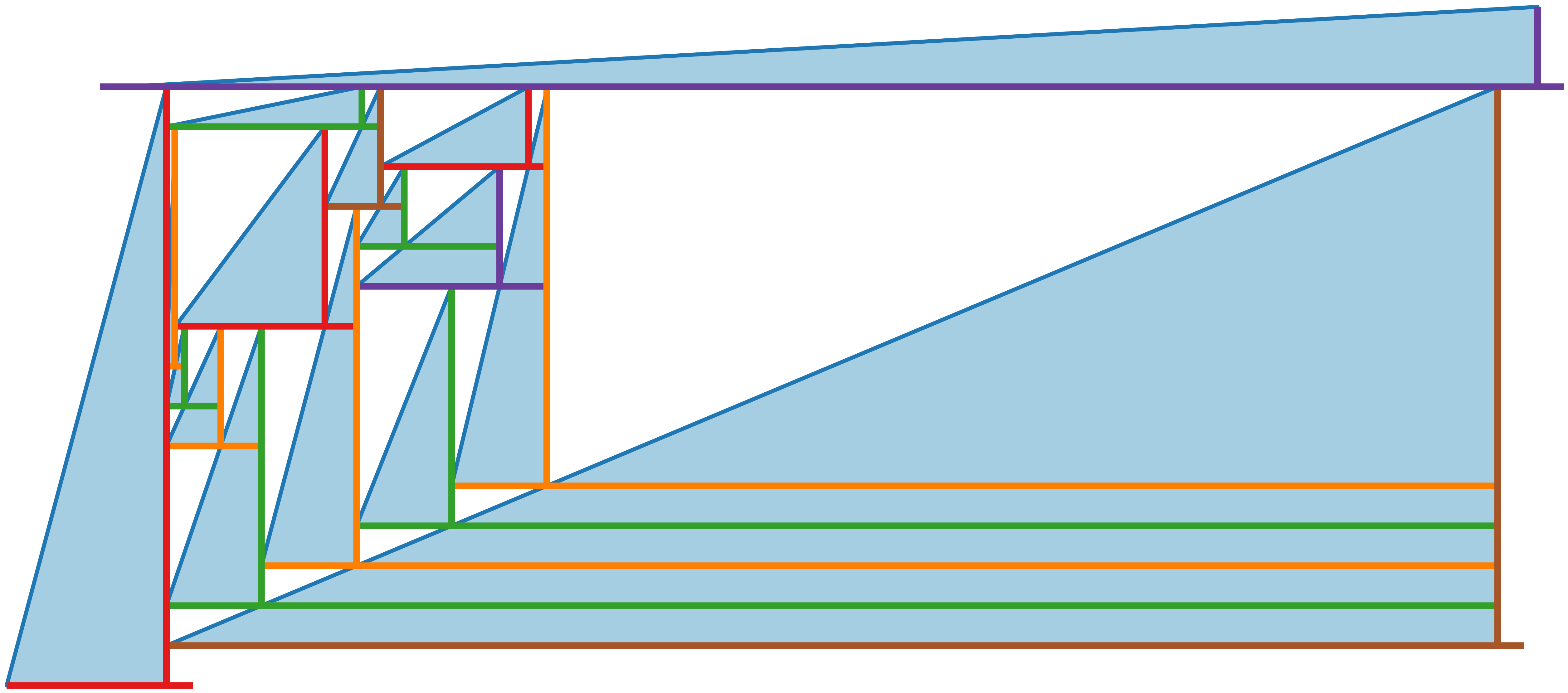
Triangle-contact representation example



Triangle-contact representation example

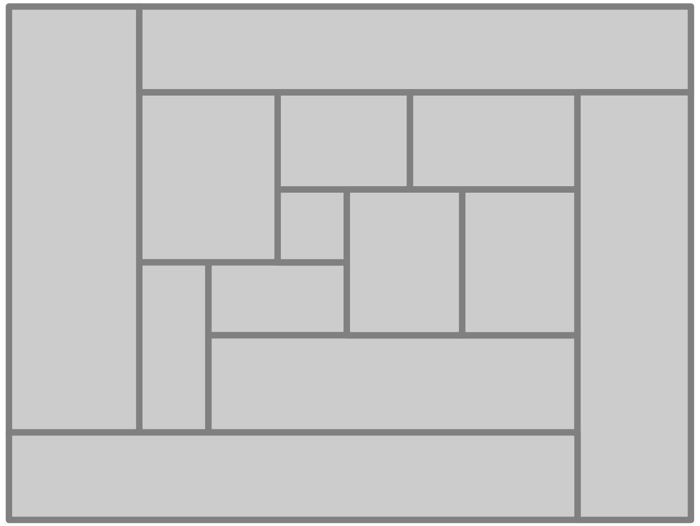


T-shape contact representation

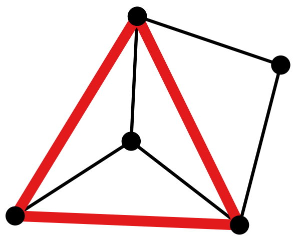


Rectangular dual

Definition.
A **rectangular dual** of a graph G is a contact representation with axis aligned rectangles s.t.
- no four rectangles share a point, and
- the union of all rectangles is a rectangle.



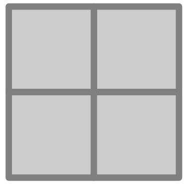
Definition.
A triangle C of G whose removal results in at least two connected components is called a **separating triangle**.



Does not have a rectangular dual.
To enclose an area we need at least four rectangles.

When does G admit a rectangular dual?

- G has no separating triangle
- G has at least 4 vertices on outer face; wlog assume this
- each inner face of G must be a triangle



:(

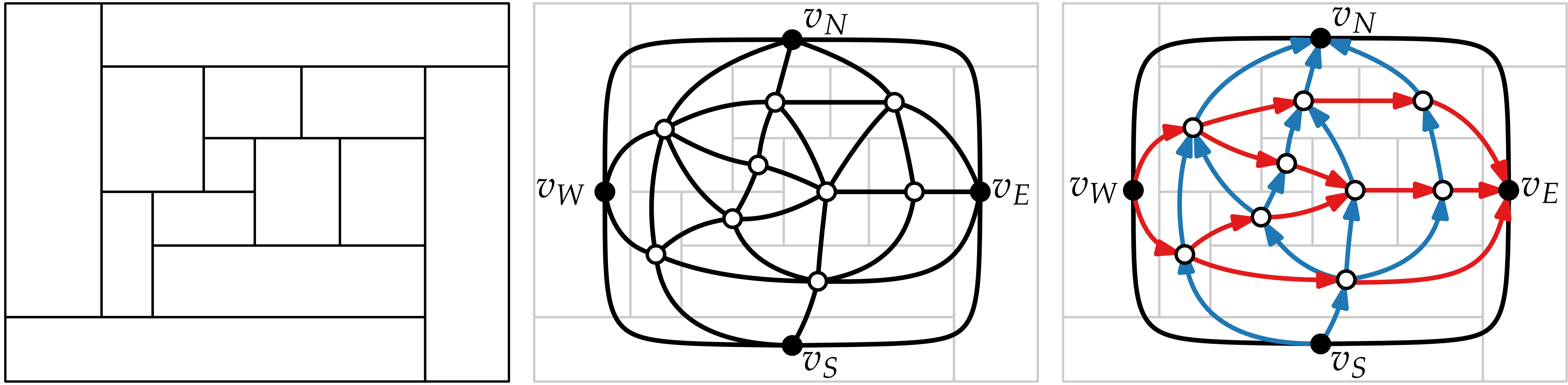
Proper triangular planar graph

Definition.
An internally triangulated, plane graph G without separating triangles and exactly four vertices on the outer face is called **properly triangulated planar (PTP)**.

