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Exact Algorithms

Sommer Term 2020

Lecture 12 Iterative Compression

Based on: [Parameterized Algorithms: §4, 4.1, 4.2]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

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Iterative Compression

Idea: Is it useful for an exact algorithm to start from a near-optimal solution?

Idea: Greedy algorithms do not solve NP-hard problems *exactly*, but might give us an incremental approach.

k-VERTEX COVER
Given: Graph $G = (V, E)$
Parameter: Integer k
Question: Does G have a vertex cover of size $\leq k$?

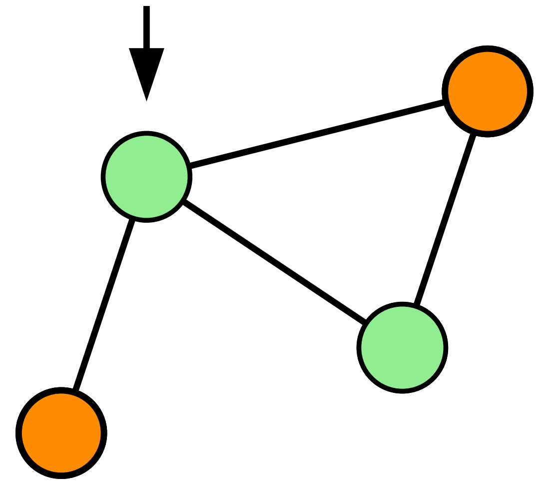
Vertex Cover

Vertex Cover with 2 vertices?

3 is easy...

But does 2 work?

Need to use other vertices!



Vertex Cover Compression

Given: Graph $G = (V, E)$, number k, vertex cover $C \subseteq V$ with $|C| = k+1$,

Find: Vertex cover $X \subseteq V$ with $|X| = k$, or answer: No.

Complexity of Vertex Cover Compression?

Not in P, otherwise Vertex Cover in P!

Vertex Cover by Compression

1 Start with a k-vertex subgraph.

In such a graph, k-vertex cover is trivial :-)

2 Add an unvisited vertex to our graph and our vertex cover.

This yields a $(k+1)$-vertex cover.

Compress to a k-vertex cover, or answer: NO.

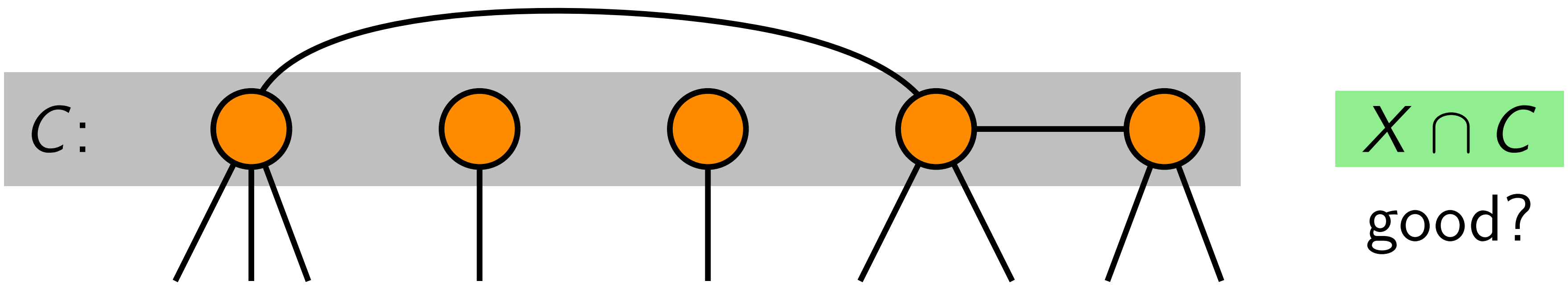
3 Profit

Runtime: $O(n)$ compression steps.

Vertex Cover Compression

Given: Vertex cover C, $|C| = k + 1$
Find: Vertex cover X, $|X| = k$

Strategy: Fix $X \cap C$, i.e., determine what to keep from C.
For each potential $X \cap C$, X is unique.



Note: $X \cap C$ is valid if

- $C \setminus X$ is independent
- $|(X \cap C) \cup N(C \setminus X)| \leq k$

Runtime: $< 2^k$ options for $X \cap C$; polytime for each one

Runtime for Vertex Cover: $O^*(2^k)$

Dominating Set

DOMINATING SET (COMPRESSION)

Given: Graph $G = (V, E)$, number k,
dominating set $D \subseteq V$, $|D| = k + 1$

Find: Dominating set $X' \subseteq V$, $|X'| = k$,
or answer: NO

Complexity of DOMINATING SET (COMPRESSION)?

