Julius-Maximilians-
UNIVERSITÄT
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Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 11 Tree Decomposition

Based on: [Parameterized Algorithms: §7.2, 7.3.1]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

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Lehrstuhl für Informatik I

(Weighted) Independent Set

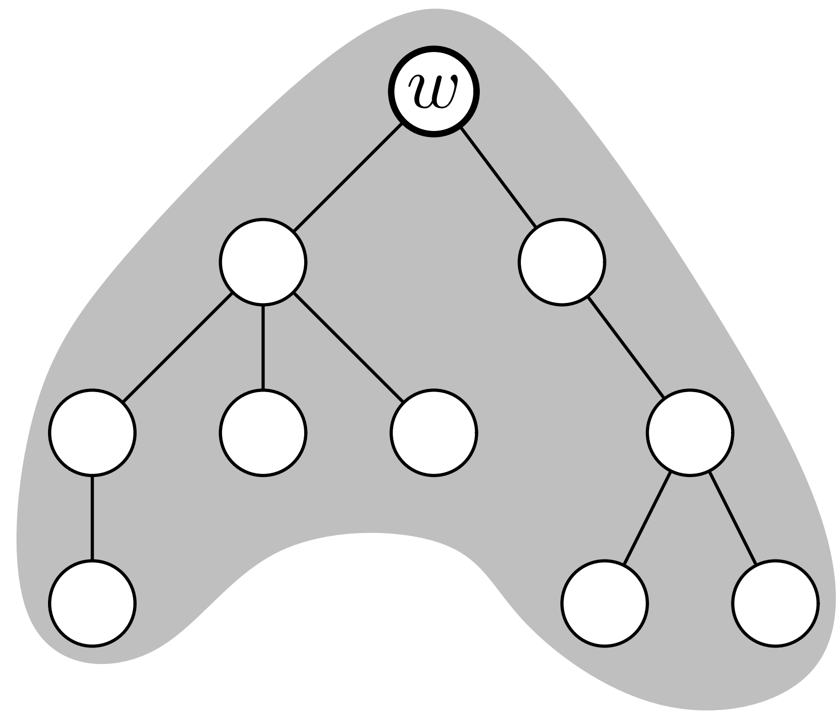
Given: graph G, weight function $\omega\colon V \to \mathbb{N}$
Question: What is the maximum weight of a set $S \subseteq V$ where no pair in S is adjacent in G?

Thm. Independent Set is NP-complete.

Thm. On trees, Independent Set can be solved in linear time.

Independent Sets in Trees

$A(w)$ = solution



Choose an arbitrary root w.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set S in $T(v)$

Let $B(v)$:= maximum weight of an independent set S in $T(v)$ *where* $v \notin S$

– If v is a leaf: $B(v) = 0$ and $A(v) = \omega(v)$

– If v has children $x_1, \ldots, x_r$:

$$B(v) = \sum_{i=1}^{r} A(x_i);\ A(v) = \max\{B(v), \omega(v) + \sum_{i=1}^{r} B(x_i)\}$$

Algorithm: Compute $A(\cdot)$ and $B(\cdot)$ bottom-up!

(s, t)-Series-Parallel Graphs

Def. A graph $G = (V, E)$ is *2-terminal* if it contains two special vertices s and t.

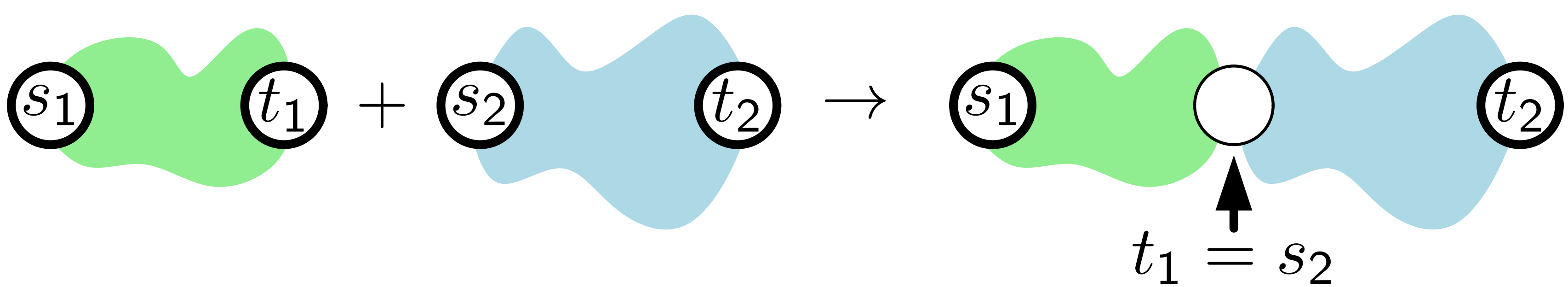
recursive definition

Series-parallel graphs have a natural tree structure!

