Julius-Maximilians-
UNIVERSITÄT WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 9.1 Fixed Parameter Tractability

Based on: [Parameterized Algorithms: §1.1, 2.2.1, 3.1]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

Lehrstuhl für Informatik I

Approaches to NP-Hard Problems

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- Parameterized Algorithms

An Example: Vertex Cover

Def. (Recall)

Let $G = (V, E)$ be an undirectred graph.
$C \subseteq V$ is a *vertex cover* of G, when, for every edge $uv \in E$, either $u \in C$ or $v \in C$.

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Prob. *k-Vertex Cover (k-VC)* – decision problem

Given: graph G, number k

Find: size $\leq k$ vertex cover in G when possible (otherwise return "NO")

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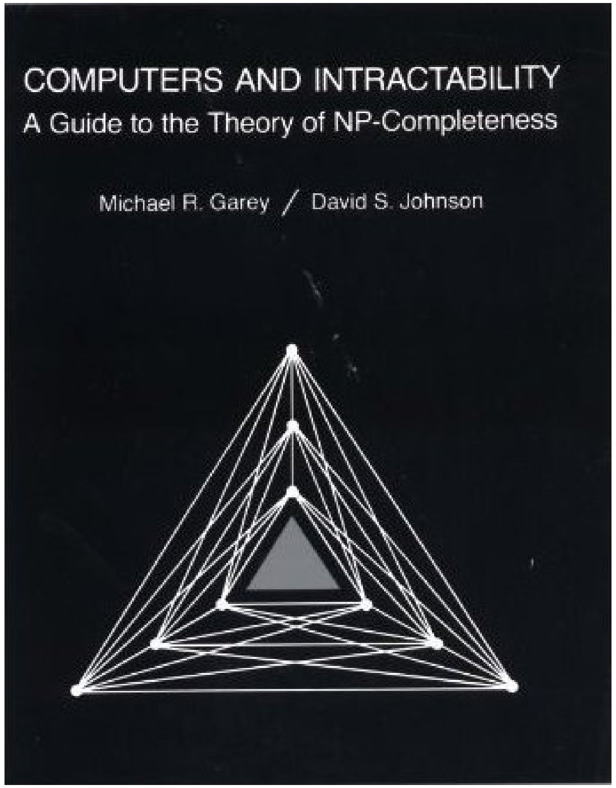

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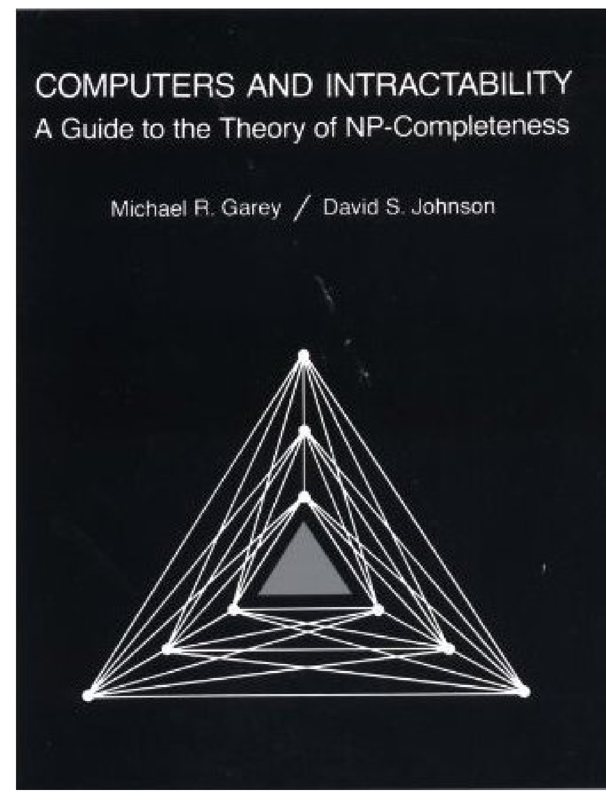

