Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 10 Kernelization

Based on: [Parameterized Algorithms: §1, 2.1]

and [Gramm, Guo, Hüffner, Niedermeier; Theory Comput. Syst. 38:373–392, 2005.
doi.org/10.1007/s00224-004-1178-y]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

Lehrstuhl für Informatik I

Complexity Theory

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Natural problems tend to be either:

- in **P**
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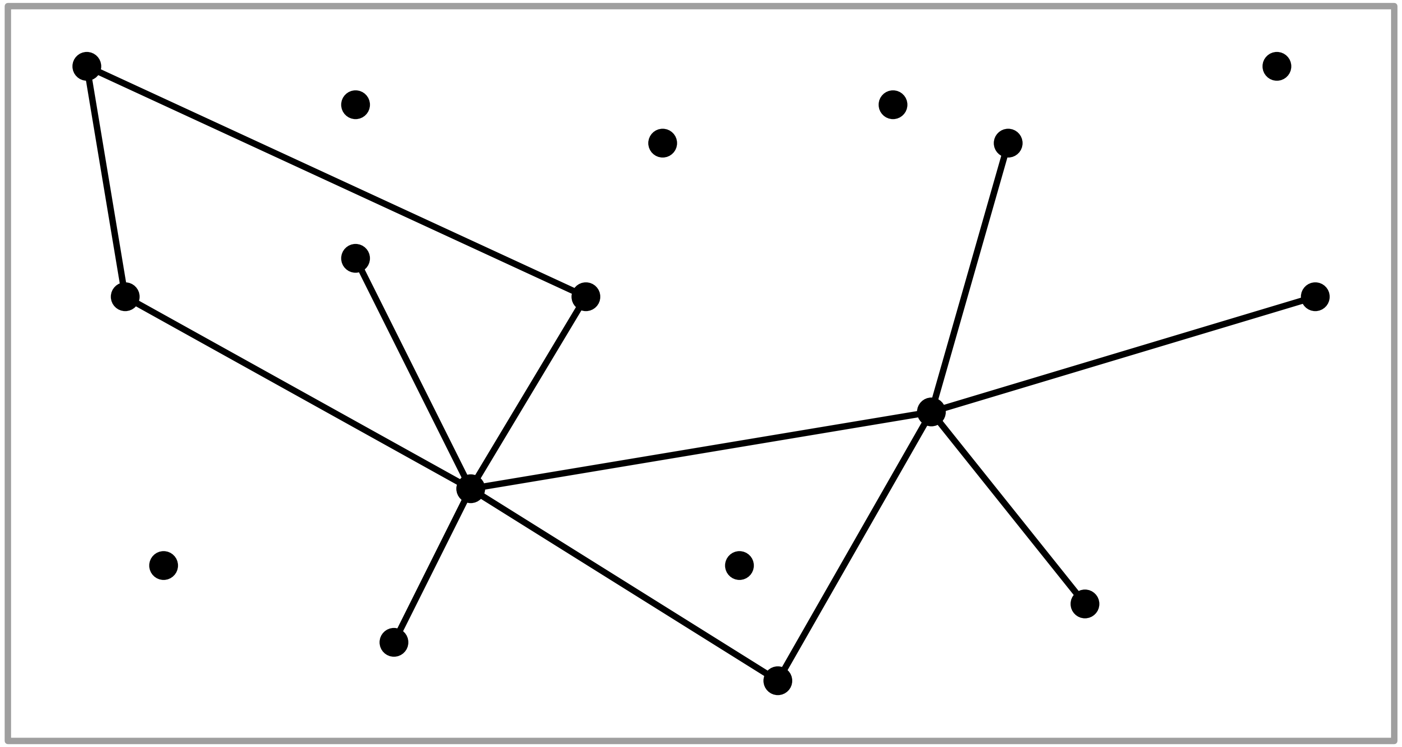
Fixed-Parameter Tractability:

- In many applications, some aspect of a problem assumed small.
- Runtime of algorithm polynomial except for this small aspect.

Vertex Cover

Minimum-size vertex set such that every edge is covered.



