Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 11 Tree Decomposition

Based on: [Parameterized Algorithms: §7.2, 7.3.1]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

Lehrstuhl für Informatik I

(Weighted) Independent Set

Given: graph G, weight function $\omega\colon V \to \mathbb{N}$
Question: What is the maximum weight of a set $S \subseteq V$ where no pair in S is adjacent in G?

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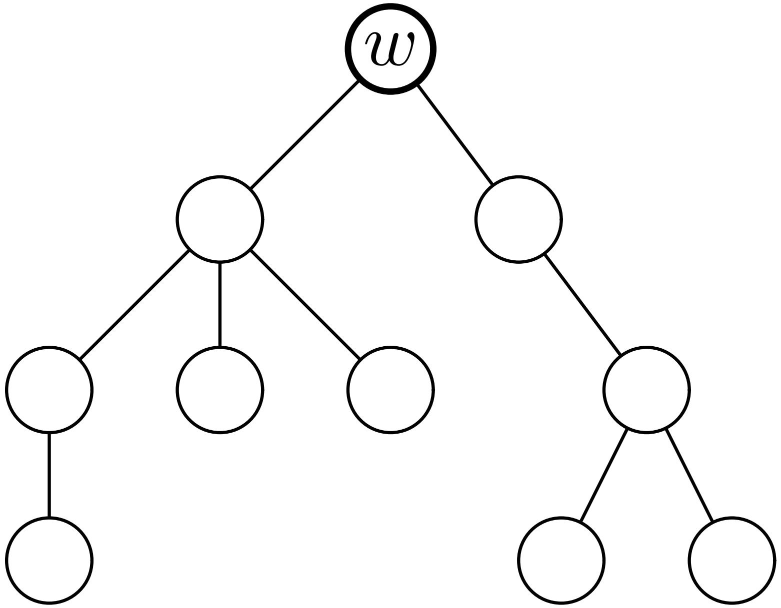
Thm. Independent Set is NP-complete.

Thm. On trees, Independent Set can be solved in linear time.

Independent Sets in Trees

Choose an arbitrary root w.



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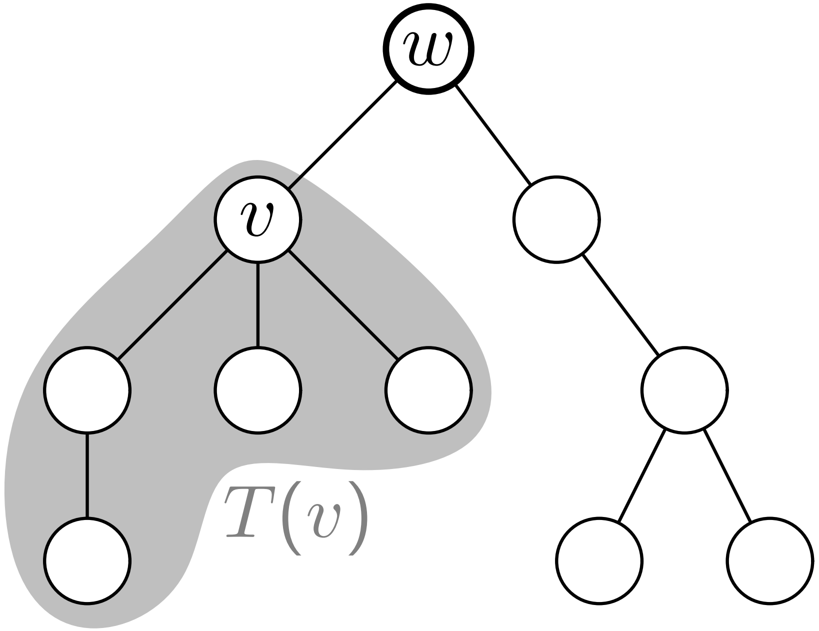

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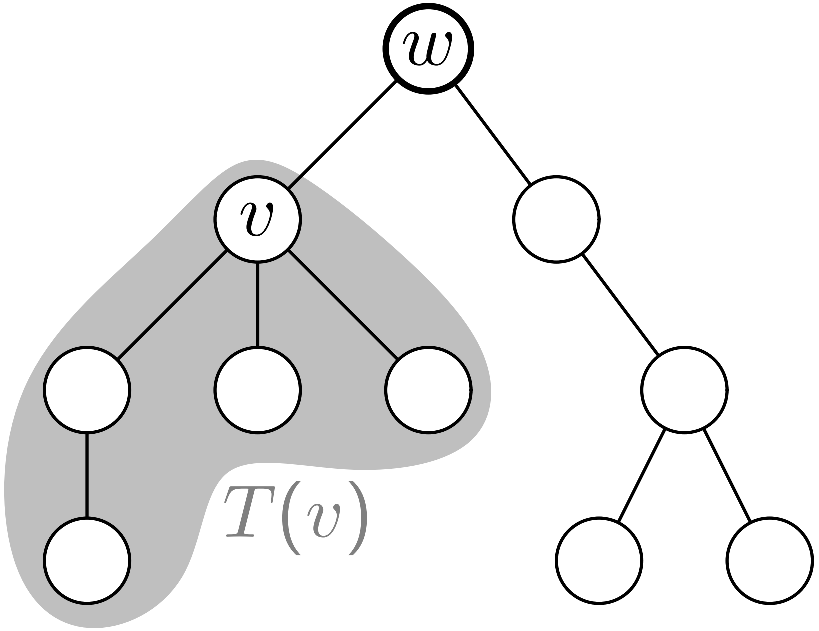

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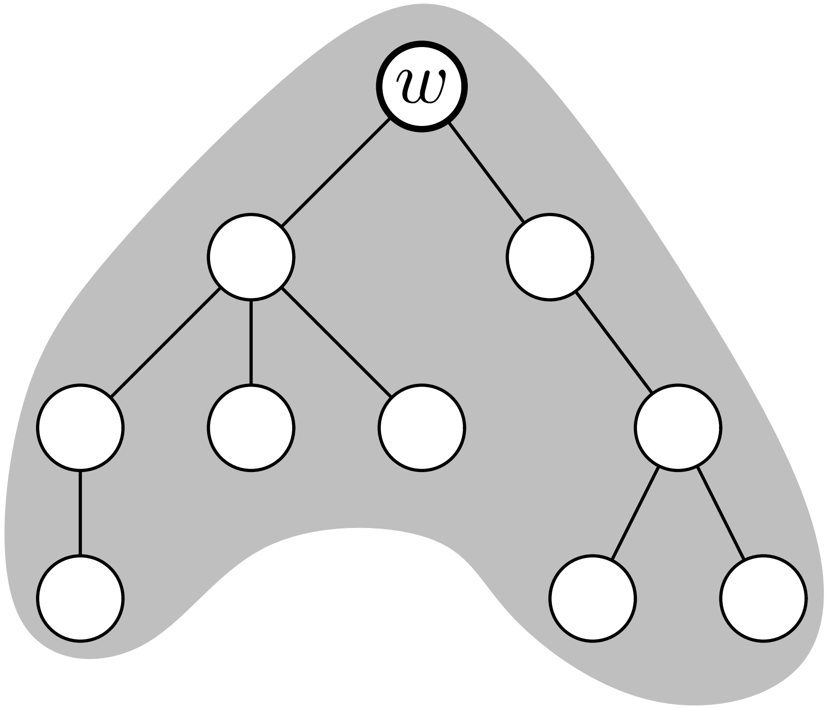
$A(w) =$ solution



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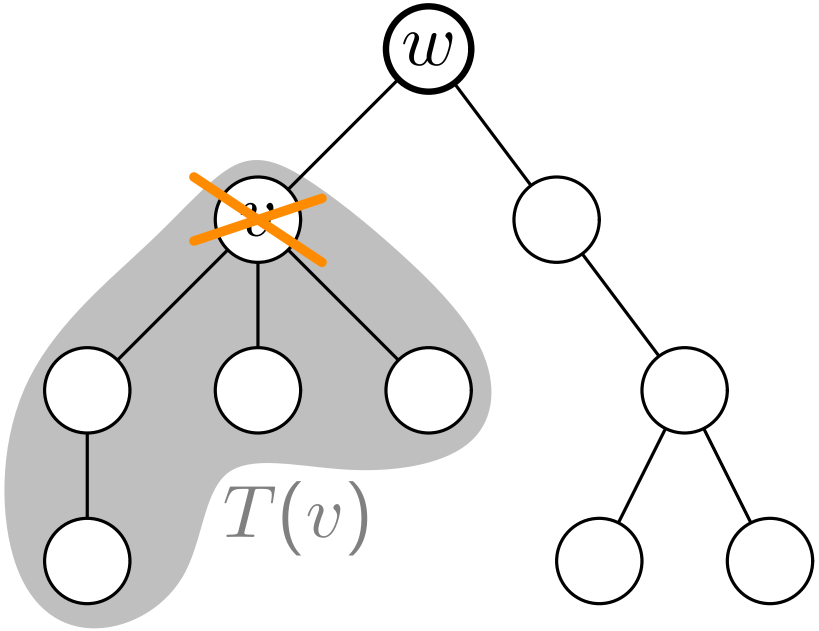

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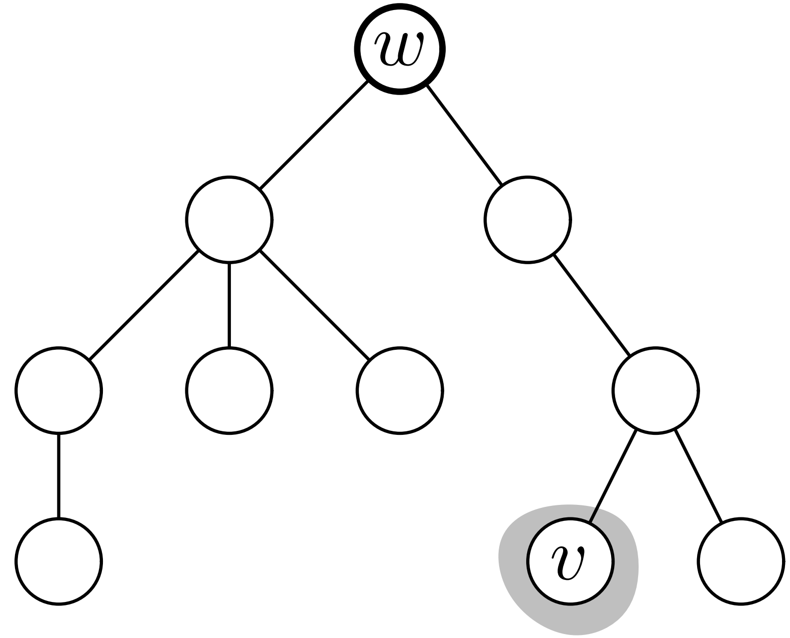

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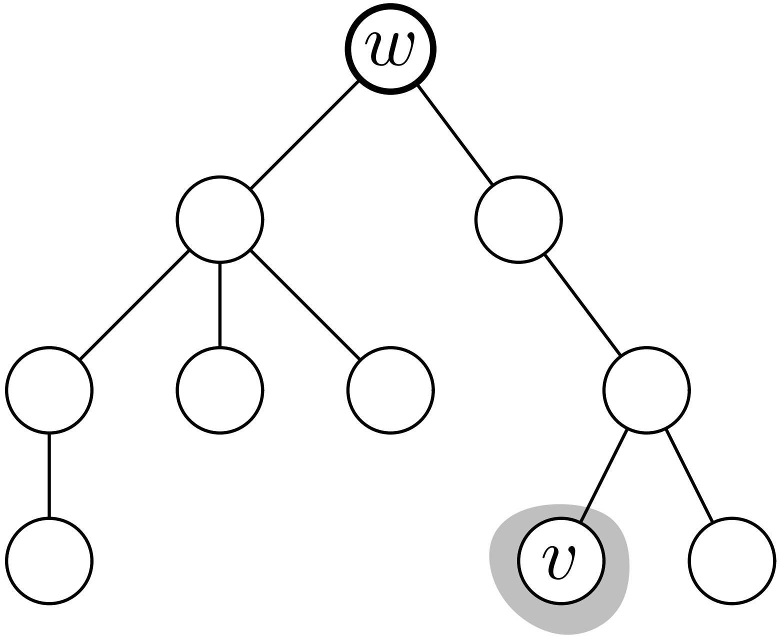
– If v is a leaf: $B(v) = 0$ and $A(v) =$



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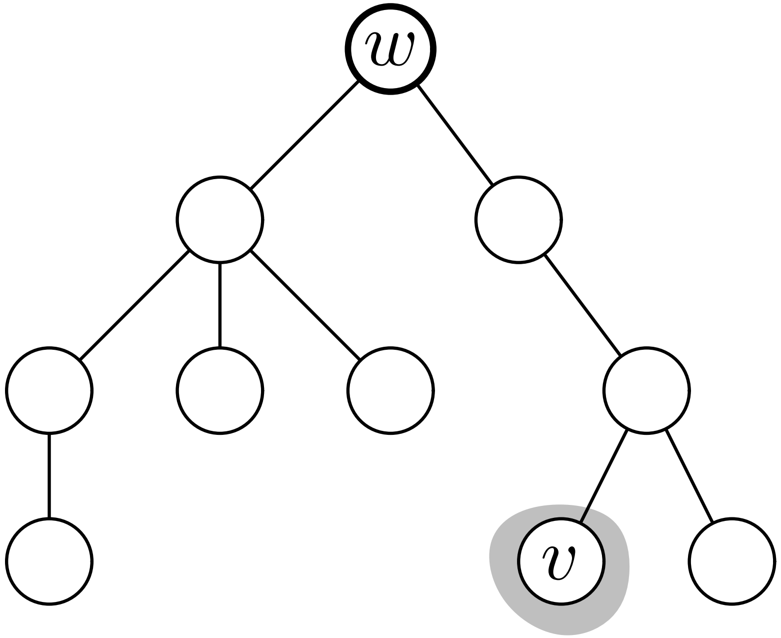

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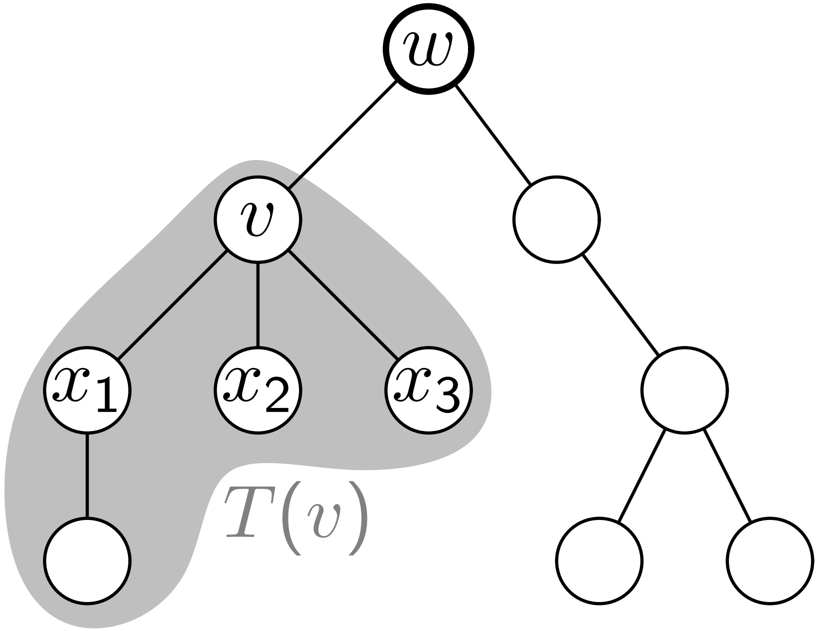

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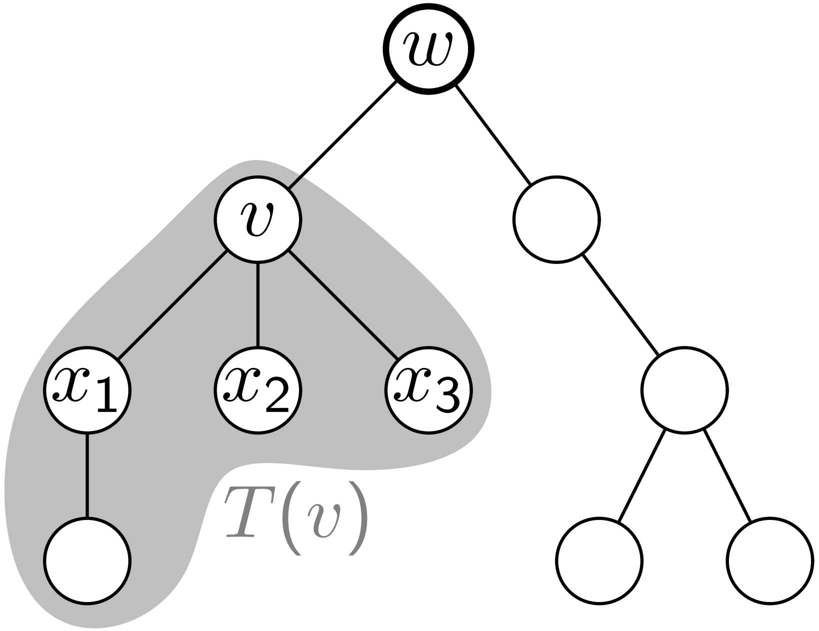

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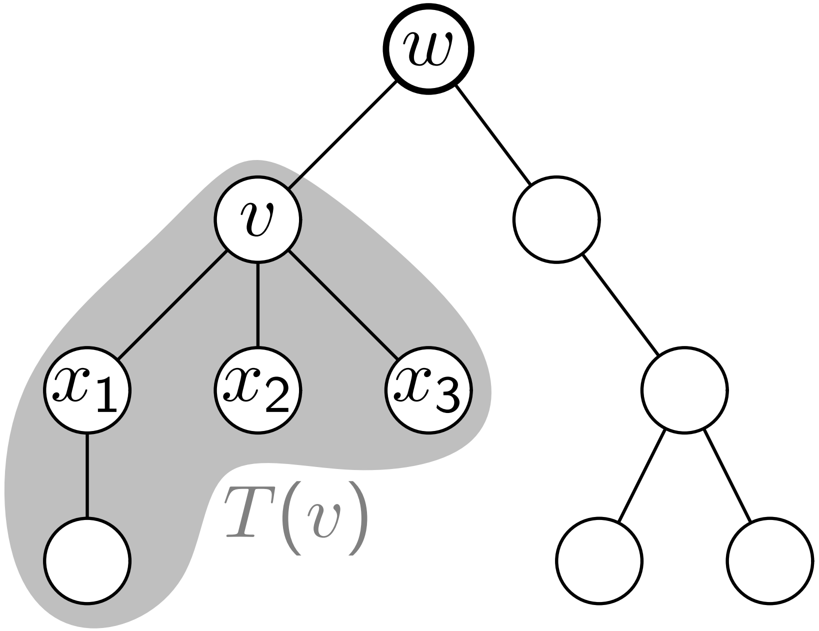

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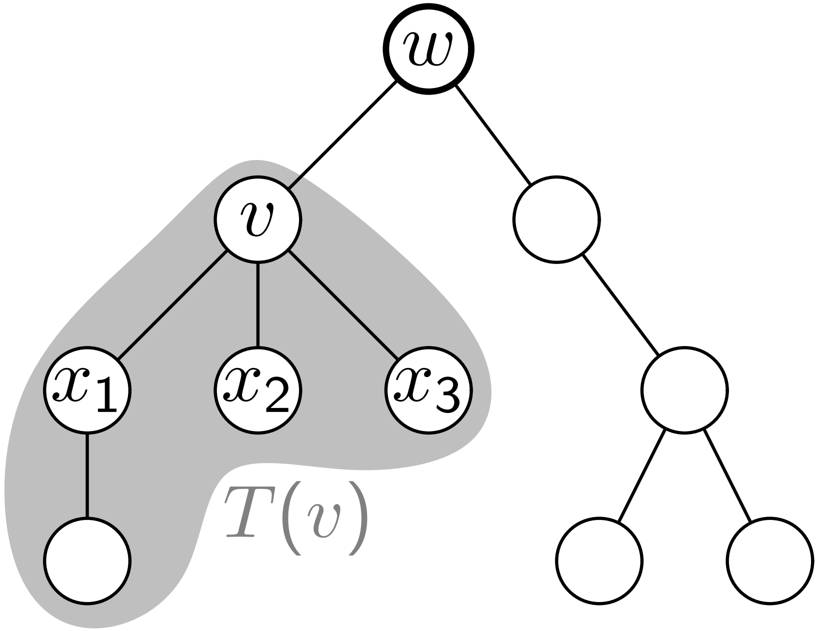