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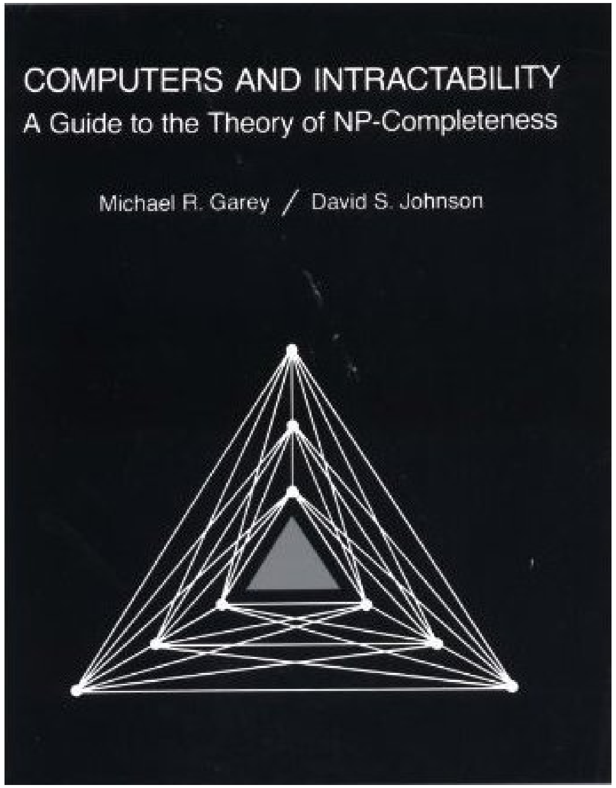

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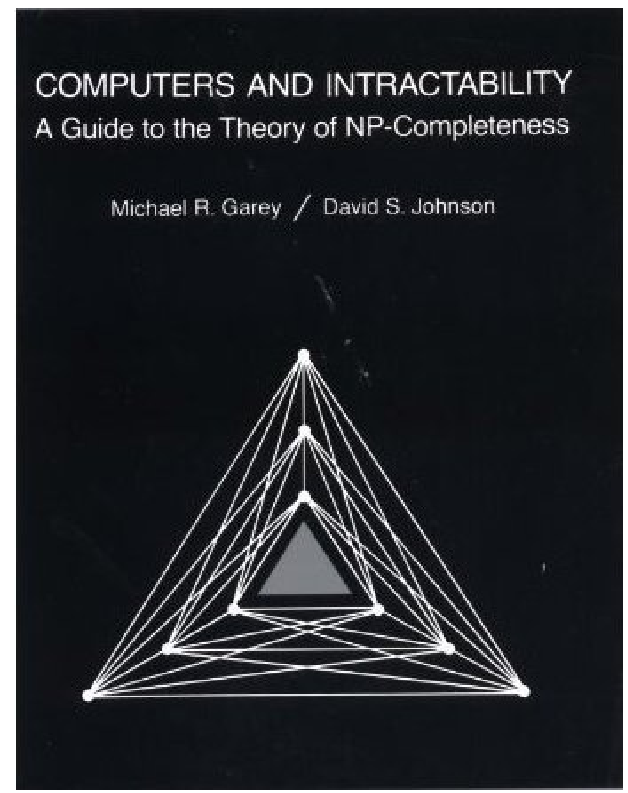

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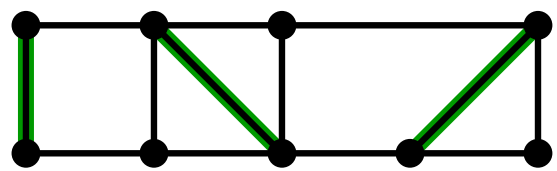

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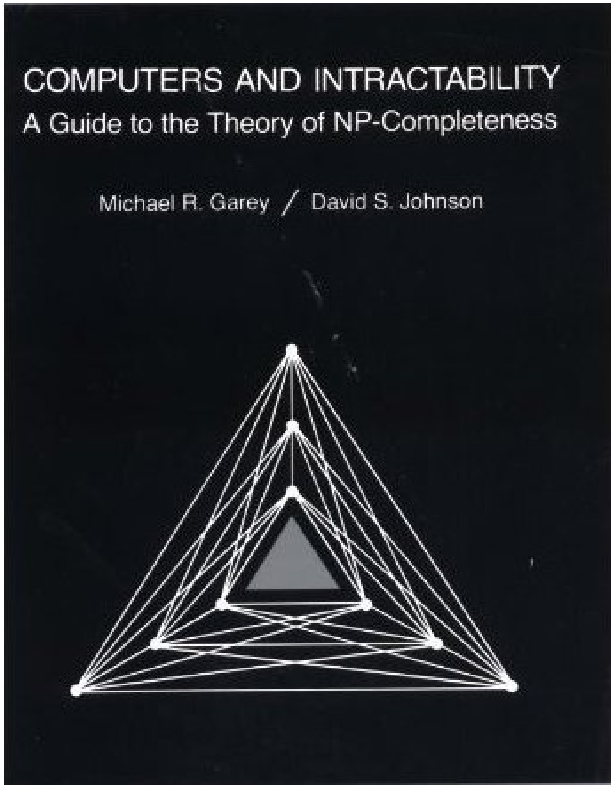

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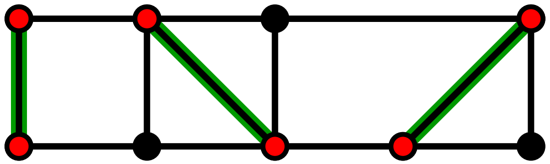