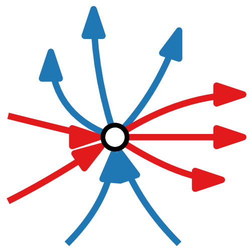
Theorem. [Koźmiński, Kinnen '85]
A graph G has a rectangular dual $\mathcal{R}$ with four rectangles on the boundary of $\mathcal{R}$ if and only if G is a PTP graph.

Regular edge labeling

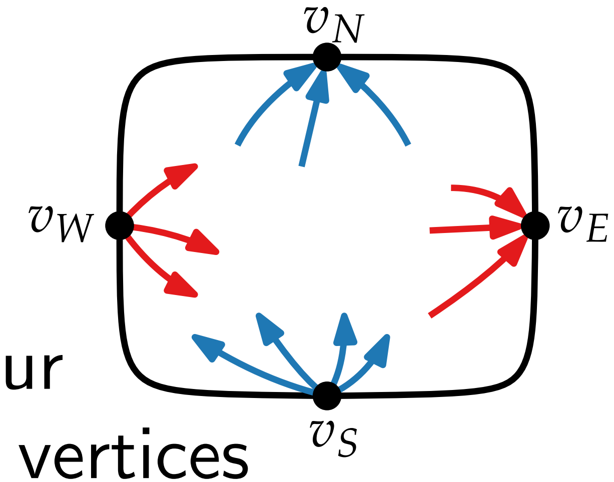
A rectangular dual gives rise to a 2-coloring and an orientation of the inner edges of G:



Definition.
A **regular edge labeling (REL)** is a 2-coloring and an orientation of inner edges of G such that



for every inner vertex

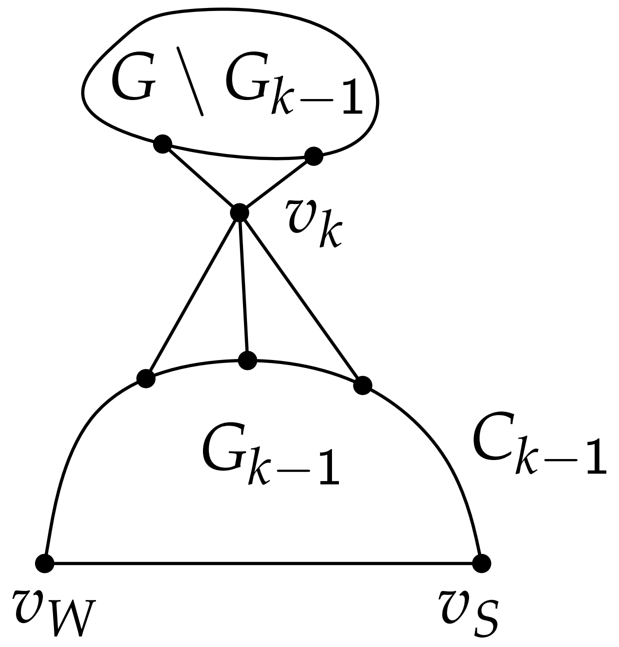


for four outer vertices

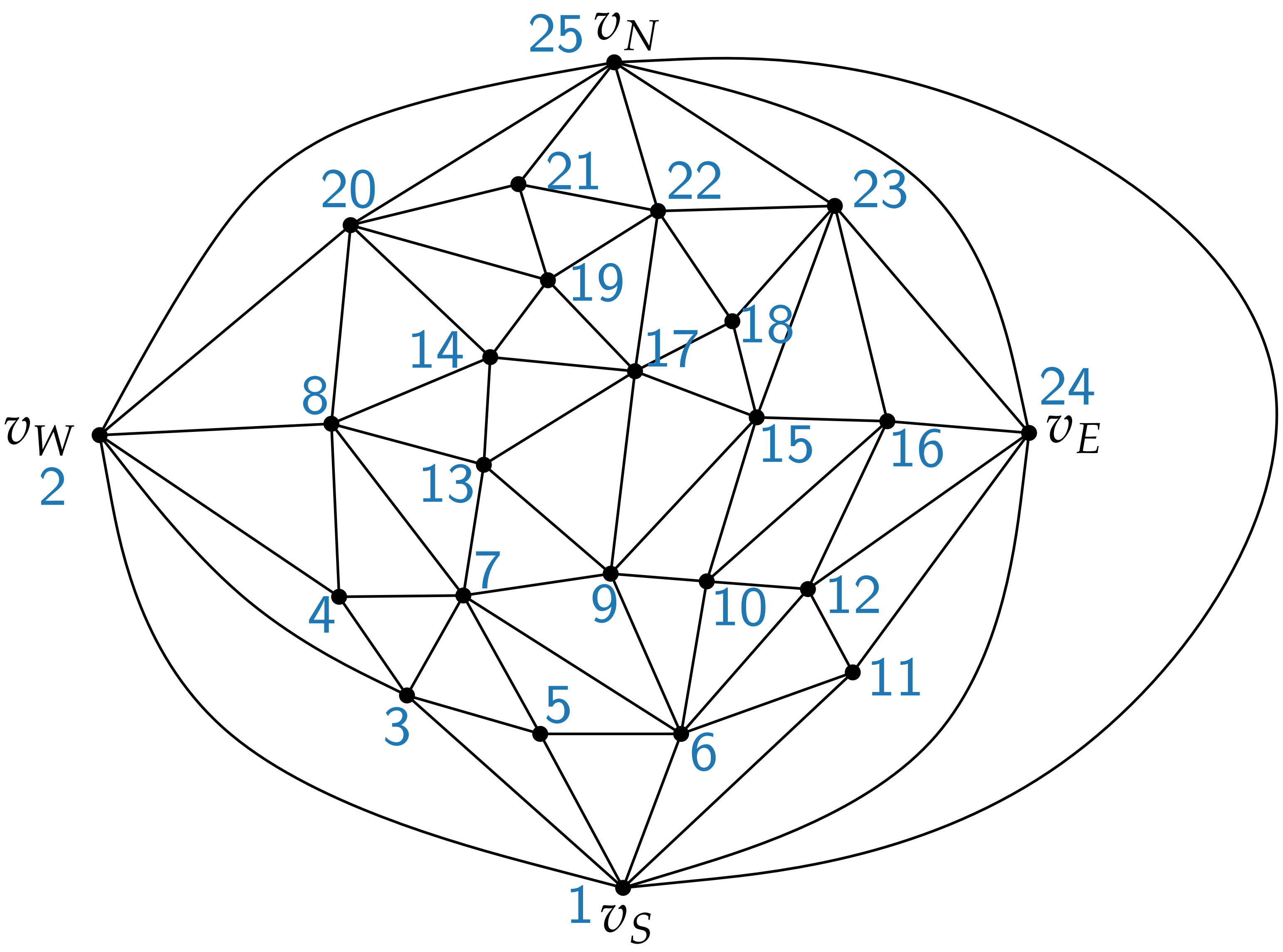
Refined canonical order

Theorem/Definition.
Let G be a PTP graph. There exists a labeling $v_1 = v_S, v_2 = v_W, v_3, \dots, v_n = v_N$ of the vertices of G such that for every $4 \le k \le n$:

- The subgraph G_{k-1} induced by $v_1, \dots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$.
- If $k \le k-2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$.