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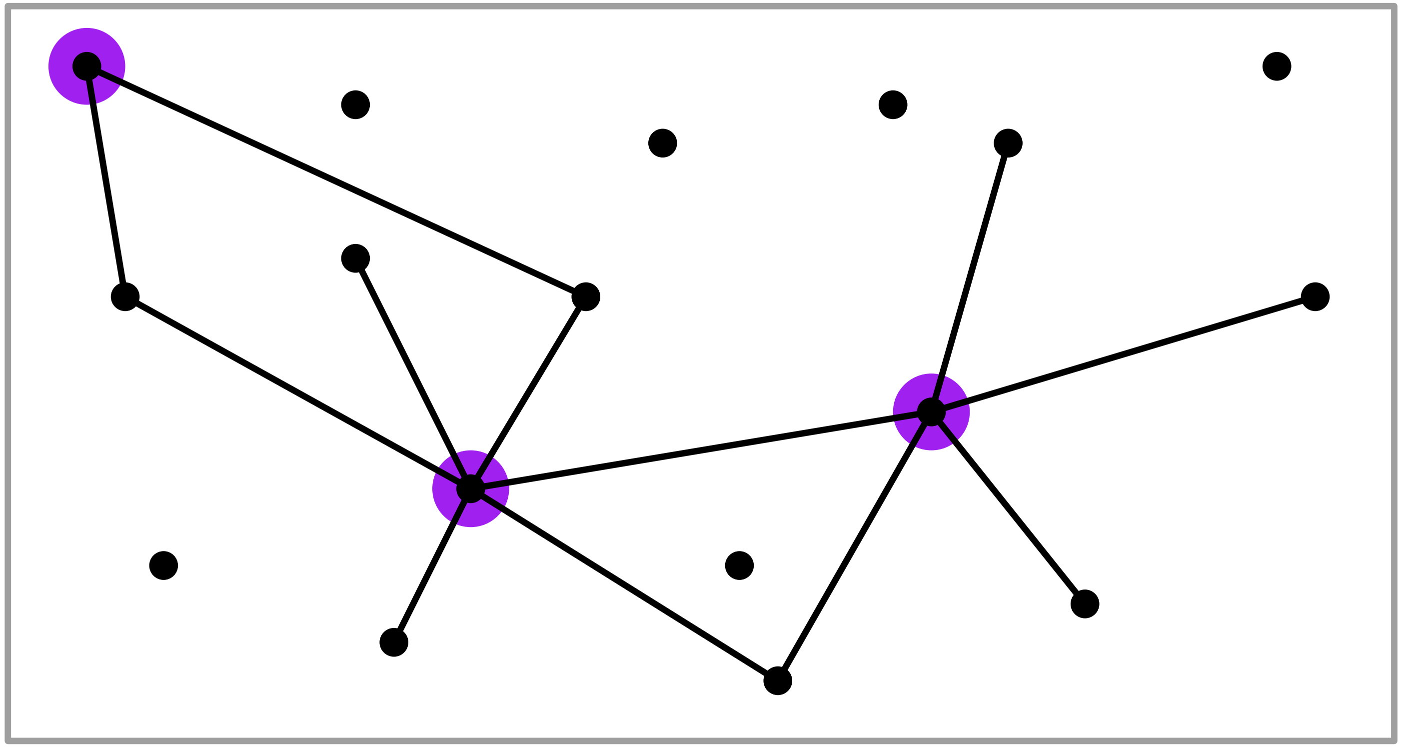

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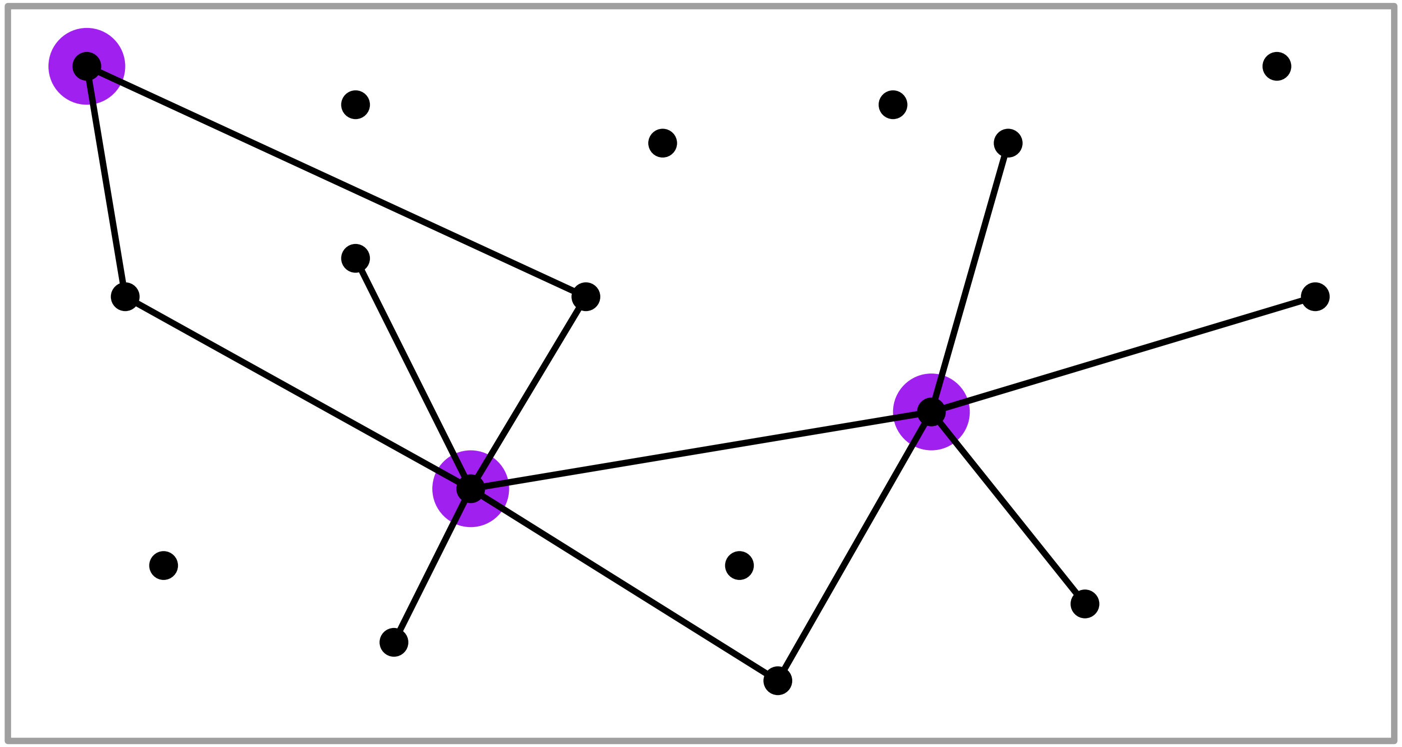