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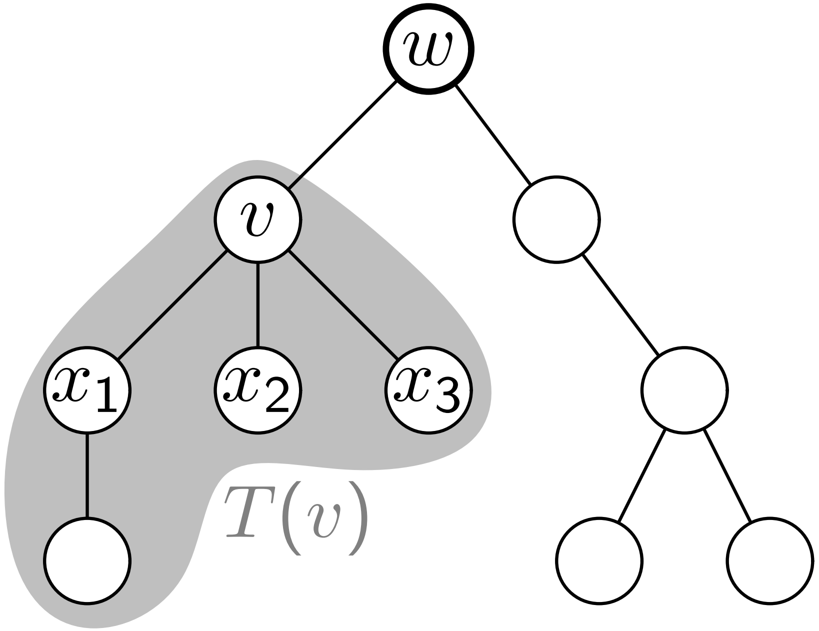

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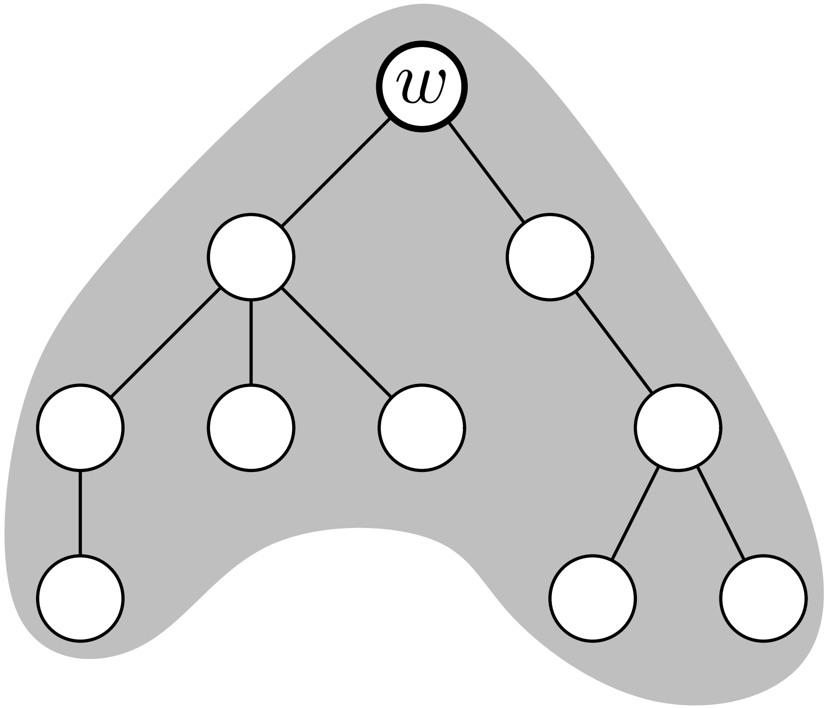
Independent Sets in Trees

$A(w) =$ solution



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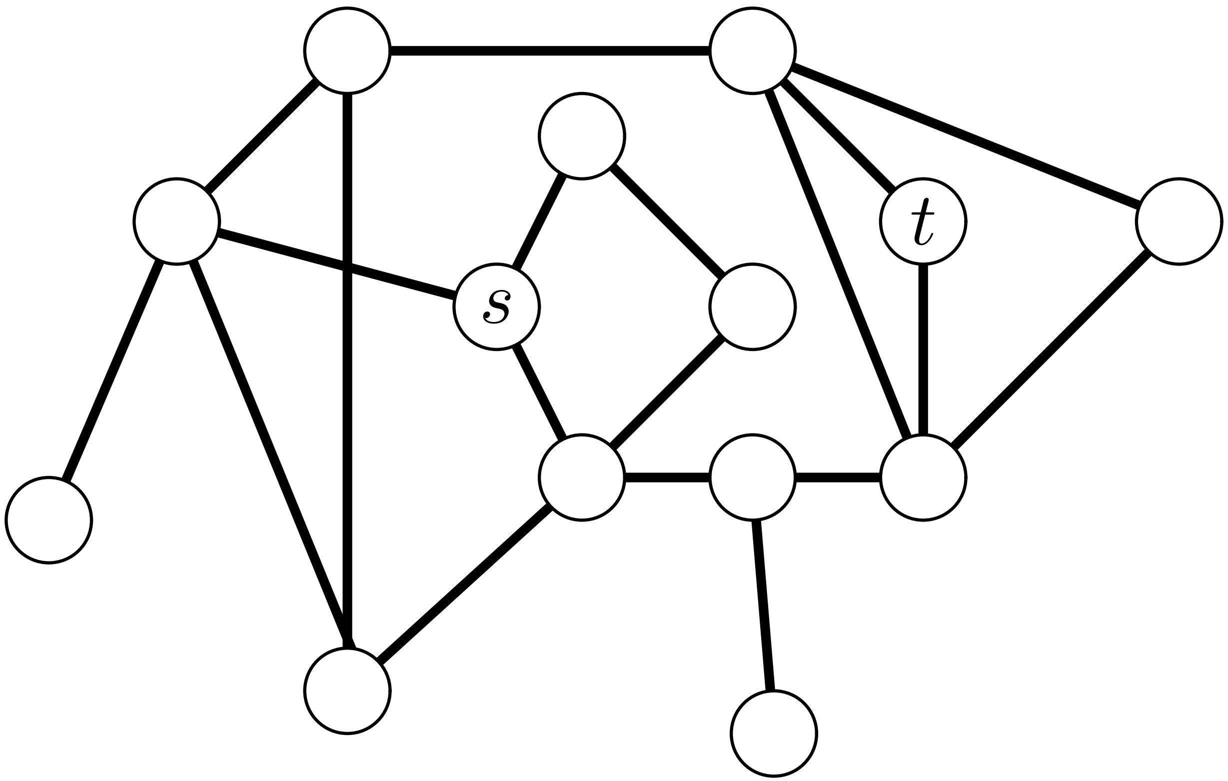
Algorithm: Compute $A(\cdot)$ and $B(\cdot)$ bottom-up!

(s,t)-Series-Parallel Graphs

Def. A graph $G = (V, E)$ is *2-terminal* if it contains two special vertices s and t.



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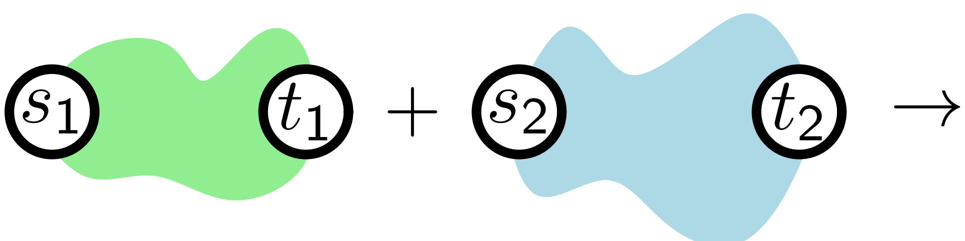

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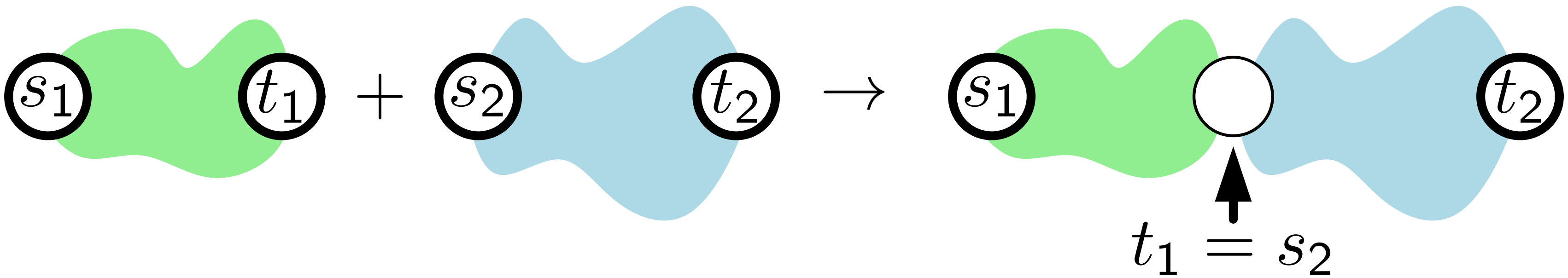

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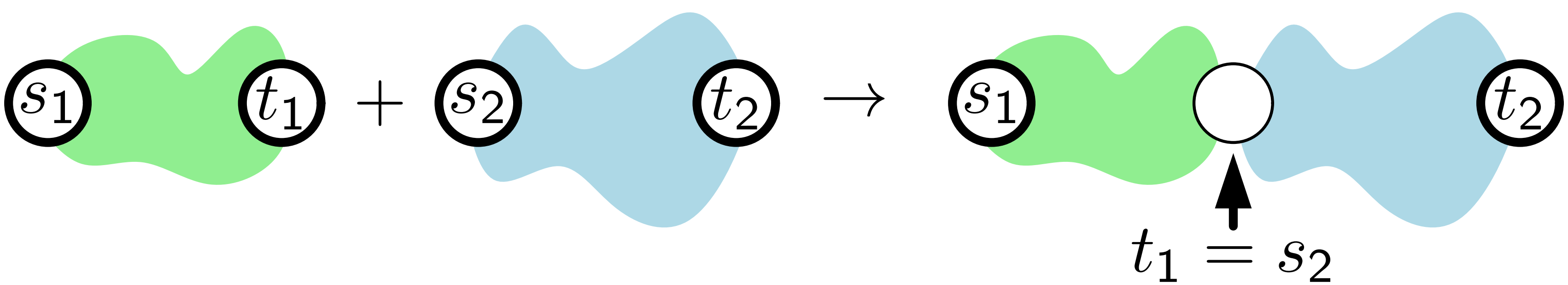
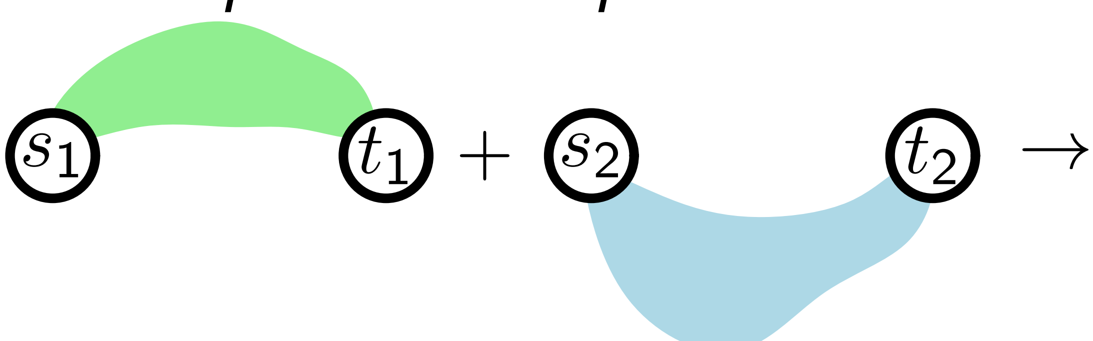


- G is a *parallel composition* of two series-parallel graphs



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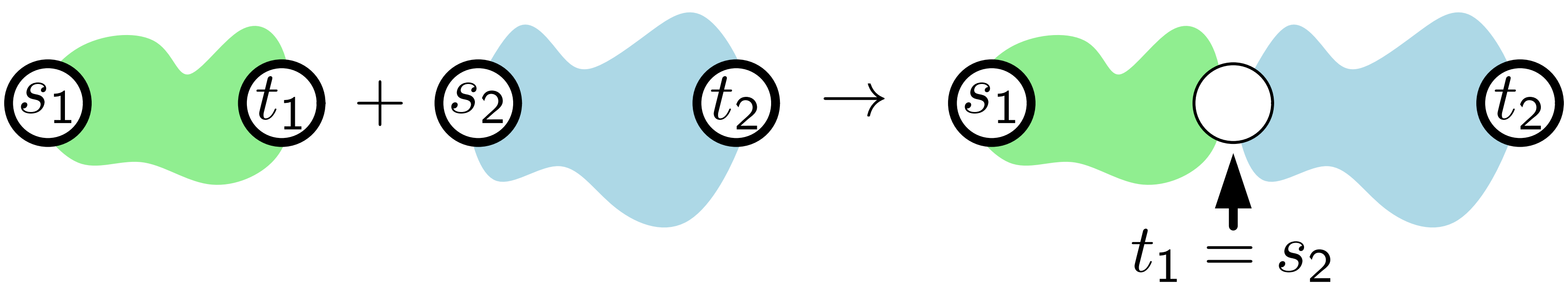

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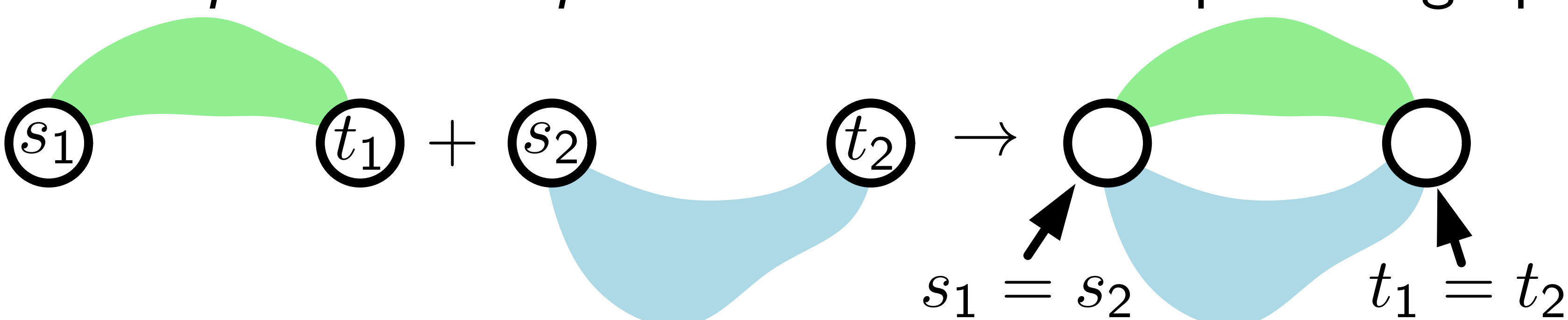

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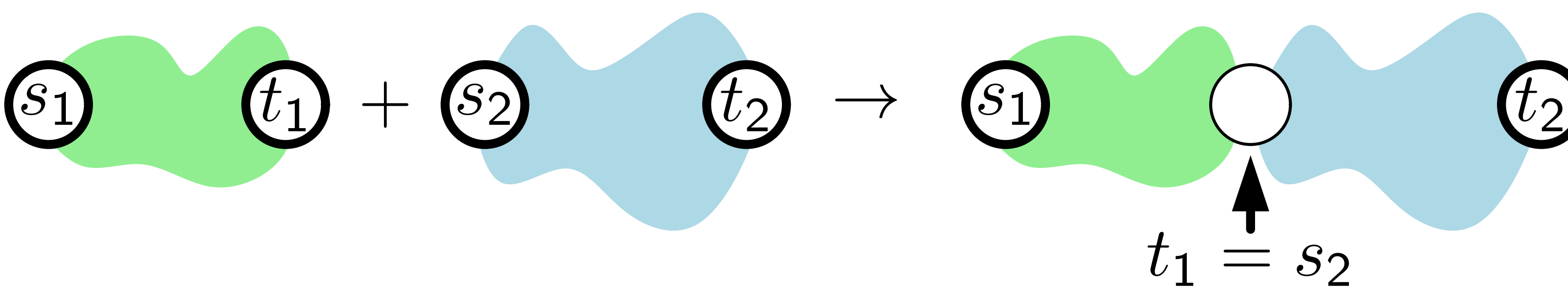

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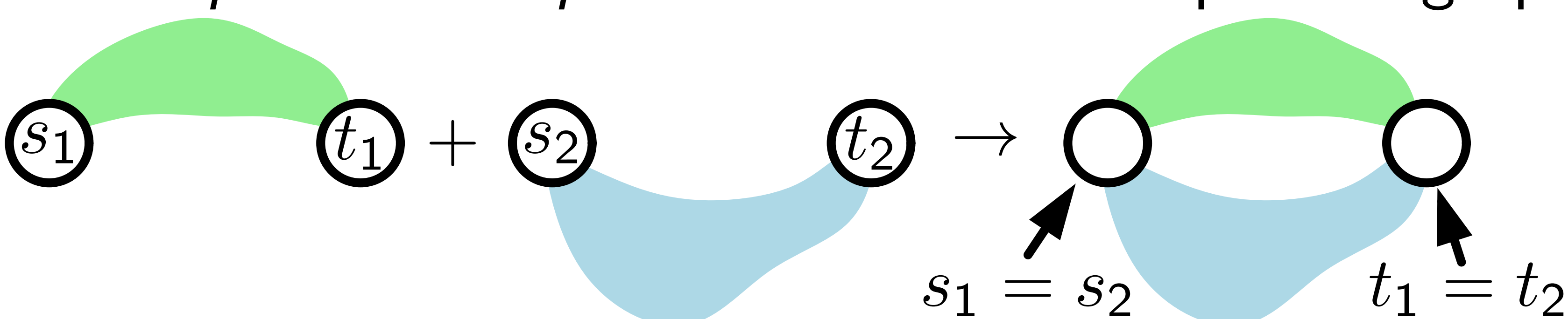

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Series-parallel graphs have a natural tree structure!

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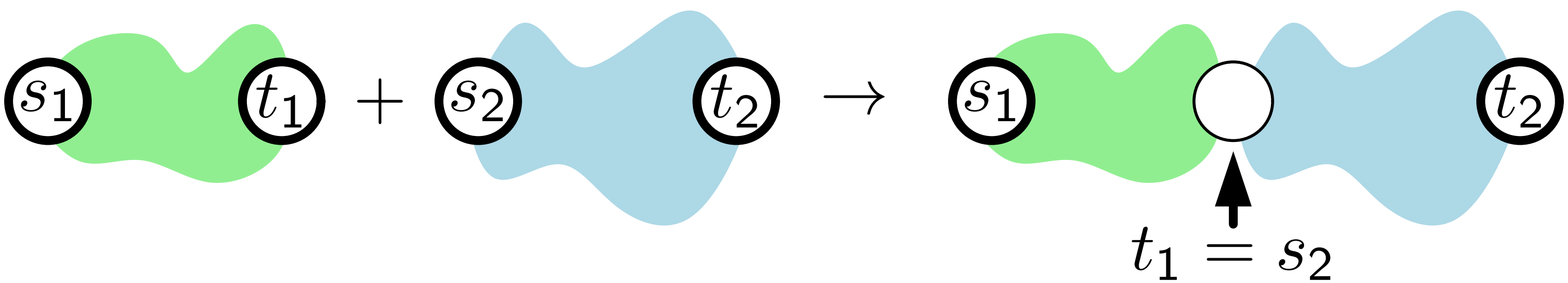

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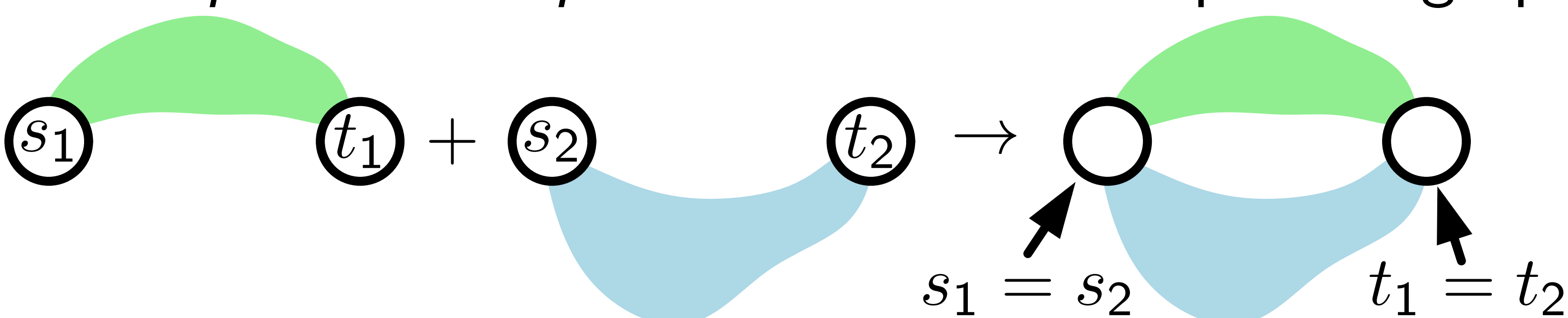
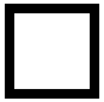


