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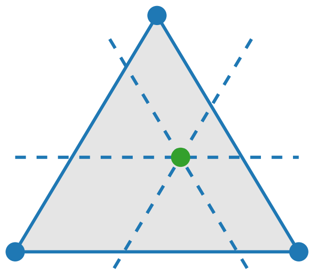
WÜRZBURG

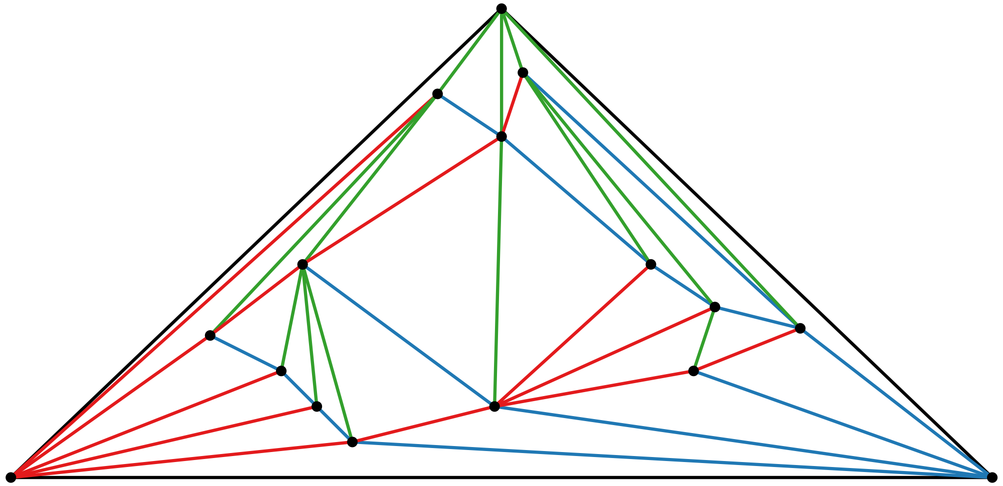
Visualisation of graphs

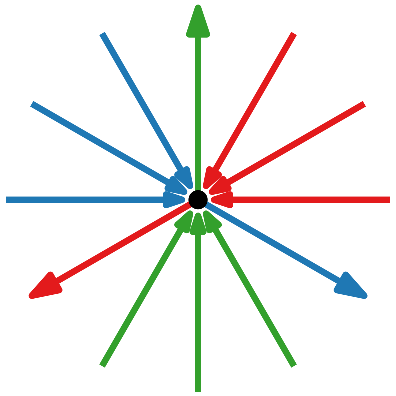
Planar straight-line drawings

Schnyder realiser

Jonathan Klawitter · Summer semester 2020







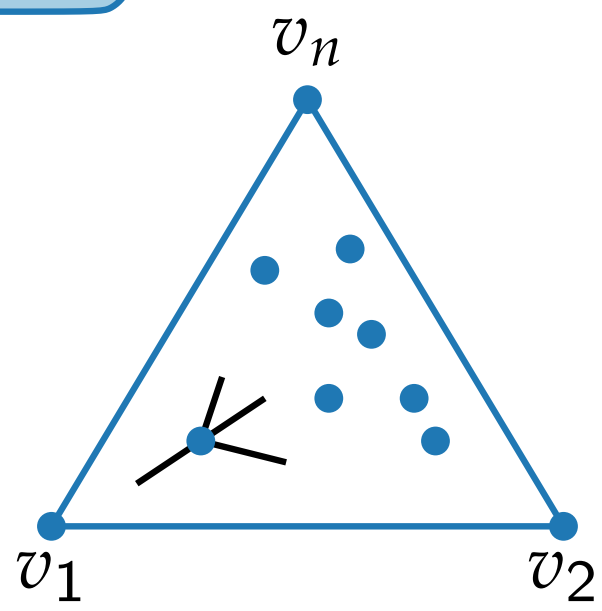
Planar straight-line drawings

Theorem. [De Fraysseix, Pach, Pollack ’90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

Theorem. [Schnyder ’90] Every n-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$.
$(2n-5) \times (2n-5)$

Idea.

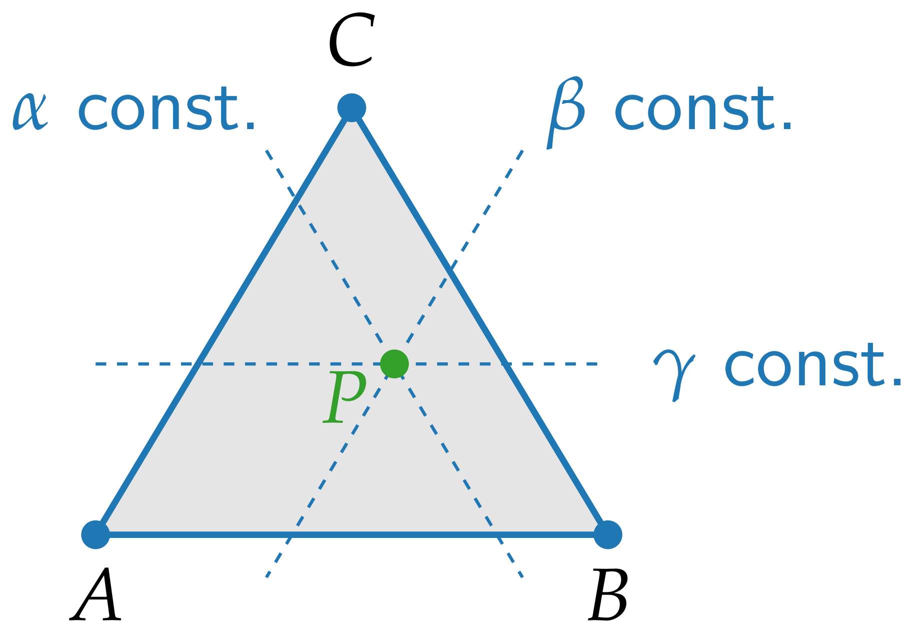
- Fix outer triangle.
- Compute coordinates of inner vertices
  - based on outer triangle
  - and how much space there has to be for other vertices
- using barycentric coordinates.