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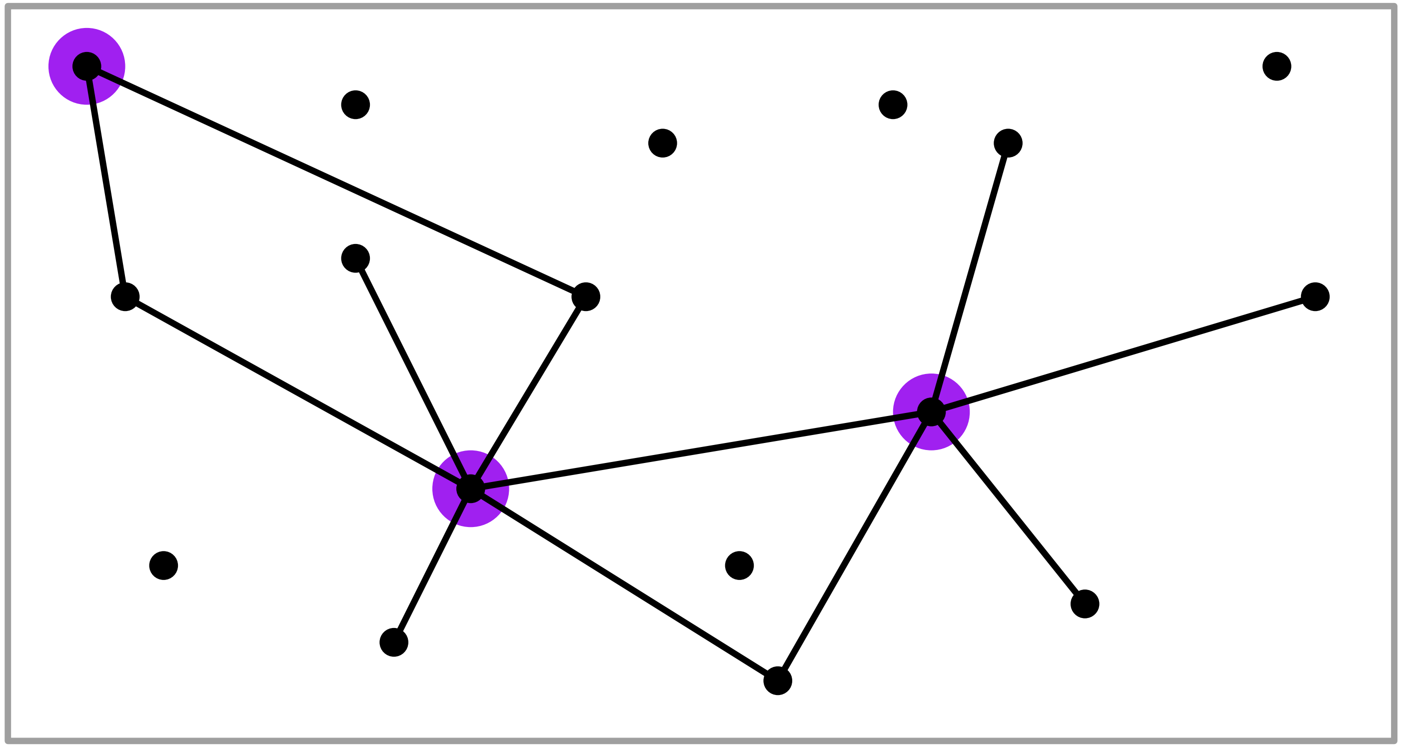