SP-Tree



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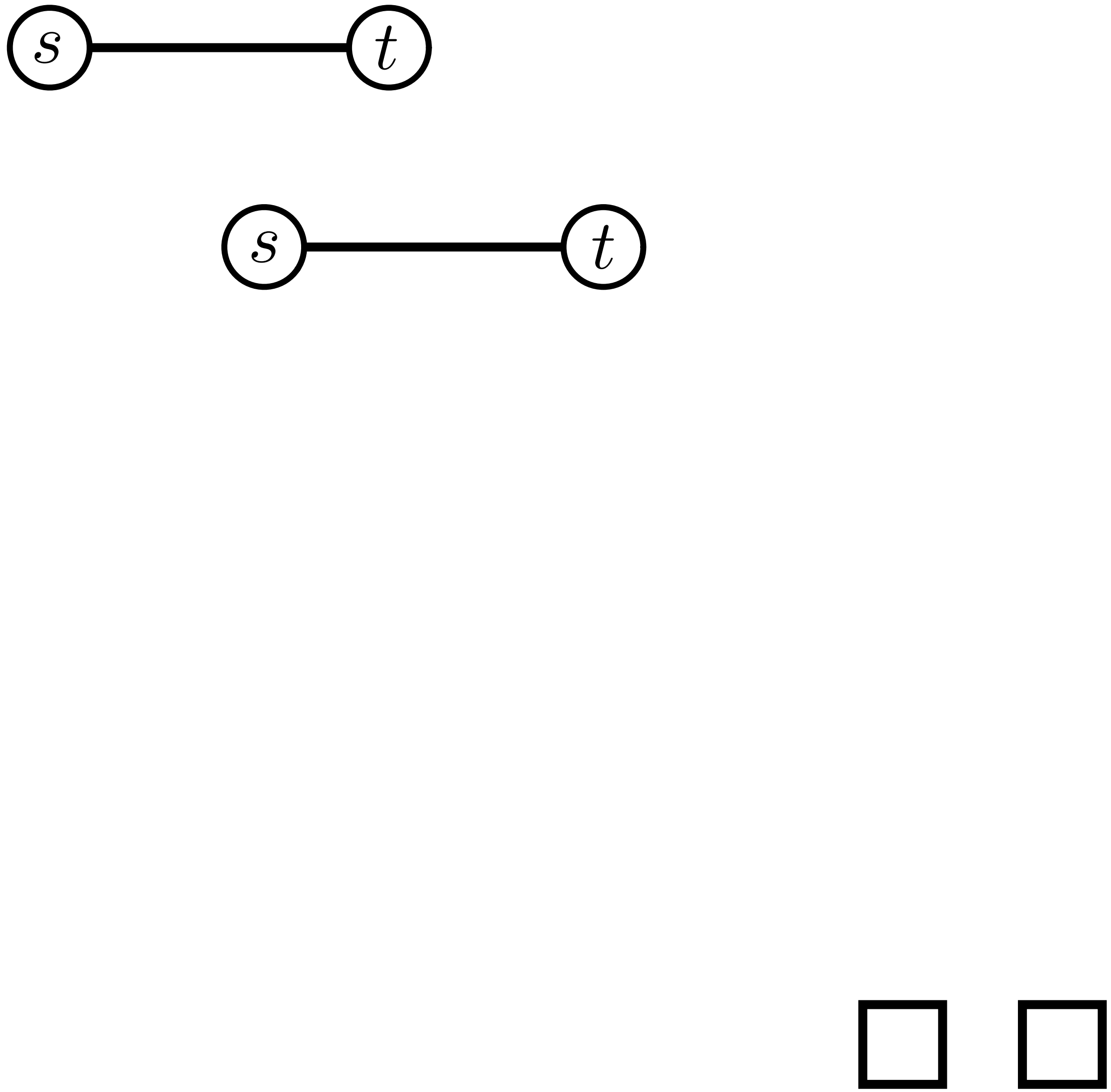

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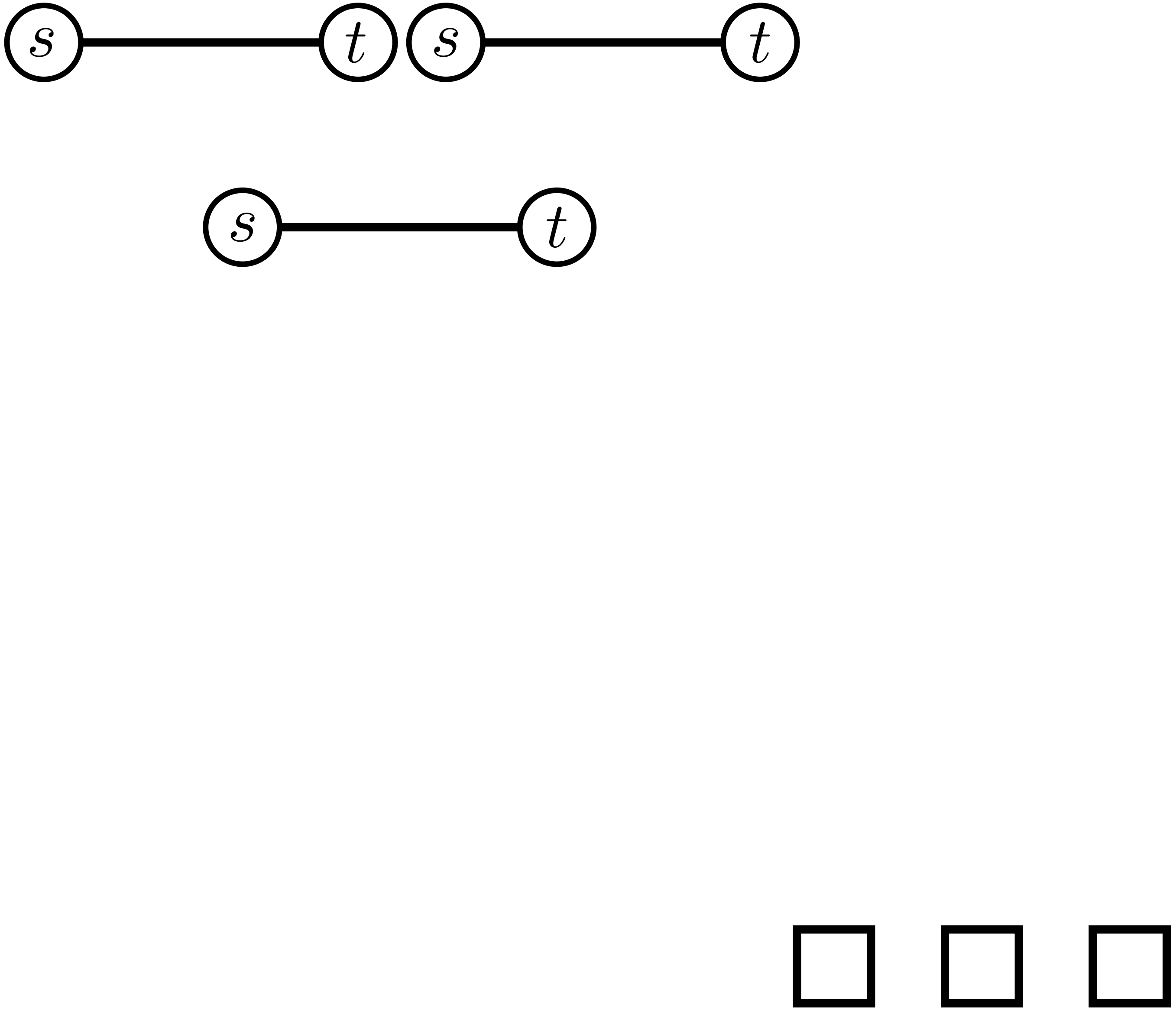

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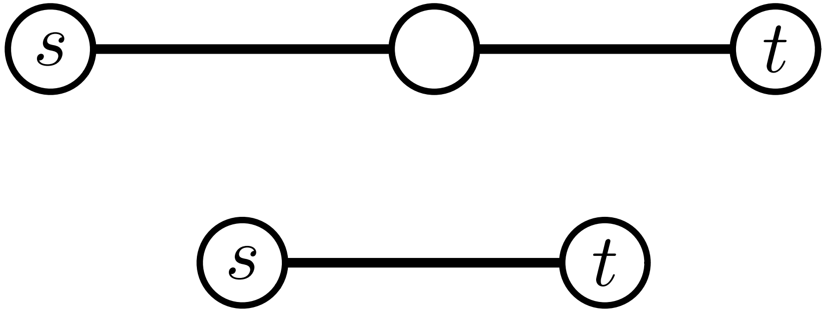

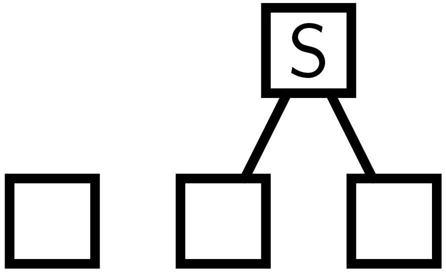

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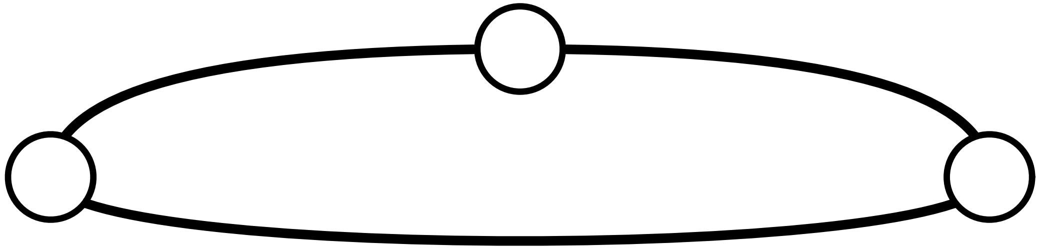

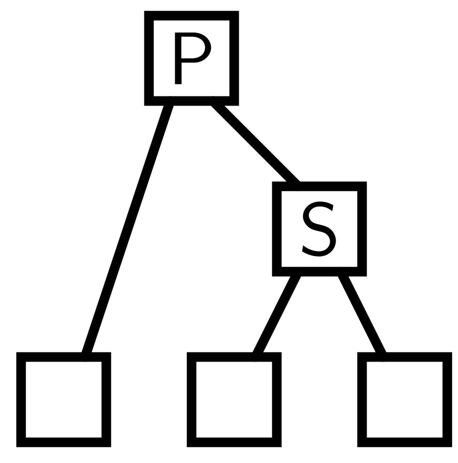

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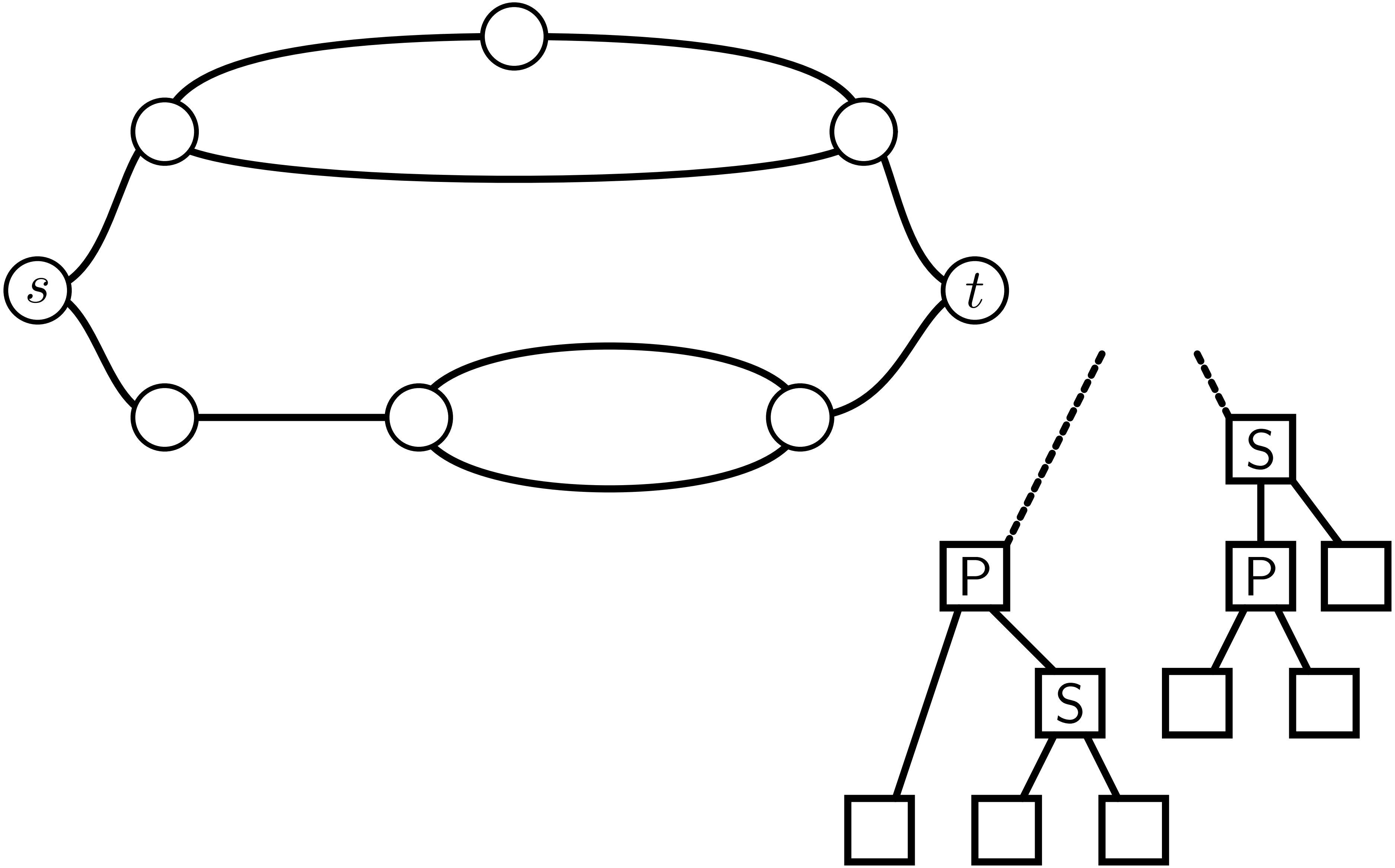

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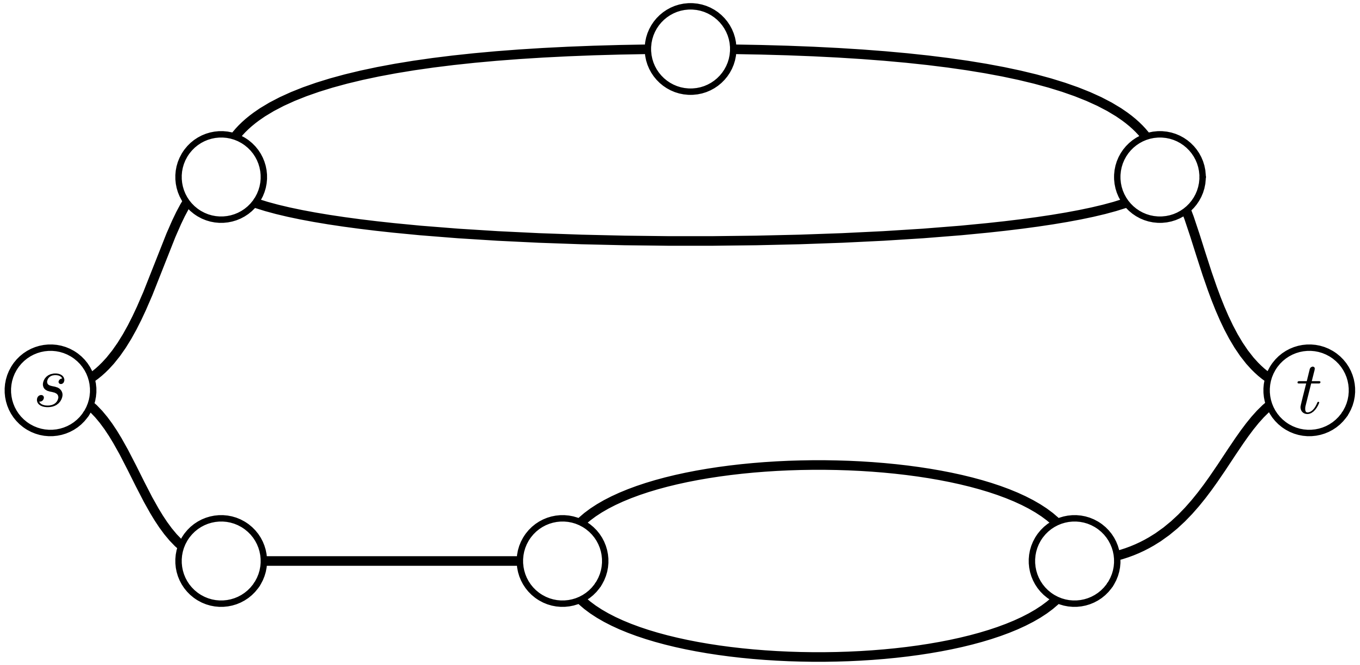
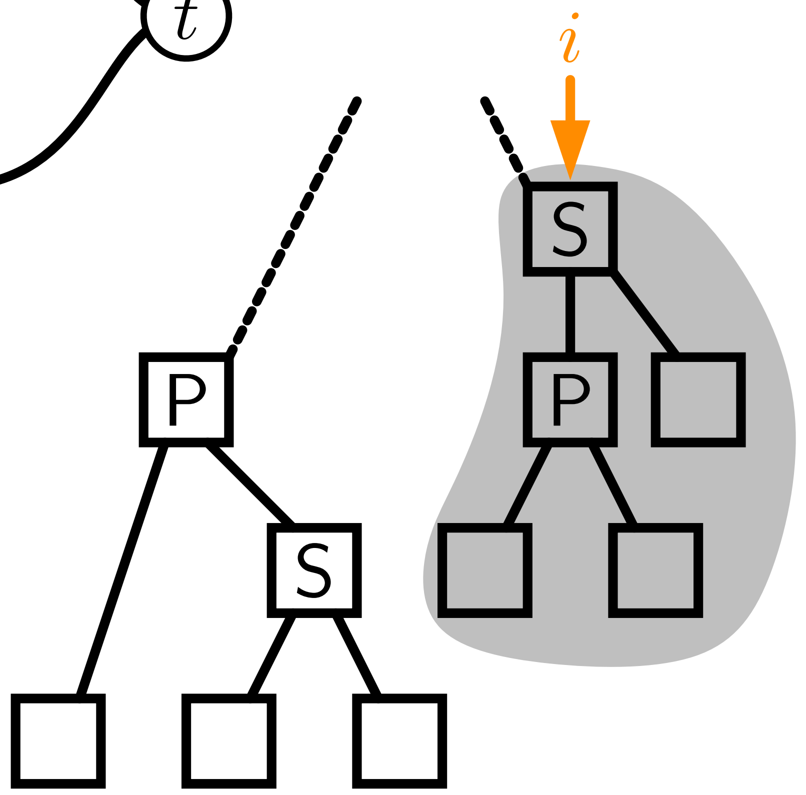
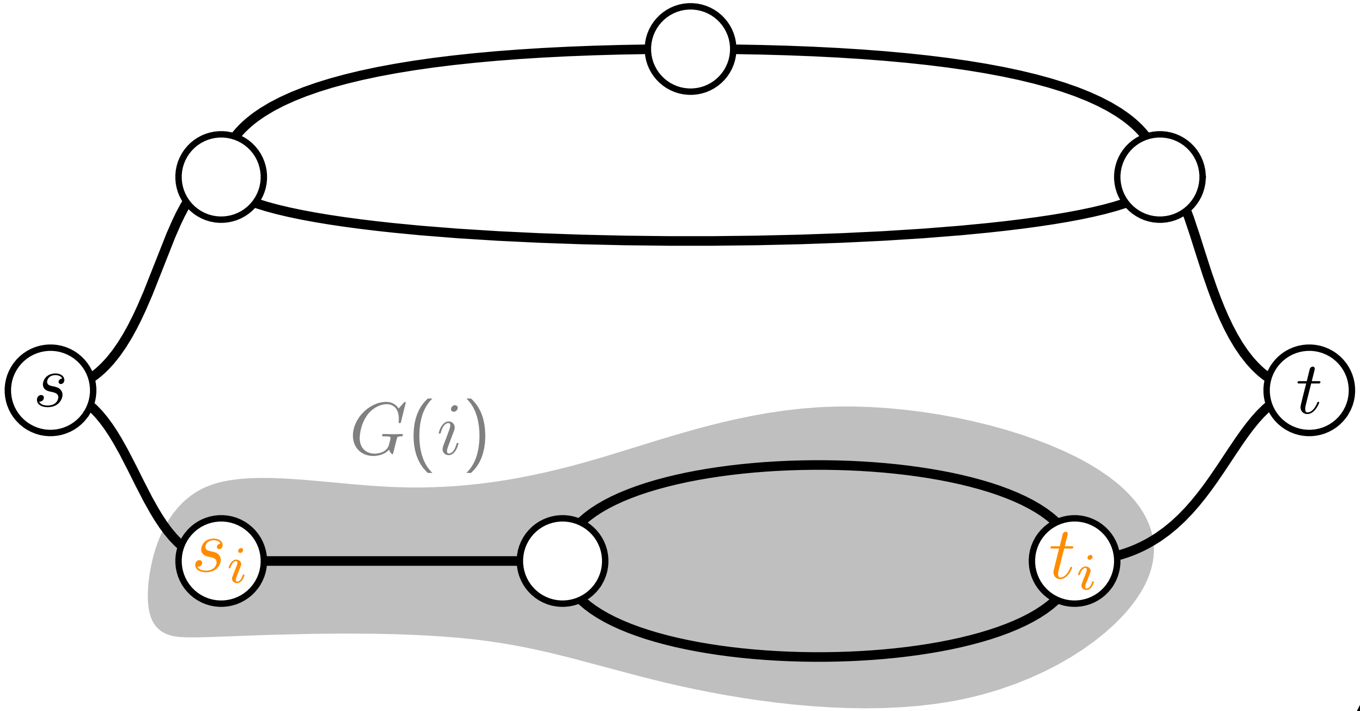
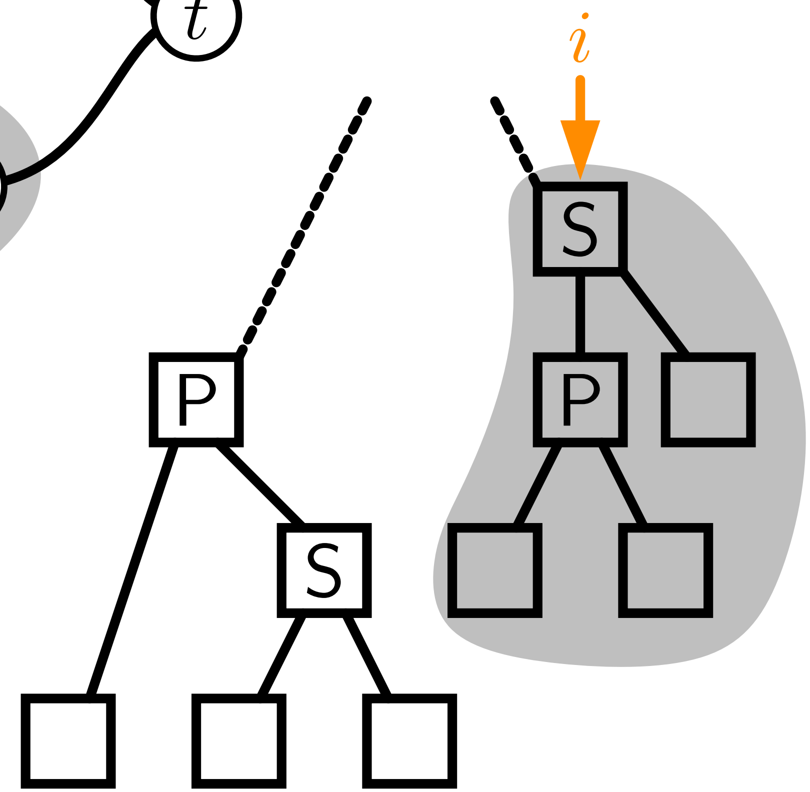