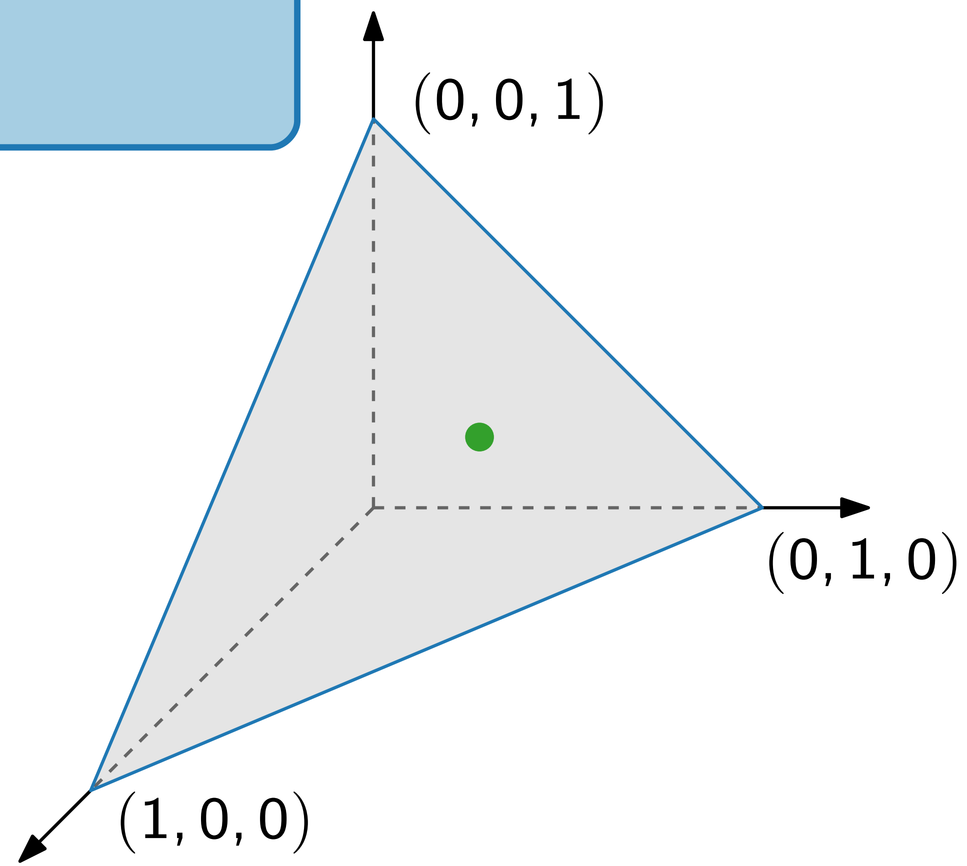


Barycentric coordinates

Definition.
Let $A, B, C, P \in \mathbb{R}^2$.
The **barycentric coordinates** of P with respect to $\triangle ABC$ are a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3_{\geq 0}$ such that

- $\alpha + \beta + \gamma = 1$
- $P = \alpha A + \beta B + \gamma C$.

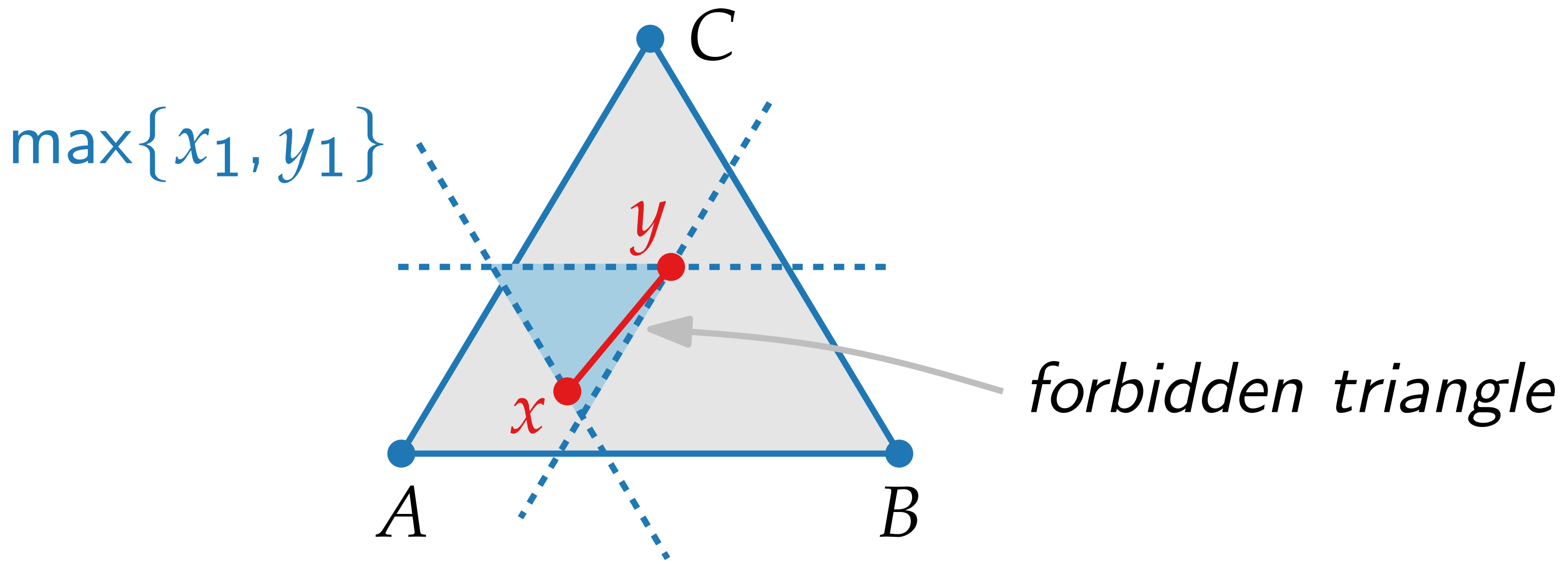




Barycentric representation

Definition.
A **barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of G; i.e. it is *injective* map $\phi \colon V \to \mathbb{R}^3_{\geq 0}$, $v \mapsto (v_1, v_2, v_3)$ with the following properties:

- $v_1 + v_2 + v_3 = 1$ for all $v \in V$
- for each $\{x, y\} \in E$ and each $z \in V \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$.



