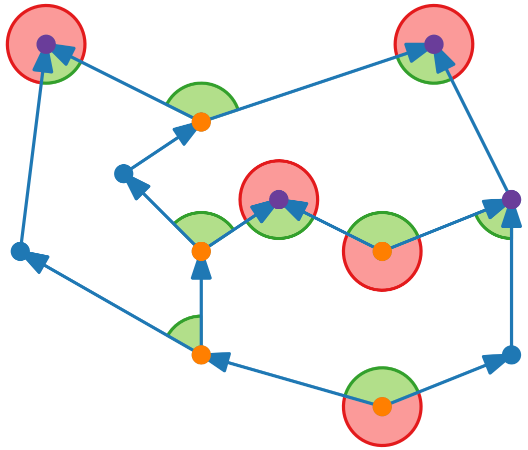
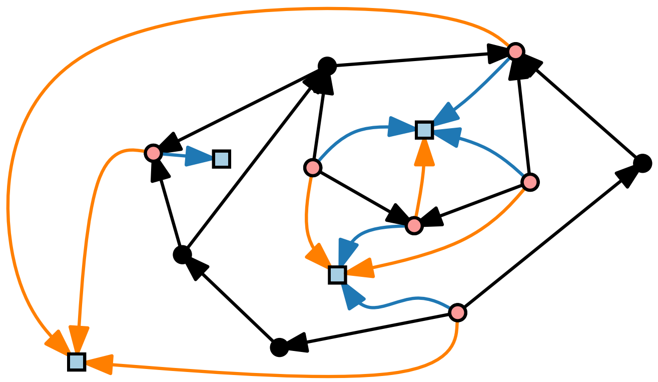
Visualisation of graphs

Upward planar drawings

Flow methods

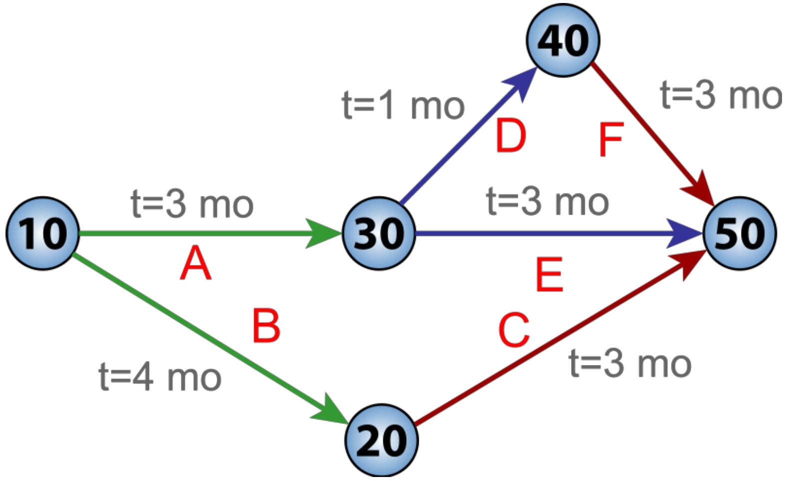
Jonathan Klawitter · Summer semester 2020



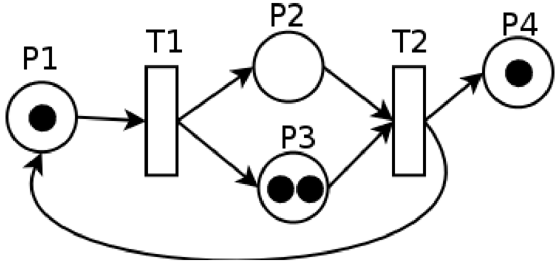


Upward planar drawings – motivation

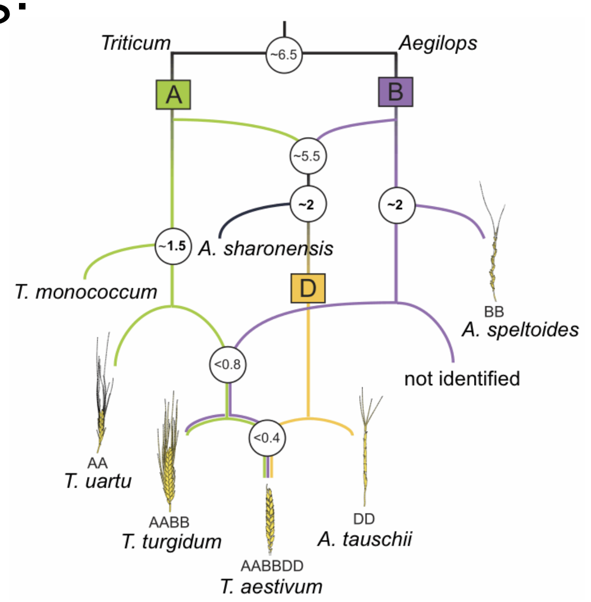
- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchie
  - ...
- Would be nice to have general direction preserved in drawing.



PERT diagram



Petri net

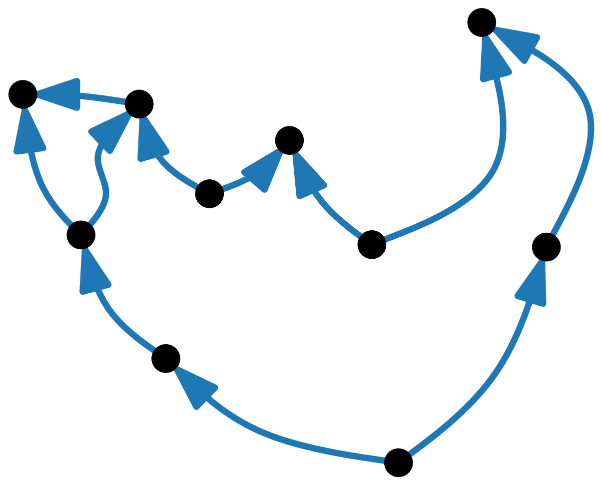


Phylogenetic network

Upward planar drawings – definition

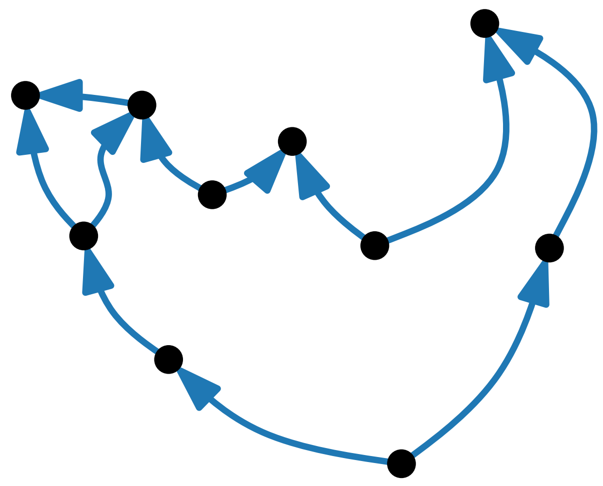
Definition.
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ (vertices = points, edges = simple curves) that is