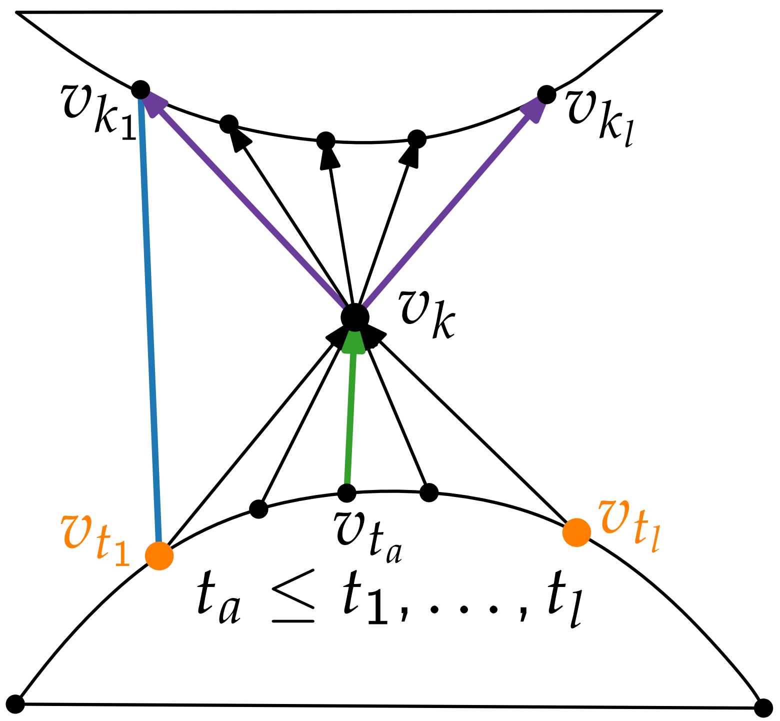
Refined canonical order example



From refined canonical order to REL

Given a refined canonical ordering of G we construct a REL as follows:

- For $i < j$, orient (v_i, v_j) from v_i to v_j;
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is **left point** of v_k and v_{t_l} is **right point** of v_k.
- **Base edge** of v_k is (v_{t_a}, v_k), where $t_a < k$ is minimal.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) **left edge** and (v_k, v_{k_l}) **right edge**.



Lemma 1.
Left edge or right edge cannot be a base edge.

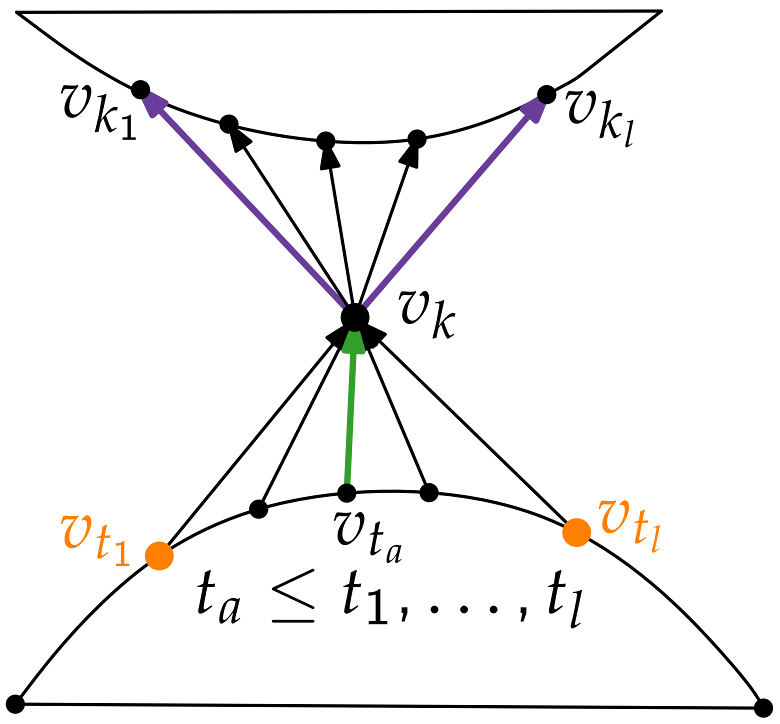
Proof. Suppose left edge (v_k, v_{k_1}) is base edge of v_{k_1}. Since G triangulated, $(v_{t_1}, v_{k_1}) \in E(G)$. Contradiction since $v_k > v_{t_1}$.

From refined canonical order to REL

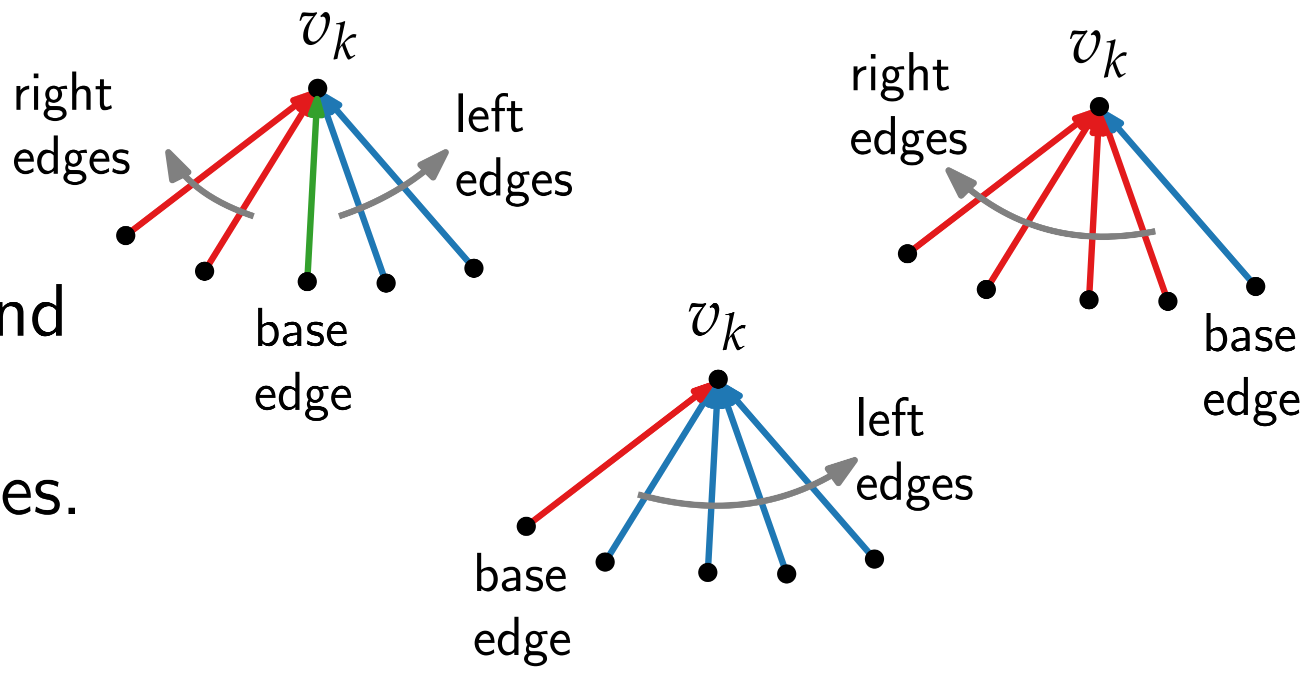
Lemma 2.
An edge is either a left edge, a right edge or a base edge.

Proof.

- Exclusive "or" follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$; v_{t_i} is right point of $v_{t_{i-1}}$:
  - v_{t_i} has at least two higher-numbered neighbors.
  - One of them is v_k; the other one is either $v_{t_{i-1}}$ or $v_{t_{i+1}}$.
  - For $1 \leq i < a-1$, it is $v_{t_{i-1}}$.
- Edges (v_{t_i}, v_k), $1 \leq i < a-1$, are right edges.
- Similarly, (v_{t_i}, v_k), for $a+1 \leq i \leq l$, are left edges.



From refined canonical order to REL



Coloring.