Let i be a node in an SP-tree.
$G(i) :=$ graph represented by the subtree rooted at i

SP-Tree





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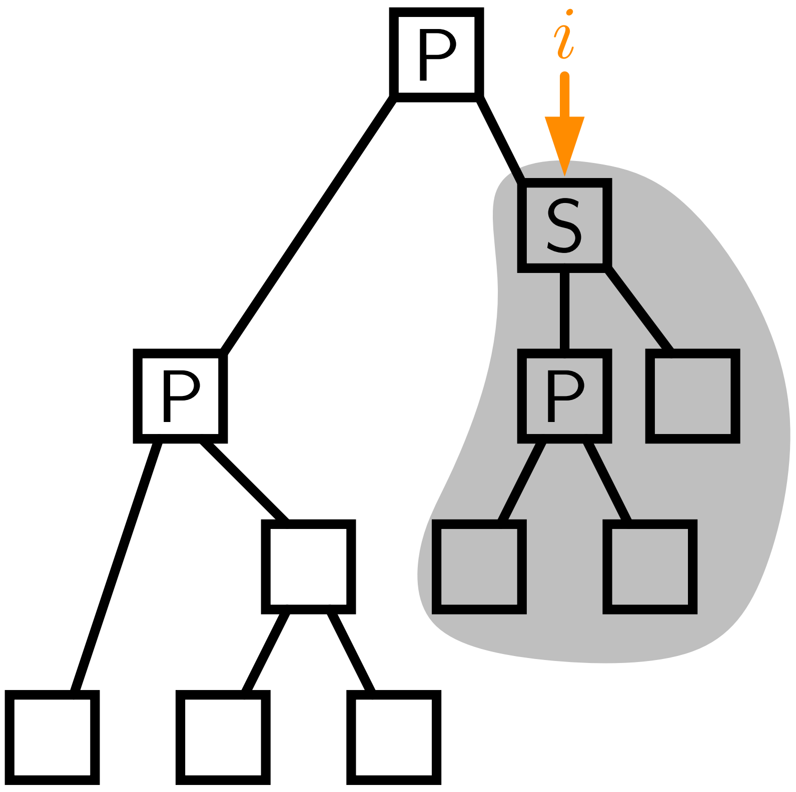
Independent Set on SP-Trees

Dynamic program on SP-tree indexed by $G(i)$



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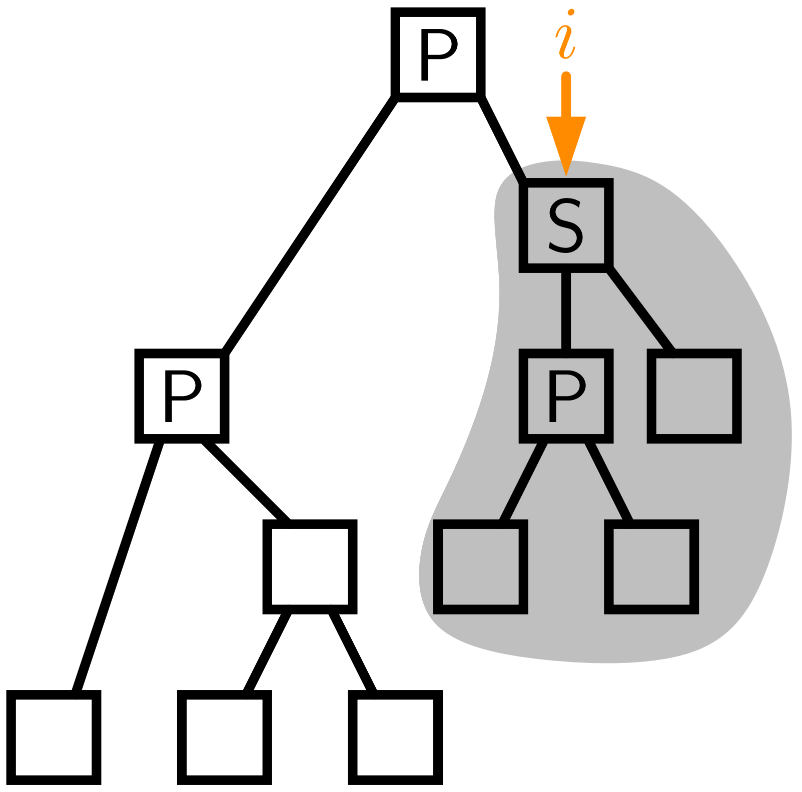

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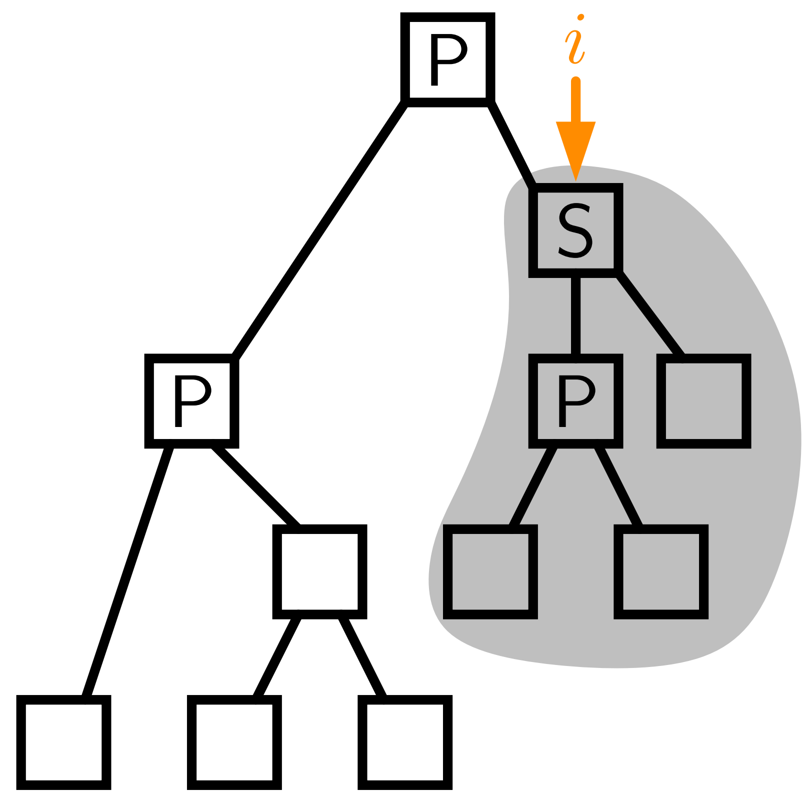

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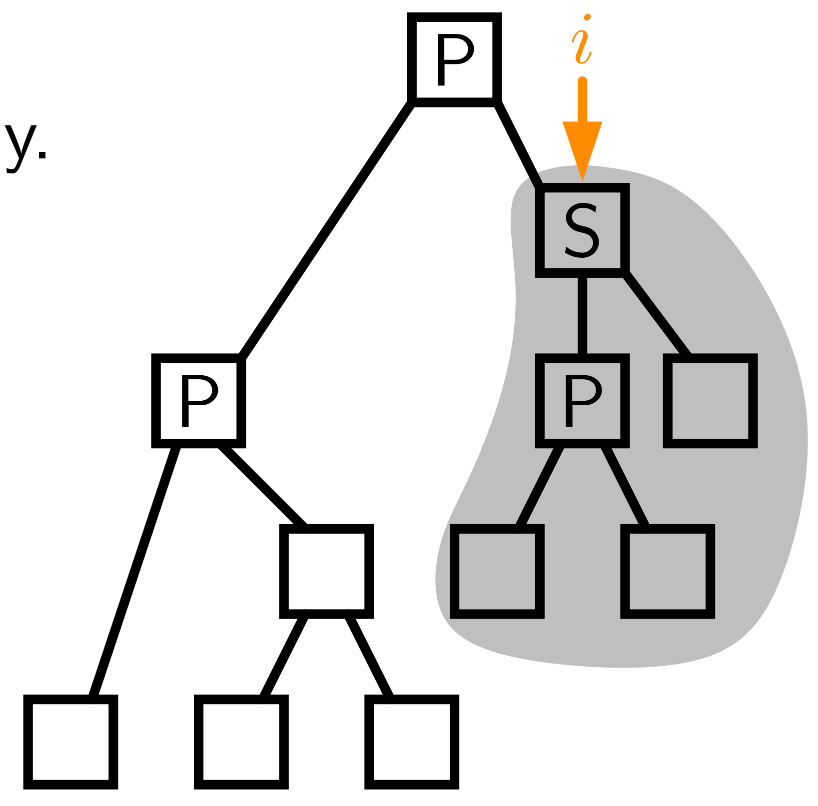

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$AA(i) :=$ maximum weight of an independent set S in $G(i)$ where $s_i \in S$ and $t_i \in S$

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$AB(i)$ and $BB(i)$ are defined similarly.

– If i is a leaf...

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$$\begin{aligned} AA(i) &= -\infty \\ AB(i) &= \omega(s_i) \\ BA(i) &= \omega(t_i) \\ BB(i) &= 0 \end{aligned}$$

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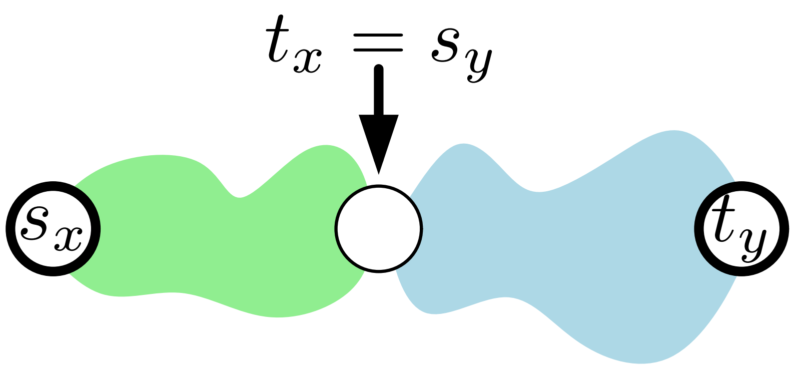

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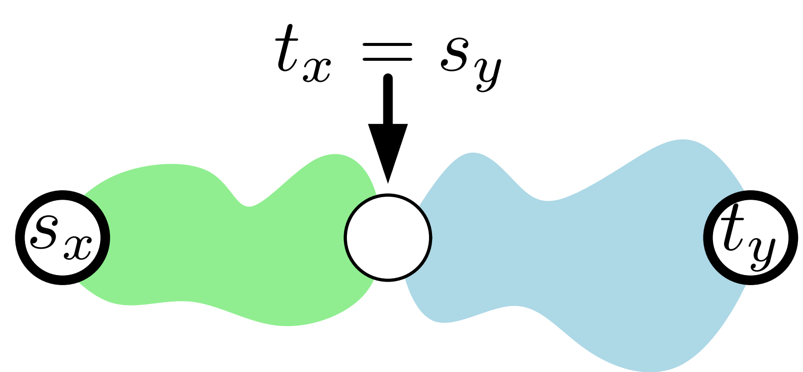

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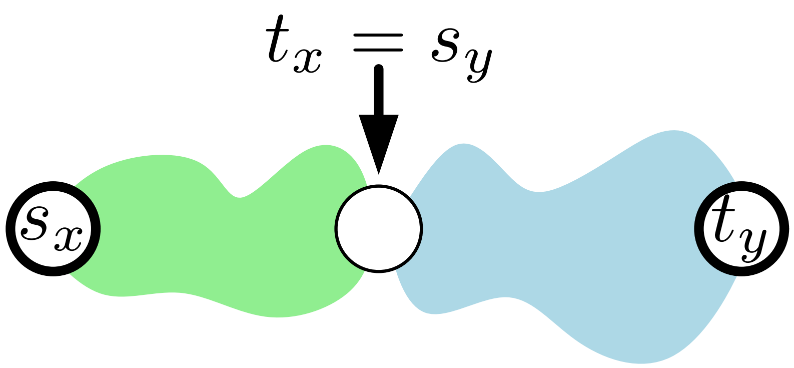

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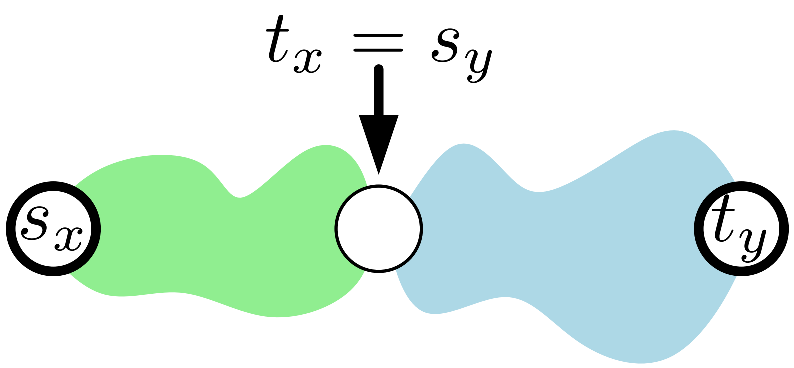

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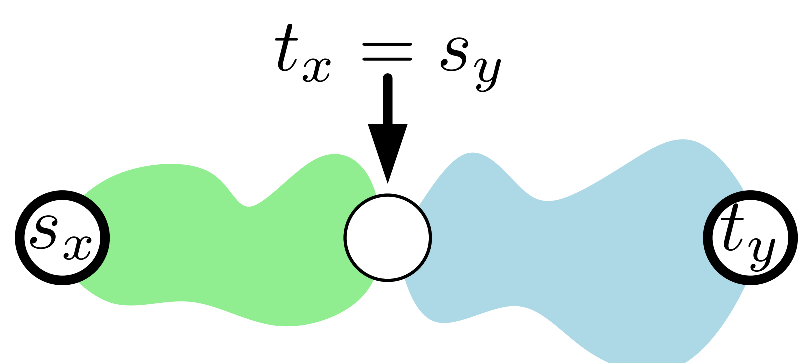

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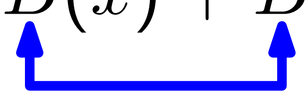

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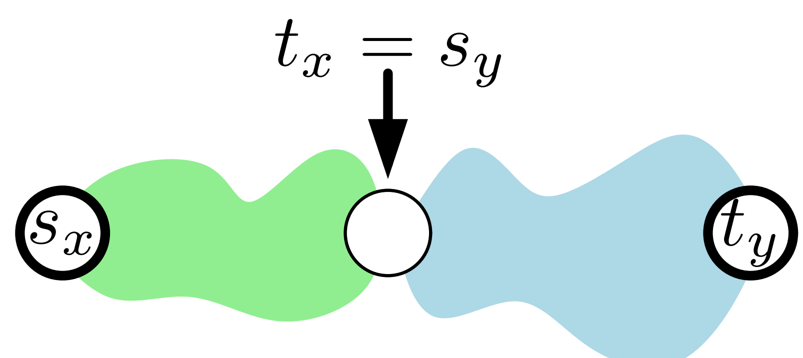

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Thm. Given an n-vertex series-parallel graph G with its SP-tree, Independent Set on G can be solved in $O(n)$ time.

Generalization?

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Example: k-terminal graph $G = (V, E, T)$, $|T| = k$

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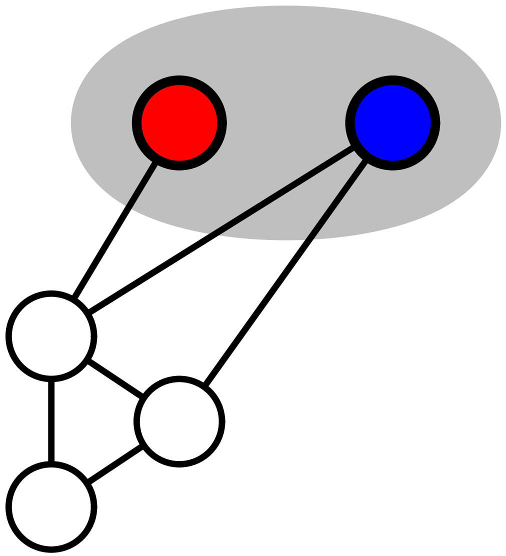

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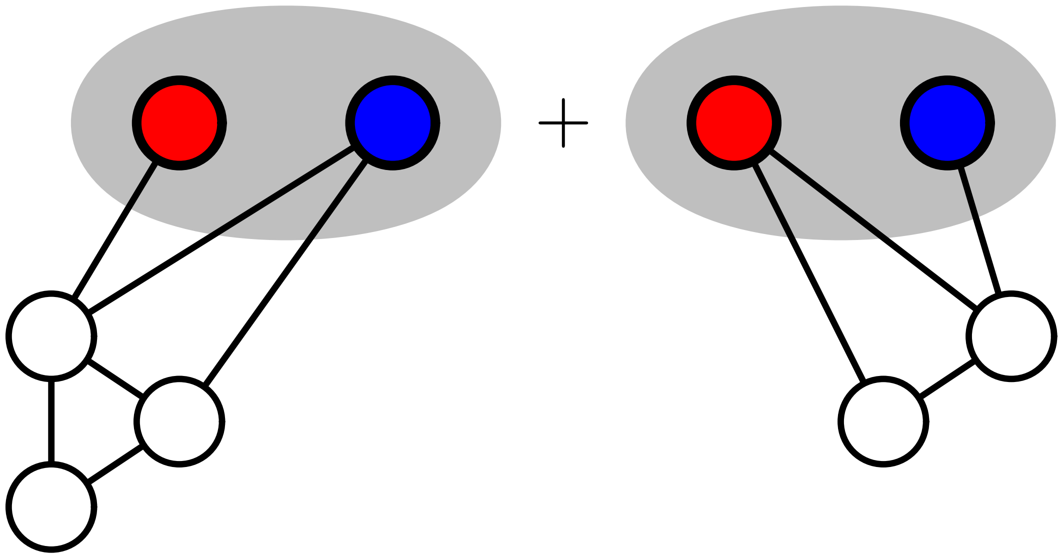

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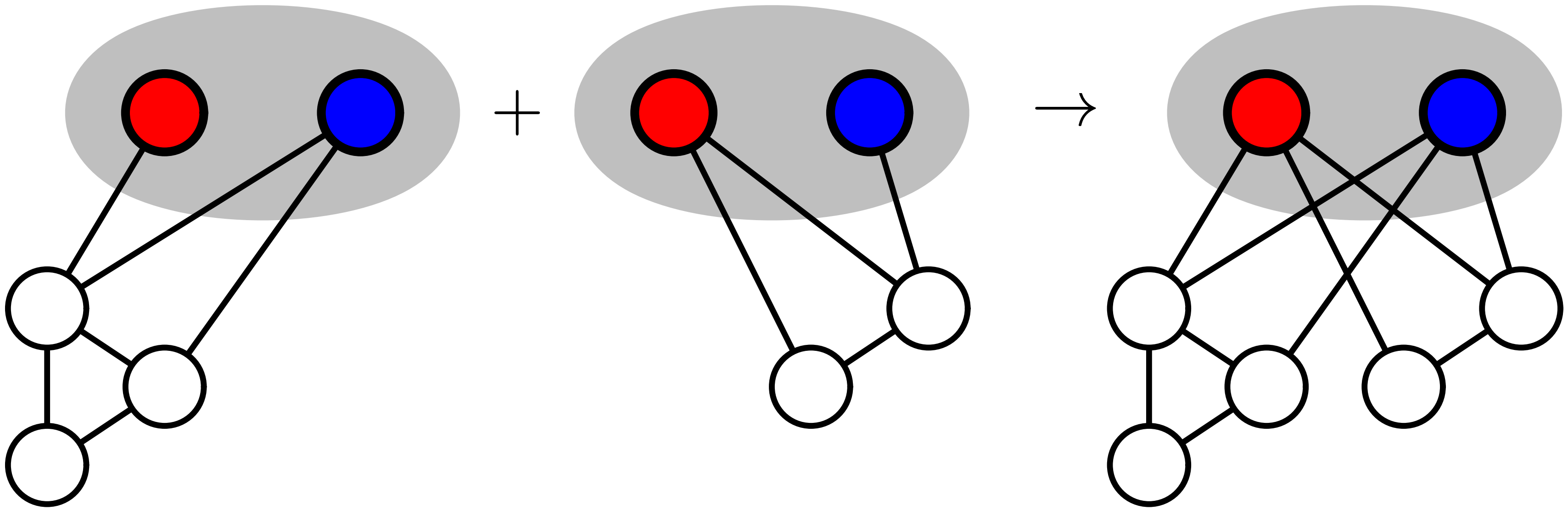
Operation: "gluing"

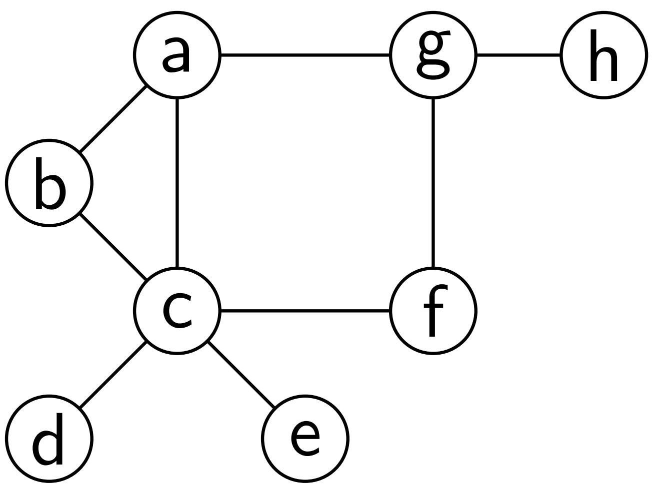


Example: Tree Decomposition

"bags"

A *tree decomposition* is a tree whose nodes map to subsets of V so that...