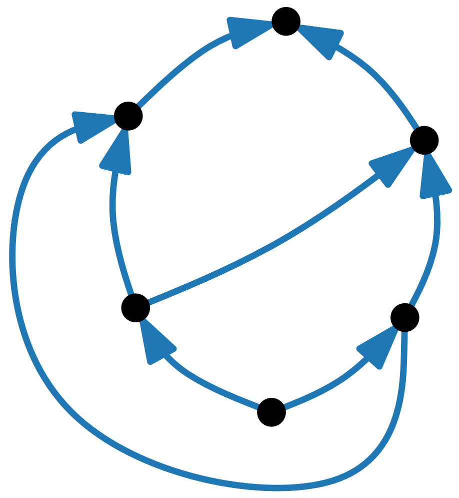
- planar and
- where each edge is drawn as an upward, y-monotone curve.

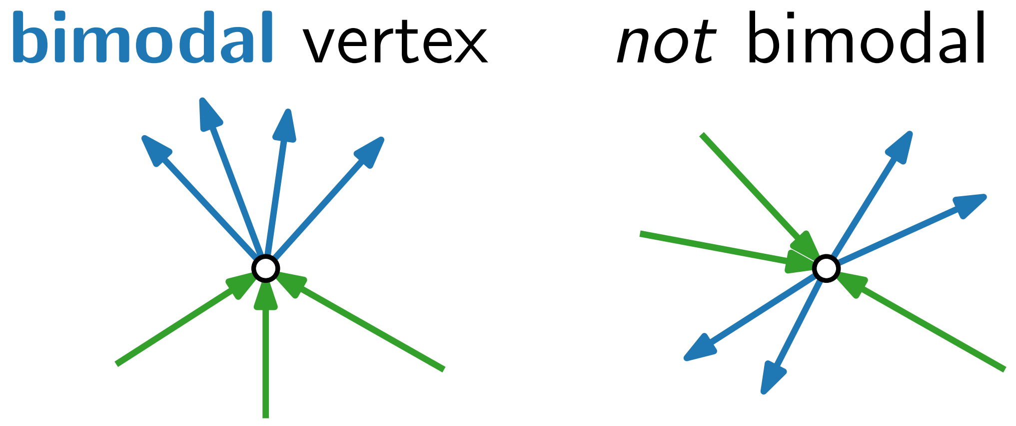


Upward planarity – necessary conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- . . . but these conditions are *not sufficient*.







Upward planarity – characterisation

Theorem 1. [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

planar: no crossings

st-digraph: acyclic digraph with a single source s and single sink t

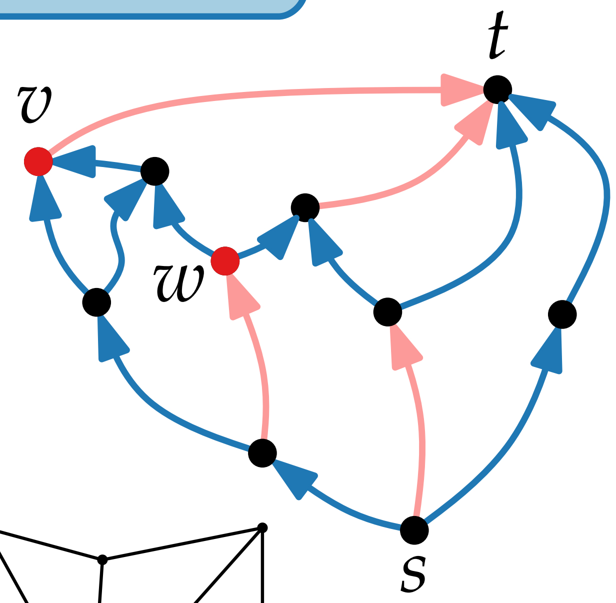
Additionally:
Embedded such that s and t are on the outerface f_0.
or:
Edge (s, t) exists.

Upward planarity – characterisation

Theorem 1. [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

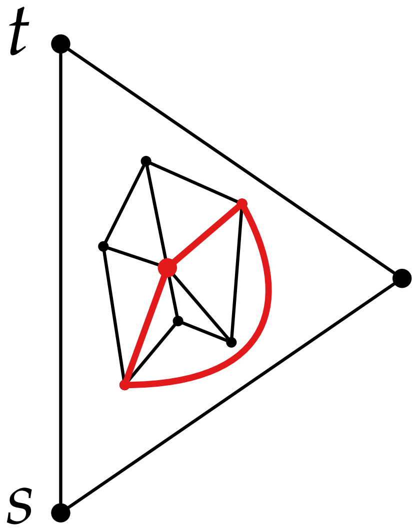
Proof.



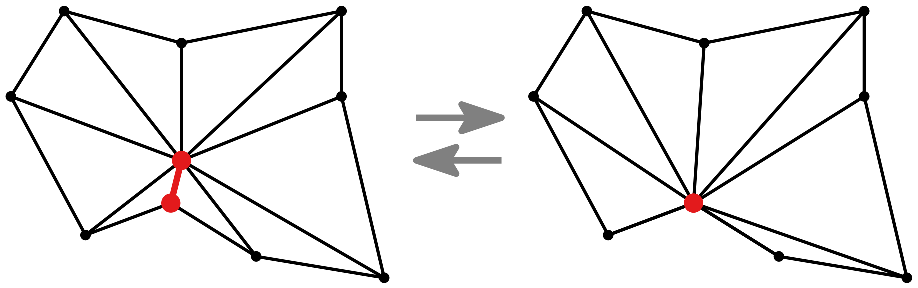
(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
(3) ⇒ (2) Triangulate & construct drawing:

Claim.
Can draw in prespecified triangle.
Apply induction.