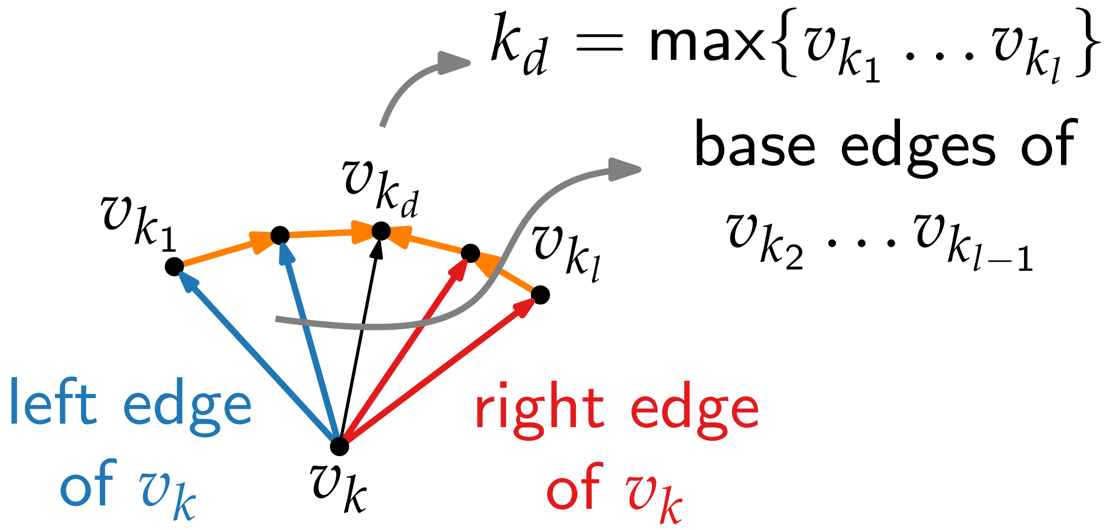
- Color right (left) edges in red (blue).
- Color a base edge (v_{t_i}, v_k) red if $i = 1$ and blue if $i = l$ and otherwise arbitrarily.

Let T_r be the red edges and T_b the blue edges.

> **Lemma 3.**
> $\{T_r, T_b\}$ is a regular edge labeling.

Proof.

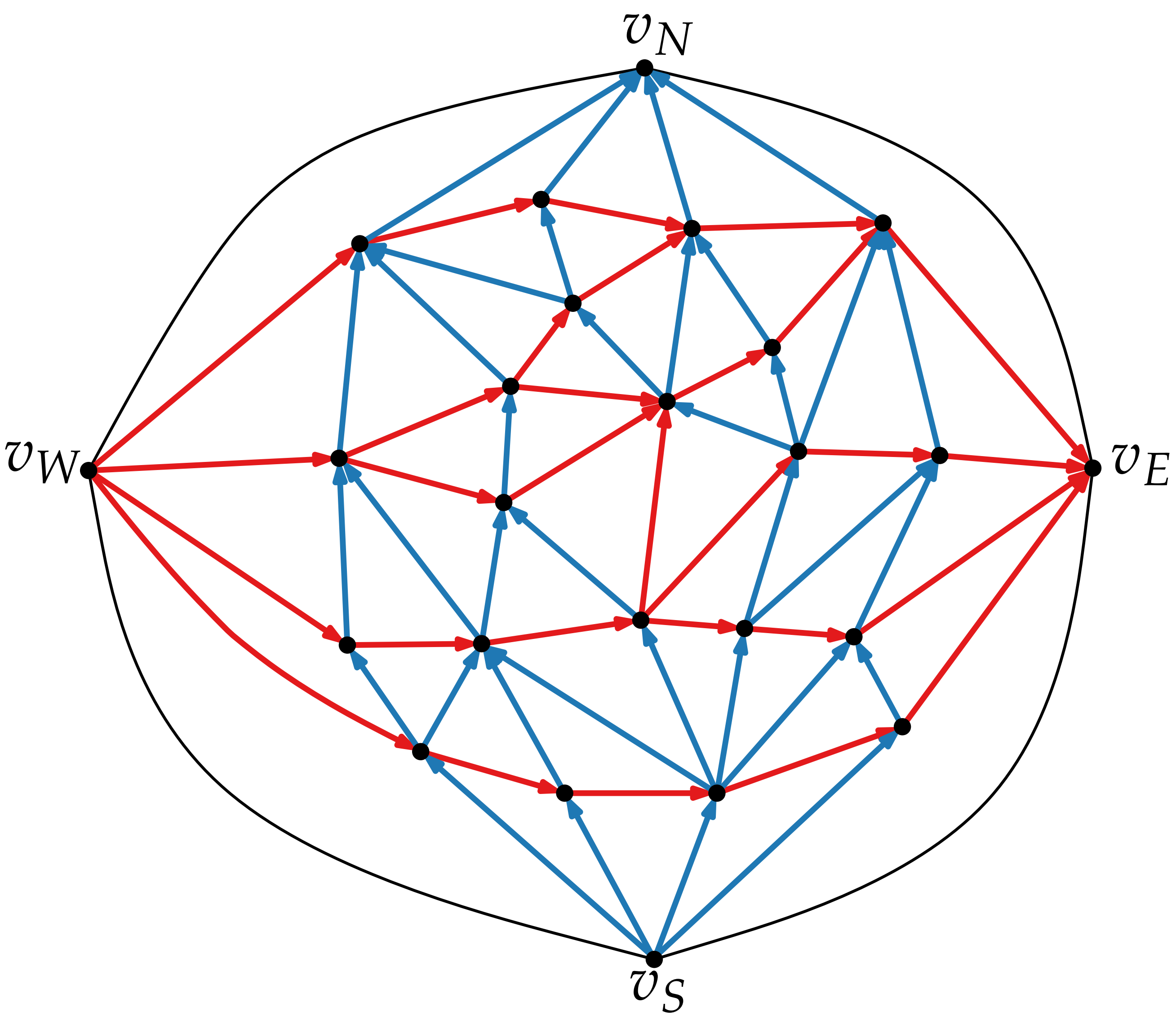
$k_l \geq 2$



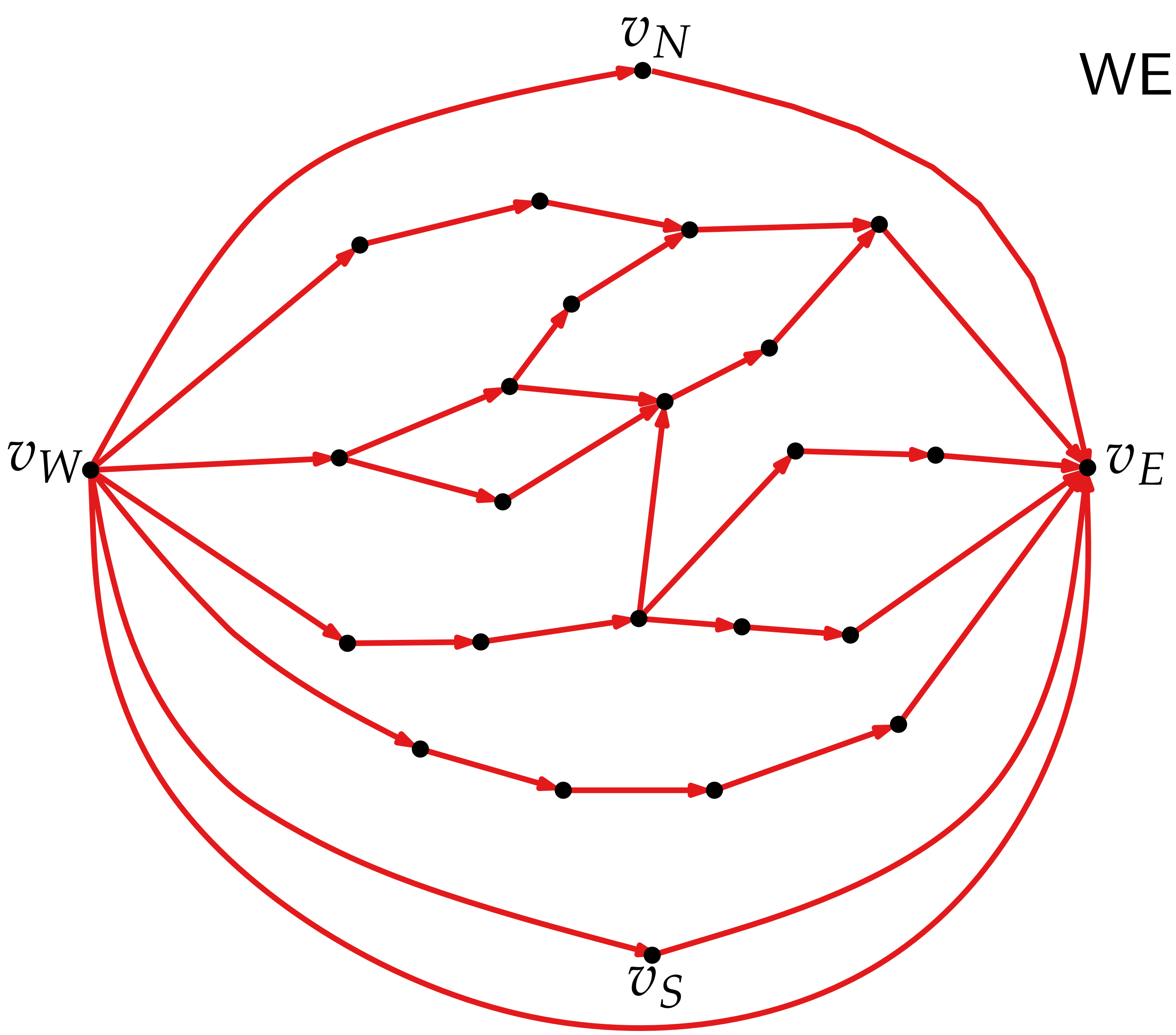
- $k_1 < k_2 < \dots < k_d$ and $k_d > k_{d+1} > \dots > k_l$
- (v_k, v_{k_i}), $2 \leq i \leq d-1$ are red
- (v_k, v_{k_i}), $d+1 \leq i \leq l-1$ are blue
- (v_k, v_{k_d}) is either red or blue