Graph $G = (V, E)$:



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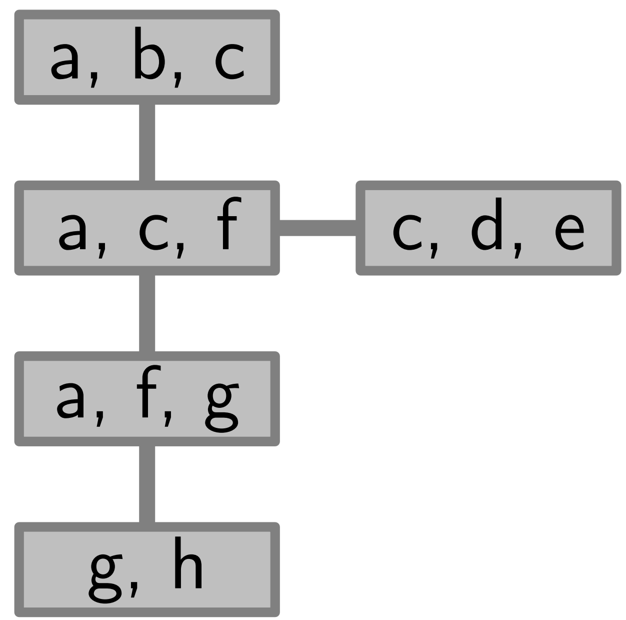

Example: Tree Decomposition

1 each vertex belongs to at least one bag

these bags are connected

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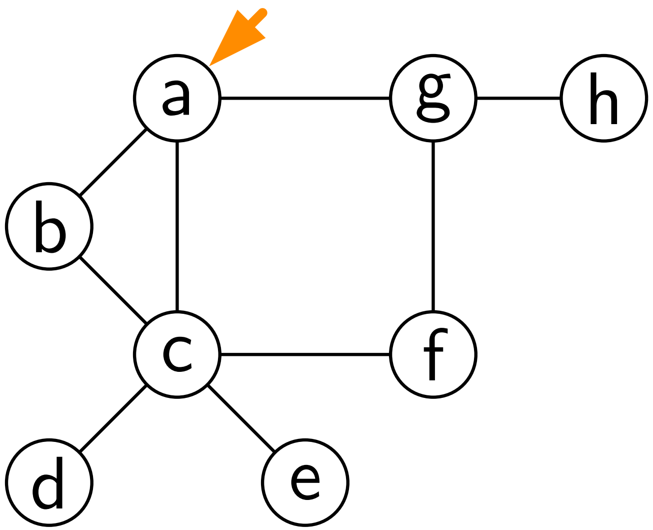

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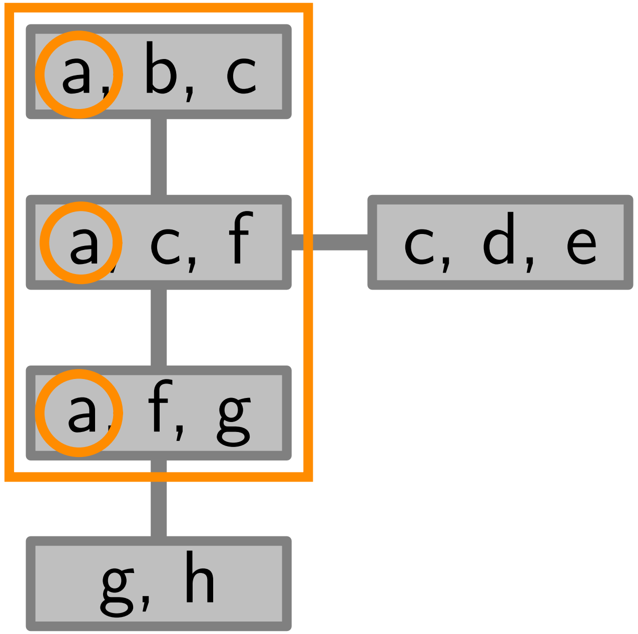

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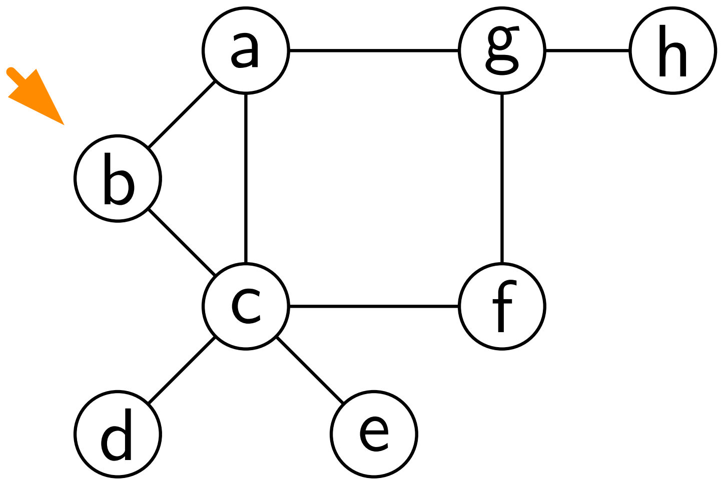

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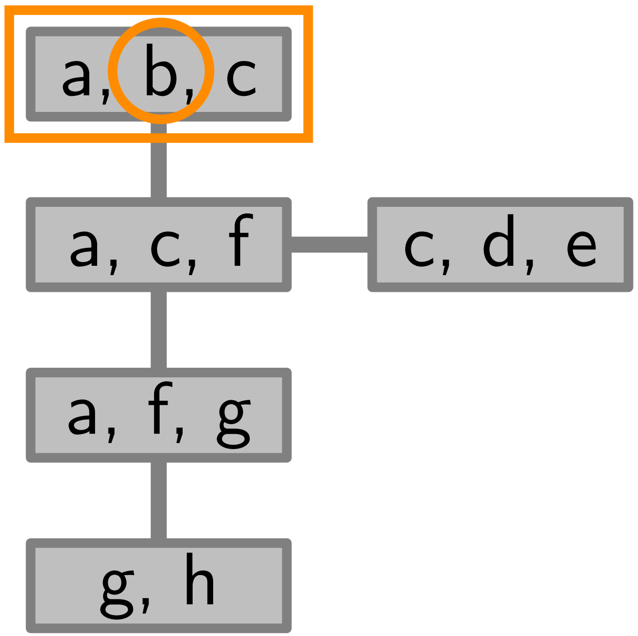

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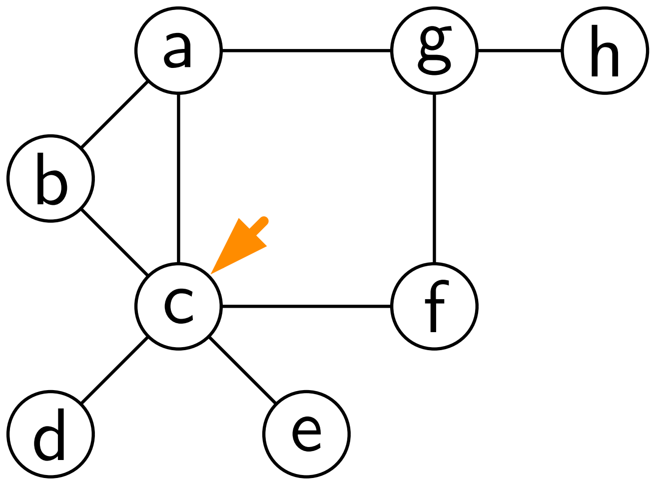

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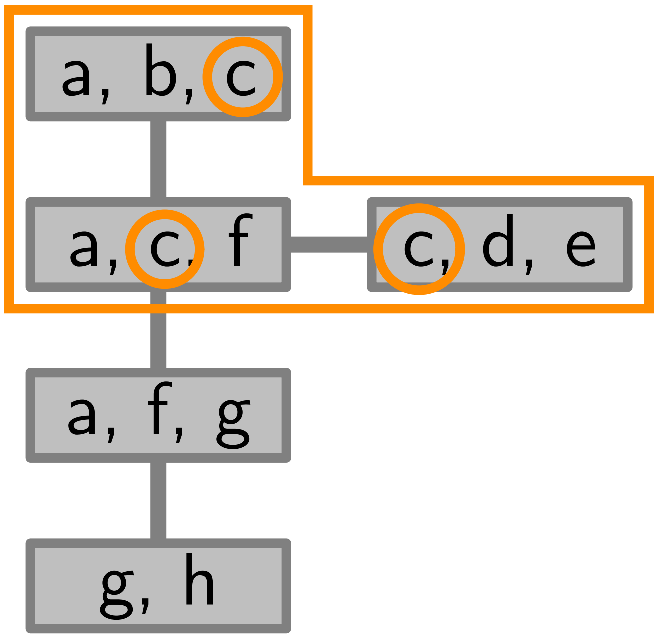

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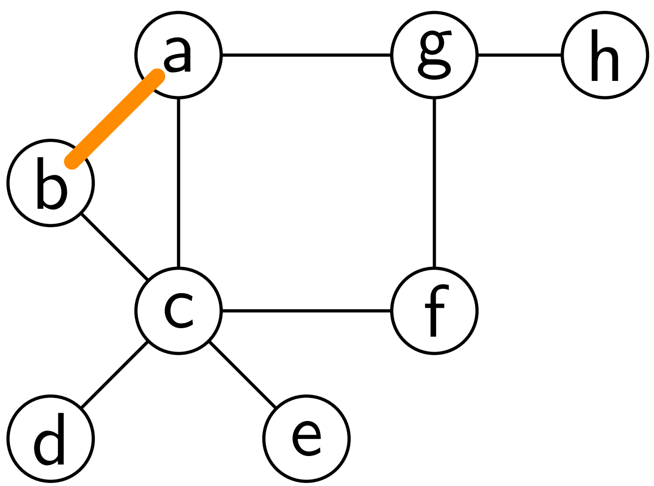

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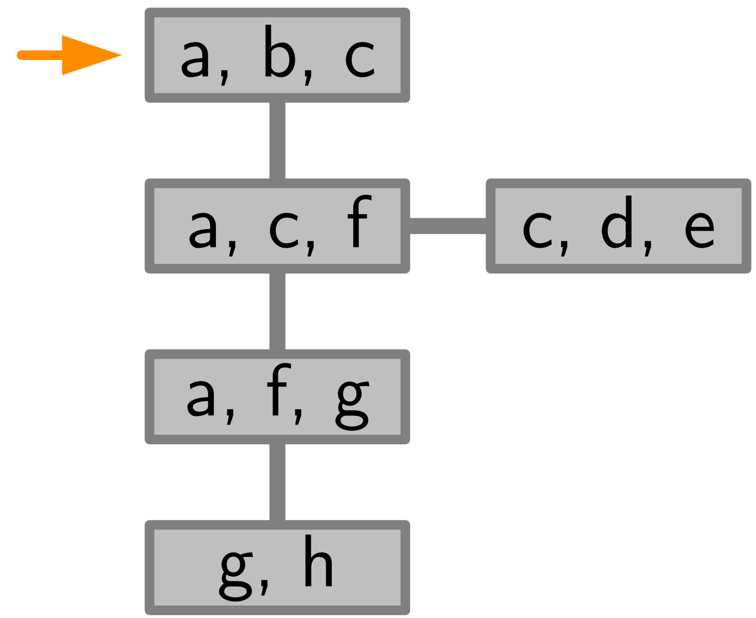

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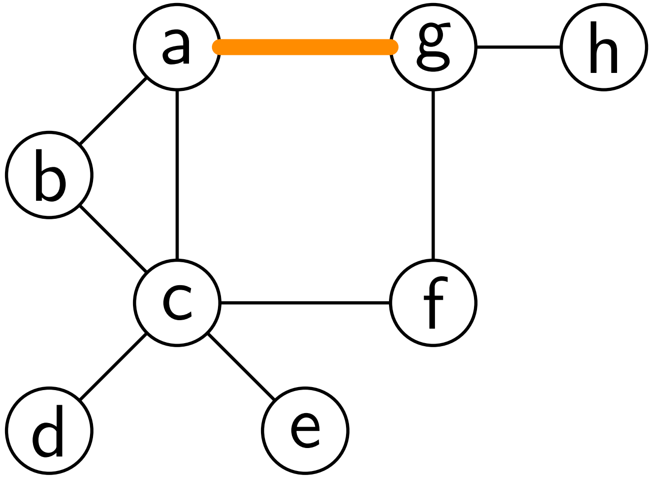

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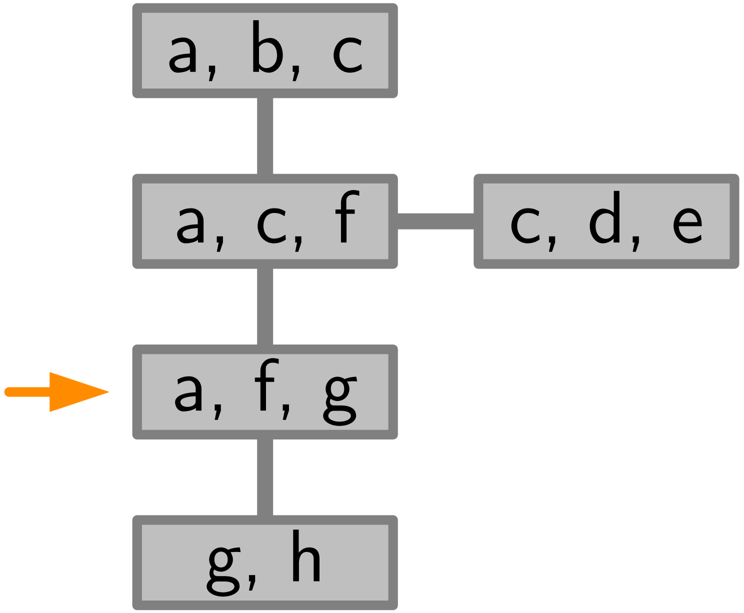

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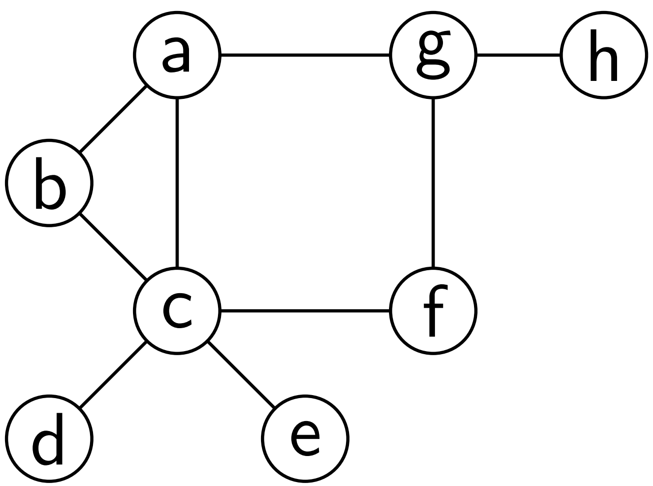

Tree Decomposition:

a, b, c,
d, e, f,
g, h

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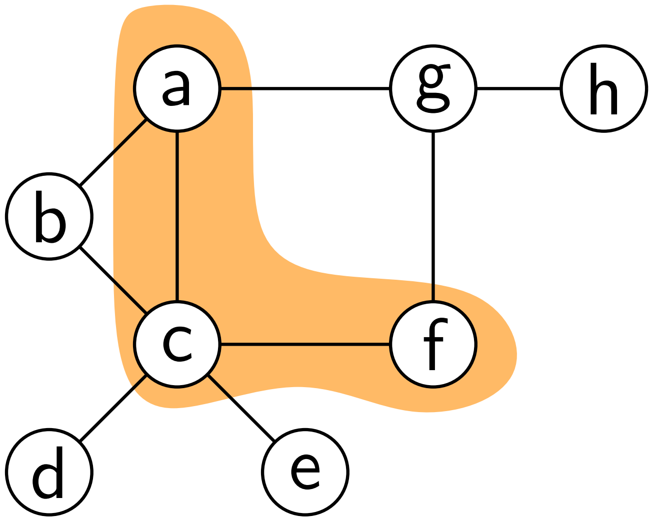

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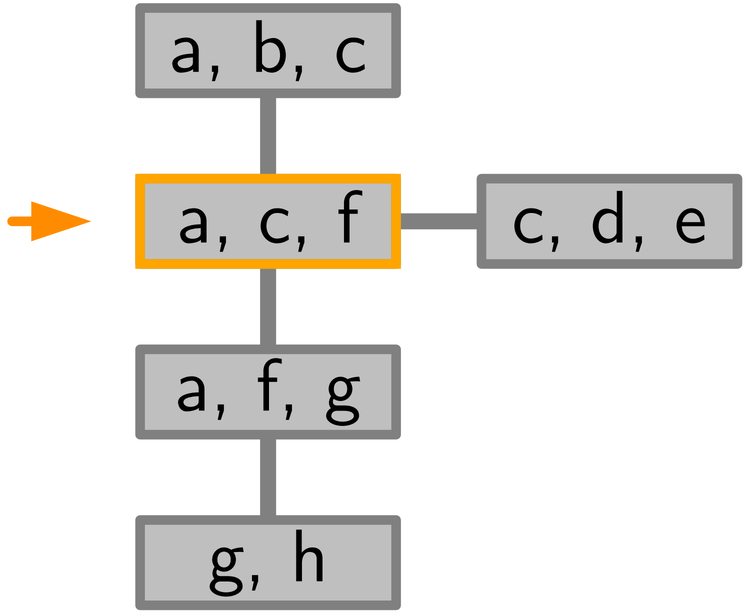

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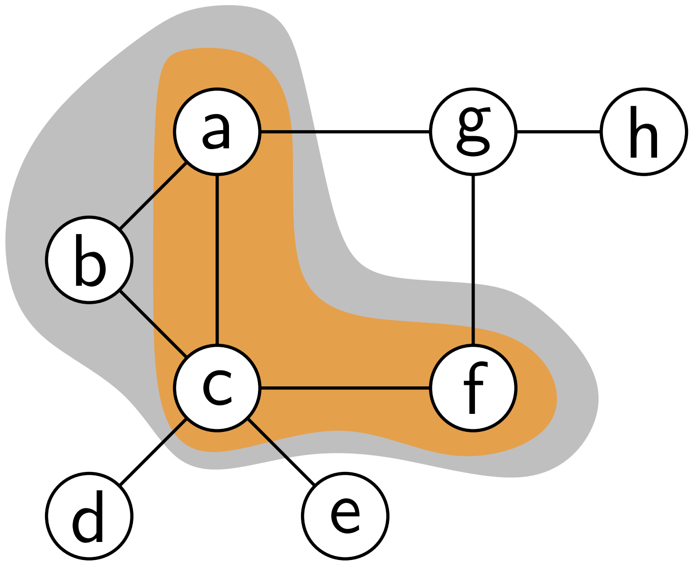