Case 1:



Case 2:



Upward planarity – complexity

Theorem. [Garg, Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Corollary.
For a *triconnected* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Theorem. [Hutton, Libow, 1996]
For a *single-source* acyclic digraph it can be tested in $\mathcal{O}(n)$ time whether it is upward planar.

The problem

Fixed embedding upward planarity testing.
Let $G = (V, E)$ be a plane digraph with the embedding given by the set of faces F and the outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

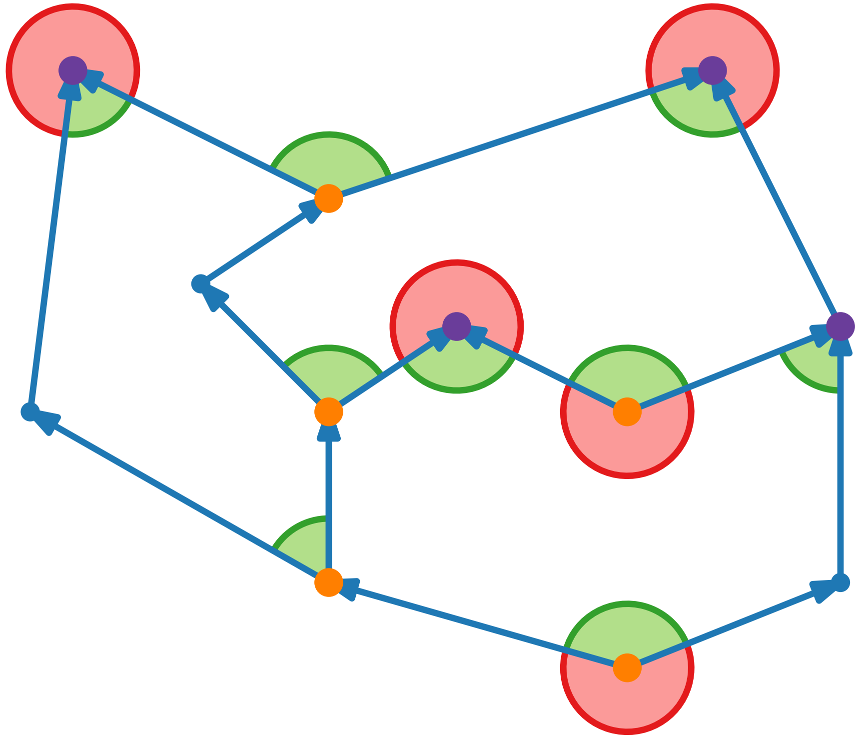
Idea.

- Find property that any upward planar drawing of G satisfies.
- Formalise property.
- Find algorithm to test property.

Angles, local sources & sinks

Definitions.

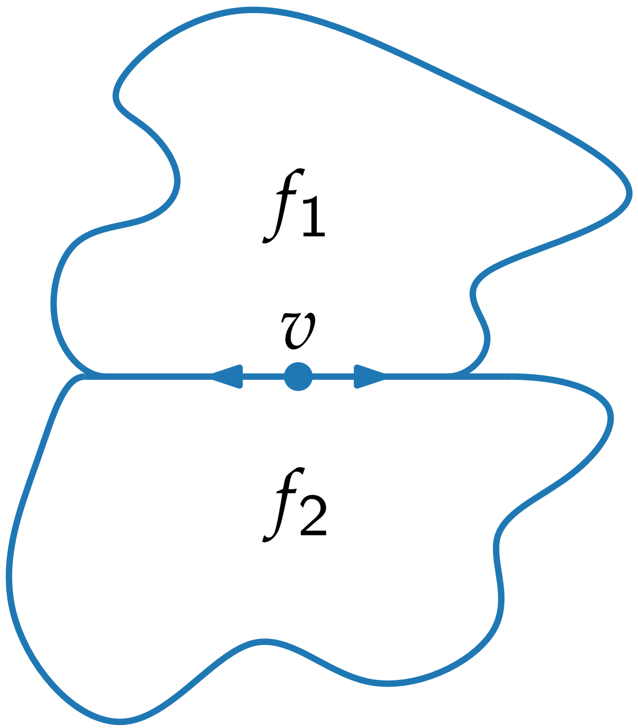
- A vertex v is a **local source** wrt to a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt to a face f if v has two incoming edges on ∂f.
- An angle α is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v) = \#$ large angles at v
- $L(f) = \#$ large angles in f
- $S(v)$ & $S(f)$ for $\#$ small angles
- $A(f) = \#$ local sources wrt to f
  $= \#$ local sinks wrt to f



Lemma 1.
$L(f) + S(f) = 2A(f)$

Assignment problem

- Vertex v is a global source for f_1 and f_2.
- Has v a **large** angle in f_1 or f_2?

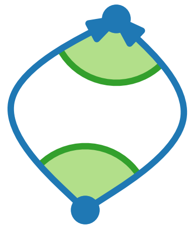


Angle relations

Lemma 2.