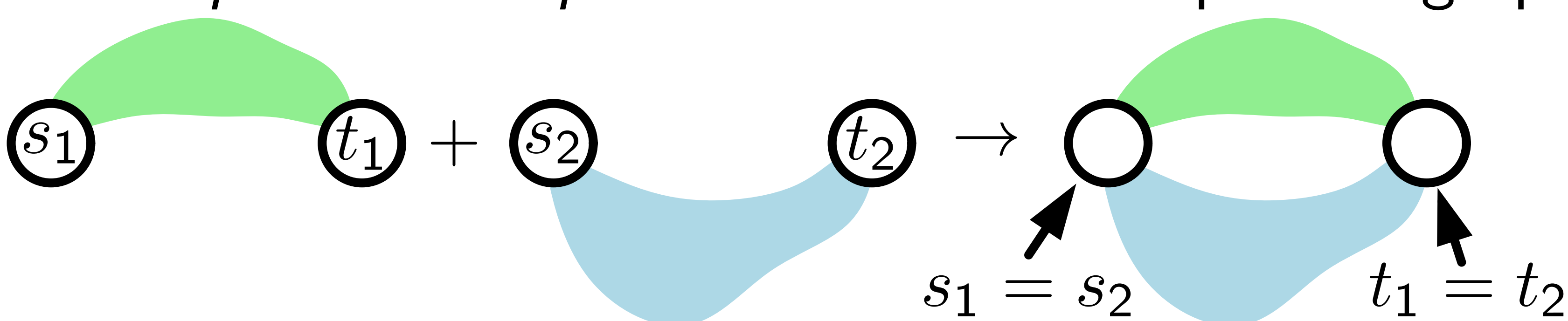
Def. A 2-terminal graph G is *series-parallel* if:

- G is a single edge (s, t)

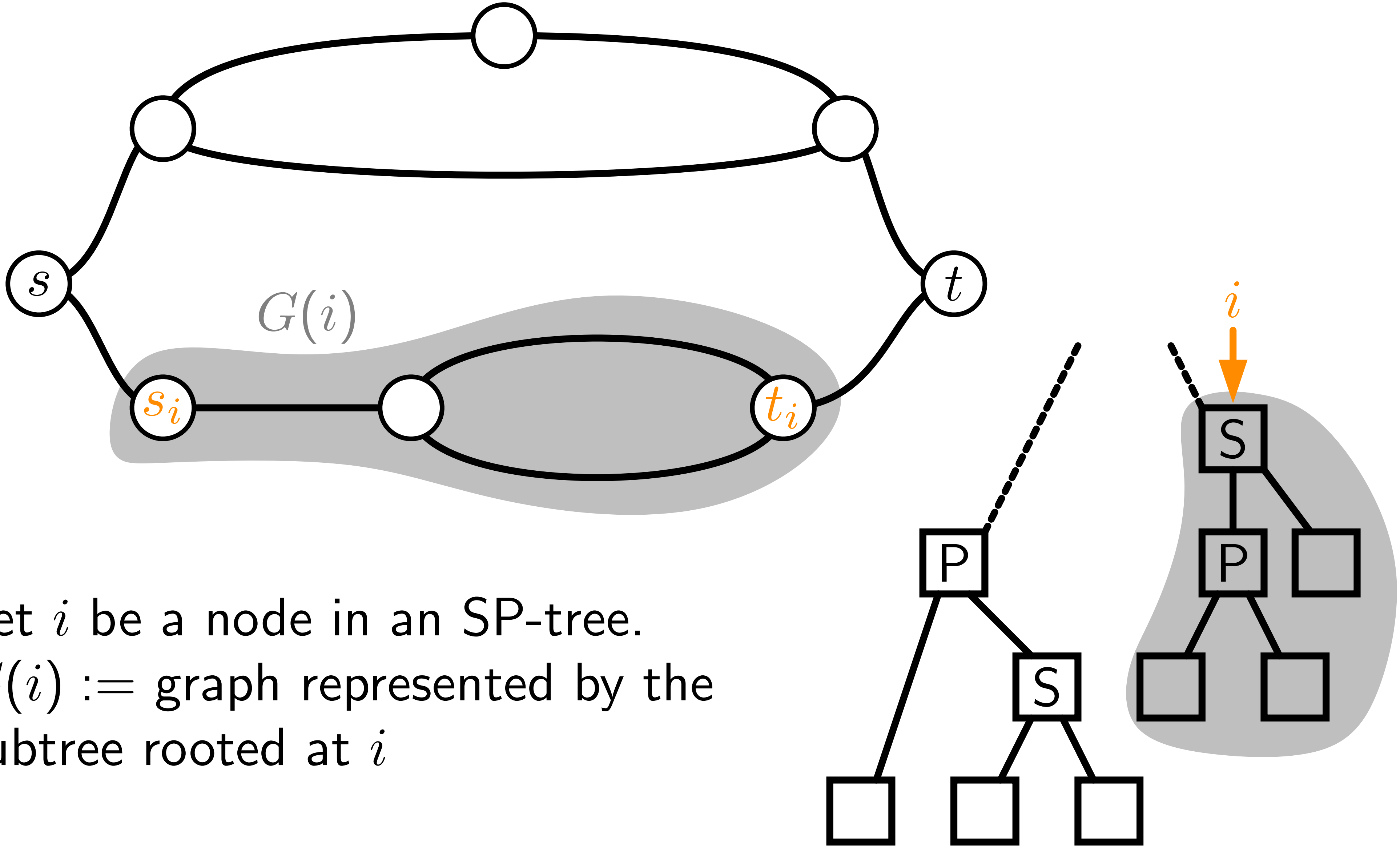
- G is a *series composition* of two series-parallel graphs



- G is a *parallel composition* of two series-parallel graphs



SP-Tree



Let i be a node in an SP-tree.
$G(i) :=$ graph represented by the subtree rooted at i

Independent Set on SP-Trees

Dynamic program on SP-tree indexed by $G(i)$

$AA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \in S$ and $t_i \in S$

$BA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \notin S$ and $t_i \in S$

$AB(i)$ and $BB(i)$ are defined similarly.