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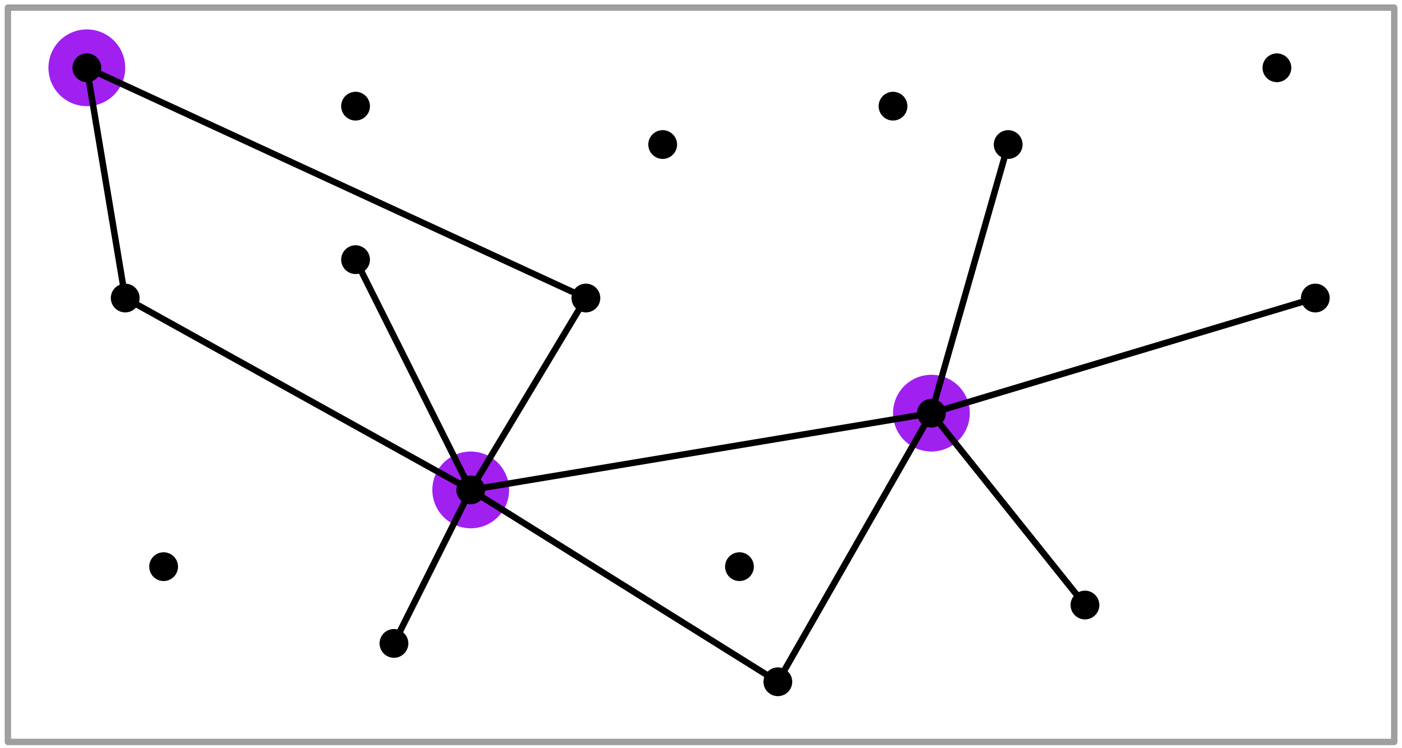

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Last time: FPT :)

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$\Rightarrow$ Not in FPT !!!!

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Requires $\Omega(n^{f(k)})$ time...
assuming FPT $\neq$ W[1]

Fixed-Parameter Complexity Theory

Fixed-Parameter Tractable

Solvable in $O(f(k)n^c)$ time – computable function f, const. c

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Evidence against Fixed-Parameter Tractability
Complexity classes FPT $\subseteq$ W[1] $\subseteq$ W[2] $\ldots$
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see Parameterized Algorithms §14.4

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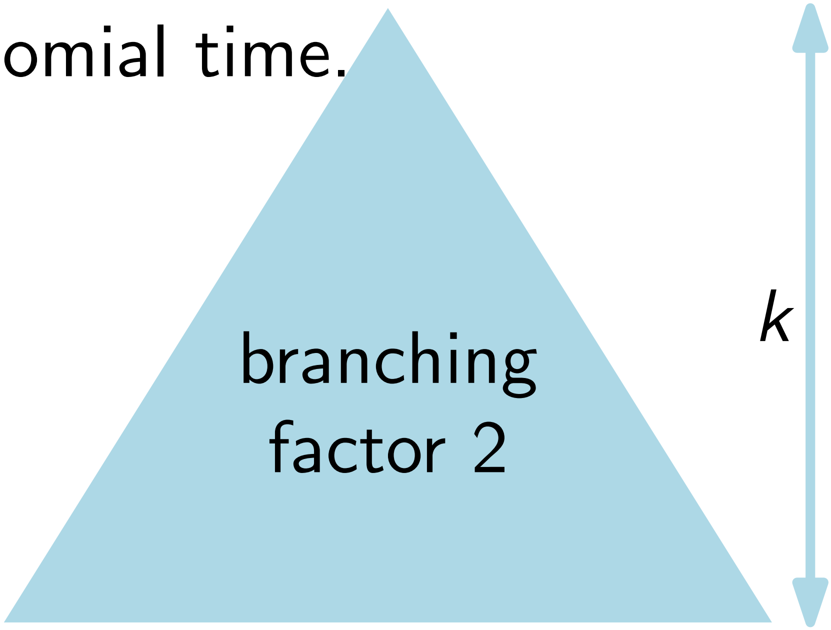

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Kernelisation

Preprocessing with quality guarantees

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$C = \{v \in V \mid \deg(v) > k\}$; **if** $|C| > k$ **then return** ("NO", $\emptyset$)

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So, we have an $f(k)$-size kernel. $\square$

Typical Form of Kernelisation

Repeat some **rules**, until no **rule** is possible

- Rules can do some necessary modification and decrease k.
- Rules can remove some part of the graph.
- Rules can output YES or NO.
- Sometimes add ‘annotations’ to the graph

Cluster Editing

Given: graph $G = (V, E)$
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Let's prove this!