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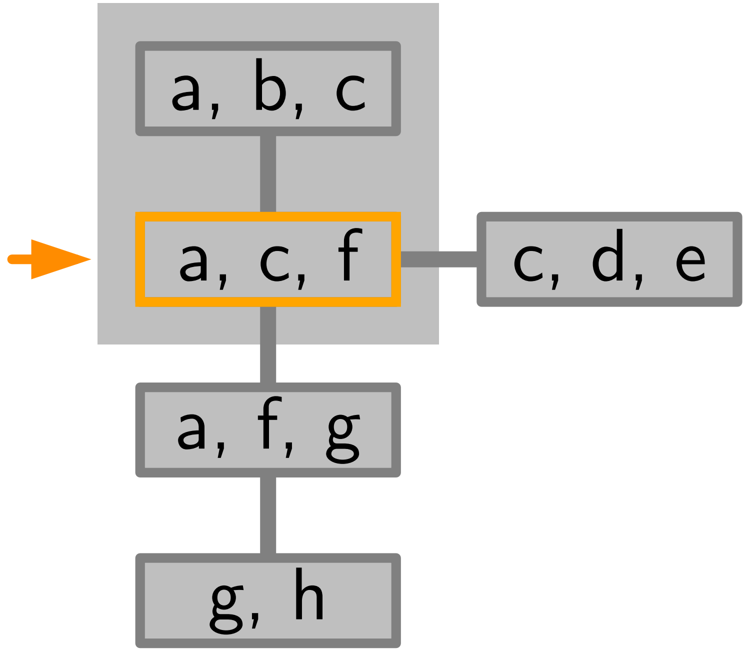

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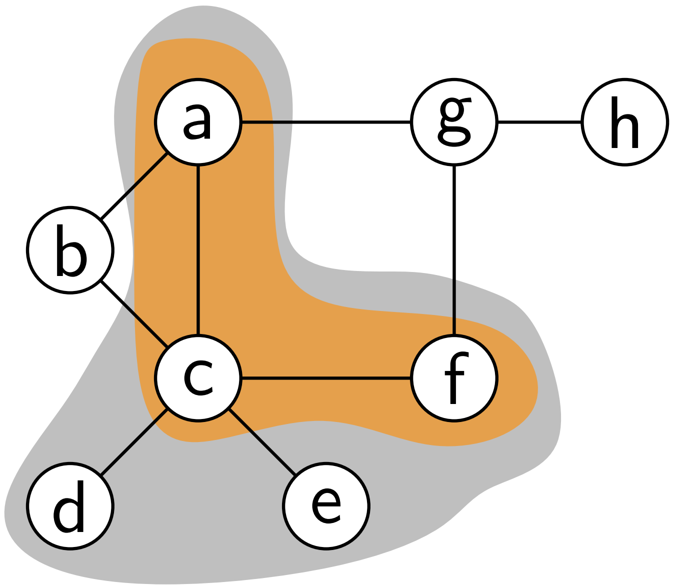

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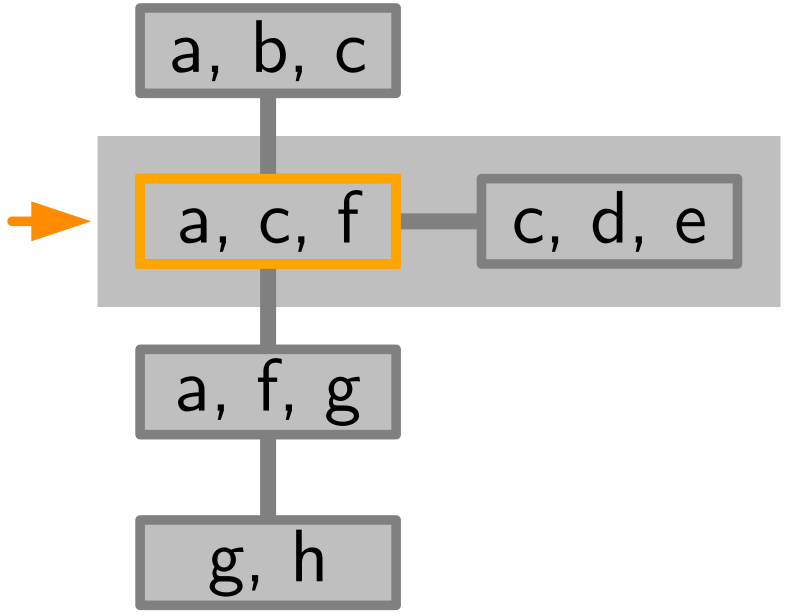

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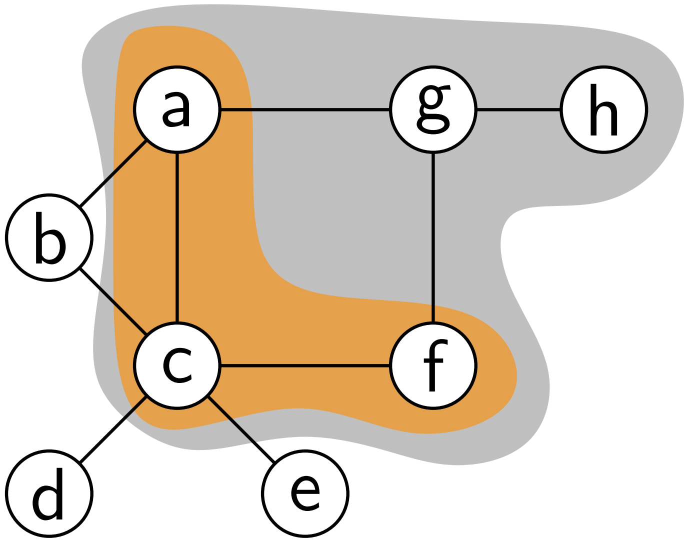

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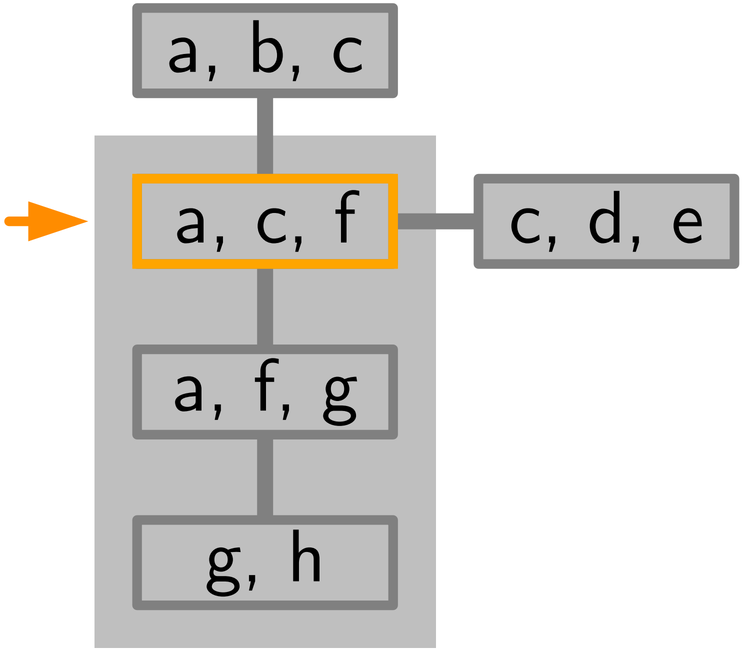

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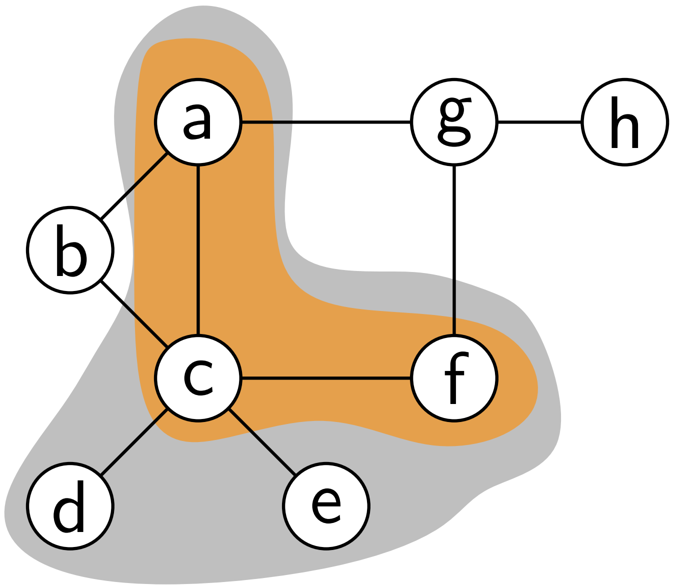

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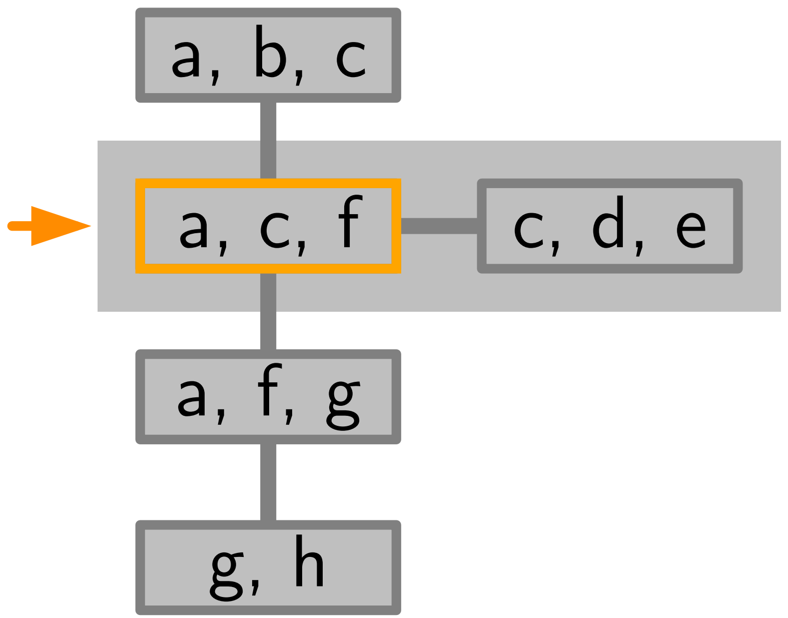

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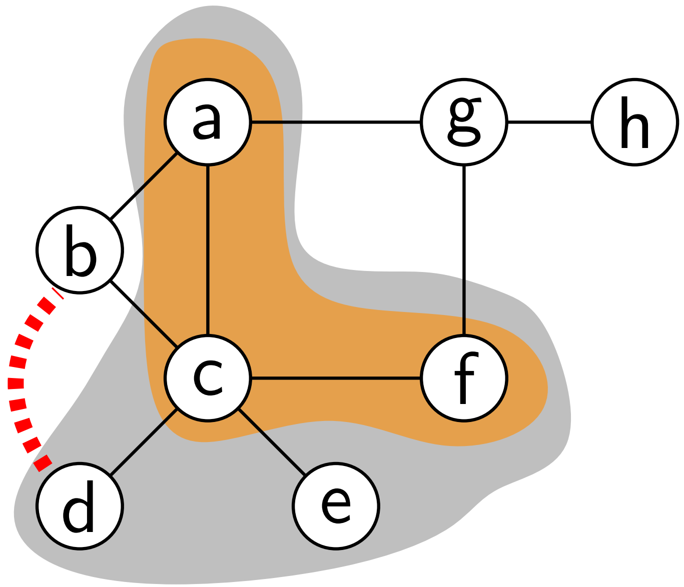

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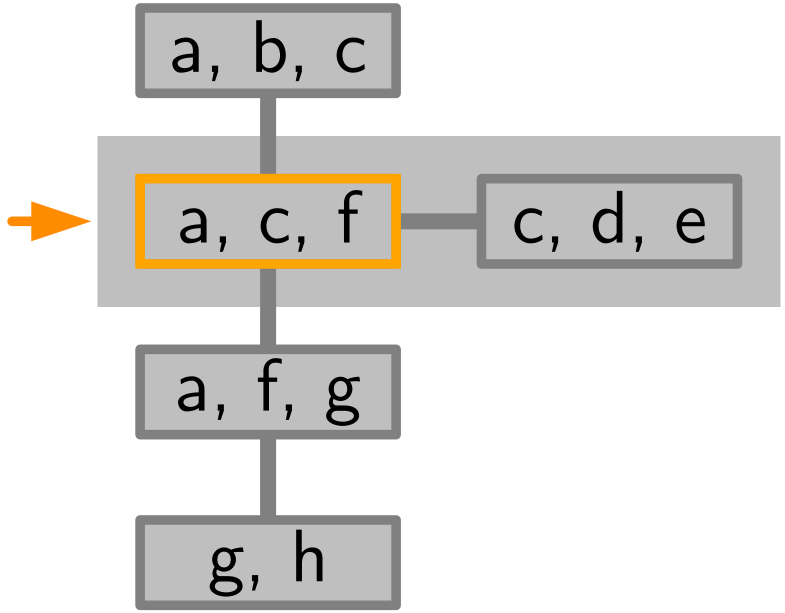

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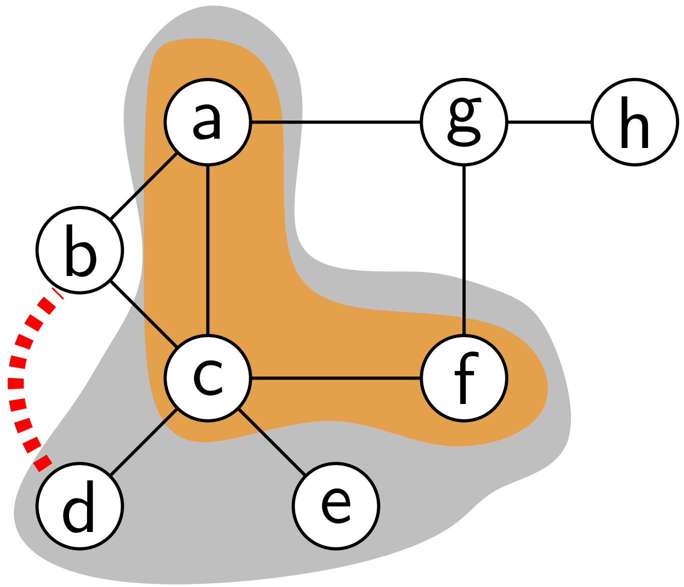

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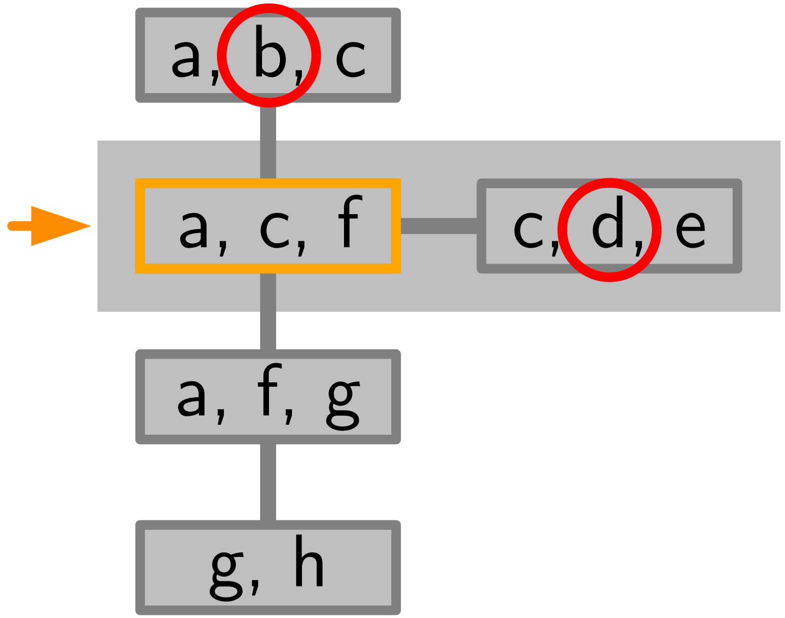

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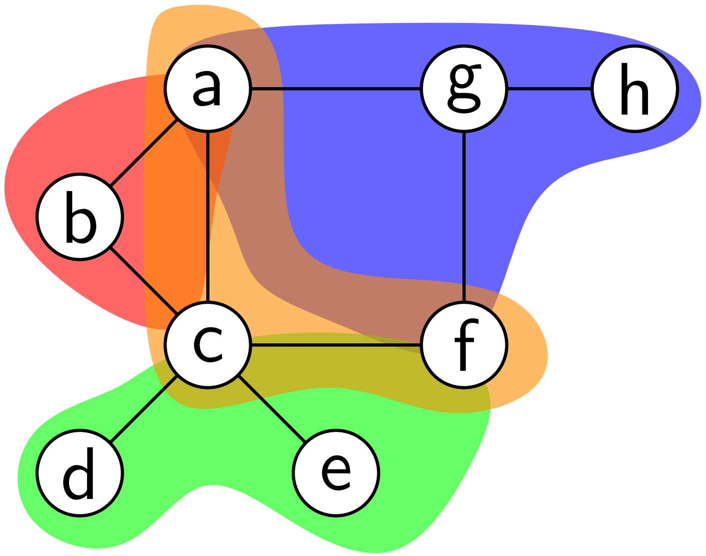

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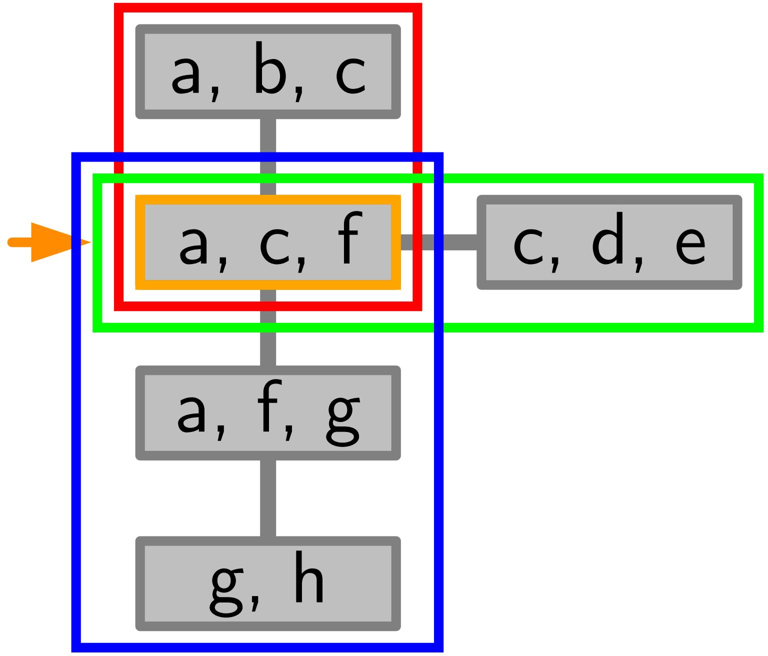

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- $\forall v \in V$ the set $\{p \in P \mid v \in X_p\}$ is connected in T.

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Exercise: Trees have treewidth 1.

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Obs. $\mathrm{tw}(G) < n$

Question: Which graphs have treewidth 0?

Exercise: Trees have treewidth 1.

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Treewidth (formal)

- a tuple $D = (X, T)$
- $T = (P, F)$ is a tree

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Thm. There is a tree decomposition of width $\text{tw}(G)$ where the tree size $|P|$ is polynomial in n.

Parameterized Problems

Given: Instance of size n and parameter k

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"natural parameterization"

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– How can we make "fixed-treewidth-tractable" algorithms?

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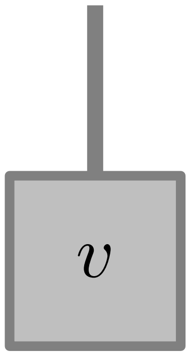

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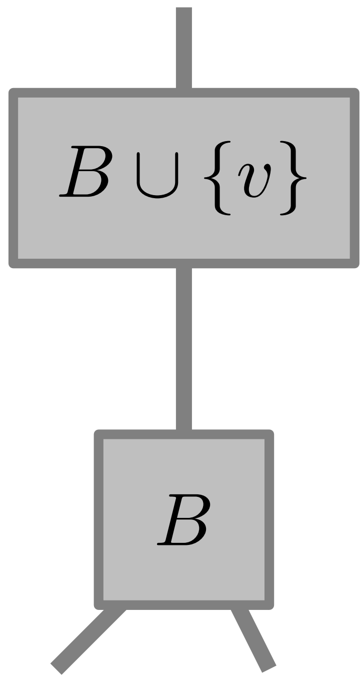

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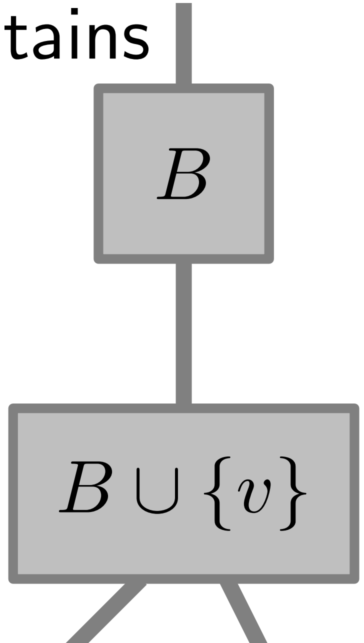

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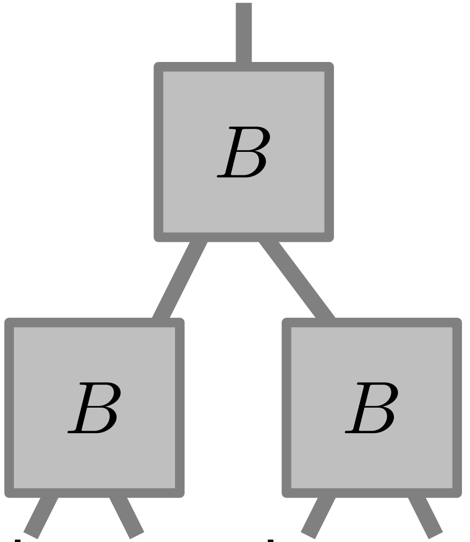
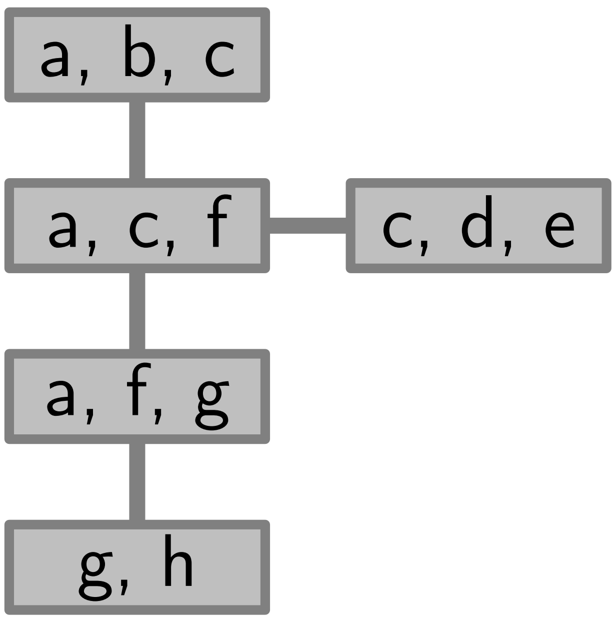
- **Join:**
  The bag has exactly two children and these three nodes have exactly the same vertices



Tool #1: Nice Tree Decompositoins

Thm. For each tree decomposition, there is a nice tree decomposition of the same width and polynomially many more bags. The nice decomposition can be constructed in polynomial time.