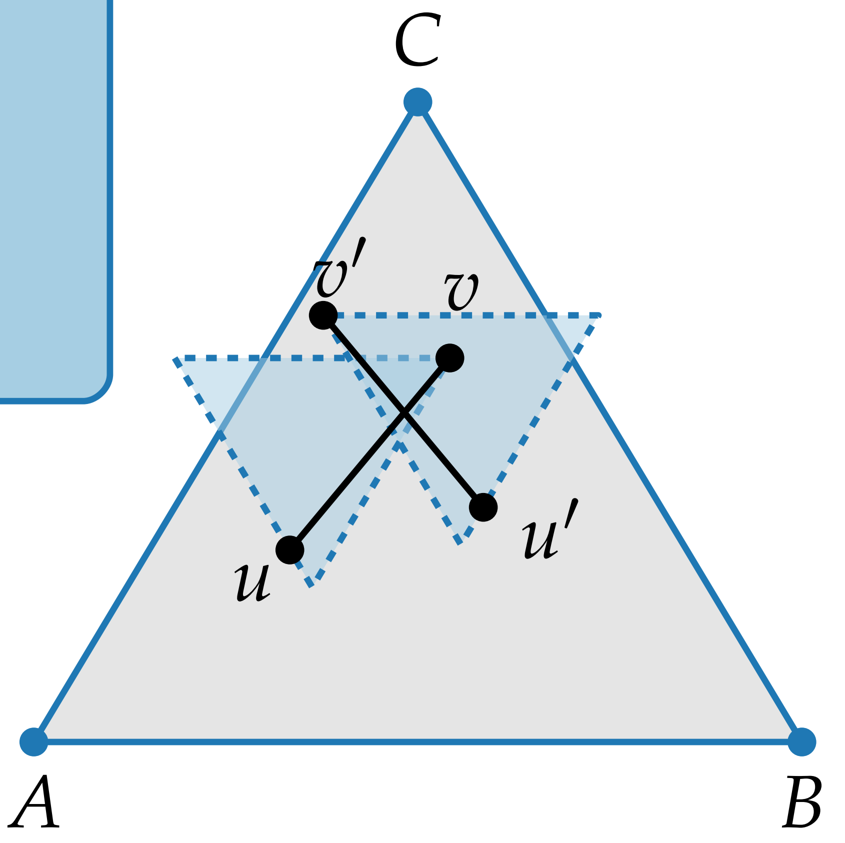
Barycentric representations & planar graphs

Lemma.
Let $\phi : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a graph $G = (V, E)$ and let $A, B, C \in \mathbb{R}^2$ in general position.
Then the mapping

$$f : v \in V \mapsto v_1 A + v_2 B + v_3 C$$

gives a **planar** drawing of G inside $\triangle ABC$.

How to get vertices on grid?



Proof.

- No vertices occur "inside" an edge
- No pair of edges $\{u, v\}$ and $\{u', v'\}$ cross:

$u'_i > u_i, v_i \quad v'_j > u_j, v_j \quad u_k > u'_k, v'_k \quad v_l > u'_l, v'_l$

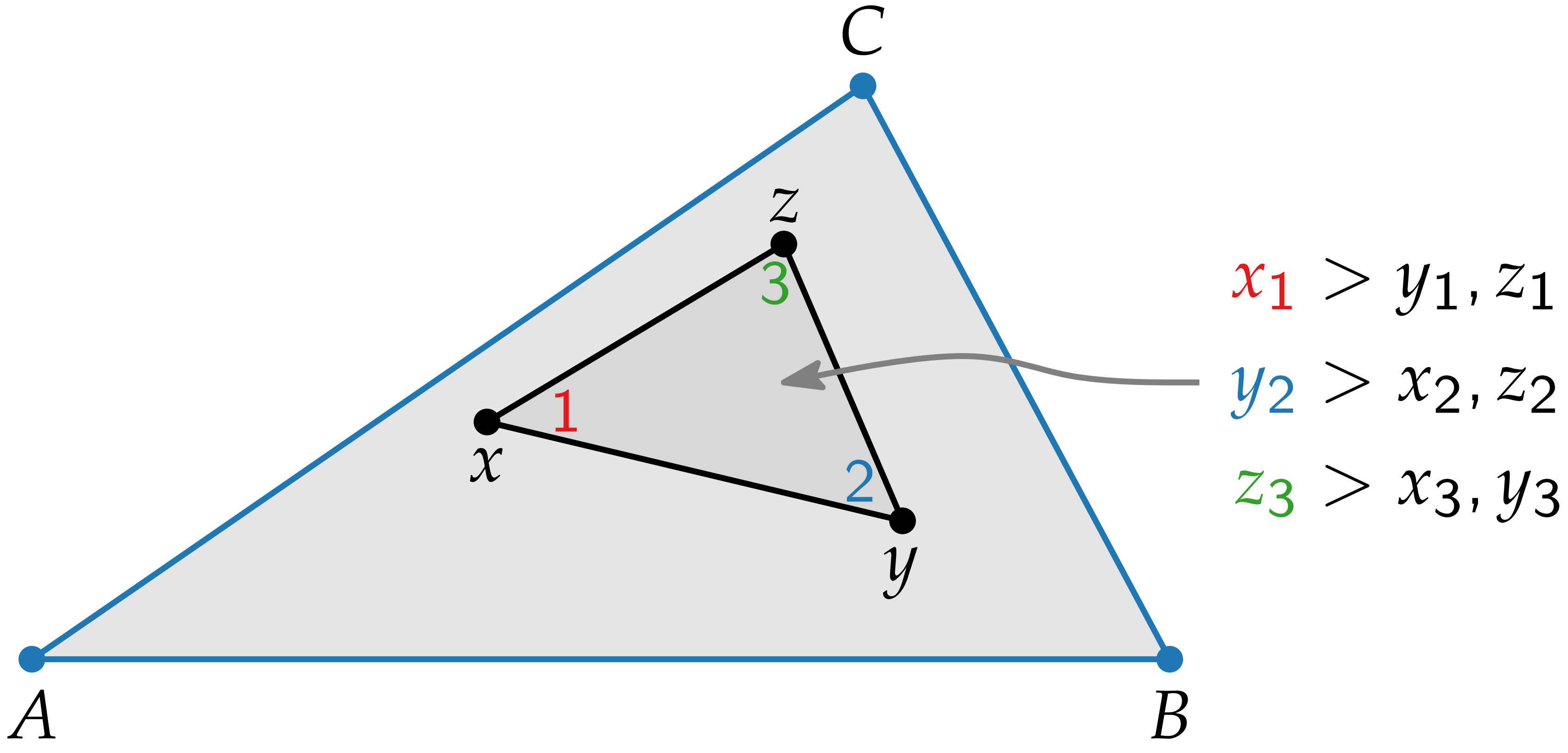
$\Rightarrow \{i, j\} \cap \{k, l\} = \emptyset$

wlog $i = j = 1 \Rightarrow u'_1, v'_1 > u_1, v_1$ $\quad\Rightarrow$ separated by straight line

Angle labeling

Observation

Let $v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a triangulated plane graph $G = (V, E)$.
We can **uniquely** label each angle $\angle(xy, xz)$ with $k \in \{1, 2, 3\}$.