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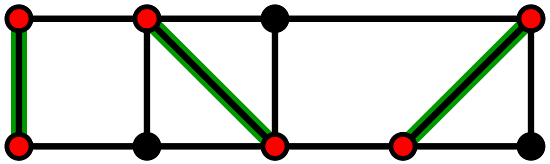

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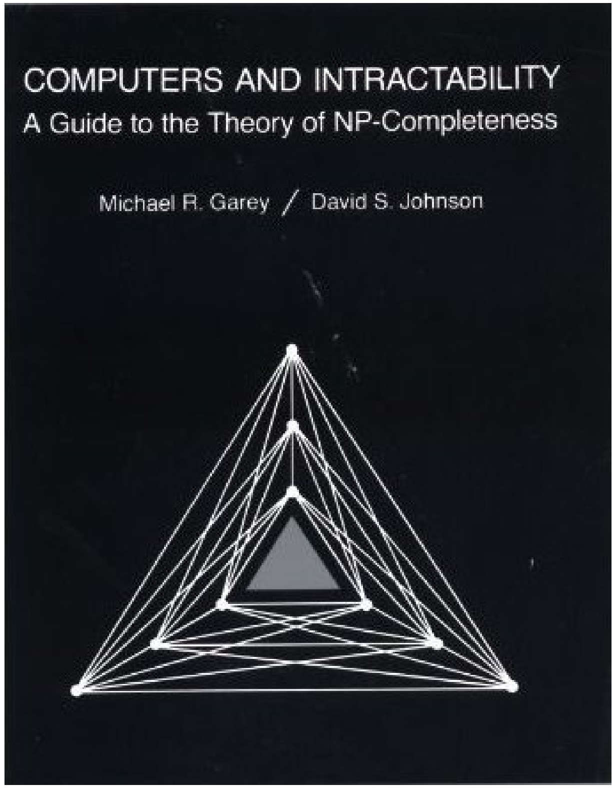

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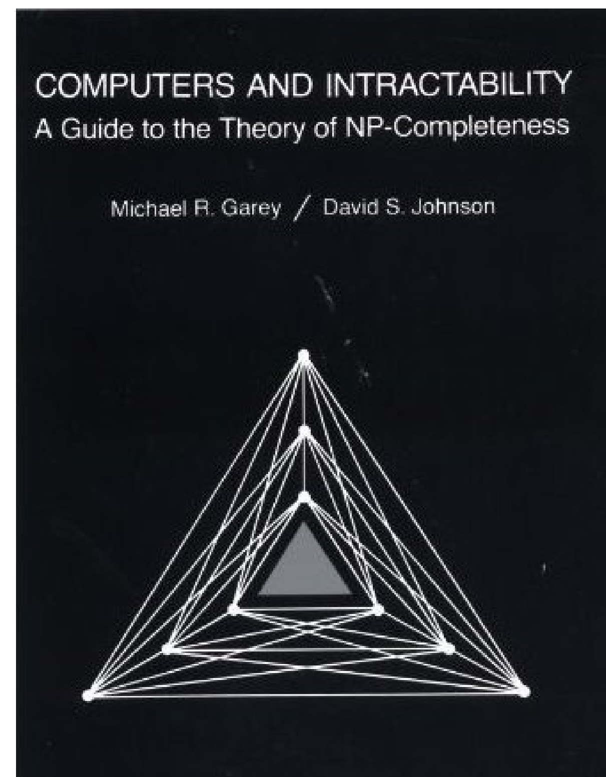

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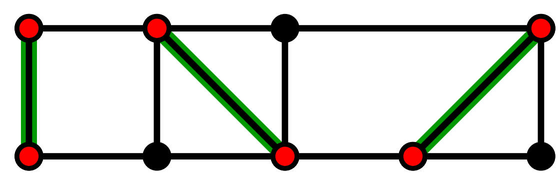

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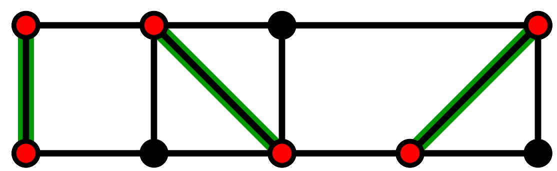