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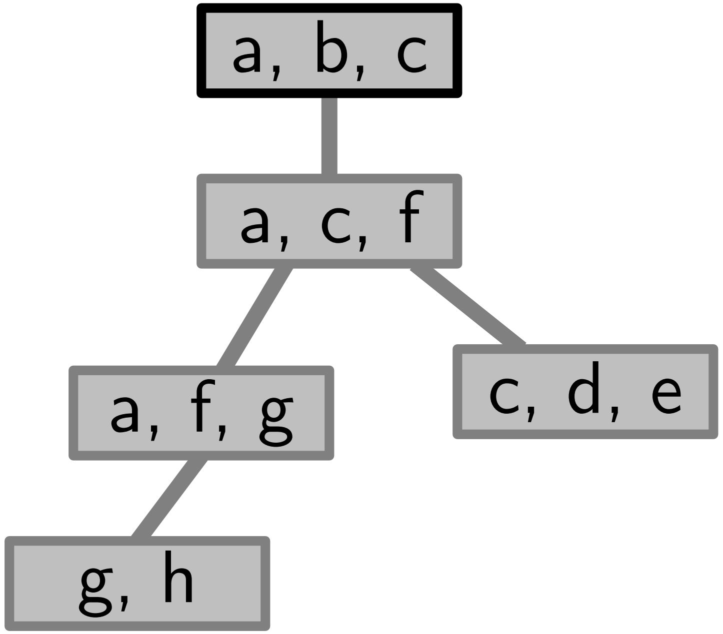

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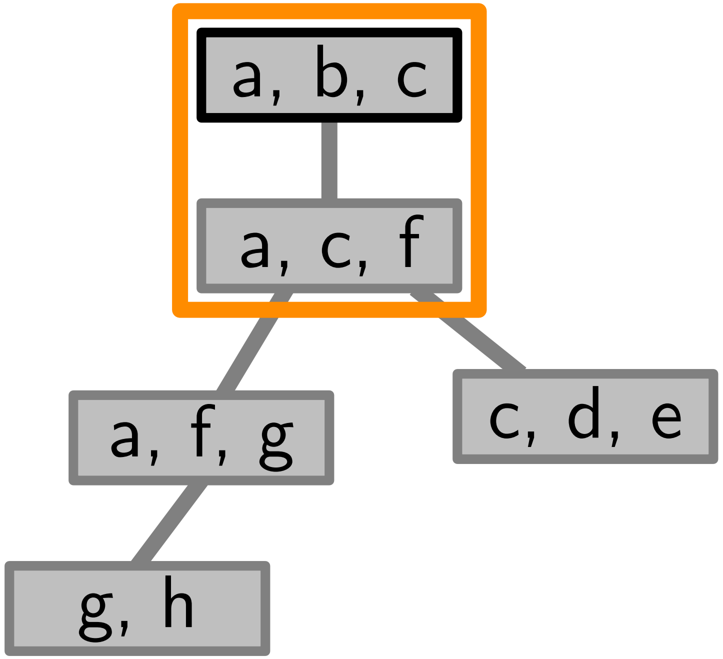

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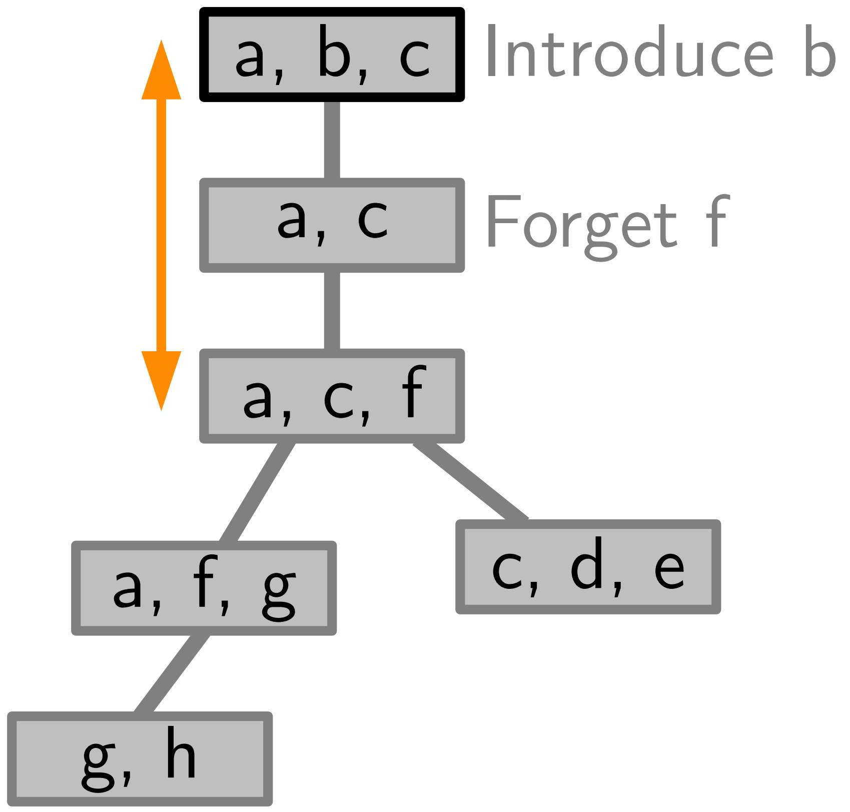

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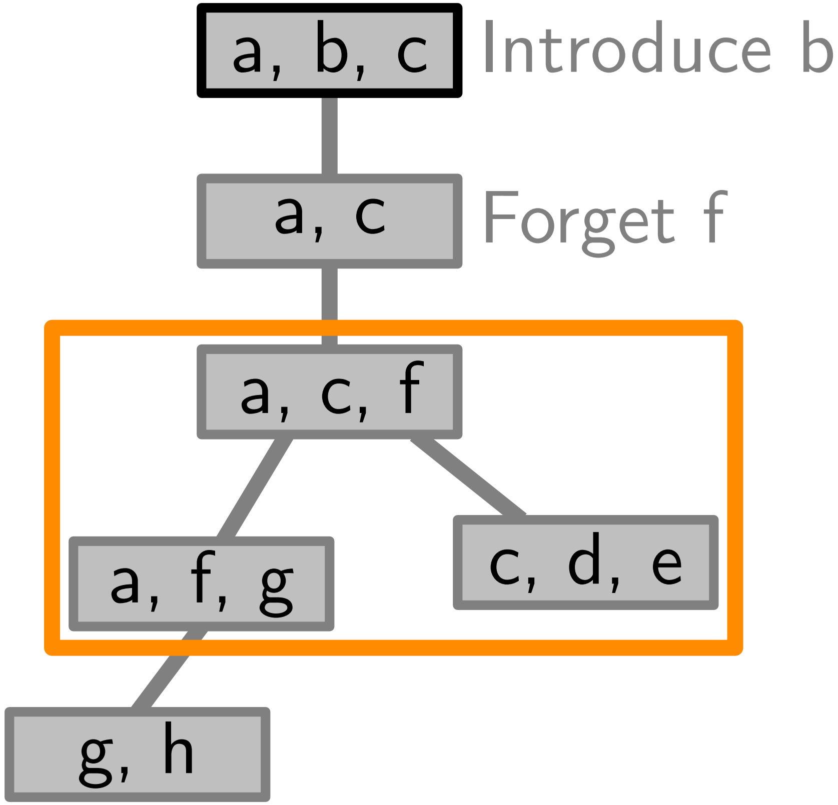

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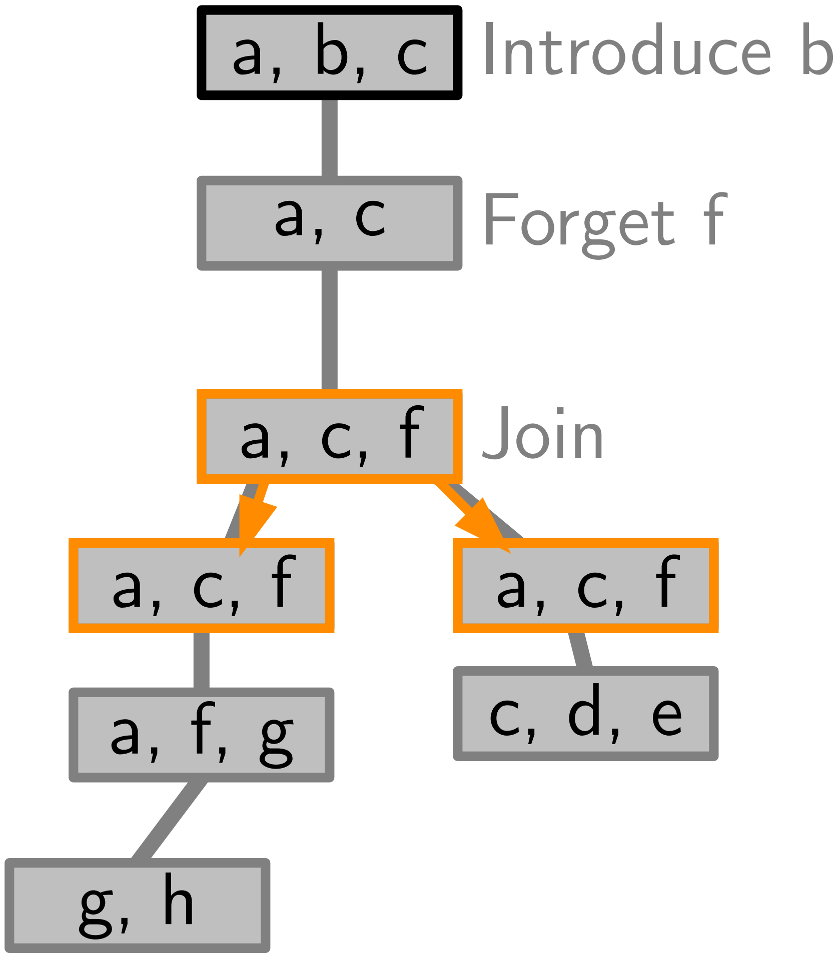

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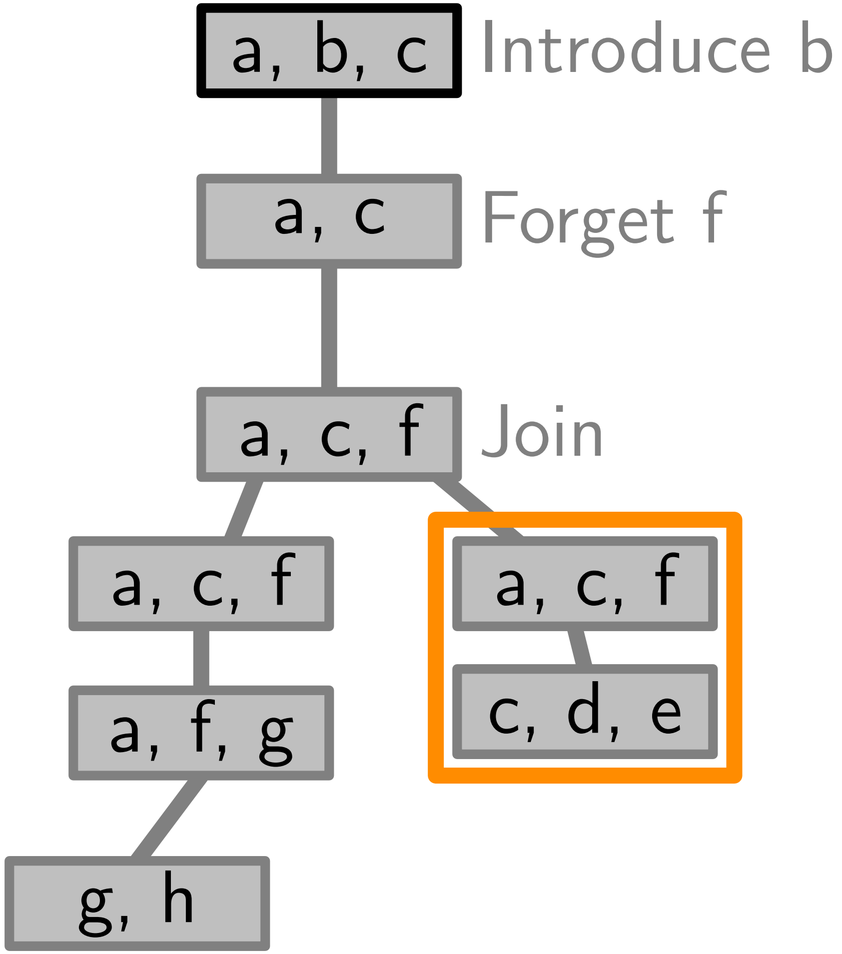

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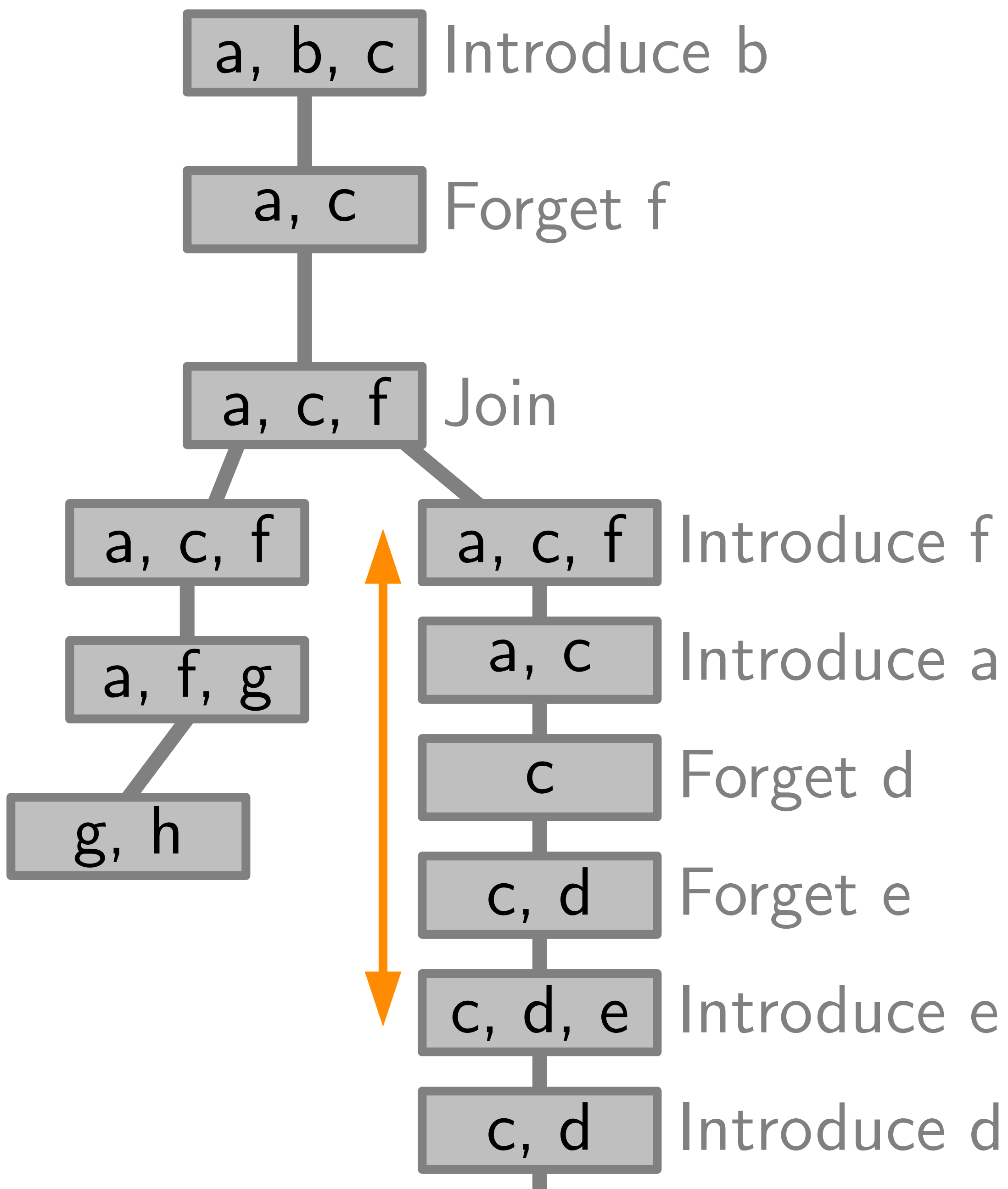

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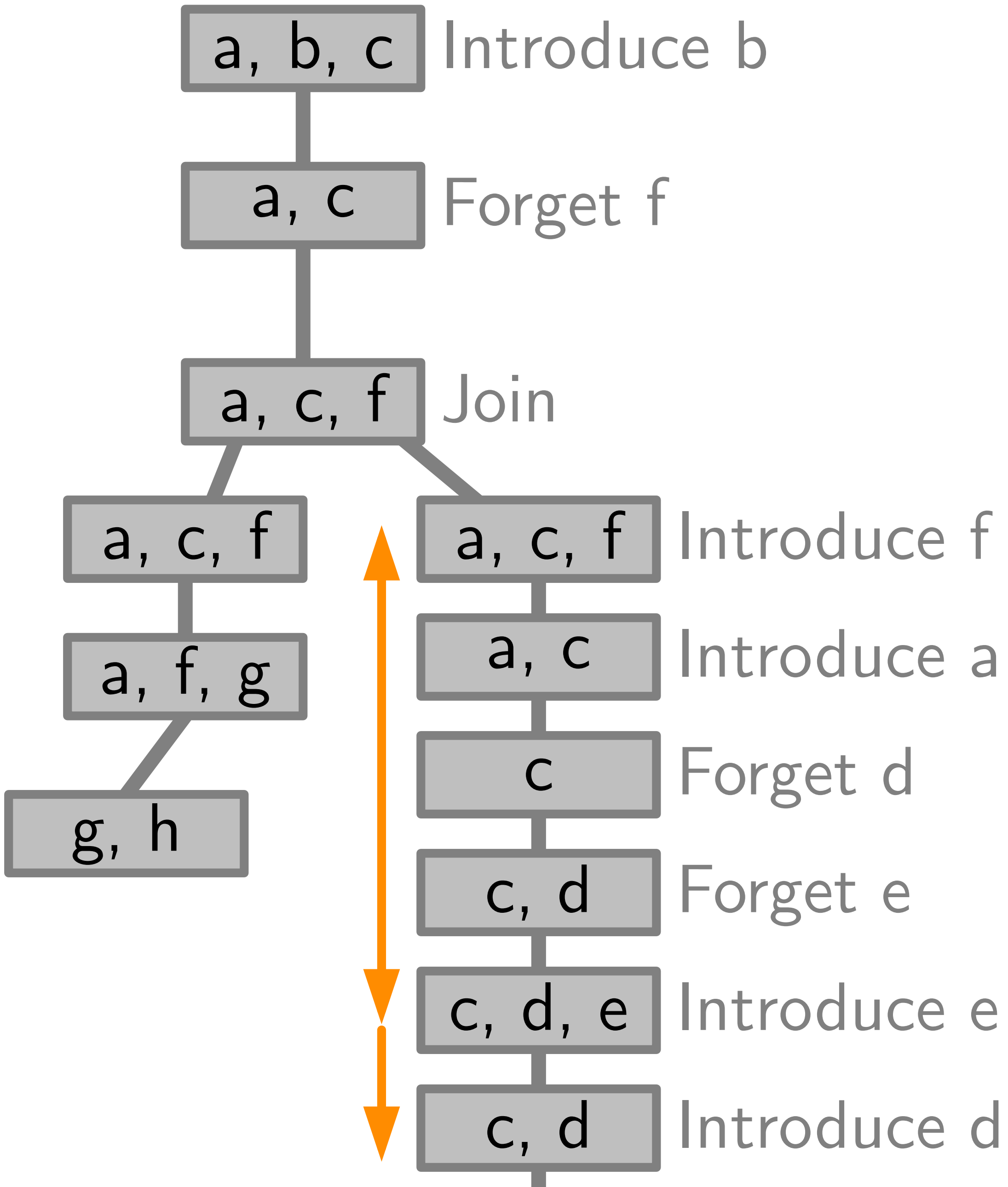