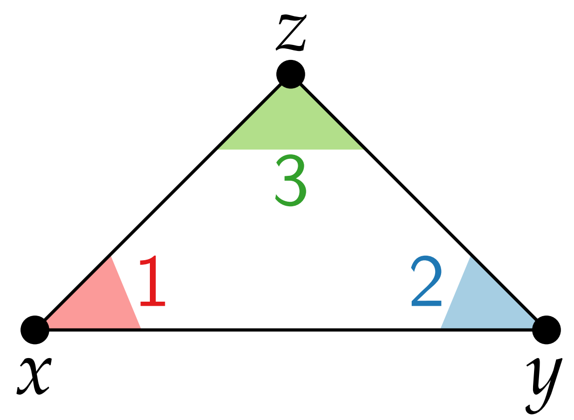


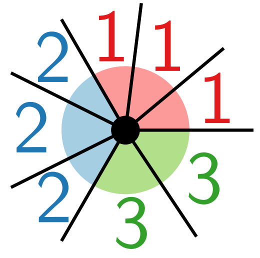
Schnyder labeling

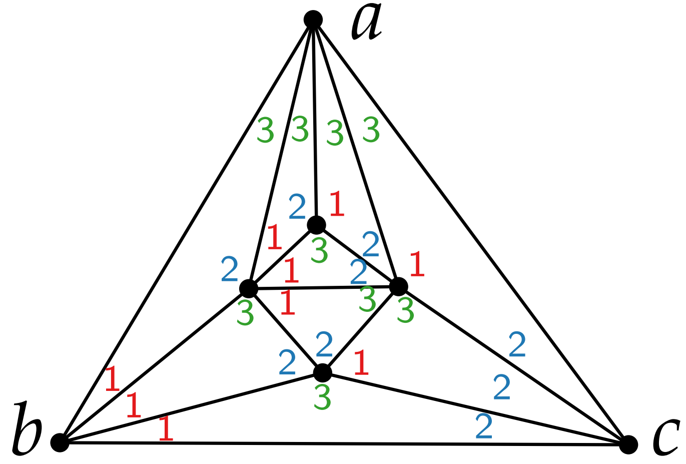
Definition.
A **Schnyder labeling** (normal labeling) of a triangulated plane graph G is a labeling of all internal angles with labels 1, 2 and 3 such that:

Faces Each internal face contain vertices with all three labels 1, 2 and 3 appearing in a counterclockwise order.

Vertices The ccw order of labels around each vertex consists of a nonempty interval of 1's followed by a nonempty interval of 2's followed by a nonempty interval of 3's.

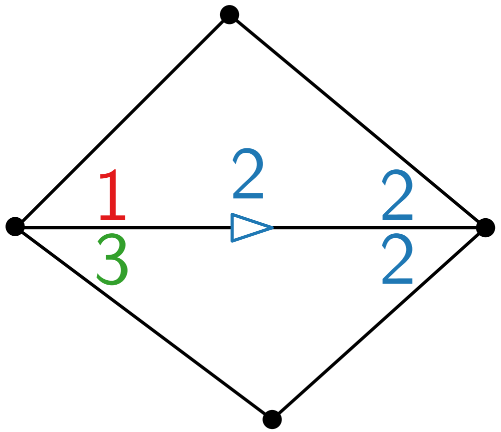


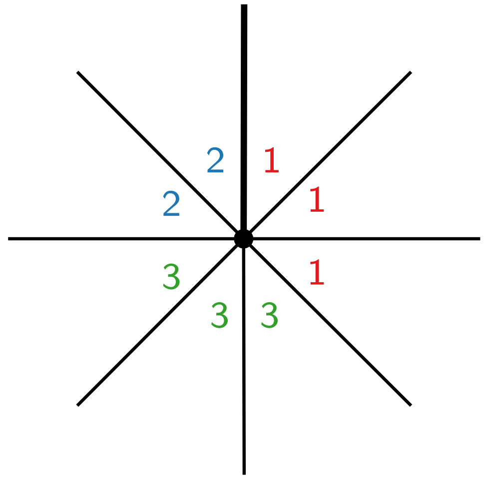




Schnyder realiser

- Schnyder labeling induces an edge labeling





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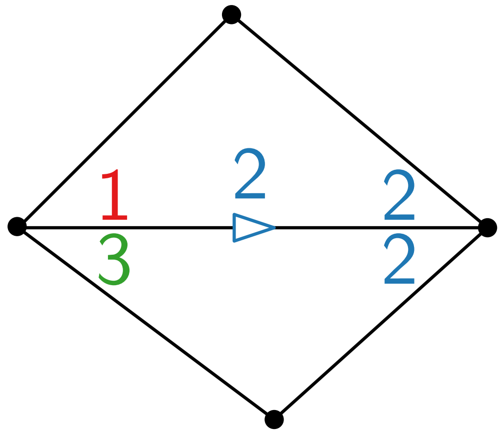