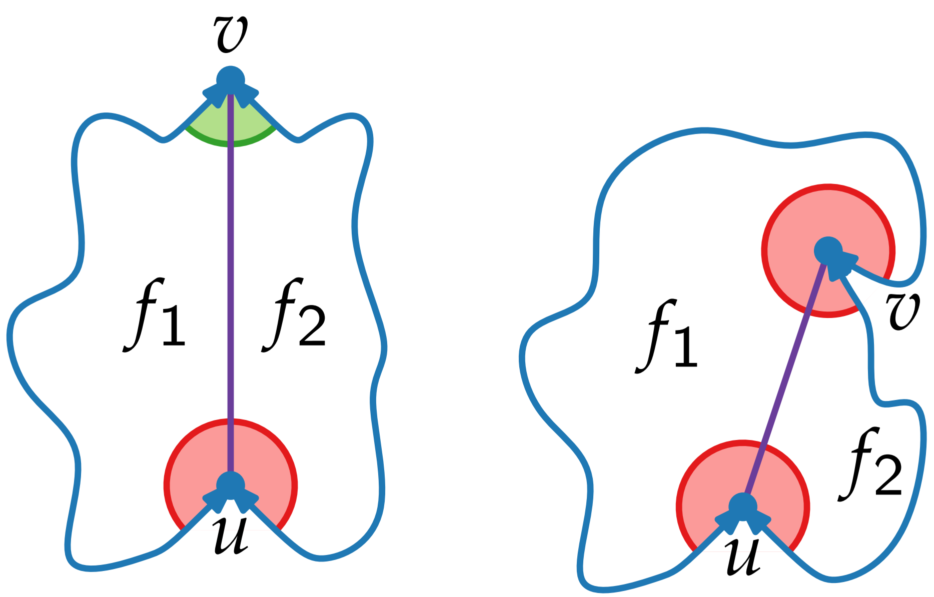
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$
- $L(f) \geq 1$

Split f with **edge** from a large angle at a "low" sink u to

- sink v with small/large angle:



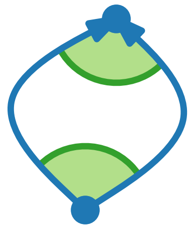
$$\begin{aligned} L(f) - S(f) &= L(f_1) + L(f_2) + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

Angle relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

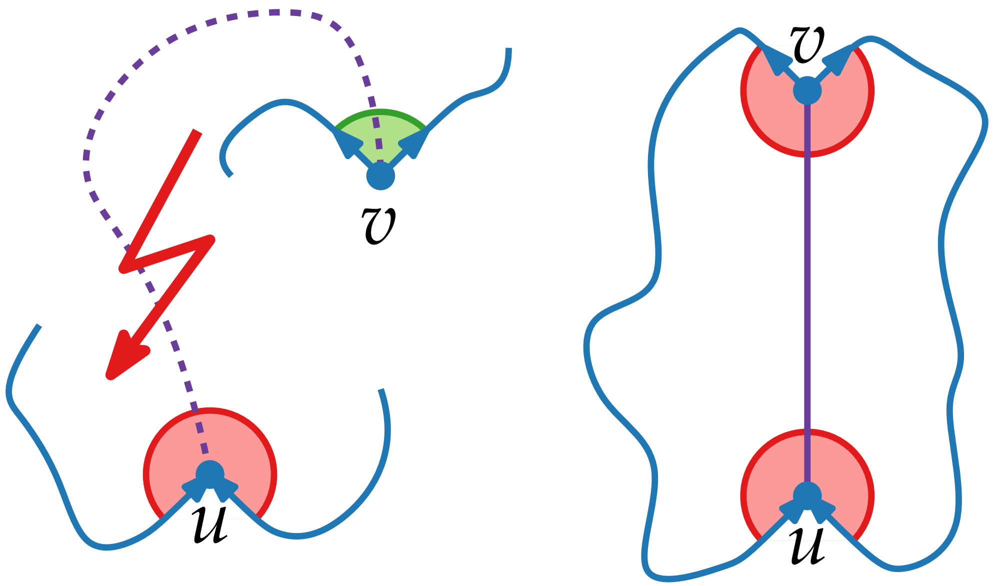
Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with **edge** from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:

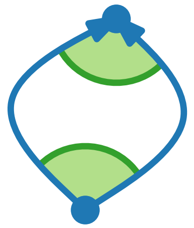


$$\begin{aligned} L(f) - S(f) &= L(f_1) + L(f_2) + 2 \\ &\quad - (S(f_1) + S(f_2)) \\ &= -2 \end{aligned}$$

Angle relations

Lemma 2.
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

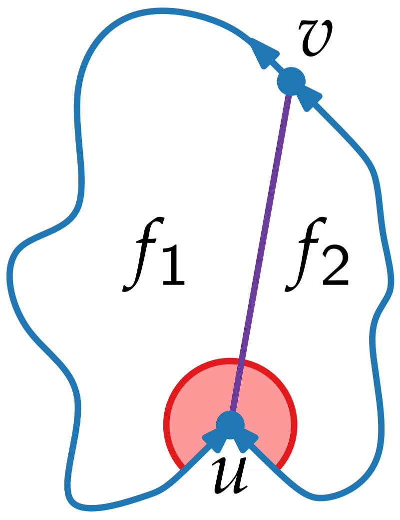
Proof by induction.

- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with **edge** from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$$\begin{aligned} L(f) - S(f) &= L(f_1) + L(f_2) + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

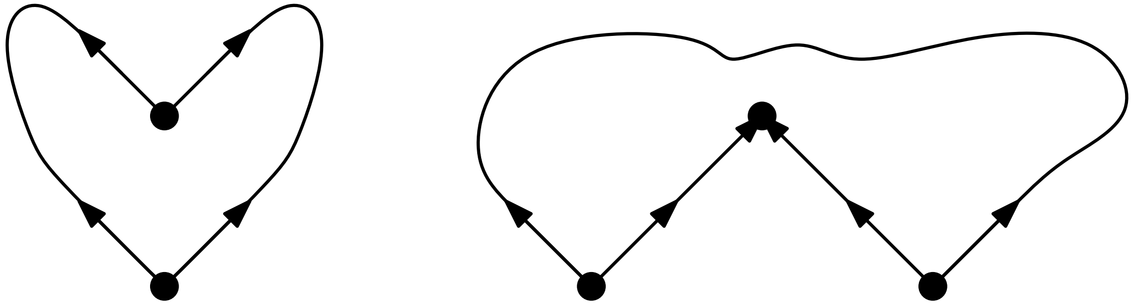
- Otherwise "high" source u exists.

Number of large angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source/sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof.
Observation and from Lemma 1: $L(f) + S(f) = 2A(f)$
and from Lemma 2: $L(f) - S(f) = \pm 2.$



Assignment of large angles to faces

- Let S and T be the sets of sources and sinks, respectively.