– If i is a leaf...

$$AA(i) = -\infty$$
$$AB(i) = \omega(s_i)$$
$$BA(i) = \omega(t_i)$$
$$BB(i) = 0$$

Independent Set on SP-Trees

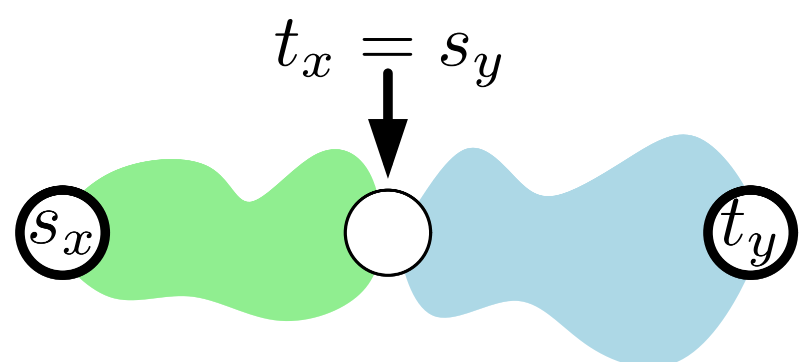
Dynamic program on SP-tree indexed by $G(i)$

$AA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \in S$ and $t_i \in S$

$BA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \notin S$ and $t_i \in S$

$AB(i)$ and $BB(i)$ are defined similarly.

– If i is a series composition with children x and y, ...



$$AA(i) = \max\{\ AB(x) + BA(y),\ AA(x) + AA(y) - \omega(t_x)\ \}$$

Independent Set on SP-Trees

Dynamic program on SP-tree indexed by $G(i)$

$AA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \in S$ and $t_i \in S$

$BA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \notin S$ and $t_i \in S$

$AB(i)$ and $BB(i)$ are defined similarly.

– Other cases omitted... (easy exercise).

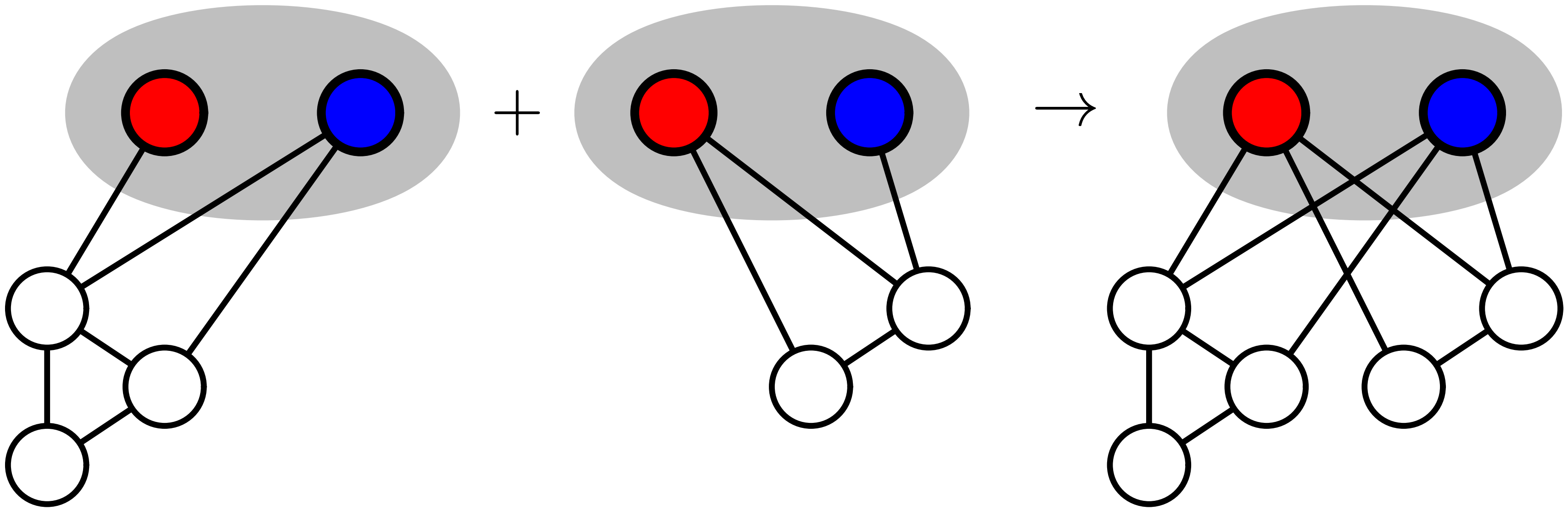
– $O(1)$ time per SP-node.

Thm. Given an n-vertex series-parallel graph G with its SP-tree, Independent Set on G can be solved in $O(n)$ time.

Generalization?