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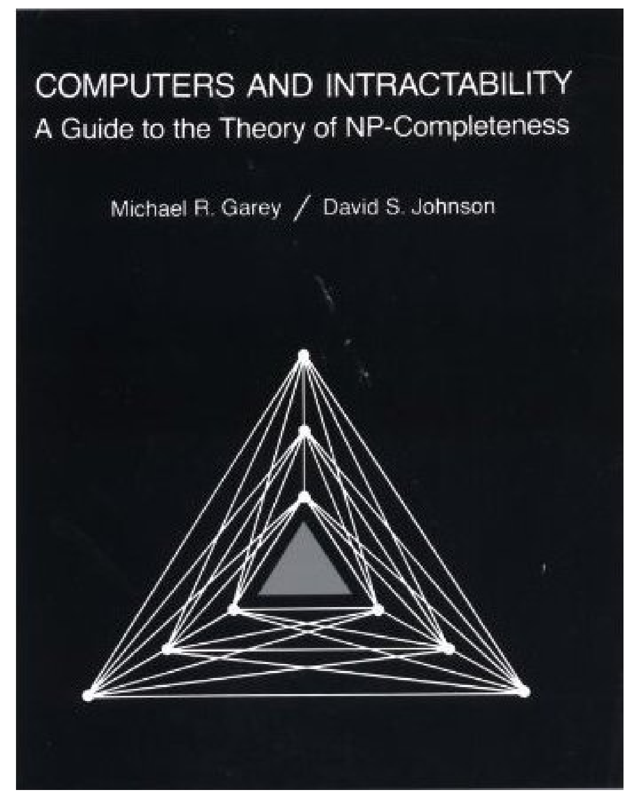

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  There is no factor-1.3606 approx. for vertex cover , assuming $\mathcal{P} \neq \mathcal{NP}$.
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An Exact Algorithm for k-VC

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Runtime. $O(n^k m)$ – This is **not** polynomial in the input size ($|G| = n + m; k$) — k is not constant as it is part of the input.

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Remark. BruteForceVC does not have this runtime.

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What if $|E| > k^2$ and every vertex has degree $\leq k$?

Obs. 3. If $|E| > k^2$ and $\Delta(G) := \max_{v \in V} \deg v \leq k$,
then G has no k-VC.

Algorithm [Buss 1993]

BussVC(Graph G, Integer k)

I) Reduce to the kernel *of the instance*

II) solve the reduced problem exactly

$(\text{vc}, C') = \text{BruteForceVC}(G', k')$

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if $|C| > k$ **then return** (“NO”, $\emptyset$)

$G' = (V', E') := G[V \setminus (C \cup L)]$ ($L =$ isolated vertices)
$k' = k - |C|$
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$O(n + m)$ time

II) solve the reduced problem exactly

(vc, C') = BruteForceVC(G', k')
return (vc, $C \cup C'$)

$O(m' \cdot (n')^{k'})$ time

Runtime.

Algorithm [Buss 1993]

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Runtime. $O\big(n + m + k^2 \cdot (2k^2)^k\big) = O\big(n + m + k^2 2^k k^{2k}\big)$

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Search Tree Algorithm

Idea. Improve phase II using a search tree.

$(G, k, \emptyset)$

Search Tree Algorithm

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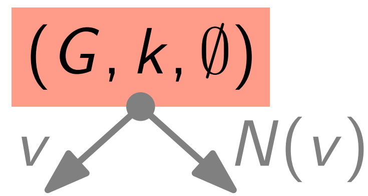

Search Tree Algorithm

Idea. Improve phase II using a search tree.

$$
\begin{tikzcd}
& (G, k, \emptyset) \arrow[dl, "v"'] \arrow[dr, "N(v)"] & \\
(G - v, k - 1, \{v\}) & & (G - N(v), k - \deg v, N(v))
\end{tikzcd}
$$