$\Rightarrow$ circular order of outgoing edges of v_k correct

From REL to st-digraphs to coordinates

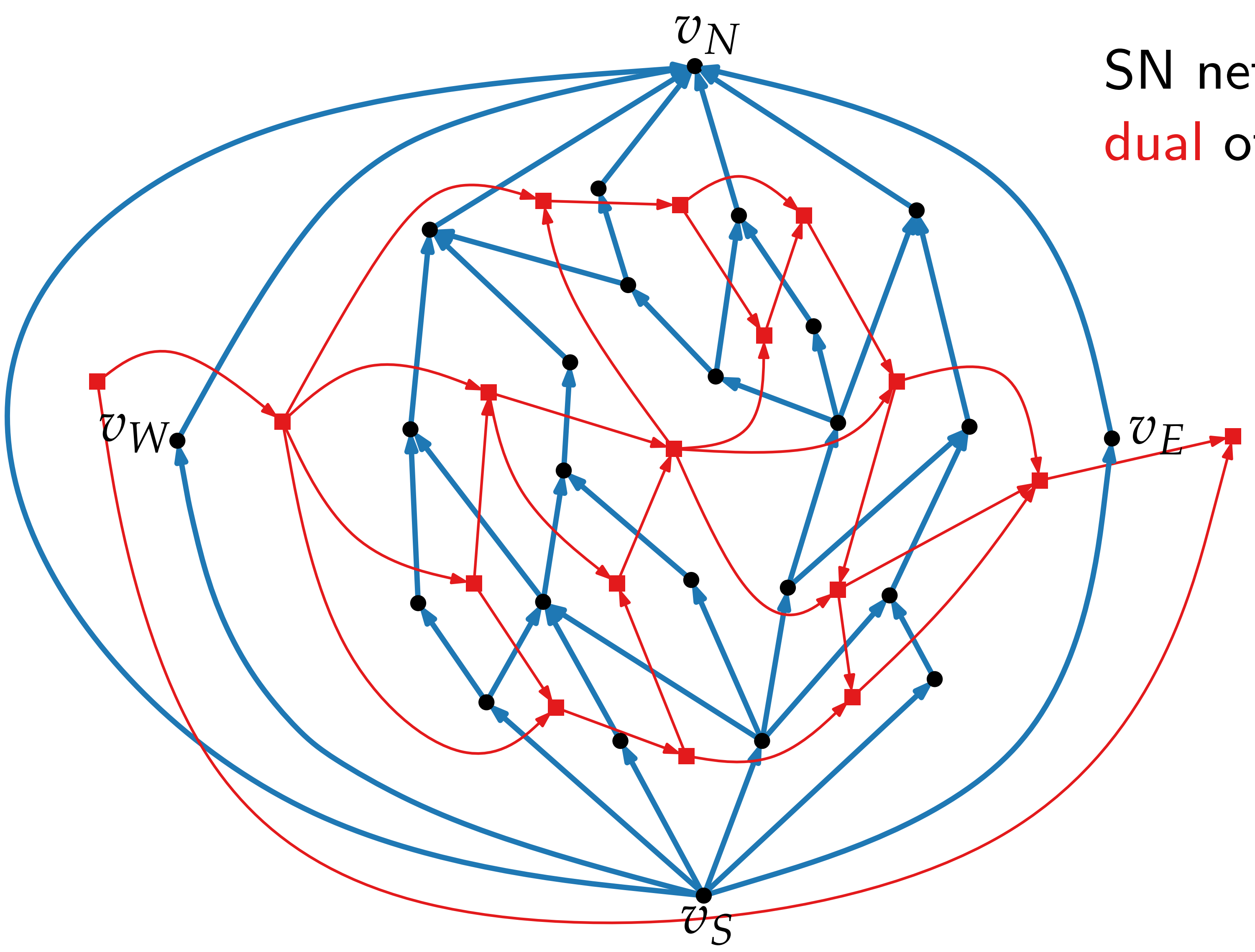


From REL to st-digraphs to coordinates



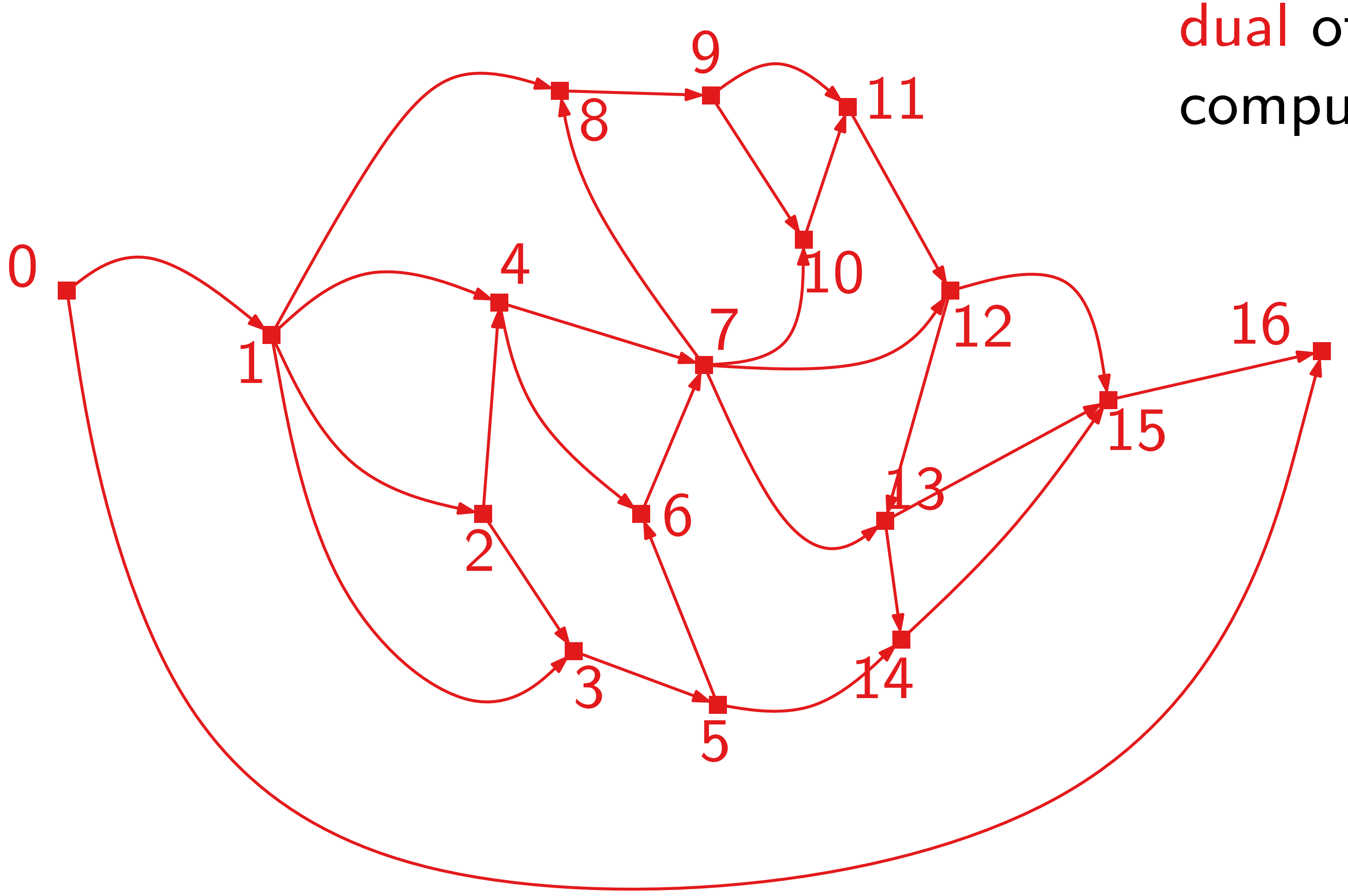
WE network G_{hor}

From REL to st-digraphs to coordinates



SN network G_{ver}
dual of G_{ver}

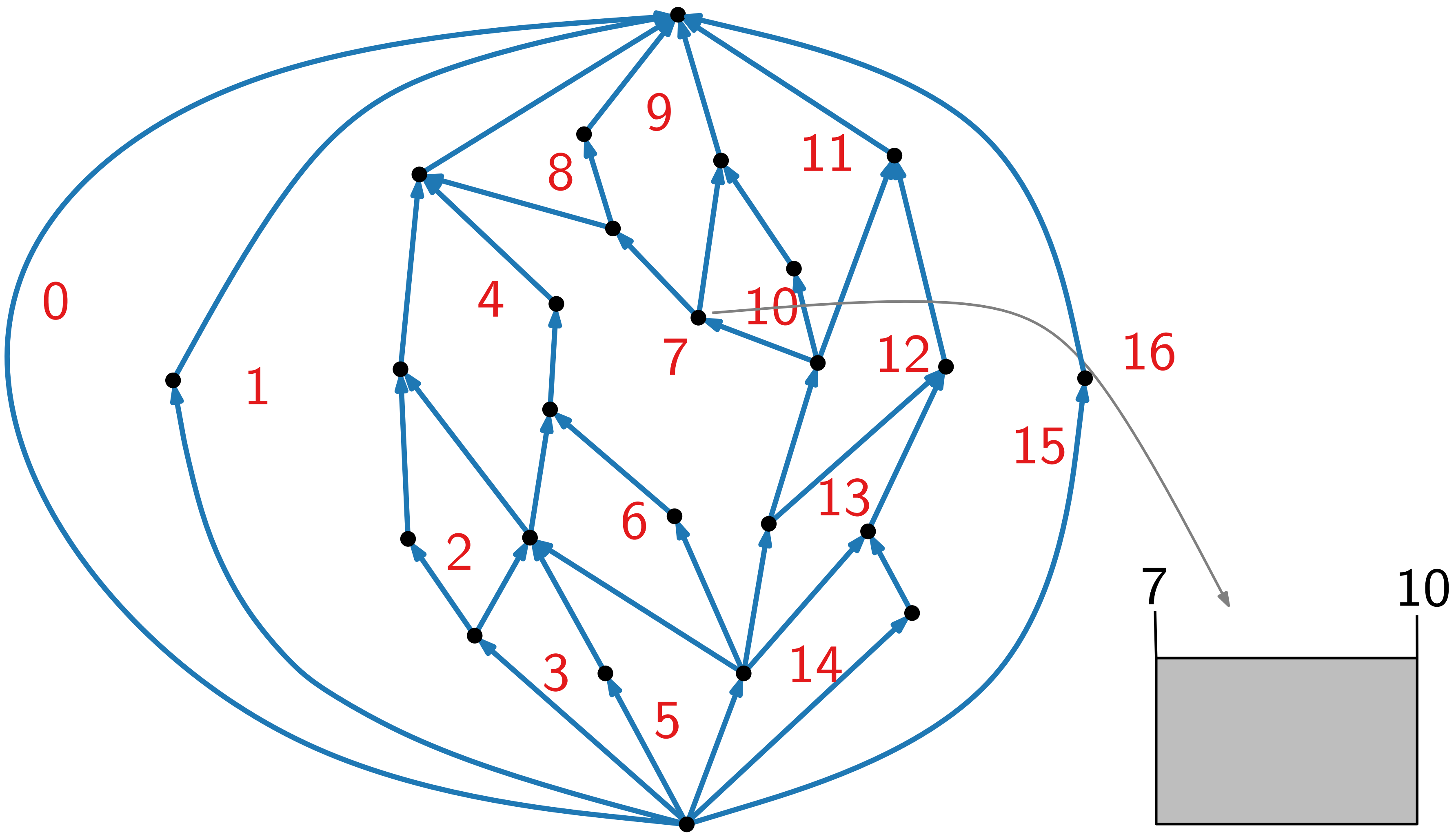
From REL to st-digraphs to coordinates



dual of G_{ver}

compute topological order

From REL to st-digraphs to coordinates

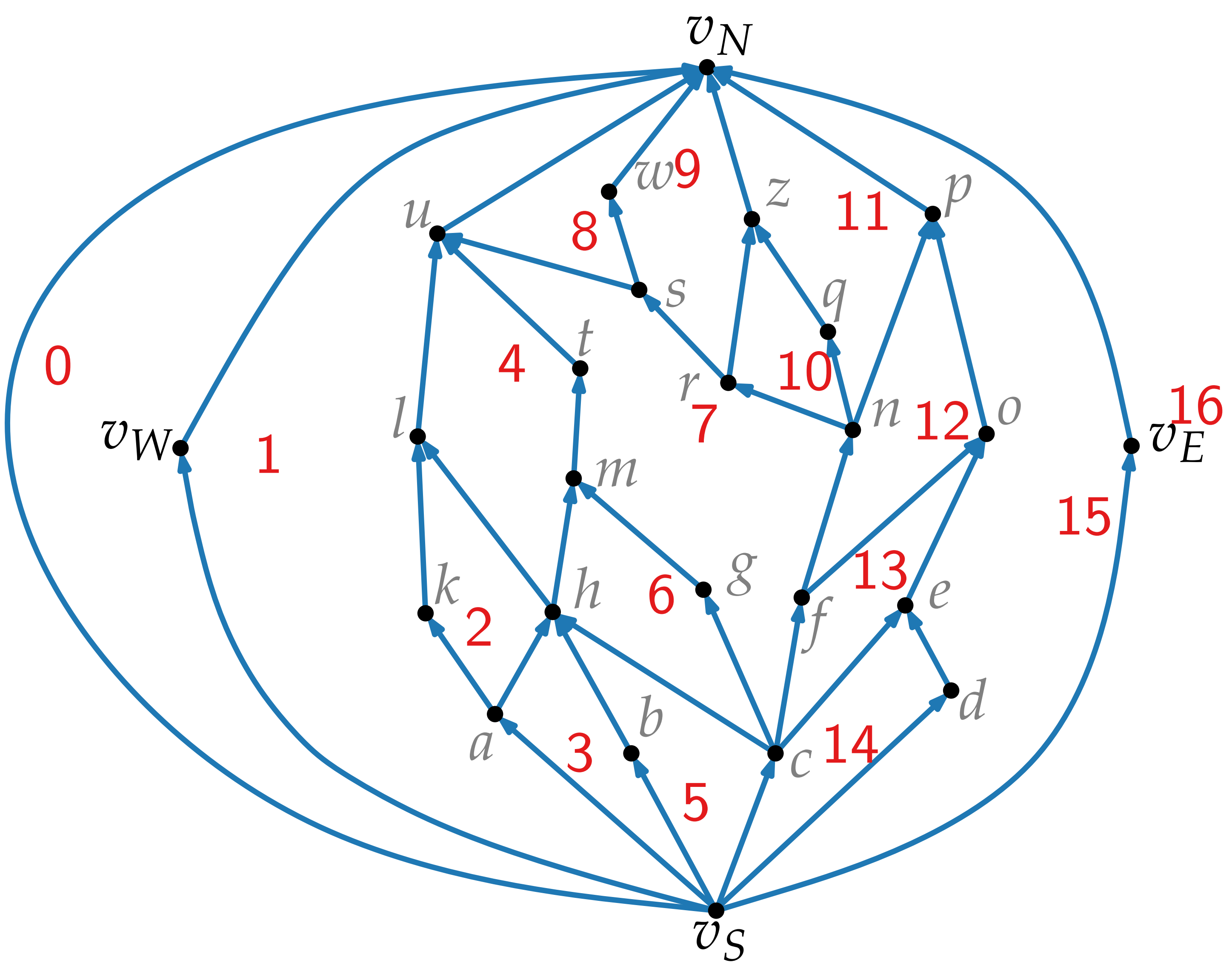