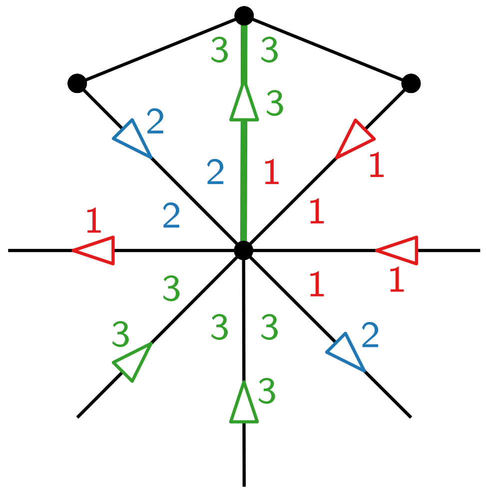

Schnyder realiser

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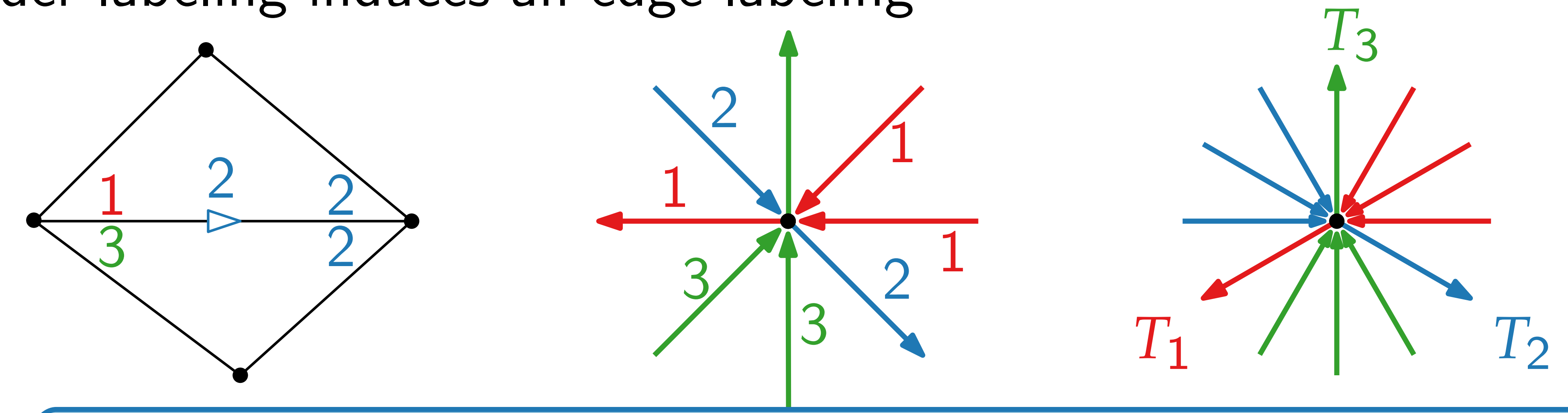


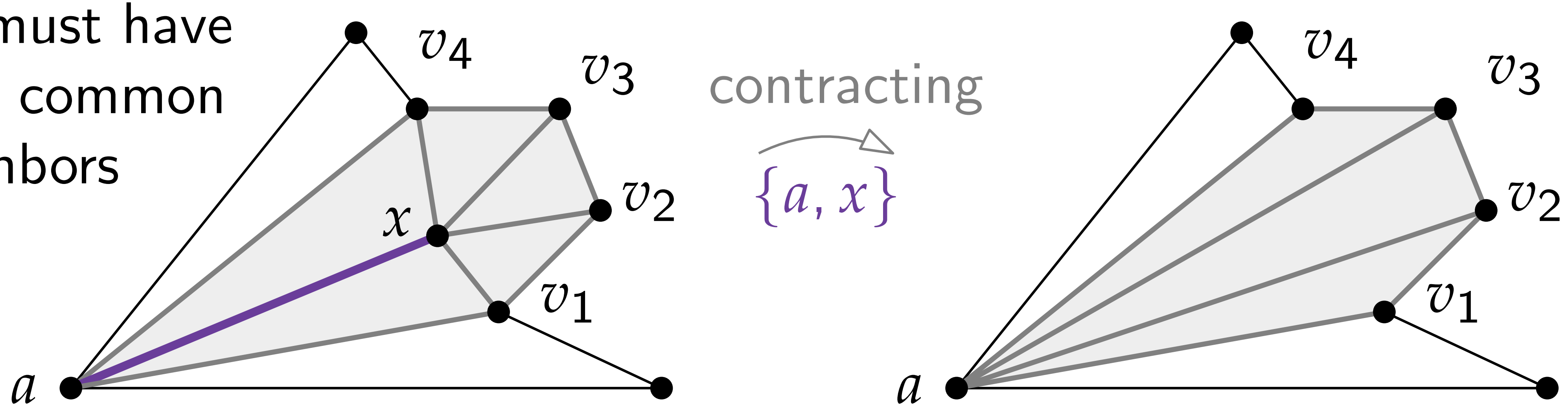
Definition.
A **Schnyder forest** or **realiser** of a triangulated plane graph $G = (V, E)$ is a partition of the inner edges of E into three sets of oriented edges T_1, T_2, T_3 such that for each inner vertex $v \in V$ holds:

- v has one outgoing edge in each of T_1, T_2, and T_3.
- The ccw order of edges around v is: leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, entering in T_2.

Schnyder realiser – existence

Lemma. [Kampen 1976]
Let G be a triangulated plane graph with vertices a, b, c on the outer face. There exists a **contractible edge** $\{a, x\}$ in G, $x \neq b, c$.

a and x must have exactly 2 common neighbors



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Theorem.
Every triangulated plane graph has a Schnyder labeling.

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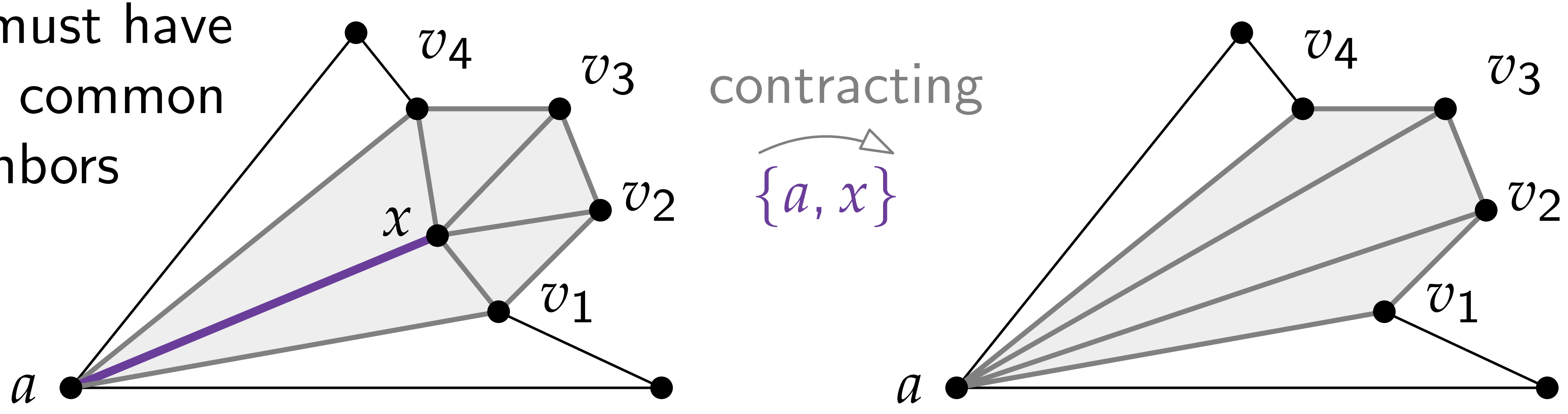