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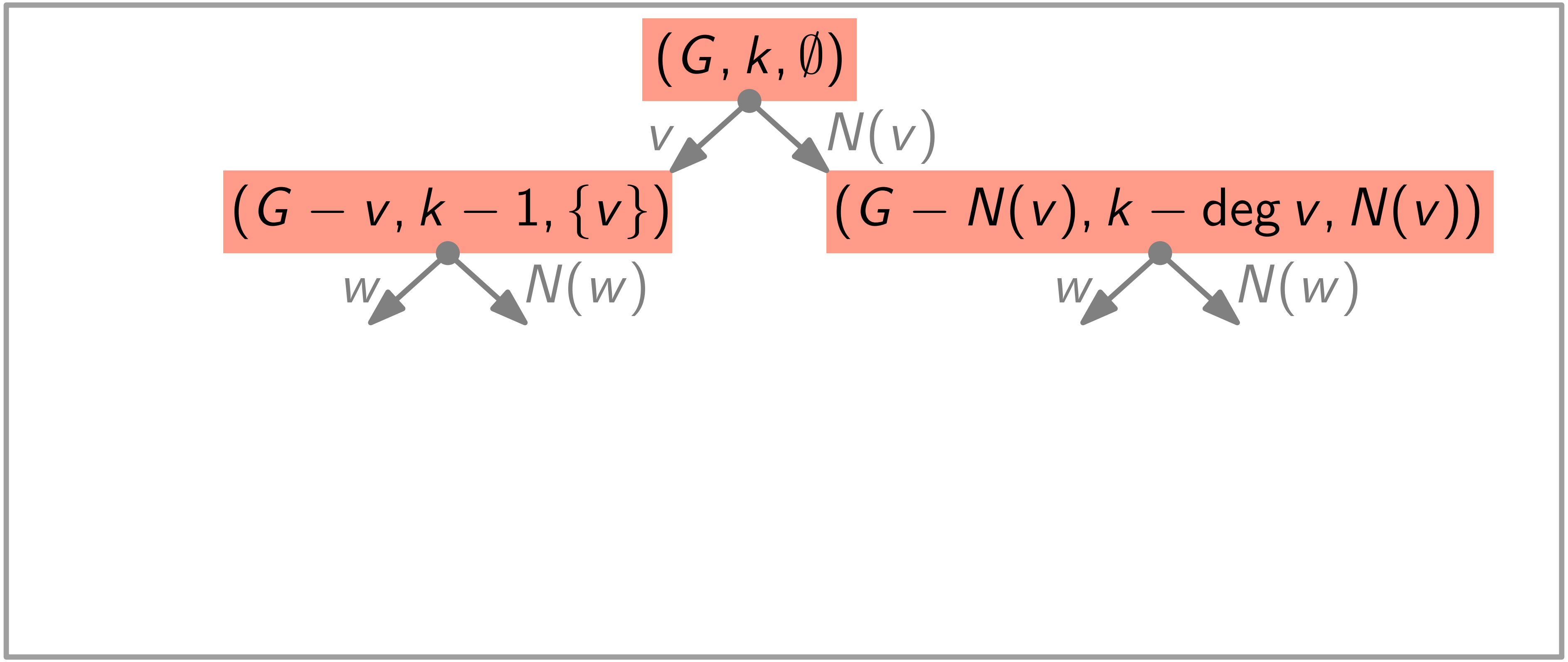

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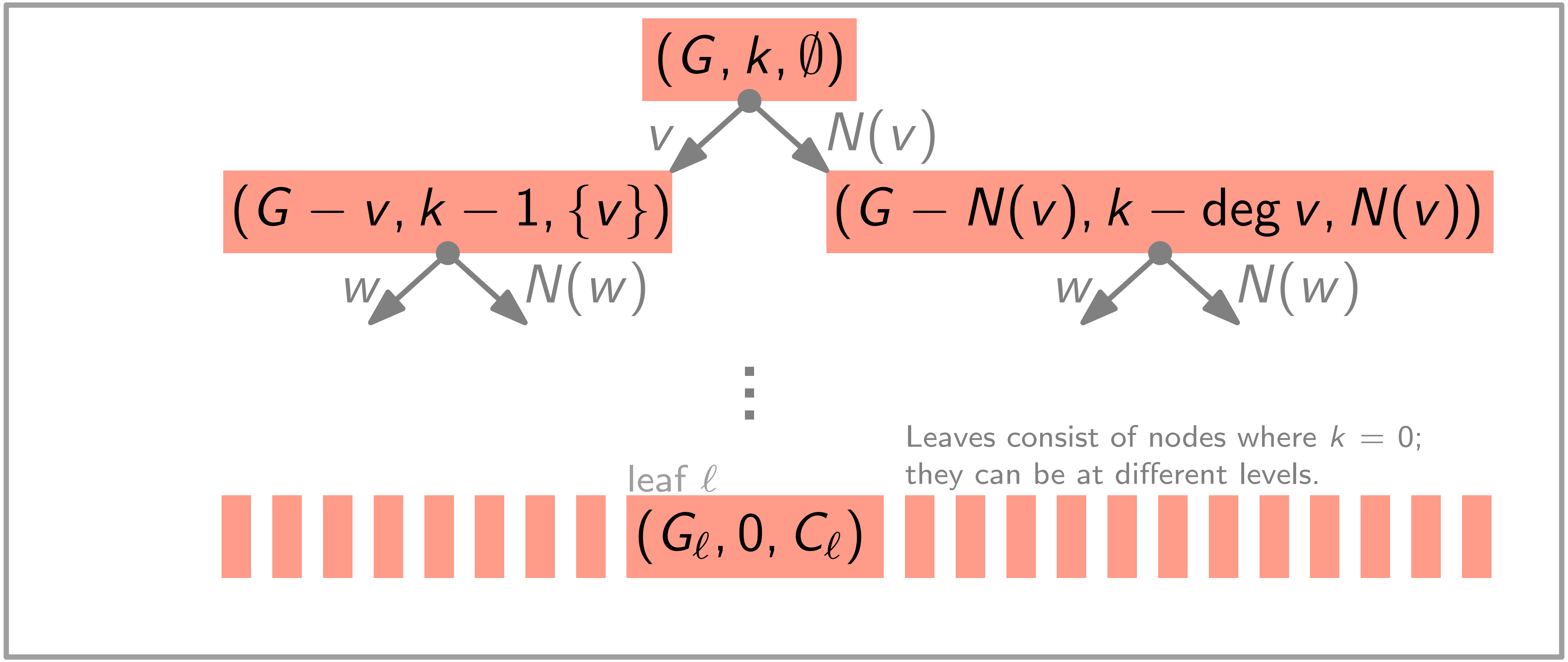