Rectangular dual algorithm

For a PTP graph $G = (V, E)$:

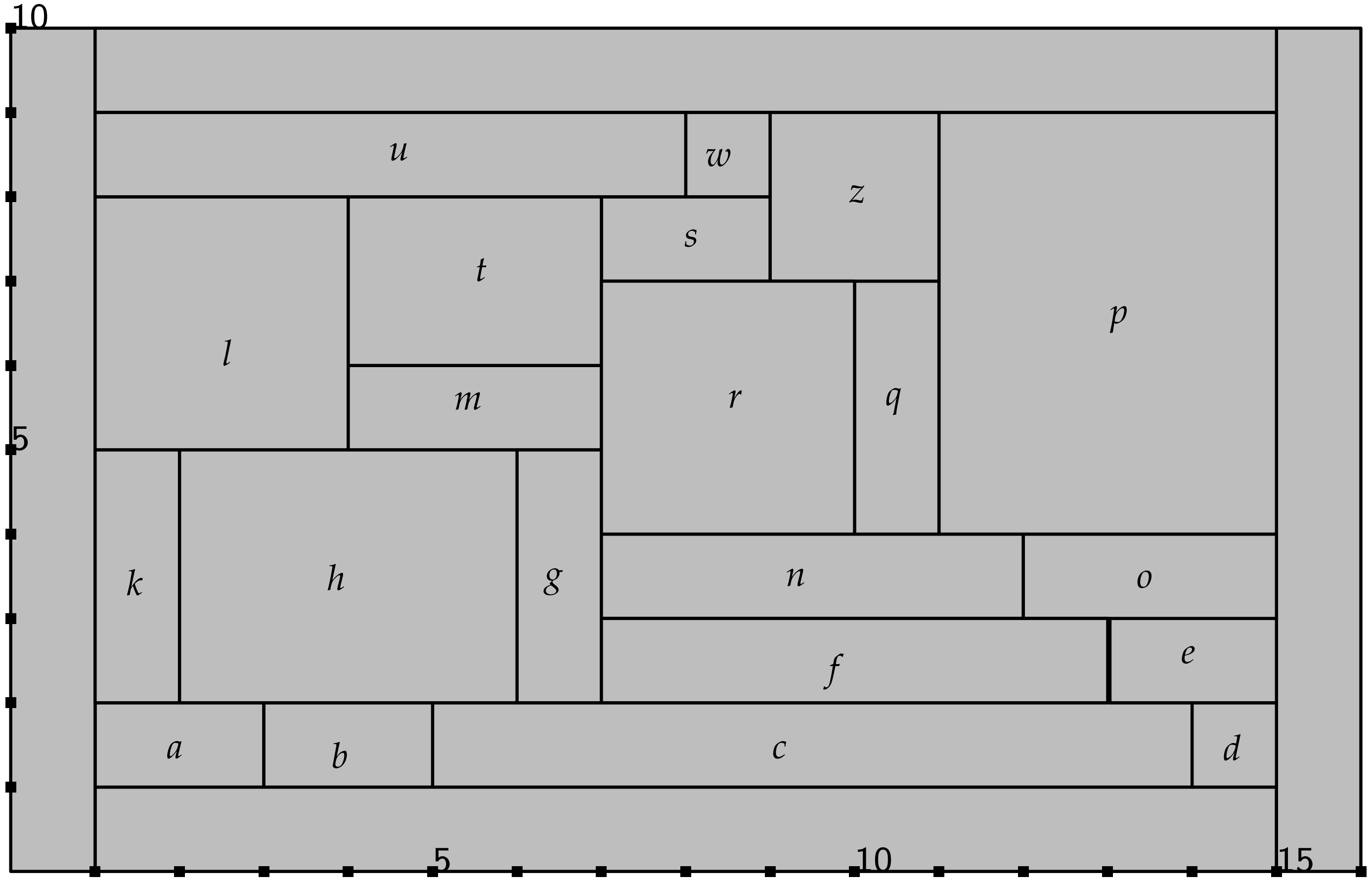
- Find a REL T_r, T_b of G;
- Construct a SN network G_{ver} of G (consists of T_b plus outer edges)
- Construct the dual $G^\star_{\text{ver}}$ of G_{ver} and compute a topological ordering f_{ver} of $G^\star_{\text{ver}}$
- For each vertex $v \in V$, let g and h be the face on the left and face on the right of v. Set $x_1(v) = f_{\text{ver}}(g)$ and $x_2(v) = f_{\text{ver}}(h)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{\text{ver}} - 1$
- Analogously compute y_1 and y_2 with G_{hor}.
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v),\ x_2(v)$ and y-coordinates $y_1(v),\ y_2(v)$.

Reading off coordinates to get rectangular dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
...
$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
...

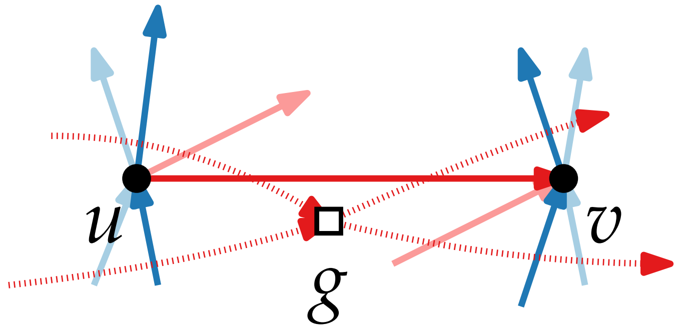
Reading off coordinates to get rectangular dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
...
$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
...