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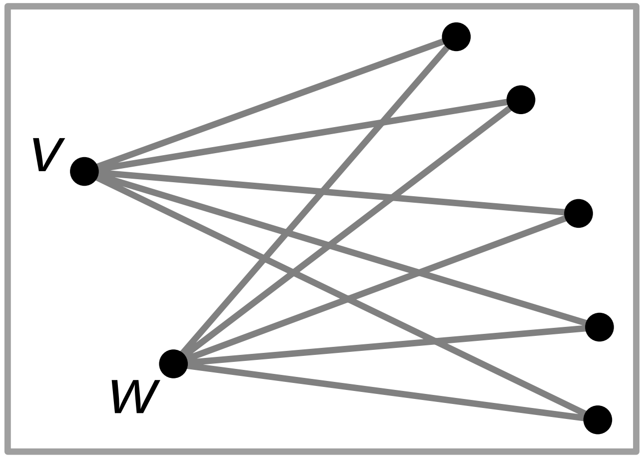
Annotate the graph:
some pairs of vertices are permanent and others are forbidden.

Rule 3: Common Neighbors

Obs. If two vertices have $k+1$ neighbors in common, they must belong to the same clique in any solution



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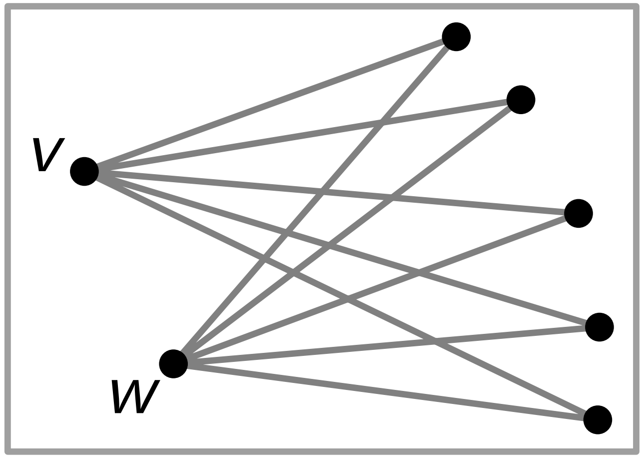

Rule 3: Let v, w be vertices with $k+1$ common neighbors. If vw is not present, add it and decrease k by 1. Set the edge vw to be **permanent**.

$$G' \leftarrow G \cup \{vw\} \qquad P \leftarrow P \cup \{vw\}$$

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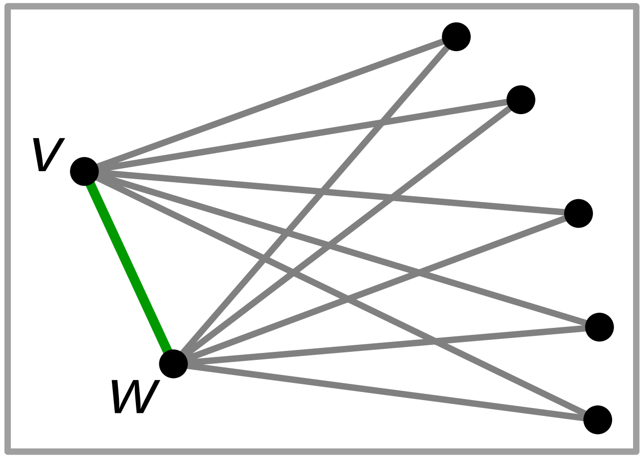

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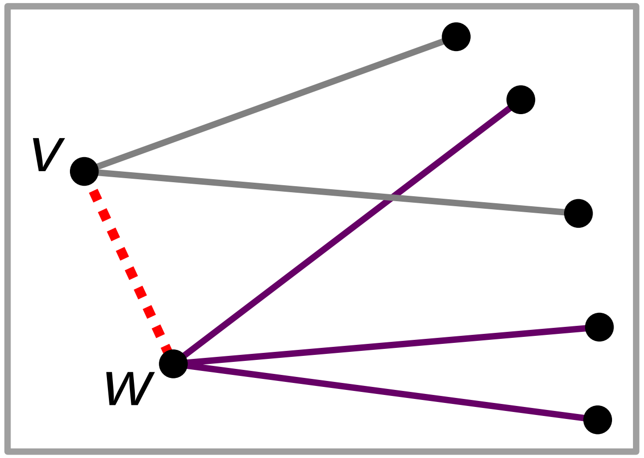

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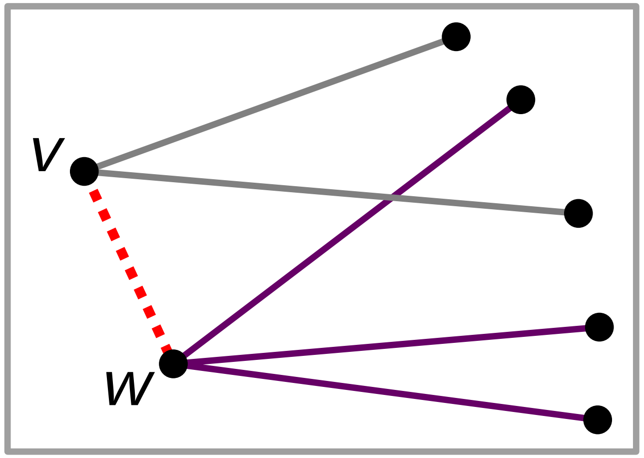