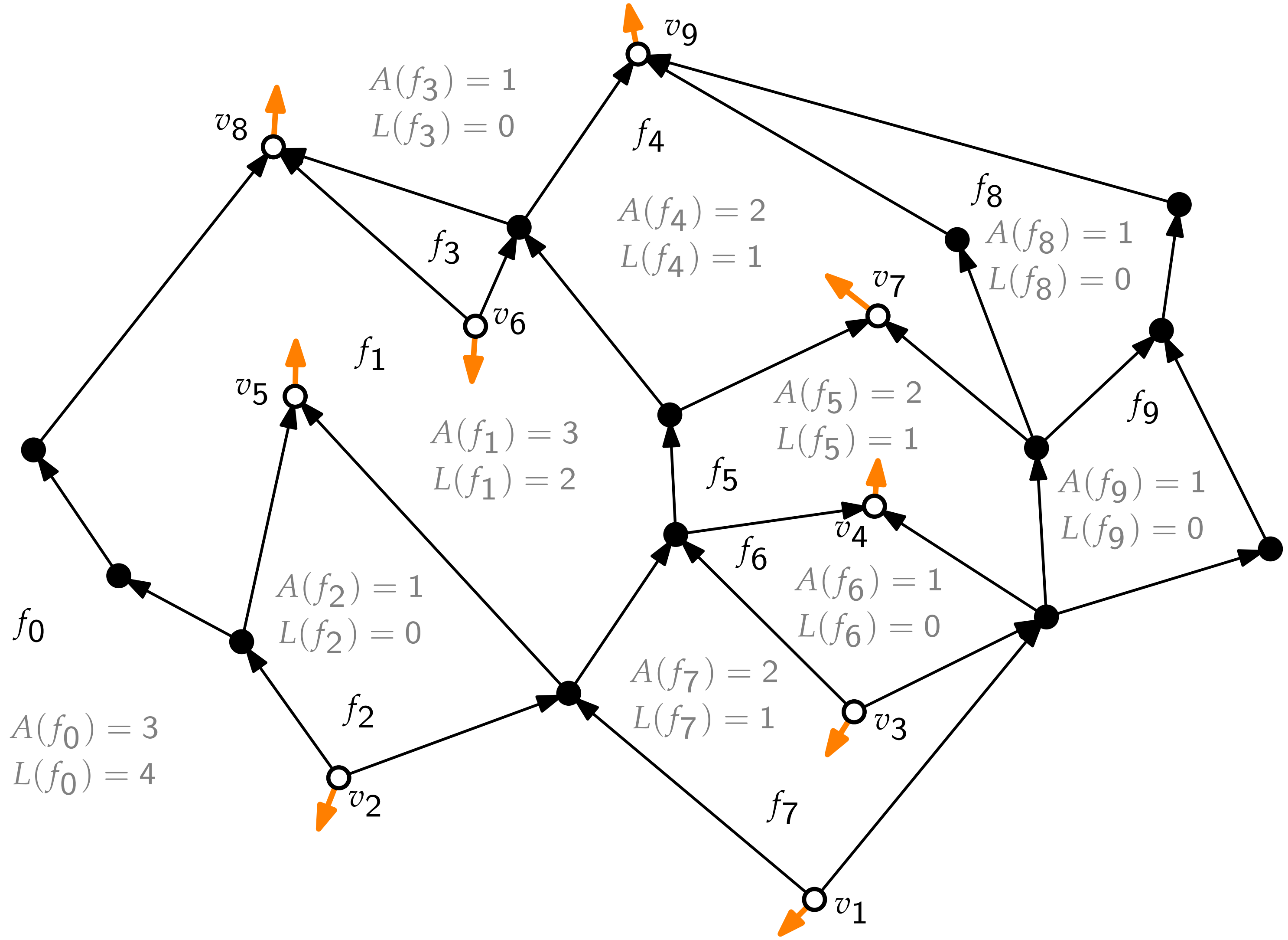
Definition.
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping where

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$

Example of angle to face assignment



∘ global sources & sinks

$A(f)$ # sources/sinks of f

assignment
$\Phi : S \cup T \to F$

Result characterisation

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Proof.

$\Rightarrow$: As constructed before.

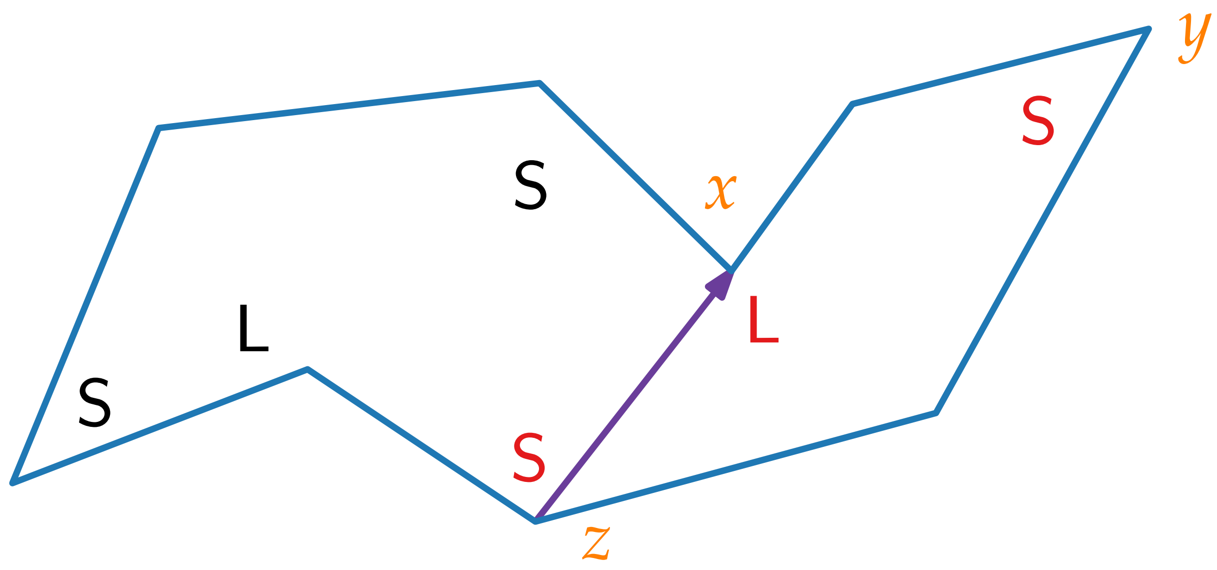
$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