Not FPT in k since DOMINATING SET is not FPT in k!

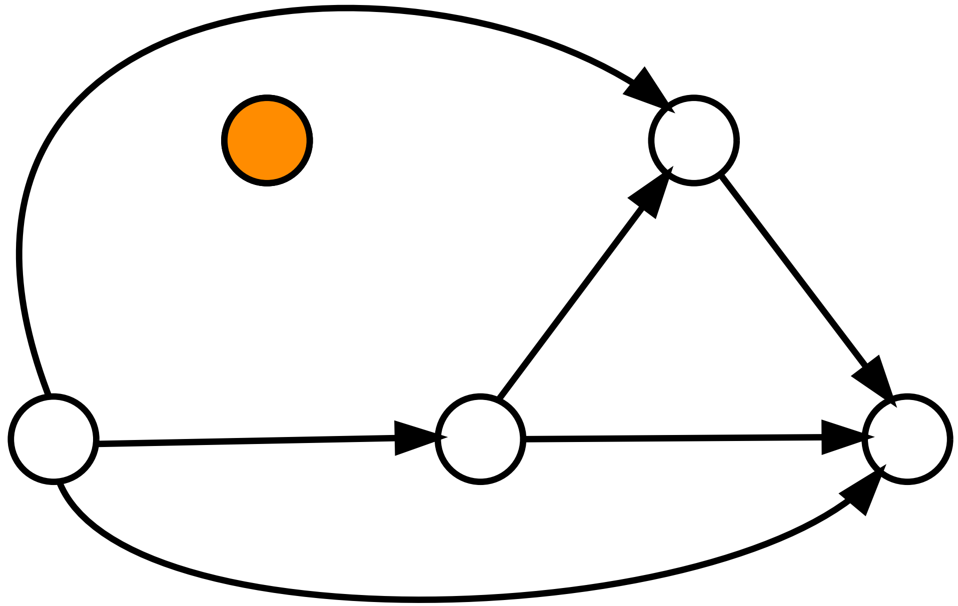
FVS in Tournaments

FEEDBACK VERTEX SET (TOURNAMENTS)

Given: Tournament $T = (V, E)$, number k

oriented clique

Question: $\exists\ X \subseteq V$ such that $|X| \leq k$ and $T[V \setminus X]$ is acyclic?

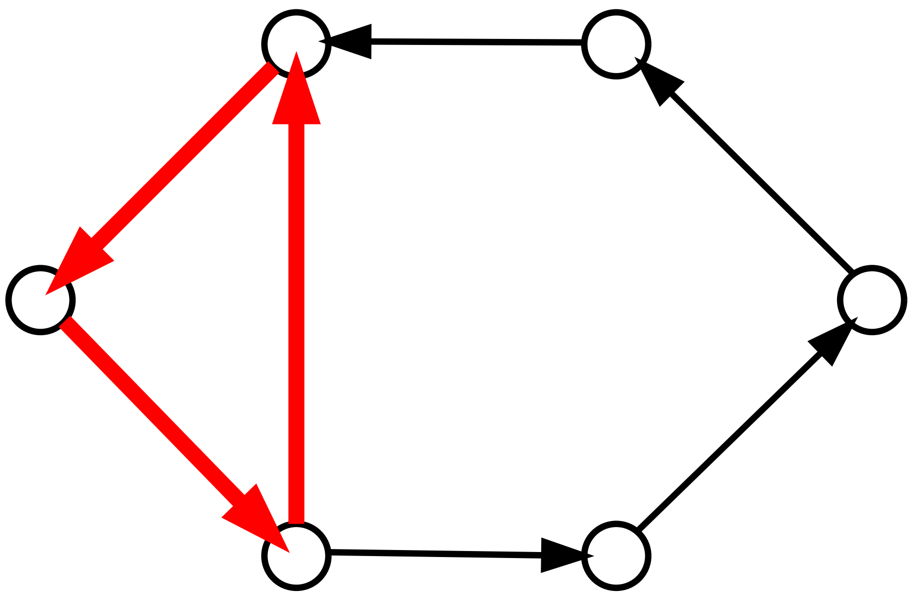


Cycles in Tournaments

Lemma. A tournament contains a cycle $\Leftrightarrow$ it contains a length-3 cycle.