Many ways to generalize the concept of having a "tree structure"

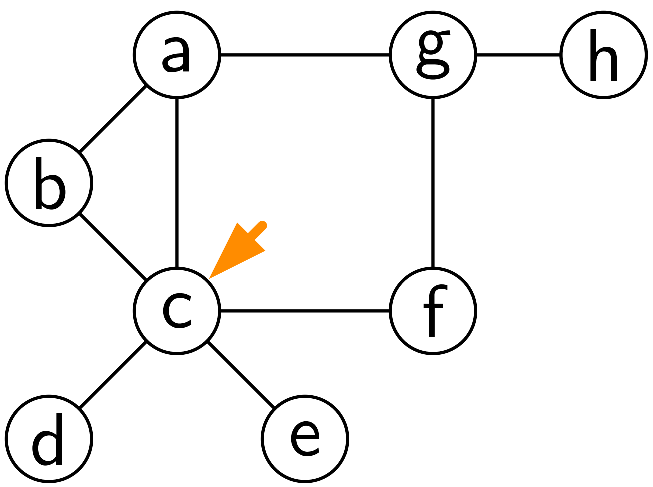
Example: k-terminal graph $G = (V, E, T)$, $|T| = k$
Operation: "gluing"



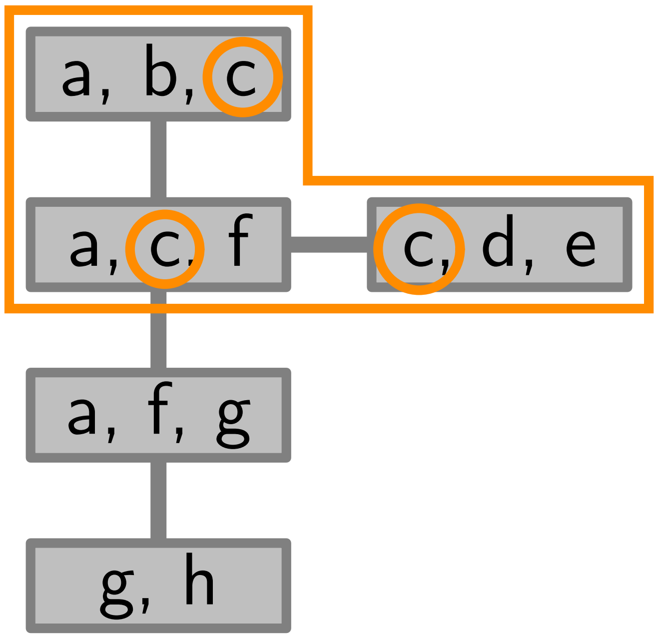
Example: Tree Decomposition

1 each vertex belongs to at least one bag
these bags are connected

Graph $G = (V, E)$:



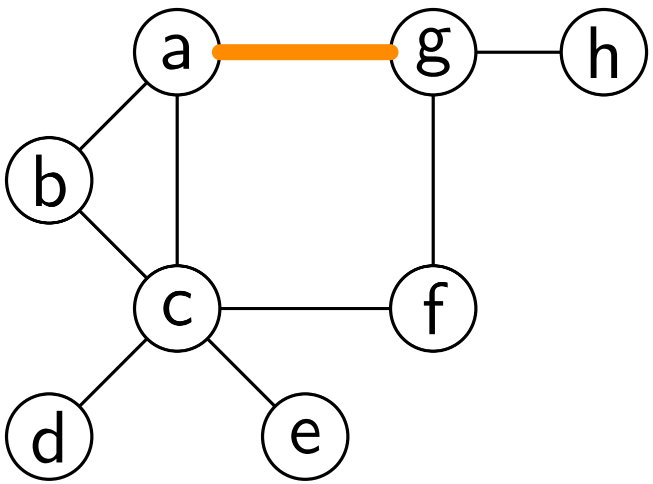
Tree Decomposition:



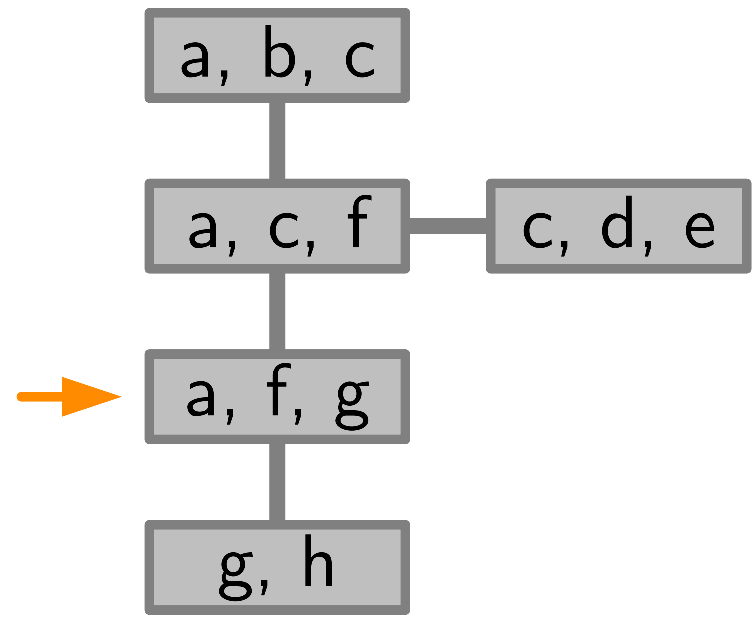
Example: Tree Decomposition

2 each edge is contained in at least one bag

Graph $G = (V, E)$:

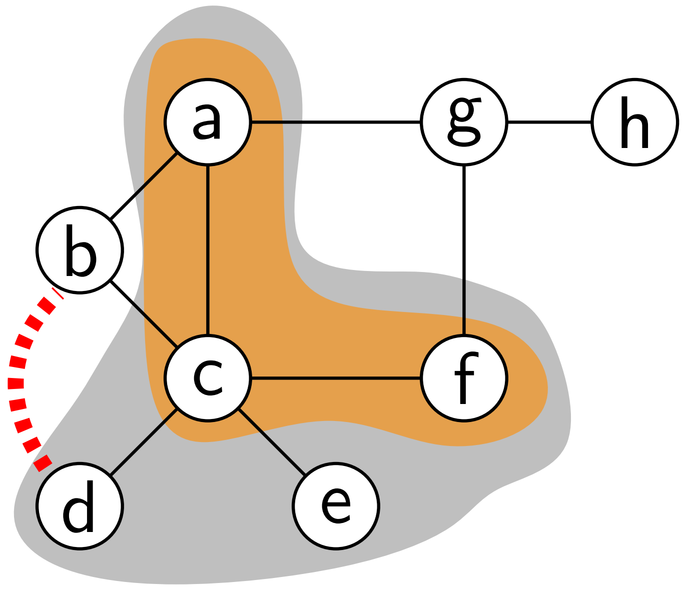


Tree Decomposition:



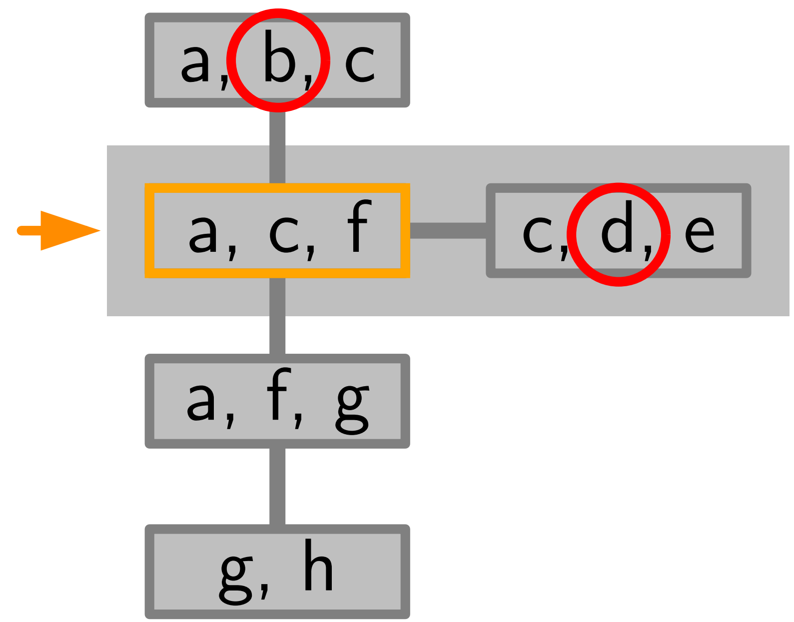
Example: Tree Decomposition

Graph $G = (V, E)$:



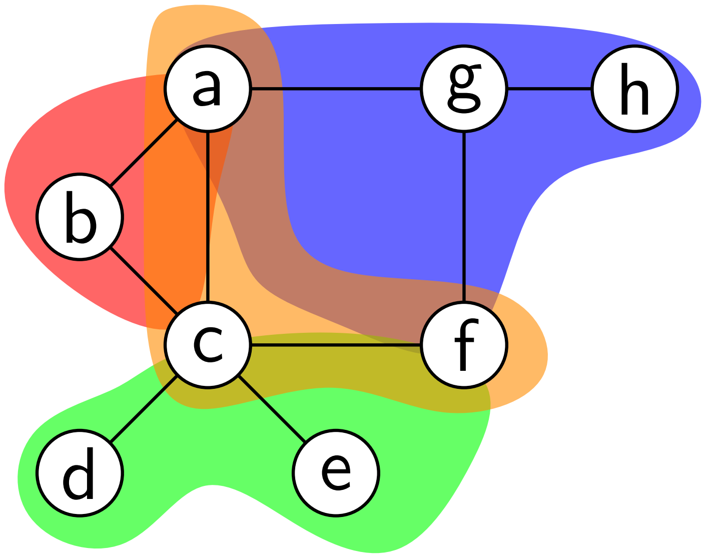
$\{b, d\} \notin E$

Tree Decomposition:

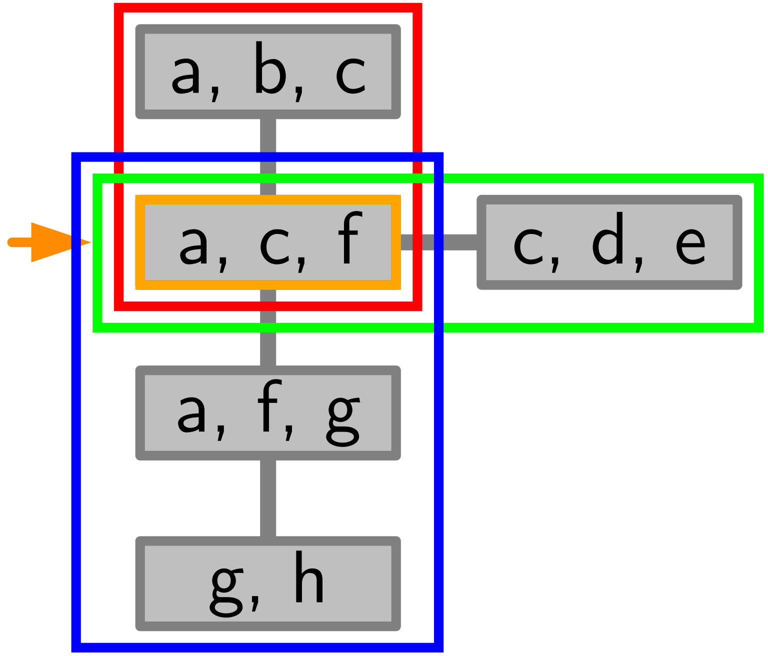


Example: Tree Decomposition

Graph $G = (V, E)$:



Tree Decomposition:



Tree Decompostion (formal)

Def. A *tree decomposition* of a graph $G = (V, E)$ is:

- a tuple $D = (X, T)$
- $T = (P, F)$ is a tree
- $X = \{X_p \mid p \in P\}$ is a set family of subsets of V ("bags"; one for each node in P)
- $\bigcup_{p \in P} X_p = V$
- $\forall \{u, v\} \in E$ there is a $p \in P$ such that $u, v \in X_p$.
- $\forall v \in V$ the set $\{p \in P \mid v \in X_p\}$ is connected in T.

Treewidth (formal)

- a tuple $D = (X, T)$
- $T = (P, F)$ is a tree

Def. Width (tree decomposition): $\max_{p \in P} |X_p| - 1$, i.e., cardinality of the largest bag -1

Def. Treewidth $\text{tw}(G)$ is the minimum width of a tree decomposition of G

Obs. $\text{tw}(G) < n$

Question: Which graphs have treewidth 0?

Exercise: Trees have treewidth 1.

Exercise: Series-parallel graphs have treewidth 2

Thm. There is a tree decomposition of width $\text{tw}(G)$ where the tree size $|P|$ is polynomial in n.

Parameterized Problems

Given: Instance of size n and parameter k

Def. Problem is FPT when solvable in $O(f(k) \cdot poly(n))$ time.

$O(f(\text{tw}(G)) \cdot poly(n))$ time.

Ex.:	k-Vertex Cover	**FPT**
	k-Independent Set	$W[1]$-**comp.**
	k-Dominating Set	$W[2]$-**comp.**
	k-Coloring	**NP-comp.** $k \geq 3$
	Independent Set (treewidth)	**FPT**
	List Coloring (treewidth)	$W[1]$-**comp.**
	Channel Assignment (treewidth)	**NP-comp.** $k \geq 3$

Computing Treewidth

TREEWIDTH
Given: Graph $G = (V, E)$, number k
Question: $\text{tw}(G) \leq k$?

Thm. TREEWIDTH is NP-complete.

k-TREEWIDTH
Given: graph $G = (V, E)$
Parameter: number k
Question: $\text{tw}(G) \leq k$?

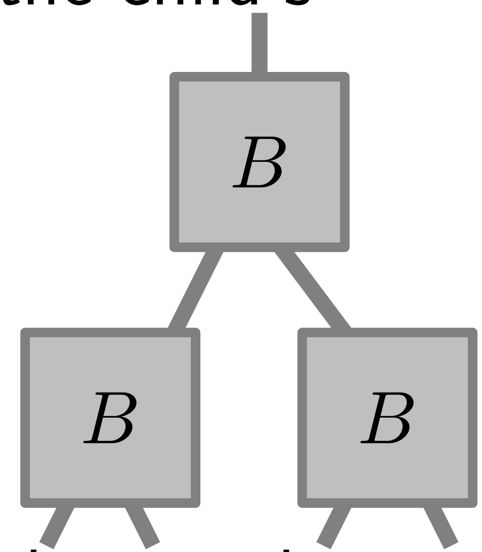
Thm. k-TREEWIDTH is FPT.

– Actually *fixed-parameter linear*: runtime $O(f(k) \cdot n)$
– Algorithm is constructive (provides an optimal tree decomp.)
– How can we make "fixed-treewidth-tractable" algorithms?