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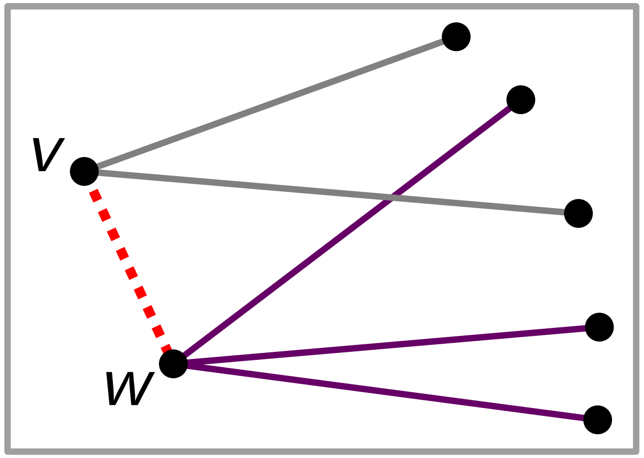

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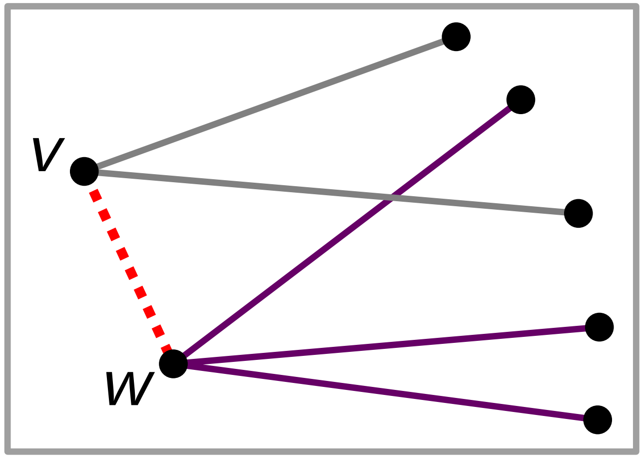

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Rule 5: If some edge is both permanent and forbidden, then there is no solution. Answer **NO**.

Rules 6 and 7: Transitivity (Triangles)

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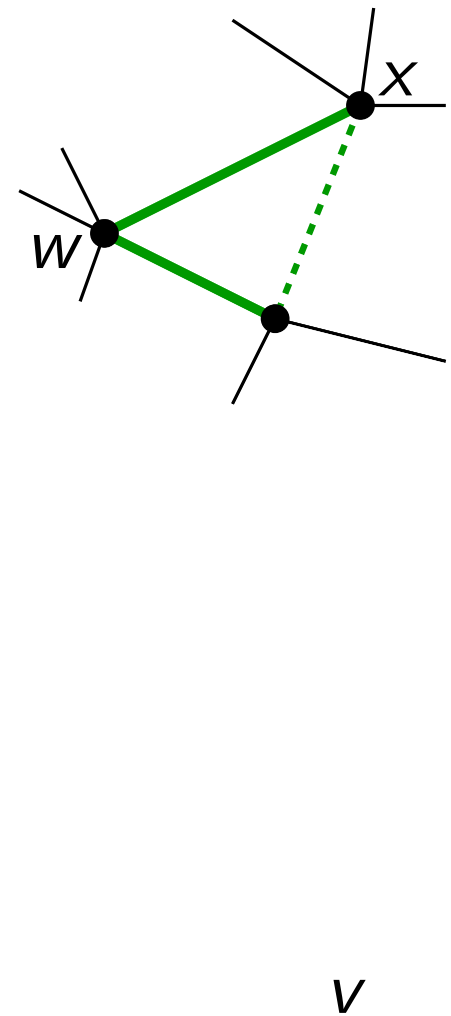

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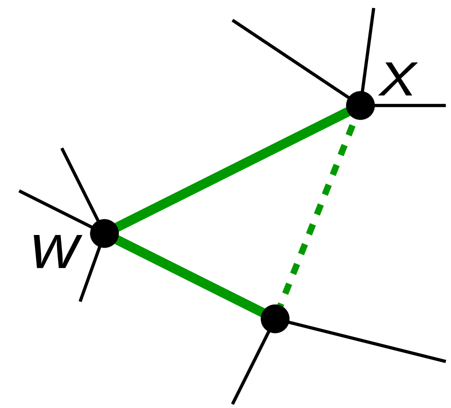

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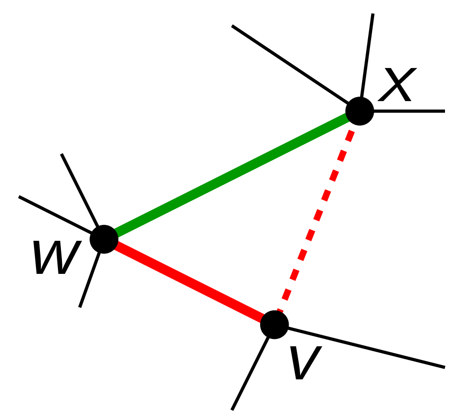

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So, "small" components are fine –
but what if we have a "big" component?