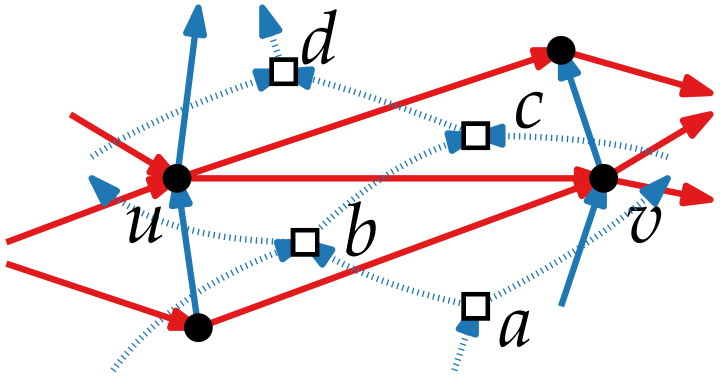
Correctness of algorithm (sketch)

- If edge (u, v) existens, then $x_2(u) = x_1(v)$



$$x_2(u) = f_{\text{ver}}(g) = x_1(v)$$

- and their veritcal segment of their rectangles overlap.



$$y_1(v) = f_{\text{hor}}(a) < y_1(u) = f_{\text{hor}}(b) <$$
$$y_2(v) = f_{\text{hor}}(c) < y_2(u) = f_{\text{hor}}(d)$$

- If path from u to v in red at least two edges long, then $x_2(u) < x_1(v)$.
- No two boxes overlap.

for details see He's paper [He '93]

Rectangular dual result

Theorem.
Every PTP graph G has a rectangular dual, which can be computed in linear time.

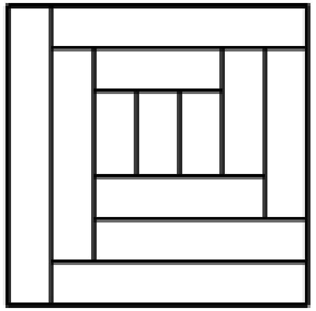
Proof.

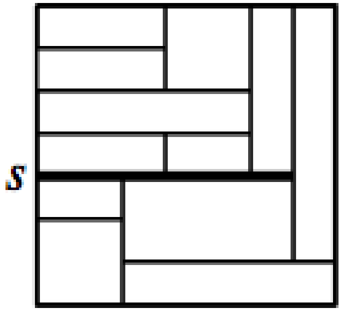
- Compute a planar embedding of G.
- Compute a refined canonical ordering of G.
- Traverse the graph and color the edges.
- Construct G_{ver} and G_{hor}.
- Construct their duals $G^{\star}_{\text{ver}}$ and $G^{\star}_{\text{hor}}$.
- Compute a topological ordering for vertices of $G^{\star}_{\text{ver}}$ and $G^{\star}_{\text{hor}}$.
- Assing coordinates to the rectangles representing vertices.

Discussion

- A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.
- A rectangular layout is **area-universal** if and only if it is **one-sided**. [Eppstein et al. SIAM J. Comp. 2012]

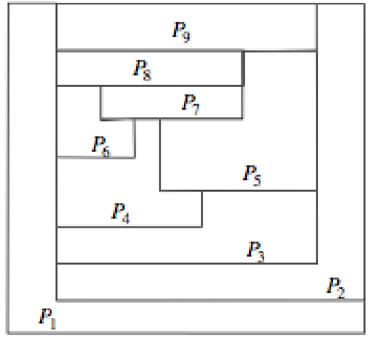
one-sided





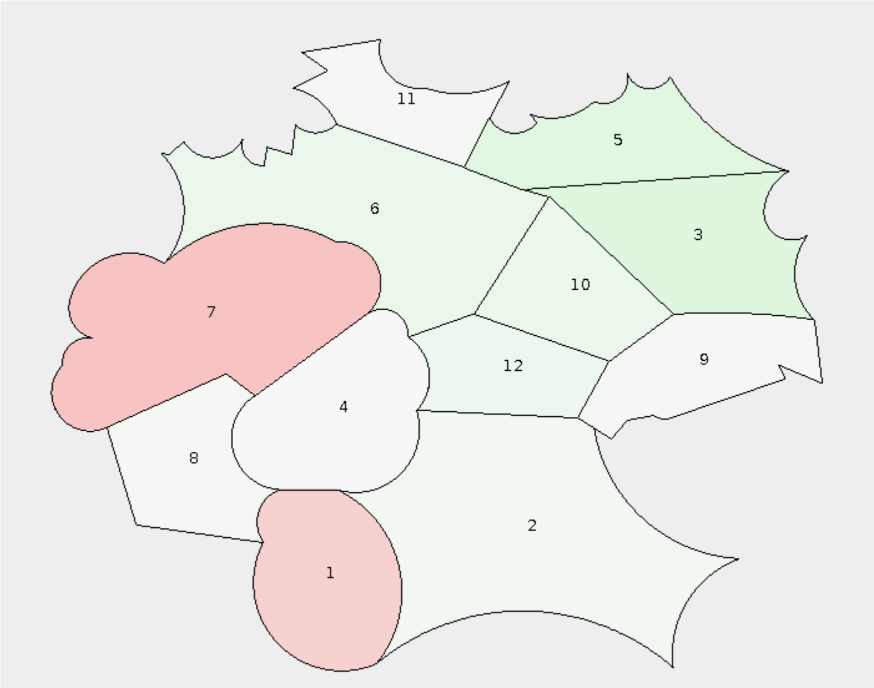
not one-sided

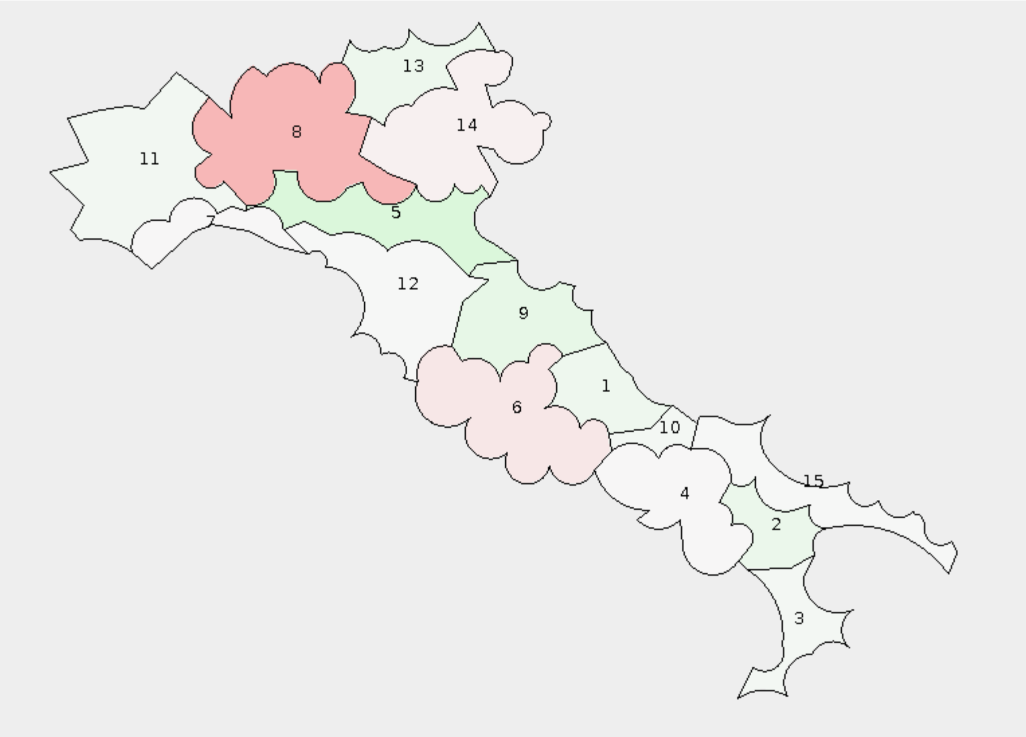
- Area universal **rectlinear** representation - possible for all planar graphs
- Alam et al. 2013: 8 sides (matches the lower bound)



Discussion

- Circular Arc Cartograms [Kämper, Kobourov, Nöllenburg. IEEE PasViz 2013]





Source: `http://cartogram.cs.arizona.edu`

Literature

Construction of triangle contact representations based on

- [de Fraysseix, de Mendez, Rosenstiehl ’94] On Triangle Contact Graphs

Construction of rectangular dual based on

- [He ’93] On Finding the Rectangular Duals of Planar Triangulated Graphs
- [Kant, He ’94] Two algorithms for finding rectangular duals of planar graphs

and originally from

- [Koźmiński, Kinnen ’85] Rectangular Duals of Planar Graphs