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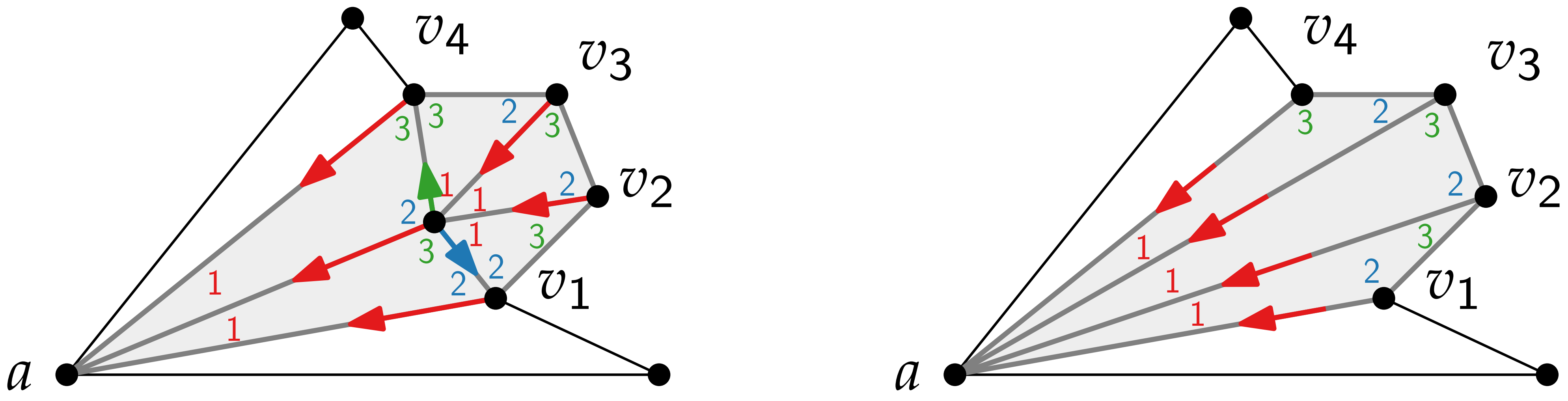
Proof by induction on # vertices via edge contractions.



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Proof also gives an algorithm to produce a Schnyder labeling.
It can be implemented in $\mathcal{O}(n)$ time . . . as exercise.

Schnyder realiser – existence

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Theorem.
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Proof also gives an algorithm to produce a Schnyder labeling.
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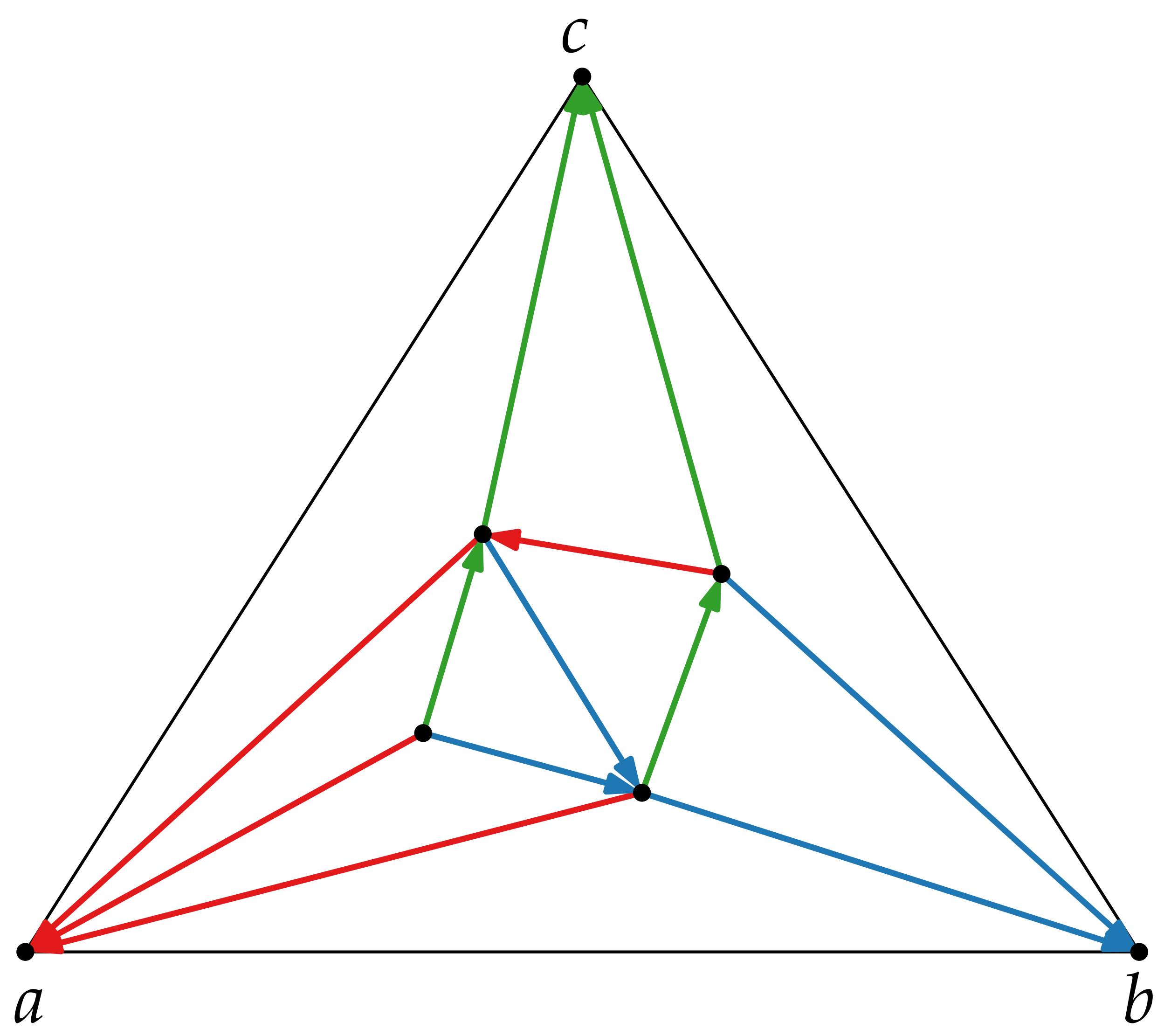
Theorem and previous construction imply:

Corollary.
Every triangulated plane graph has a Schnyder realiser.