Can we use this lemma algorithmically?
FVS must contain ≥ 1 vertex of each 3-cycle.

Algorithm: Branch $(1, 1, 1)$.



Theorem. FVS in Tournaments can be solved in $O^*(3^k)$ time.

FVS in Tournaments

FVS (Tournaments) Compression

Given: Tournament $T = (V, E)$, number k, feedback vertex set $S \subseteq V$ with $|S| \leq k + 1$

Question: $\exists\ X \subseteq V$ so that $|X| \leq k$ and $T \setminus X$ is acyclic?

Strategy: Fix $X \cap S$. Let $R = X \cap S$ and $F = S \setminus X$.
Then: delete R and *forbid* F.

Disjoint FVS (Tournaments) Compression

Given: Tournament $T' = (V', E')$, number k', feedback vertex set $S' \subseteq V$ with $|S'| \leq k' + 1$

Question: $\exists X \subseteq V' \setminus S'$ so that $|X| \leq k'$ and $T' - X$ is acyclic?

Reduction: FVS-T Comp. by 2^k calls to Disj.-FVS-T Comp.
For each call, set $T' := T - R$, $S' := F$, $k' := |F| - 1$.

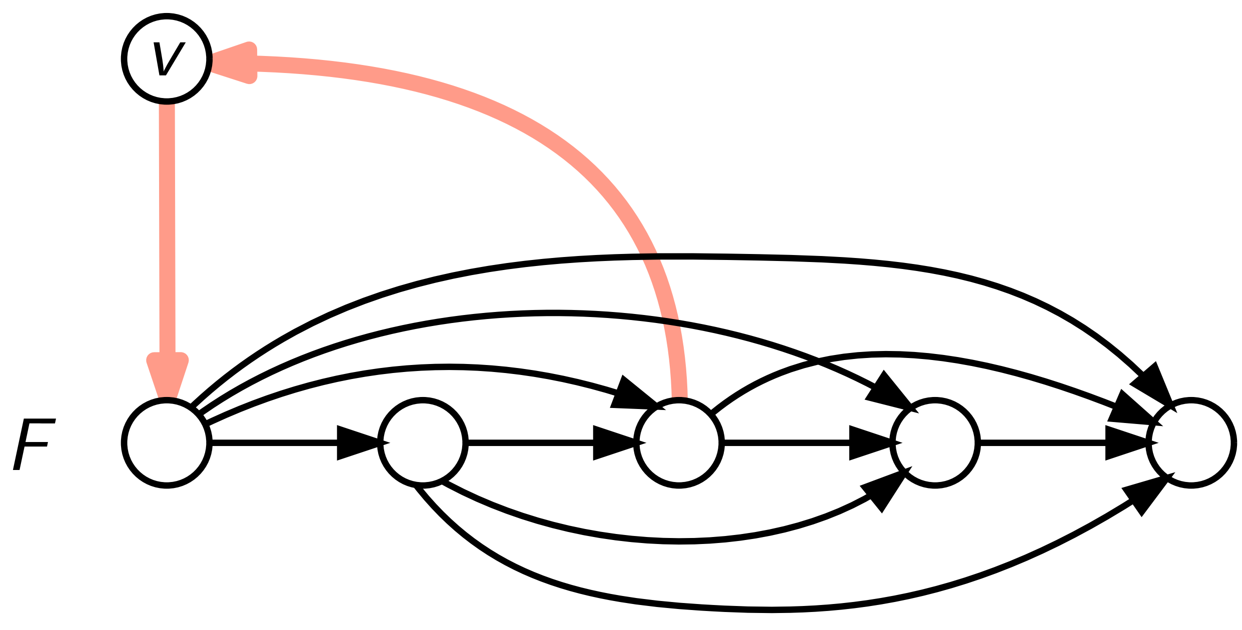
Disjoint FVS in Tournaments

Reduction Rule: If $T'[F]$ is not acyclic, answer: $\textsc{No}$.

Obs. Acyclic tournaments have a total order on vertices.

Let $A := V' \setminus F$ be the set of *chooseable* vertices.