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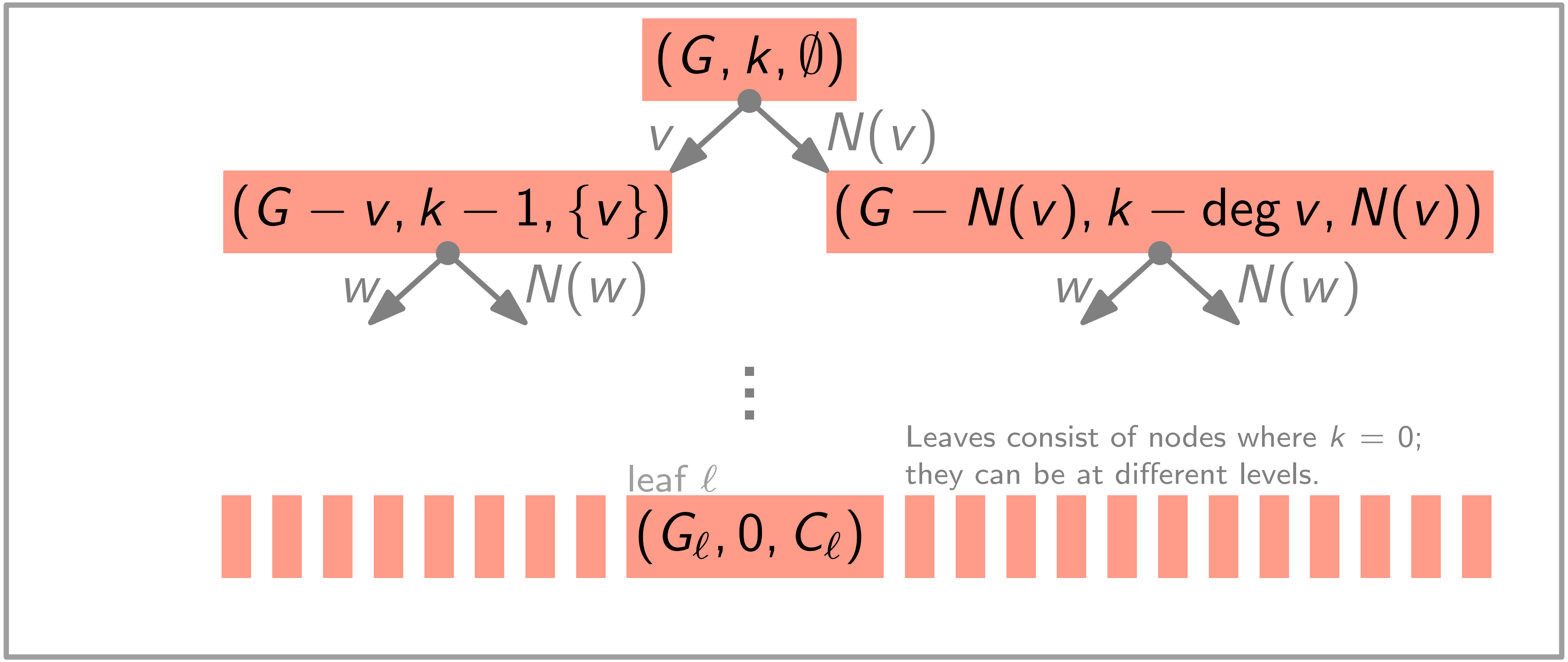

YES:

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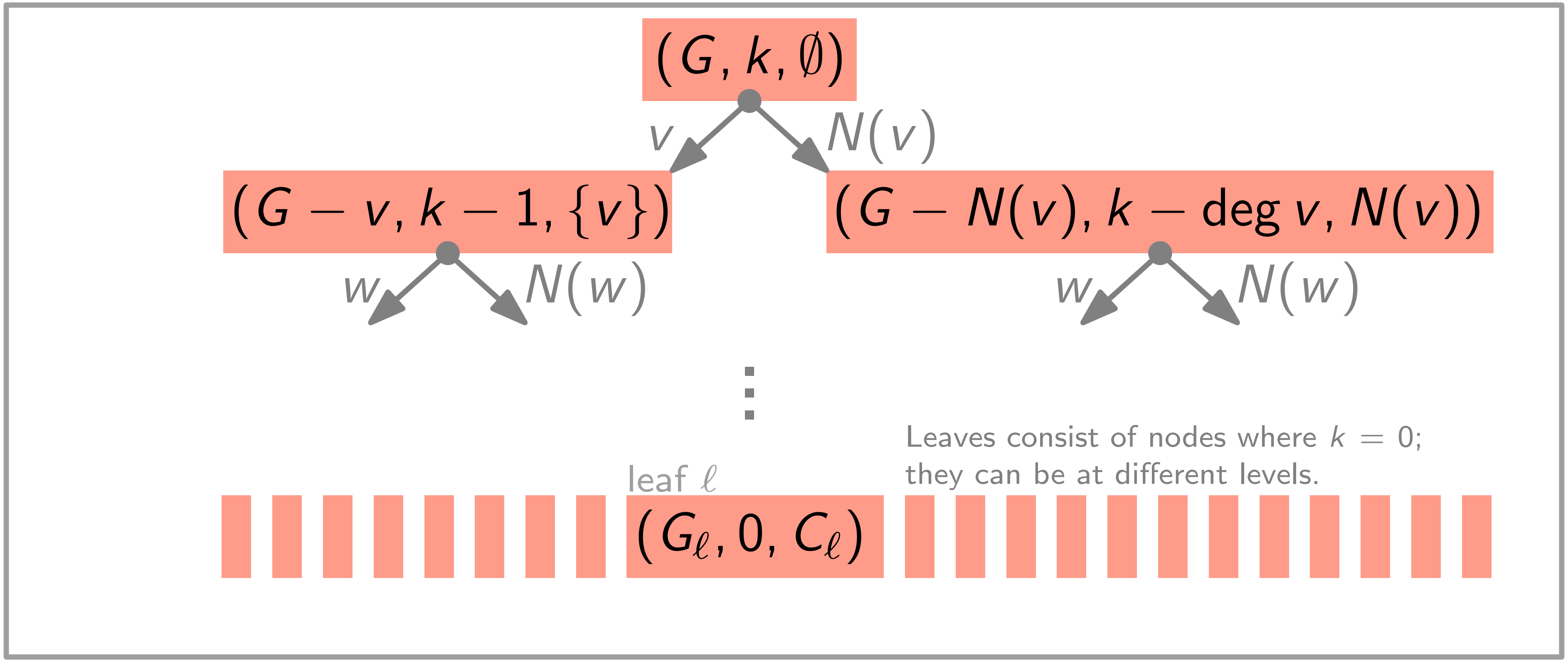

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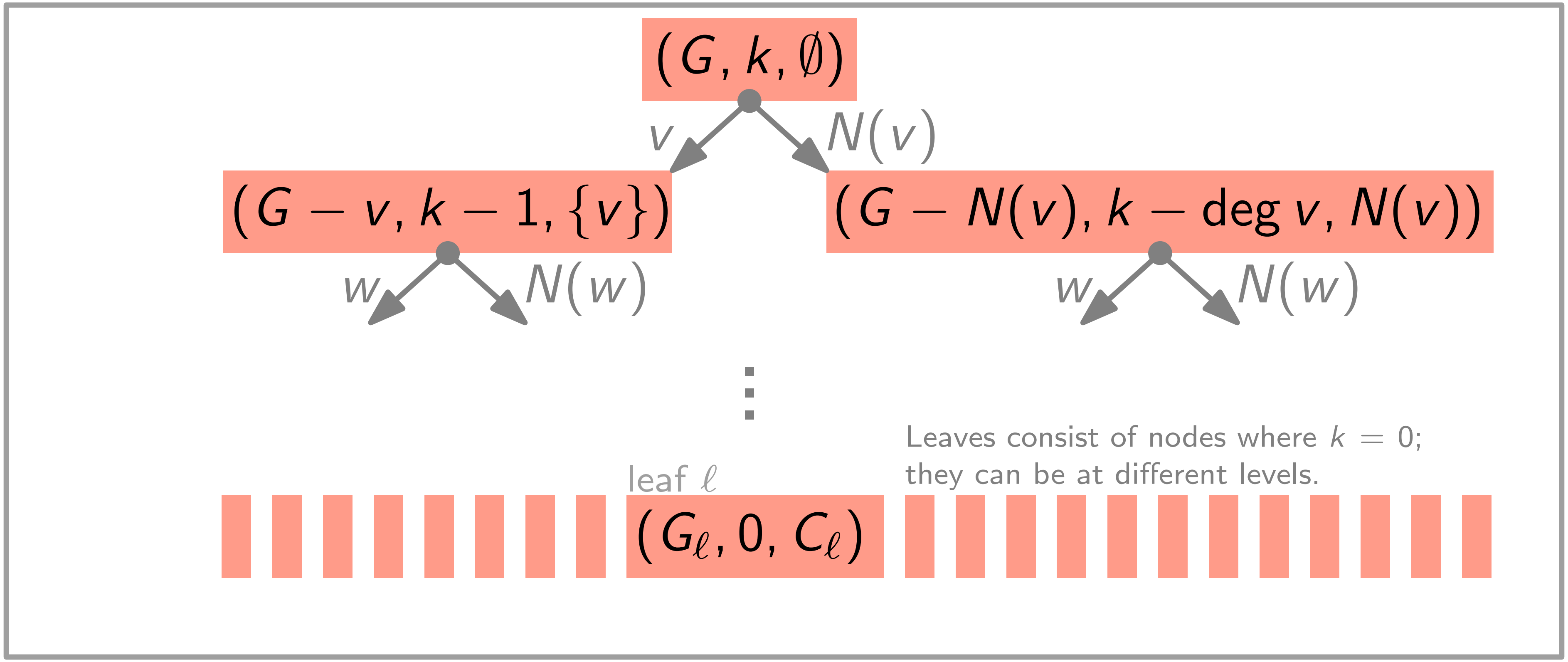