Schnyder realiser – properties

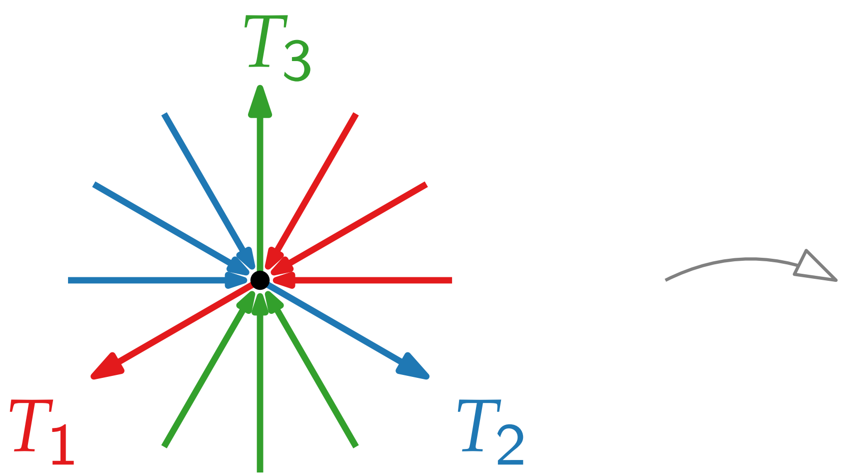


- For each v there exists a directed red, blue, green path from v to a, b, c, respectively.
- No monochromatix cycle exists
- Each monochromatic subgraph is a tree!
- The sinks of red/blue/green trees are the vertices a, b, c.

This is ensured by construction via contraction operation.
(Bonus: Can construct all valid Schnyder realiser.)

Schnyder drawing

- How to get from Schnyder realiser to barycentric representation



$$f : v \in V \mapsto v_1 A + v_2 B + v_3 C$$

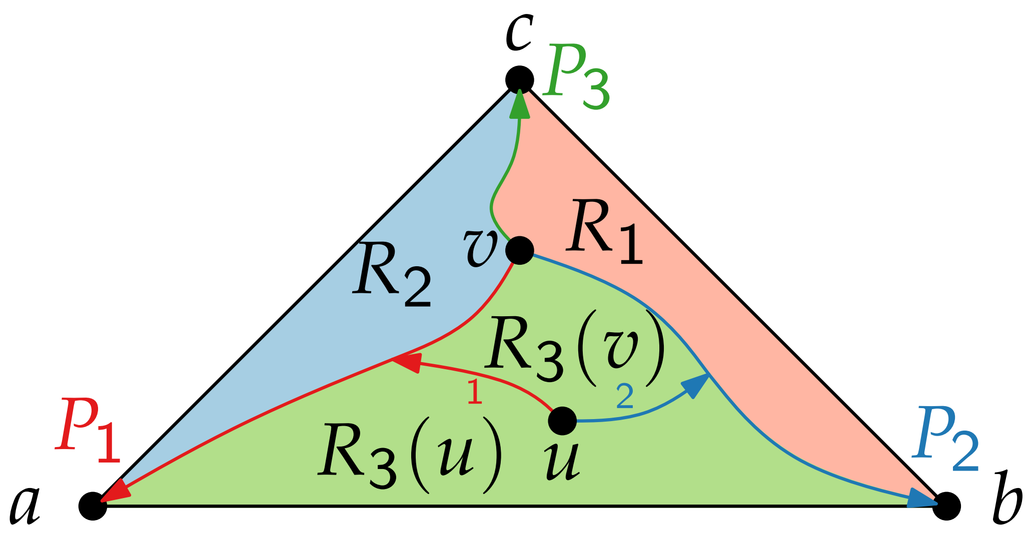
Face regions

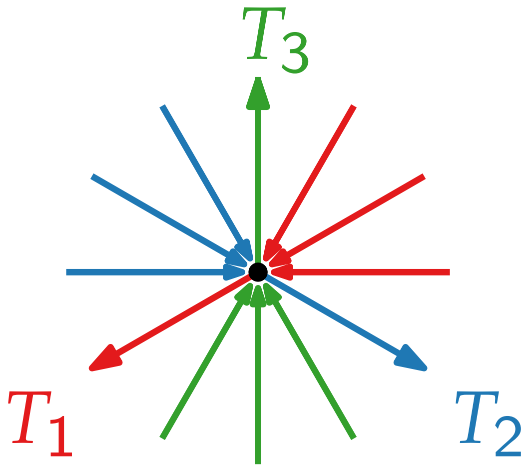
- $P_i(v)$ path from v to source of T_i
- $R_1(v)$, $R_2(v)$, $R_3(v)$ are sets of faces

Lemma.

- Paths $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at vertex v.
- For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

Proof ...





Schnyder drawing

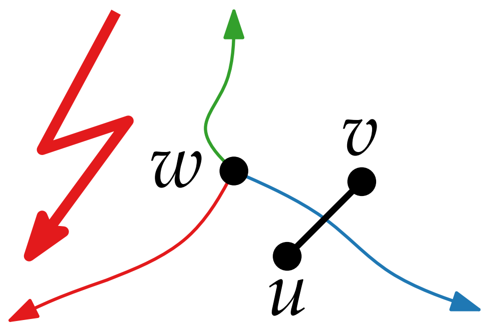
- Let barycentric coordinates of $v \in G \setminus \{a, b, c\}$ be (v_1, v_2, v_3), where $v_1 = |R_1(v)|/(2n-5)$, $v_2 = |R_2(v)|/(2n-5)$ and $v_3 = |R_3(v)|/(2n-5)$.

- Set
  - $A = (2n-5, 0)$
  - $B = (0, 2n-5)$
  - $C = (0, 0)$

Theorem.
The mapping

$$f \colon v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G, which thus gives a planar straight-line drawing of G in a $(2n-5) \times (2n-5)$ grid.



Proof.
- Condition 1: $v_1 + v_2 + v_3 = 1$
- Condition 2: For each edge $\{u, v\}$ and vertex $w \neq u, v$ at least one of three is true: $w_1 > u_1, v_1$, $w_2 > u_2, v_2$, $w_3 > u_3, v_3$.