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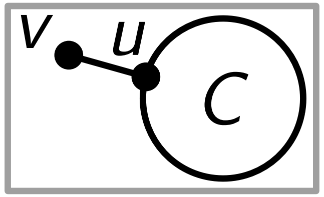

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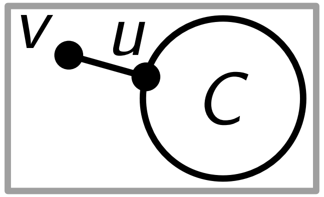

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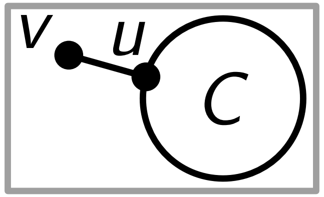

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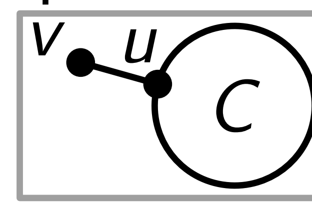

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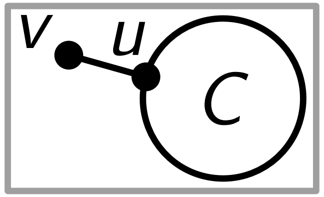

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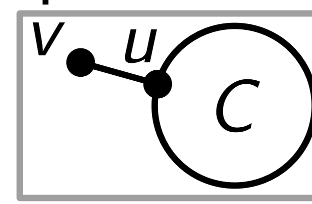

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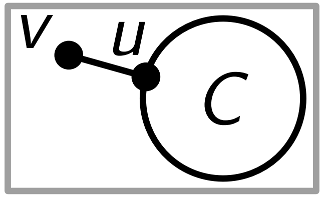

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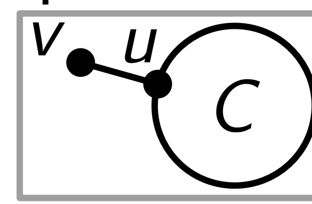

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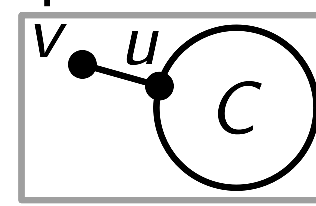

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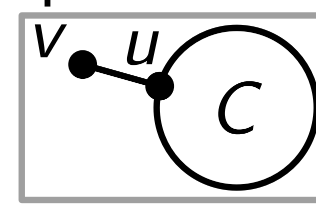

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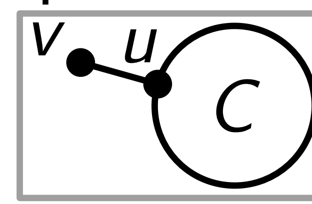

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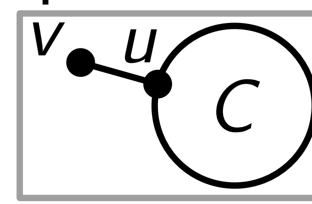

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Exercise: Find a contradiction for the case $k = 1$!

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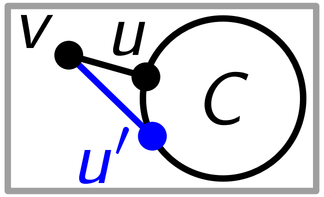

Case 1B: v is adjacent to some $u' \in C \setminus \{u\}$. Then

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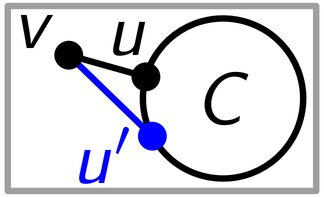

Case 1B: v is adjacent to some $u' \in C \setminus \{u\}$. Then

$$|C| \geq |V|/d >$$

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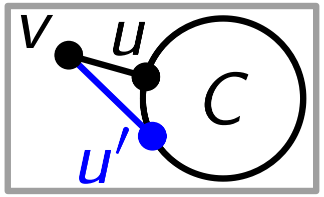

Case 1B: v is adjacent to some $u' \in C \setminus \{u\}$. Then

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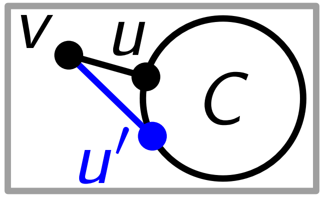

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v has at most k neighbors in C (otherwise)

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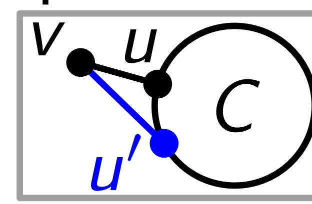

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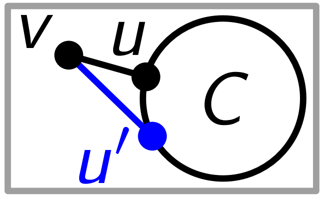

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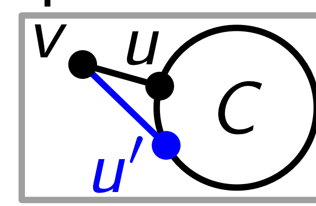

Case 1B: v is adjacent to some $u' \in C \setminus \{u\}$. Then

$$|C| \geq |V|/d > 2k+1$$

v has at most k neighbors in C (otherwise G is a NO)

$\Rightarrow$ u and v have $k+1$ uncommon neighbors ↯ Rule 4

Analysis – Case 2

Case 2: $a > 0$

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Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
$|C| \geq |V|/(d+1) \geq |V|/k > 2k + 1$

Case 2A: $a = k$

Case 2B: $a < k$

Analysis – Case 2

Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
$|C| \geq |V|/(d+1) \geq |V|/k > 2k+1$

Case 2A: $a = k \Rightarrow V =$

Case 2B: $a < k$

Analysis – Case 2

Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
$|C| \geq |V|/(d+1) \geq |V|/k > 2k+1$

Case 2A: $a = k \Rightarrow V = C$, and G' consists of one clique.