Tool #1: Nice Tree Decompositoins

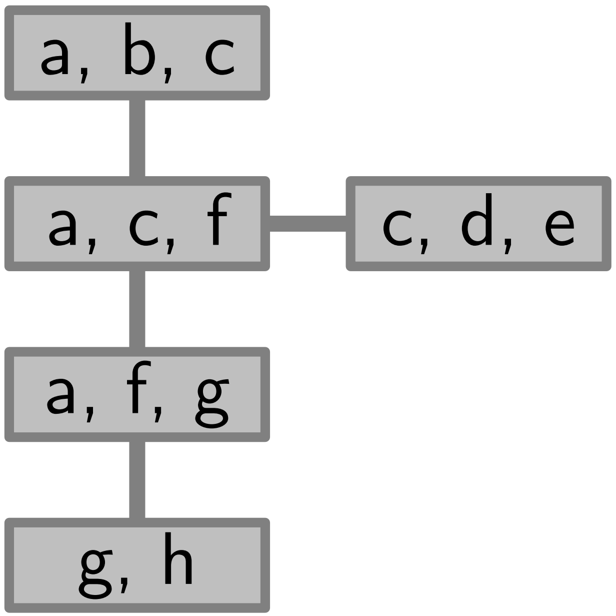
In a *nice* tree decomp., one bag is marked as the root and there are only 4 types of bags:

- **Leaf:** the bag is a leaf and contains only one vertex
- **Introduce:**
  The bag has exactly one child and contains the child's vertices and exactly one new vertex.
- **Forget:**
  The bag has exactly one child and contains one vertex fewer than the child.
- **Join:**
  The bag has exactly two children and these three nodes have exactly the same vertices

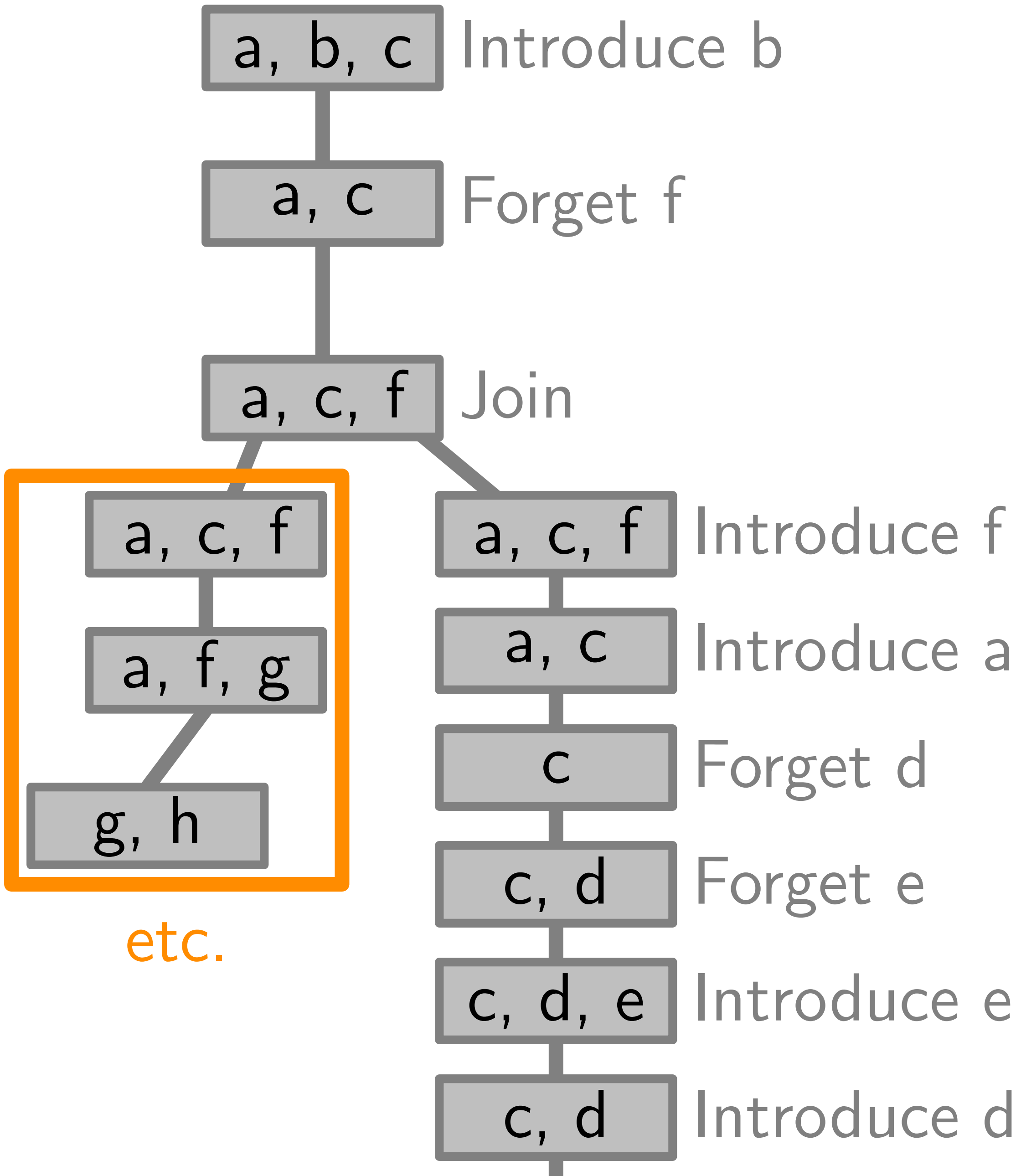


Tool #1: Nice Tree Decompositoins

Thm. For each tree decomposition, there is a nice tree decomposition of the same width and polynomially many more bags. The nice decomposition can be constructed in polynomial time.