Weak barycentric representation

Definition.
A **weak barycentric representation** of a graph $G = (V, E)$ is an *injective* map $v \in V \mapsto (v_1, v_2, v_3) \in \mathbb{R}^3$ with the following properties:

- $v_1 + v_2 + v_3 = 1$ for every $v \in V$
- for every $\{x, y\} \in E$ and every $z \in V \setminus \{x, y\}$ there is $k \in \{1, 2, 3\}$ with $(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$ and $(y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$.

i.e., either $y_k < z_k$ or $y_k = z_k$ and $y_{k+1} < z_{k+1}$

Weak barycentric representation

Definition.
A **weak barycentric representation** of a graph $G = (V, E)$ is an *injective* map $v \in V \mapsto (v_1, v_2, v_3) \in \mathbb{R}^3$ with the following properties:

- $v_1 + v_2 + v_3 = 1$ for every $v \in V$
- for every $\{x, y\} \in E$ and every $z \in V \setminus \{x, y\}$ there is $k \in \{1, 2, 3\}$ with $(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$ and $(y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$.

i.e., either $y_k < z_k$ or $y_k = z_k$ and $y_{k+1} < z_{k+1}$

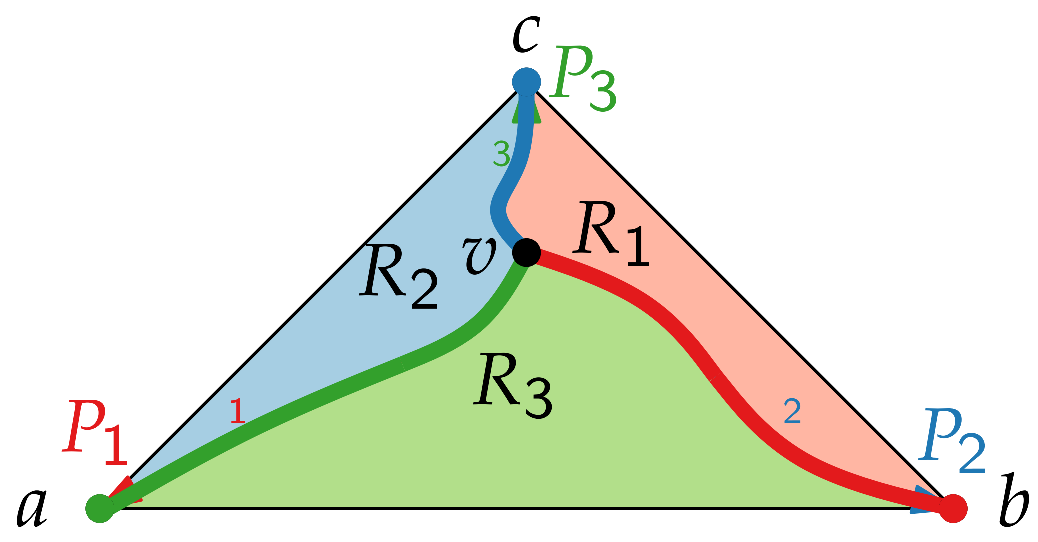
A weak barycentric representation still provides a planar drawing.

Proof is similar to before.. and thus an exercise.

New barycentric coordinates

- Set $v'_i = |V(R_i(v))| - |P_{i-1}(v)|$
- Additionally, for outer vertices set
  - $a'_1 = n - 2$
  - $a'_2 = 1$
  - $a'_3 = 0$

  and analogously for b' and c'



Lemma.
For inner vertices $u \neq v$ it holds that

$$u \in R_i(v) \Rightarrow (u'_i, u'_{i+1}) <_{\text{lex}} (v'_i, v'_{i+1})$$

Schnyder drawing

Theorem.
The mapping

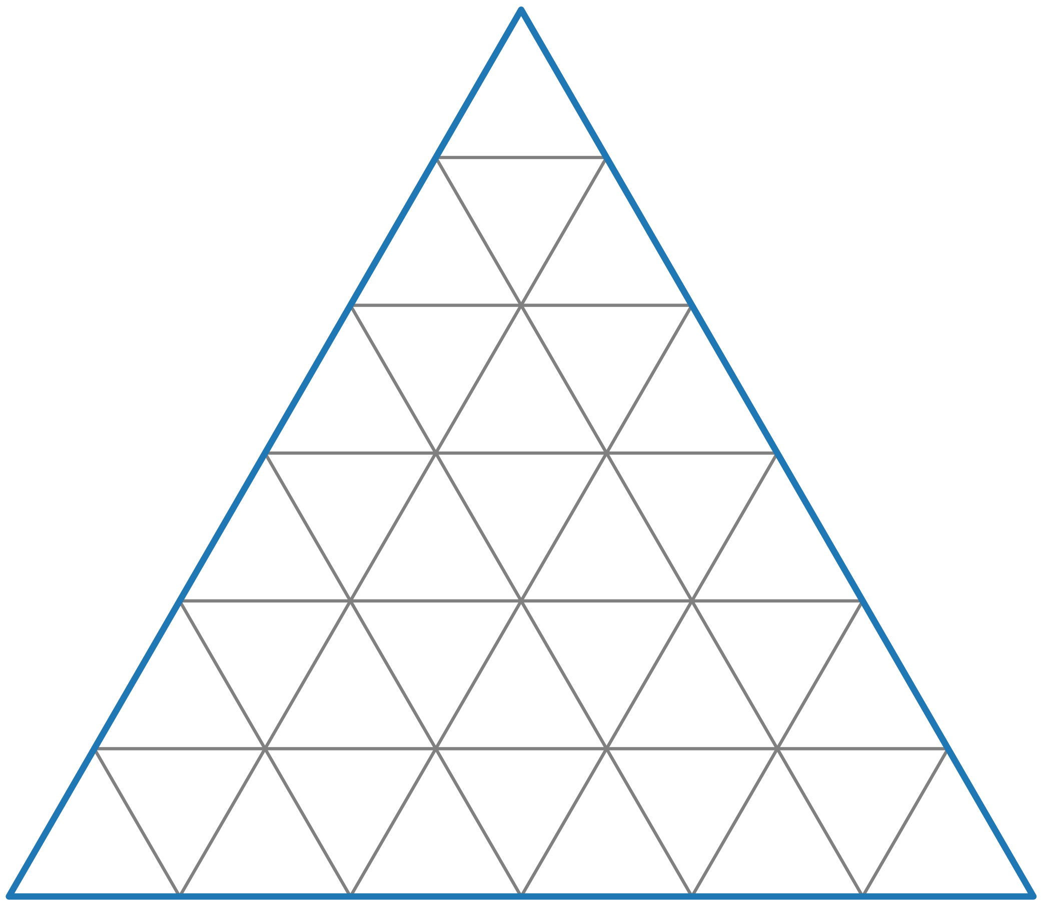
$$f : v \mapsto \frac{1}{n-1}(v'_1, v'_2, v'_3)$$

is a weak barycentric represenation of G.

Remarks.

- By setting $A = (n-1, 0)$, $B = (0, n-1)$, $C = (0, 0)$, one obtains a planar straight-line drawing of G on an $(n-2) \times (n-2)$ grid.
- To calculate all the coordinates, a constant number of tree traversals are enough – exercise.

Why do vertices land on a grid?



Literature

- [PGD Ch. 4.3] for detailed explanation of shift method
- [Sch90] Schnyder "Embedding planar graphs on the grid" 1990 – original paper on Schnyder realiser method