Case 2B: $a < k$

Analysis – Case 2

Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
$|C| \geq |V|/(d+1) \geq |V|/k > 2k+1$

Case 2A: $a = k \Rightarrow V = C$, and G' consists of one clique.
Since $a \geq 1$, $\exists uv \notin E$.

Case 2B: $a < k$

Analysis – Case 2

Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
$|C| \geq |V|/(d+1) \geq |V|/k > 2k+1$

Case 2A: $a = k \Rightarrow V = C$, and G' consists of one clique.
Since $a \geq 1$, $\exists uv \notin E$. Recall that $|V| > (2k+1) \cdot k$.

Case 2B: $a < k$

Analysis – Case 2

Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
$|C| \geq |V|/(d+1) \geq |V|/k > 2k+1$

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$\Rightarrow u$ and v have $\geq k+1$ common neighbors.

Case 2B: $a < k$

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Case 2: $a > 0 \Rightarrow k > d$. Let $C \subset V$ be the largest clique in G'.
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Case 2A: $a = k \Rightarrow V = C$, and G' consists of one clique.
Since $a \geq 1$, $\exists uv \not\in E$. Recall that $|V| > (2k+1) \cdot k$.
$\Rightarrow$ u and v have $\geq k+1$ common neighbors. ↯ Rule 3

Case 2B: $a < k$

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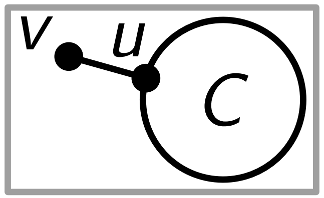
Case 2B: $a < k \Rightarrow \exists uv \in E$: $u \in C, v \in V \setminus C$



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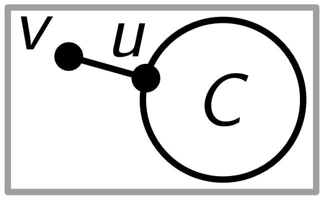
Case 2B: $a < k \Rightarrow \exists uv \in E \colon u \in C, v \in V \setminus C$
Recall that $|C| >$



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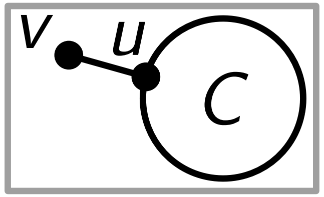

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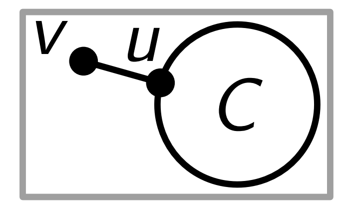
Case 2B: $a < k \Rightarrow \exists uv \in E \colon u \in C, v \in V \setminus C$
Recall that $|C| > 2k+1$.
Note: u has $\geq 2k+1-a \overset{??}{-} 1$ neighbors in C.



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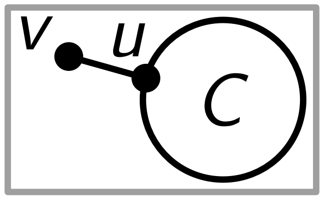

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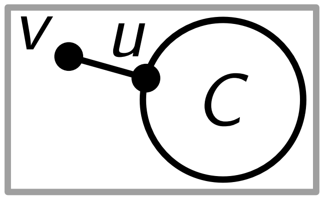

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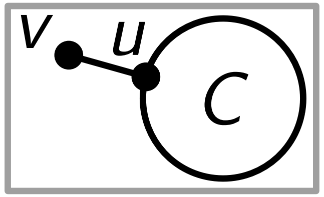

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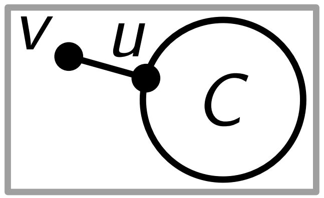

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$\Rightarrow x \geq 2k+1-a-d =$

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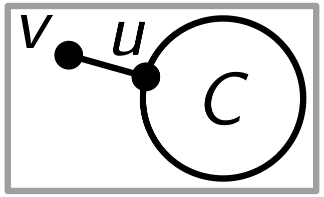

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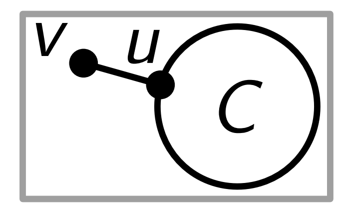

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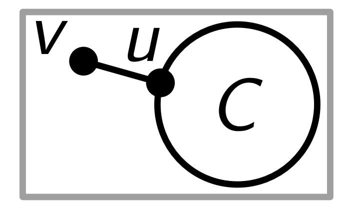

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Exercise:

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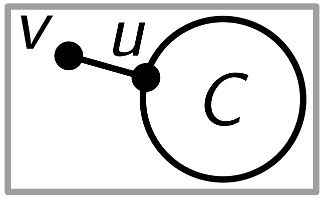

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Exercise: Find a simple branching rule for Cluster Editing!