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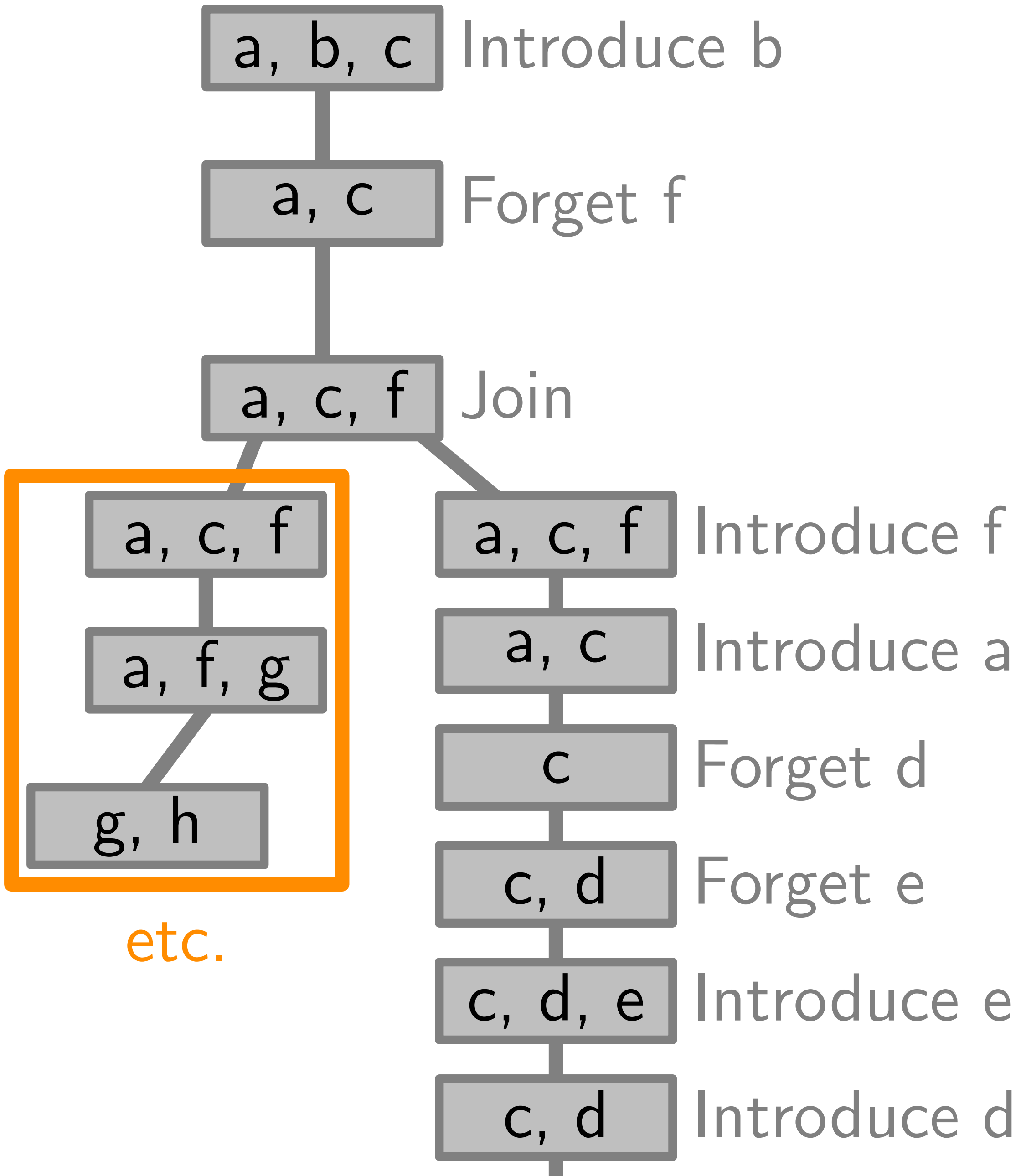

Tool #2: DP (on Nice Tree Decompositions)

Thm. k-Treewidth is FPT

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Corollary. For FPT-Algorithms it suffices to consider nice tree decompositions.

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Corollary. For FPT-Algorithms it suffices to consider nice tree decompositions.

Strategy: Build a recurrence for each type of bag, and use dynamic programming.

Independent Set Using Nice Tree Decomp.

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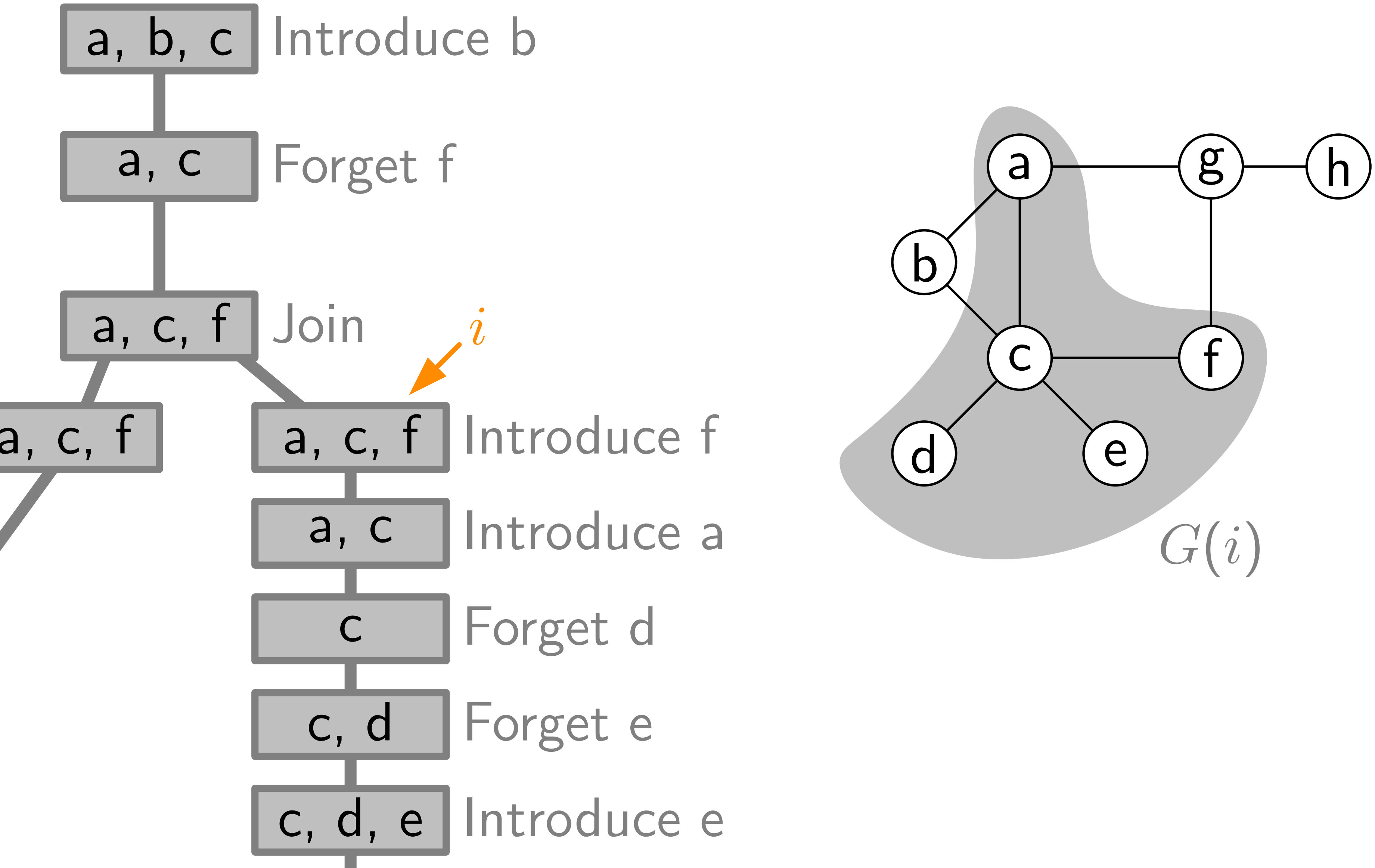

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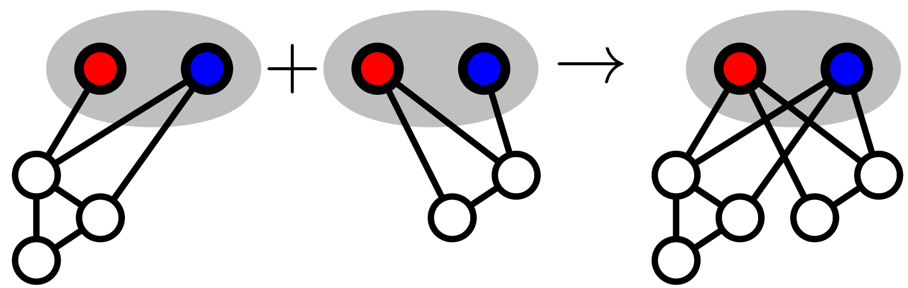

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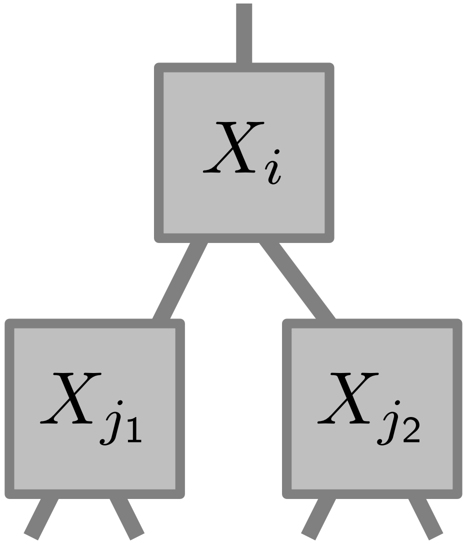

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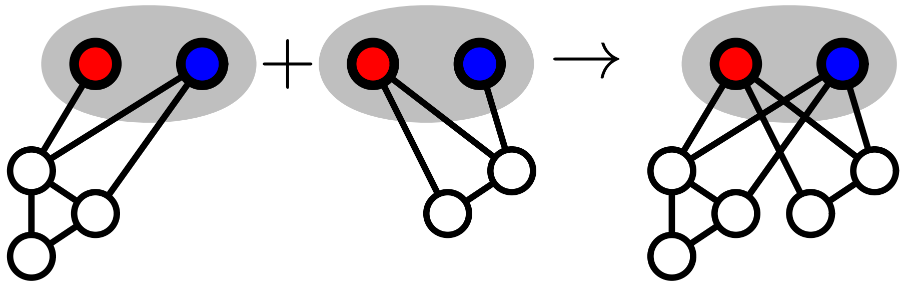

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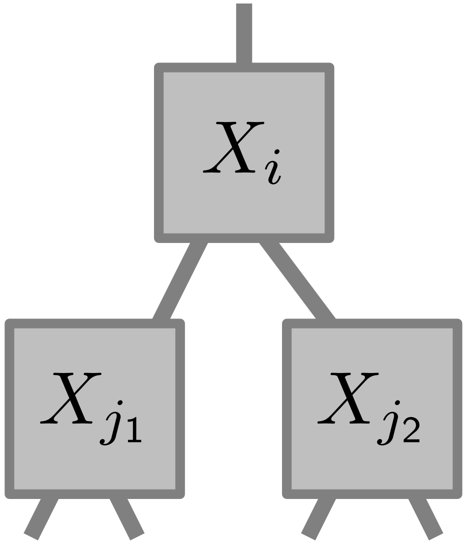

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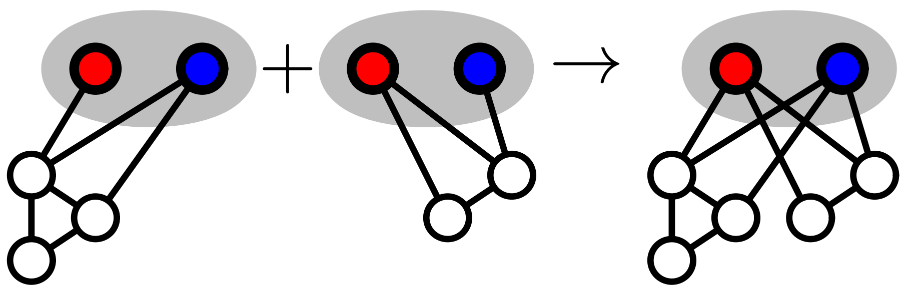

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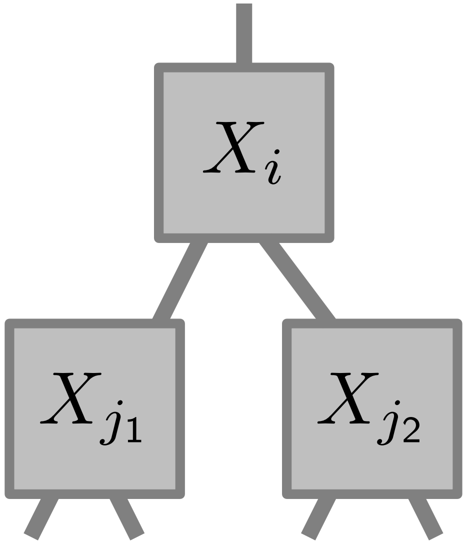

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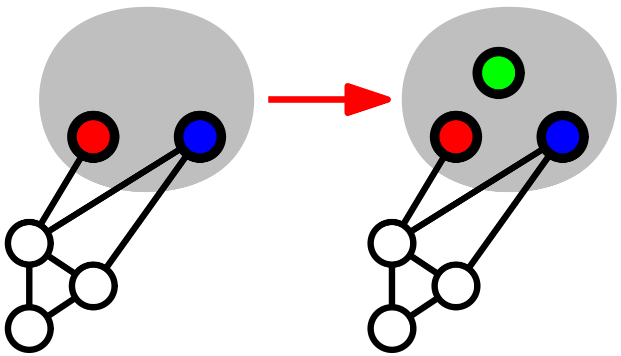

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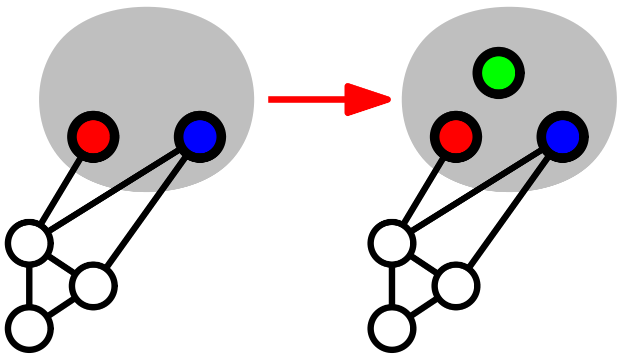

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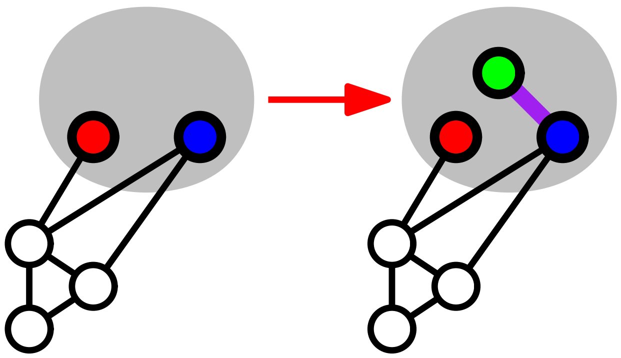

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Otherwise if v has neighbors in S, $R(i, S) = -\infty$

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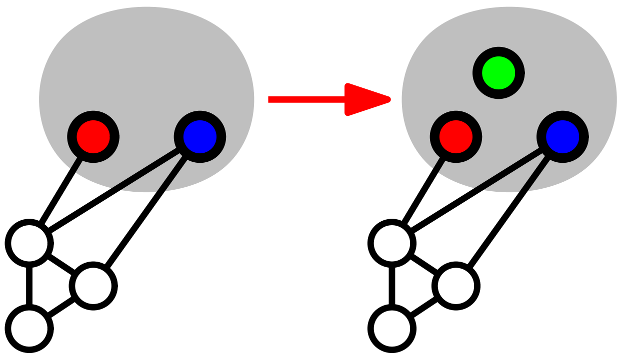

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For each $S \subseteq X_j$: $R(i, S) = R(j, S)$.

Otherwise if v has neighbors in S, $R(i, S) = -\infty$

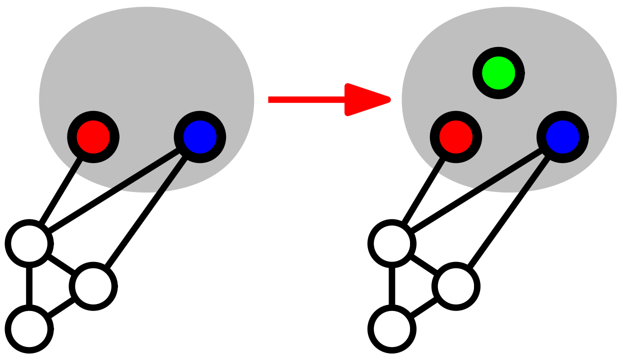
if v has no neighbors in S,

Independent Set Using Nice Tree Decomp.

Let $G(i)$:= graph induced by the vertices in the subtree at i.

For bag i and $S \subseteq X_i$, let:

$R(i, S) :=$ maximum weight of an independent set I in $G(i)$ with $I \cap X_i = S$.



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$$R(i, S) = R(j, S \setminus \{v\}) + \omega(v)$$

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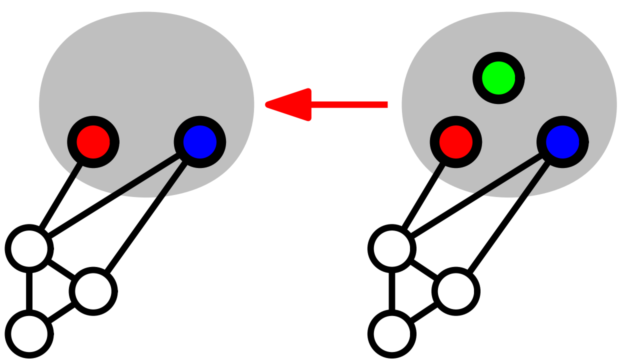

If i is a **Forget** ...

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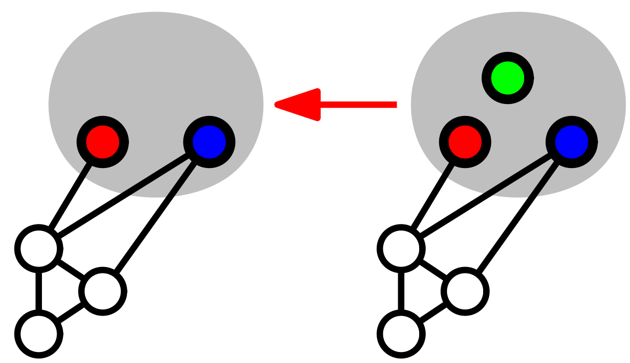

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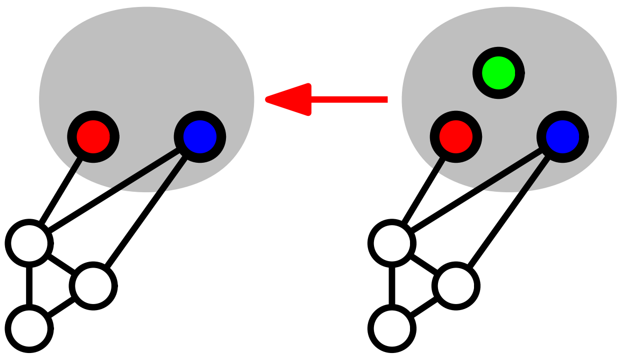

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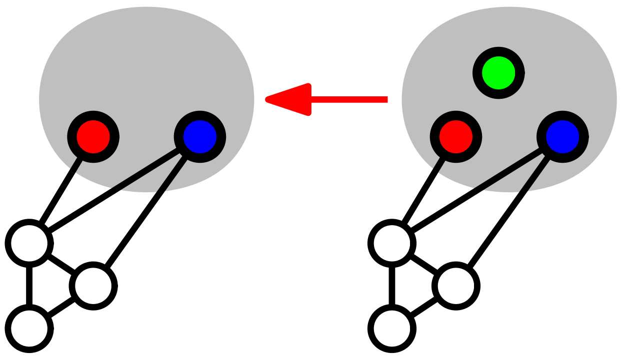

$$R(i, S) = \max\{\ R(j, S), R(j, S \cup \{v\})\ \}$$

Independent Set Using Nice Tree Decomp.

Let $G(i) :=$ graph induced by the vertices in the subtree at i.

For bag i and $S \subseteq X_i$, let:

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Algorithm: Compute $R(i, S)$ for all i and corresponding S.

Independent Set Using Nice Tree Decomp.

Let $G(i) :=$ graph induced by the vertices in the subtree at i.

For bag i and $S \subseteq X_i$, let:

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Algorithm: Compute $R(i, S)$ for all i and corresponding S.

Runtime: **?**