Refinement algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

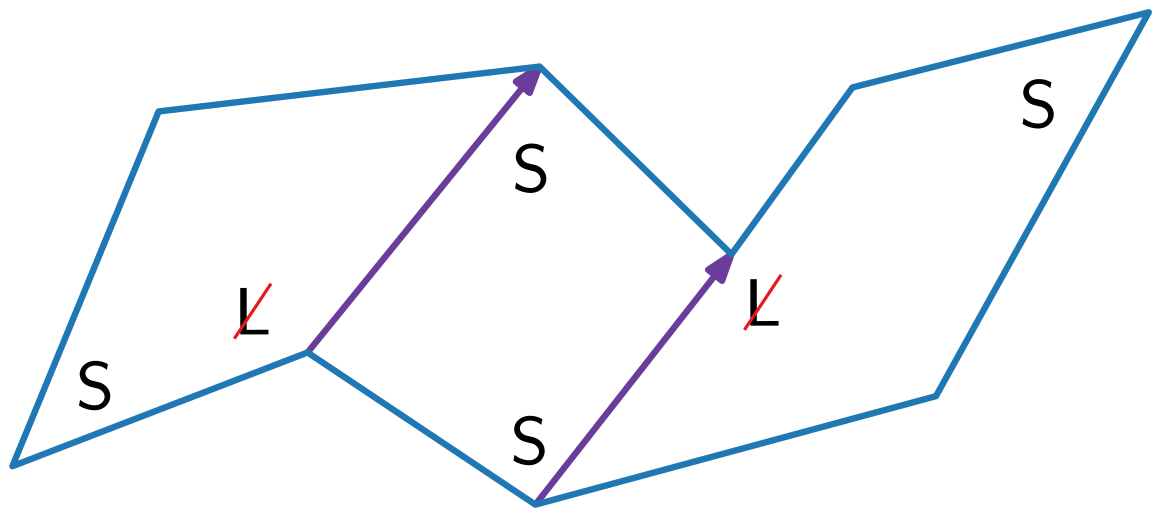
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
  - x source $\Rightarrow$ insert edge (z, x)



Refinement algorithm – $\Phi, F, f_0 \to$ st-digraph

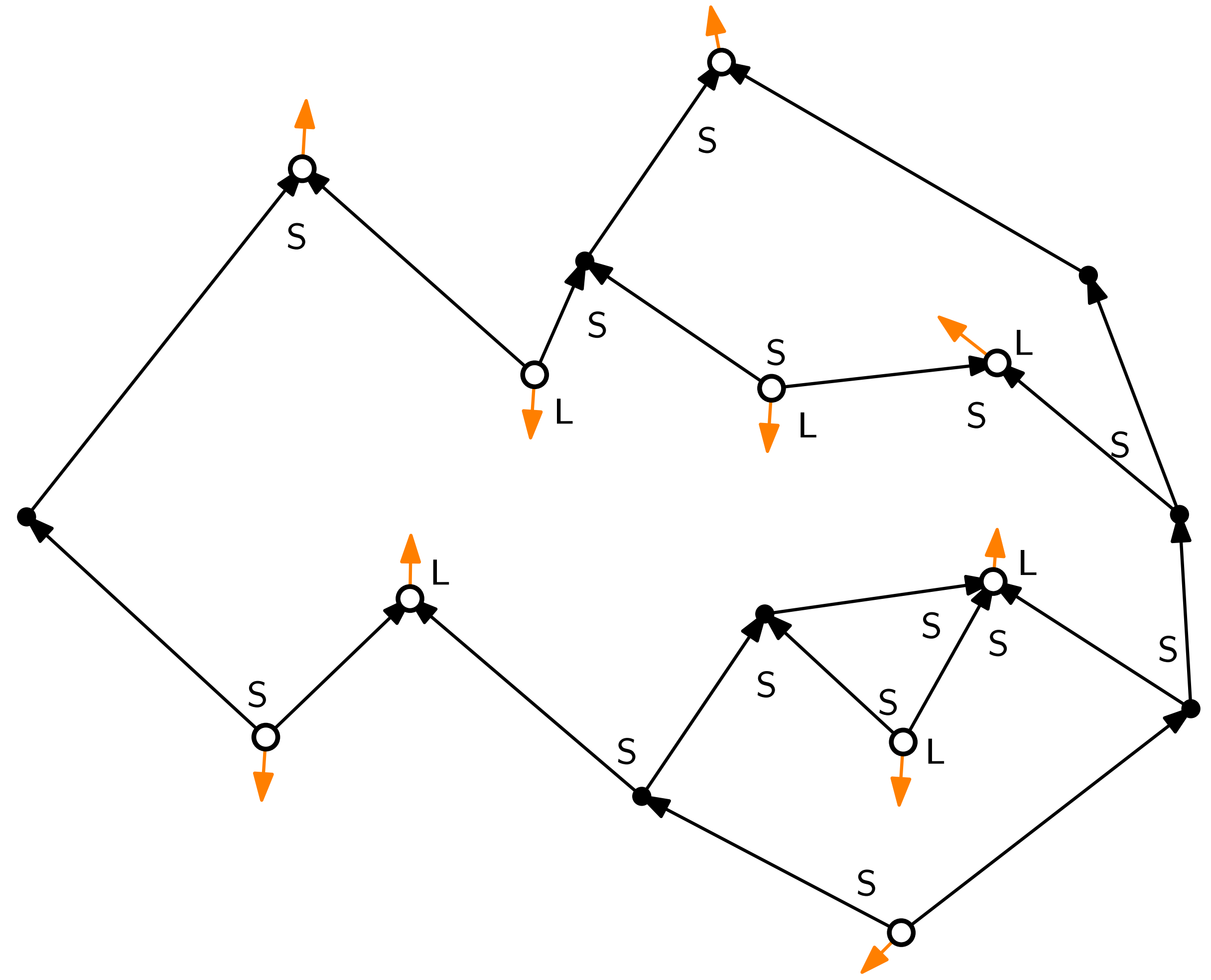
Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
  - x source $\Rightarrow$ insert edge (z, x)
  - x sink $\Rightarrow$ insert edge (x, z).
- Refine outer face f_0.