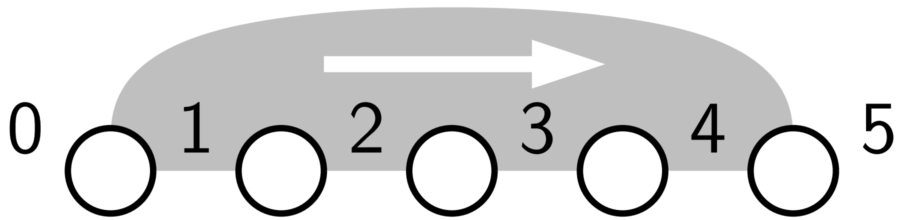
Reduction Rule: Let $v \in A$. If $T'[F \cup \{v\}]$ is not acyclic, then delete v and set $k' \leftarrow k' - 1$.



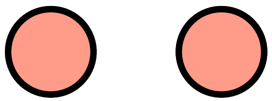
Disjoint FVS in Tournaments (cont'd)

$T = (V, E)$, $S = F \,\dot\cup\, R$ is FVS, $A = V \setminus S$

F:



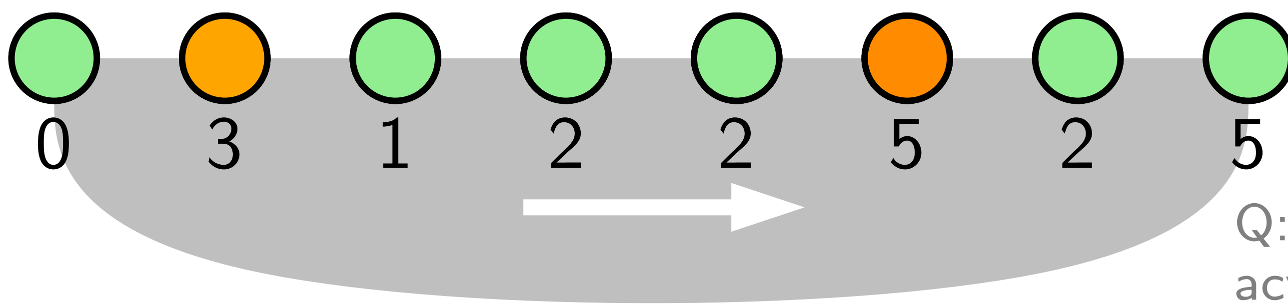
R:



Find: longest monotonically increasing subsequence

→ easy DP exercise for polytime

A: X:



Q: Why is A acyclic, too??

Find: minimum FVS $X \subseteq A$ of $T[F \cup A]$

$\Rightarrow X \cup R$ is a FVS of T!

FVS in Tournaments by Compression

Start with any k-vertex subgraph G_k of G, and set $S = V(G_k)$.

Iteratively add vertices: $n - k$

Partial graph gains a vertex, and so does S; now $|S| \leq k + 1$.

// Compress using Disj. FVS-T Compression: $O(2^k)$

For each $F \subseteq S$:

Remove $R_F = S \setminus F$. poly(n)

Apply reduction rules.

Compute longest increasing subsequence via labels defined before.

$\rightsquigarrow$ min. FVS X_F of $T[F \cup A]$

If some set X_F fulfills $|X_F| + |R_F| \leq k$, then $S \leftarrow X_F \cup R_F$.

Otherwise, answer No.

Theorem. FVS in Tournaments can be solved in $O^*(2^k)$ time.

FVS in general graphs

Theorem. FVS has a kernel with $O(k^2)$ vertices and edges.

see Parameterized Algorithms §9.1

Kernel brute-force $\rightsquigarrow$ runtime … $2^{O(k^2)}\text{poly}(n)$

Iterative Compression $\rightsquigarrow$ $O^*(5^k)$ time

see Parameterized Algorithms §4.3

Current “best” algorithm: $O^*(3^k)$ randomised (Monte Carlo)

see Parameterized Algorithms §11.2.1

Lemma. If FVS $\leq k$, then treewidth $\leq k+1$.

Summary: Iterative Compression

- Show: (Problem) solvable
  if (Problem)-Compression solvable
- Show: (Problem)-Compression solvable
  if Disjoint-(Problem) solvable
- Solve: Disjoint-(Problem) (This is the hard part!)

What makes the Disjoint-(Problem) easier?

- Size-$(k+1)$ solution Y is given.
- Complement of Y has special properties (here: acyclic).
- Splitting $Y = F \uplus R$ implies special properties of F (here: acyclic).