Tool #1: Nice Tree Decompositoins



Tool #2: DP (on Nice Tree Decompositions)

Thm. k-Treewidth is FPT

Thm. Tree decompositions can be made *nice* in poly-time.

Corollary. For FPT-Algorithms it suffices to consider nice tree decompositions.

Strategy: Build a recurrence for each type of bag, and use dynamic programming.

Independent Set Using Nice Tree Decomp.

Let $G(i)$:= graph induced by the vertices in the subtree at i.

For bag i and $S \subseteq X_i$, let:

$R(i, S) :=$ maximum weight of an independent set I in $G(i)$ with $I \cap X_i = S$.

If i is a **Leaf** ...

Sei $X_i = \{v\}$.

$R(i, \{v\}) = \omega(v)$

$R(i, \emptyset) = 0$

Independent Set Using Nice Tree Decomp.

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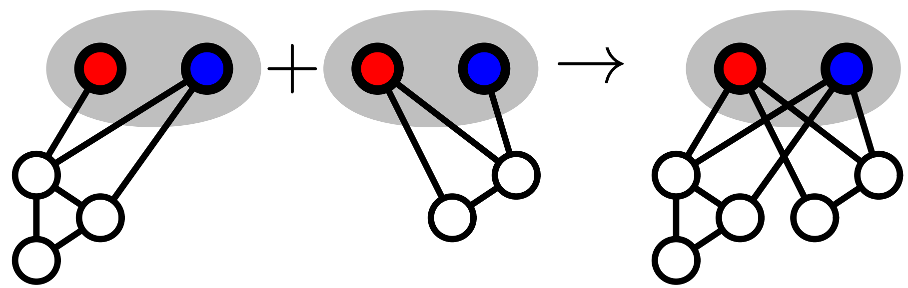
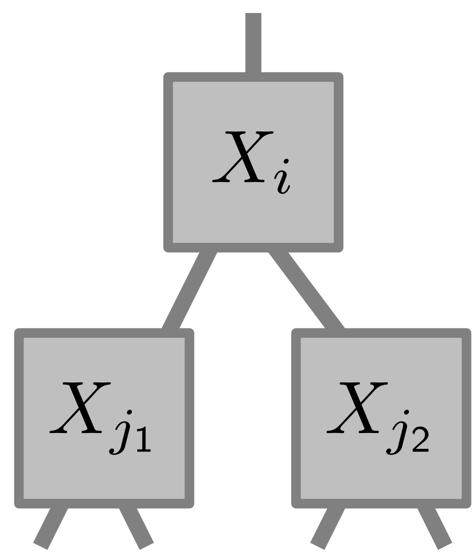


If i is a **Join** ...

with children j_1 and j_2

$$R(i, S) = R(j_1, S) + R(j_2, S) - \sum_{v \in S} \omega(v)$$



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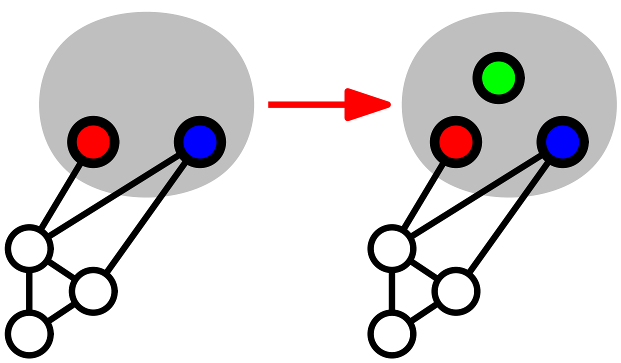

If i is an **Introduce** ...

with child j and $X_i = X_j \cup \{v\}$

For each $S \subseteq X_j$: $R(i, S) = R(j, S)$.

Otherwise if v has neighbors in S, $R(i, S) = -\infty$

if v has no neighbors in S,

$$R(i, S) = R(j, S \setminus \{v\}) + \omega(v)$$

Independent Set Using Nice Tree Decomp.

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For bag i and $S \subseteq X_i$, let:

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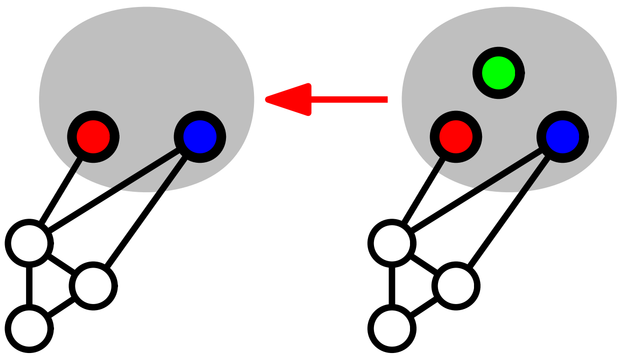

If i is a **Forget** ...

with child j and $X_i = X_j \setminus \{v\}$

$R(i,S) =$ max$\{ R(j,S), R(j, S \cup \{v\}) \}$

Independent Set Using Nice Tree Decomp.

Let $G(i) :=$ graph induced by the vertices in the subtree at i.

For bag i and $S \subseteq X_i$, let:

$R(i, S) :=$ maximum weight of an independent set I in $G(i)$ with $I \cap X_i = S$.

Algorithm: Compute $R(i, S)$ for all i and corresponding S.

Runtime: **?**