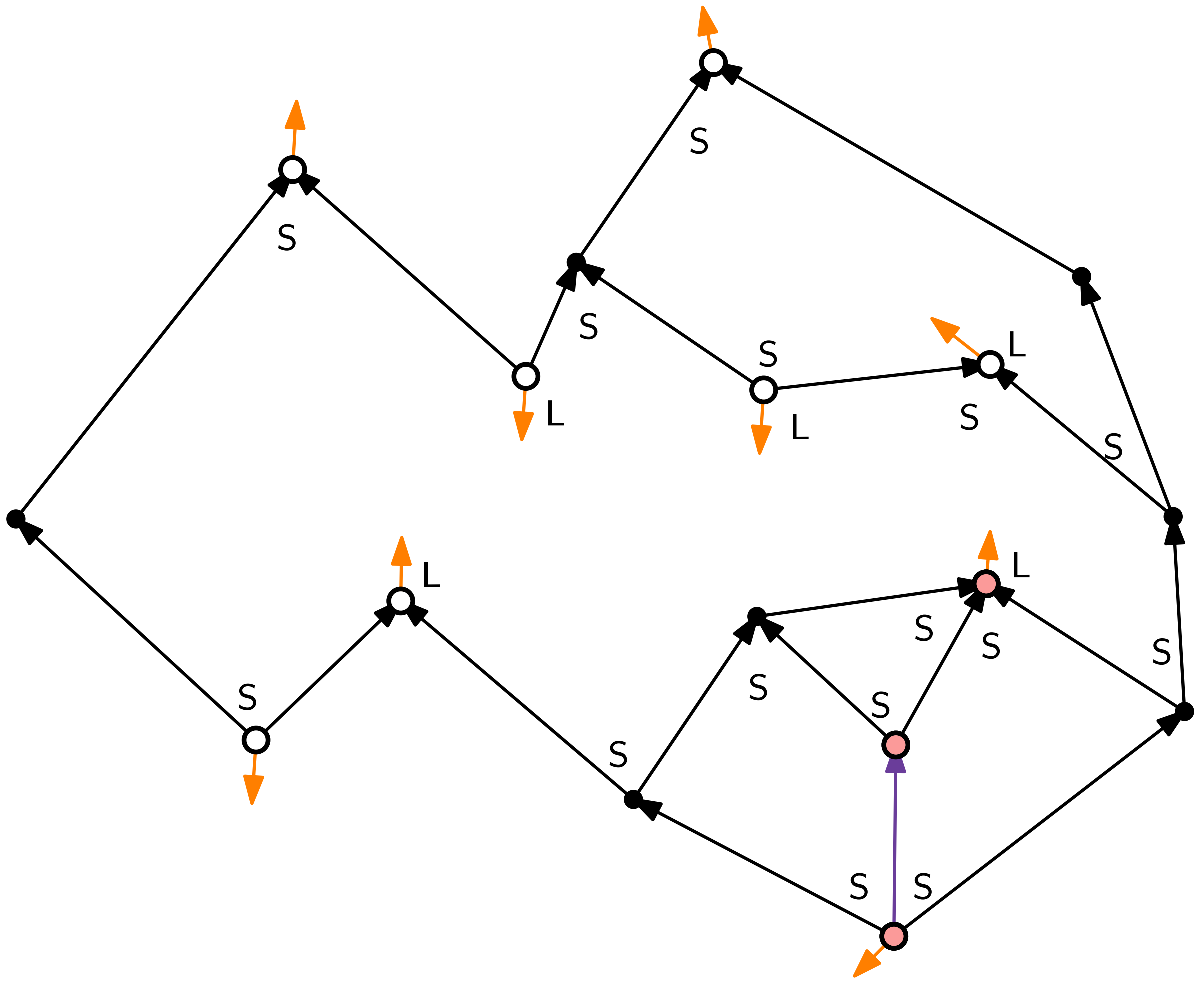


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

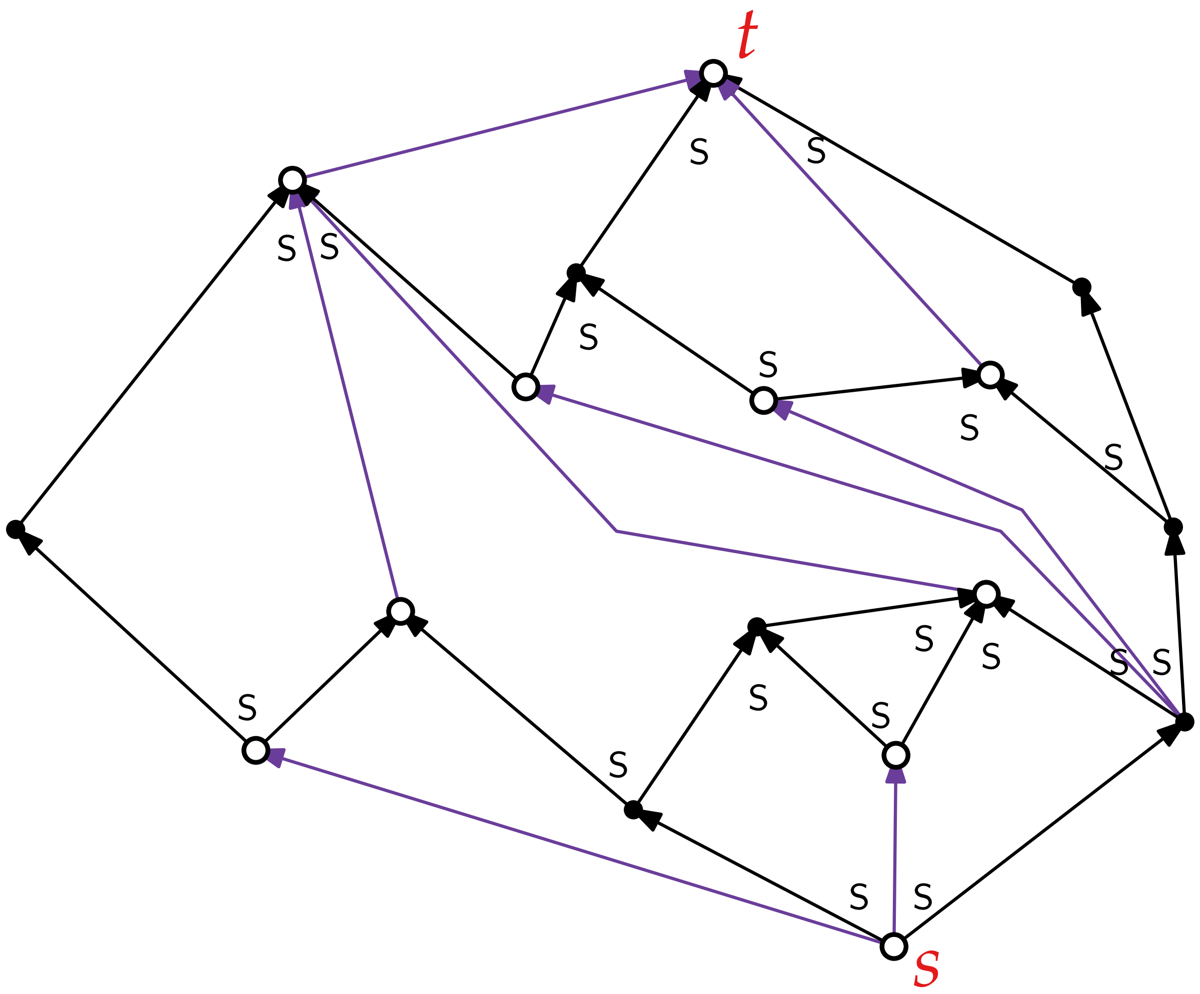
Refinement example



Refinement example



Refinement example



Result upward planarity test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).
- If G bimodal and Φ exists, refine G to plane st-digraph H.
- Draw H upward planar.
- Deleted edges added in refinement step.

Finding a consistent assignment

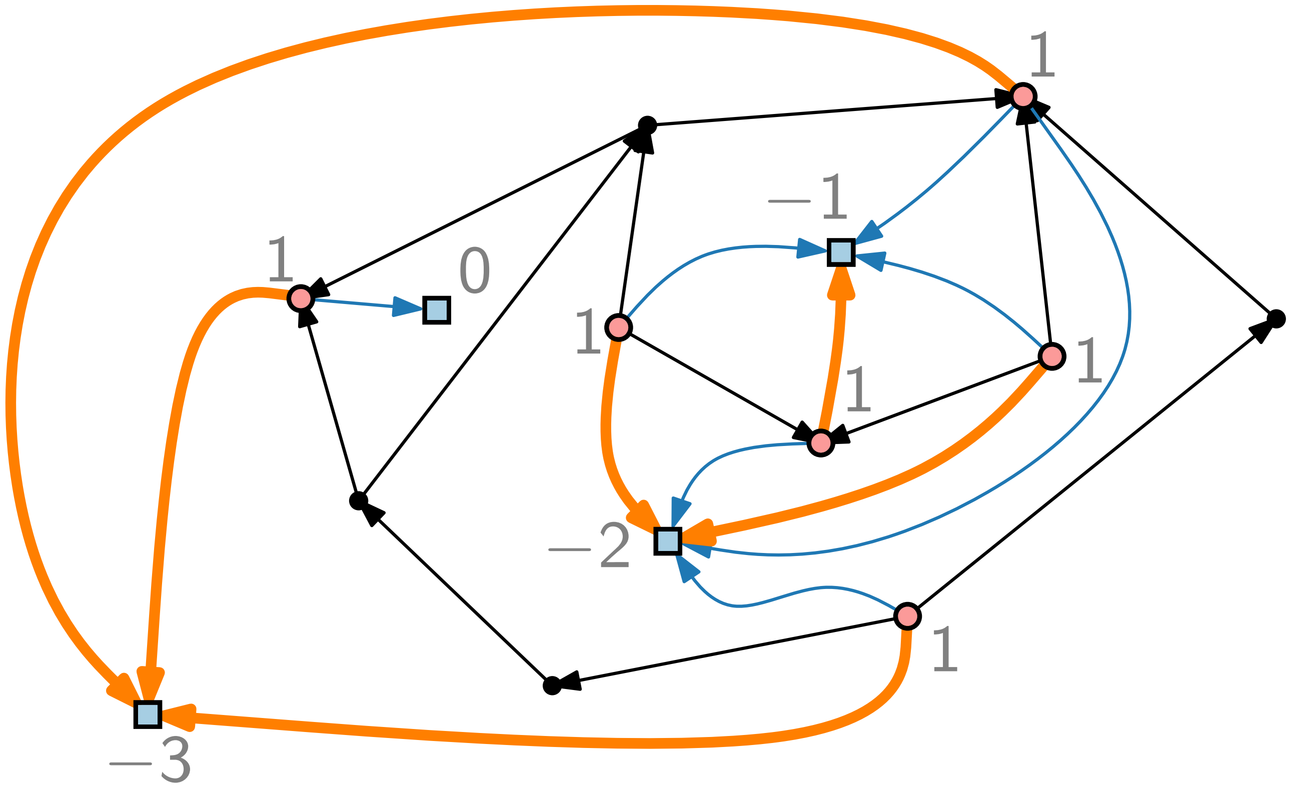
Idea.
Flow $(v, f) = 1$ from global source/sink v to the incident face f its large angle gets assigned to.

Flow network.

$N_{F,f_0}(G) = ((W, E'); \ell; u; d)$

- $W = \{v \in V \mid v \text{ source or sink}\} \cup F$
- $E' = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $d(p) = \begin{cases} 1 & \forall p \in W \cap V \\ -(A(p) - 1) & \forall p \in F \setminus \{f_0\} \\ -(A(p) + 1) & p = f_0 \end{cases}$

Example.



Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]
- Finding assignment in Theorem 2 can be sped up to $\mathcal{O}(n + r^{1.5})$ where $r = \#$ sources/sinks.
  [Abbasi, Healy, Rextin 2010]
- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, ...

Literature

- [GD Ch. 6] for detailed explanation

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg, Tamassia '95] On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton, Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10] Improving the running time of embedded upward planarity testing