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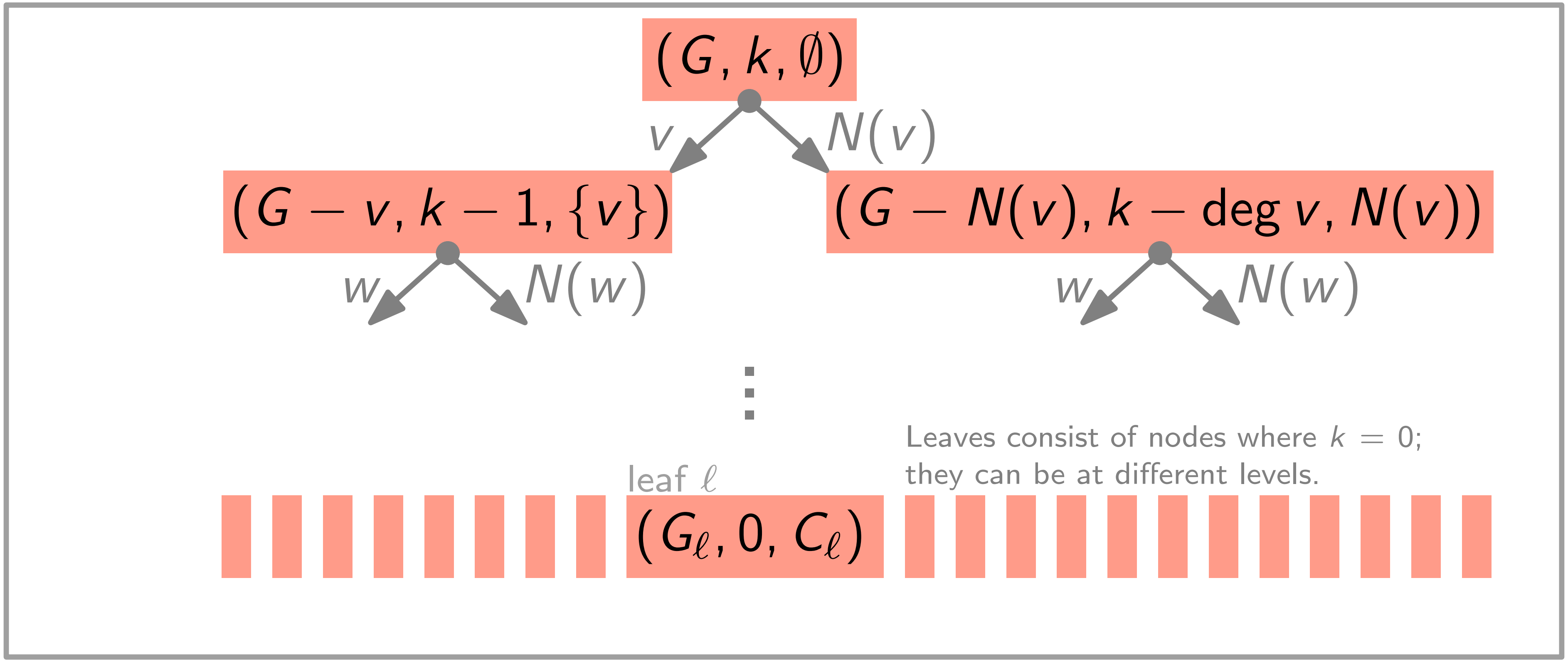

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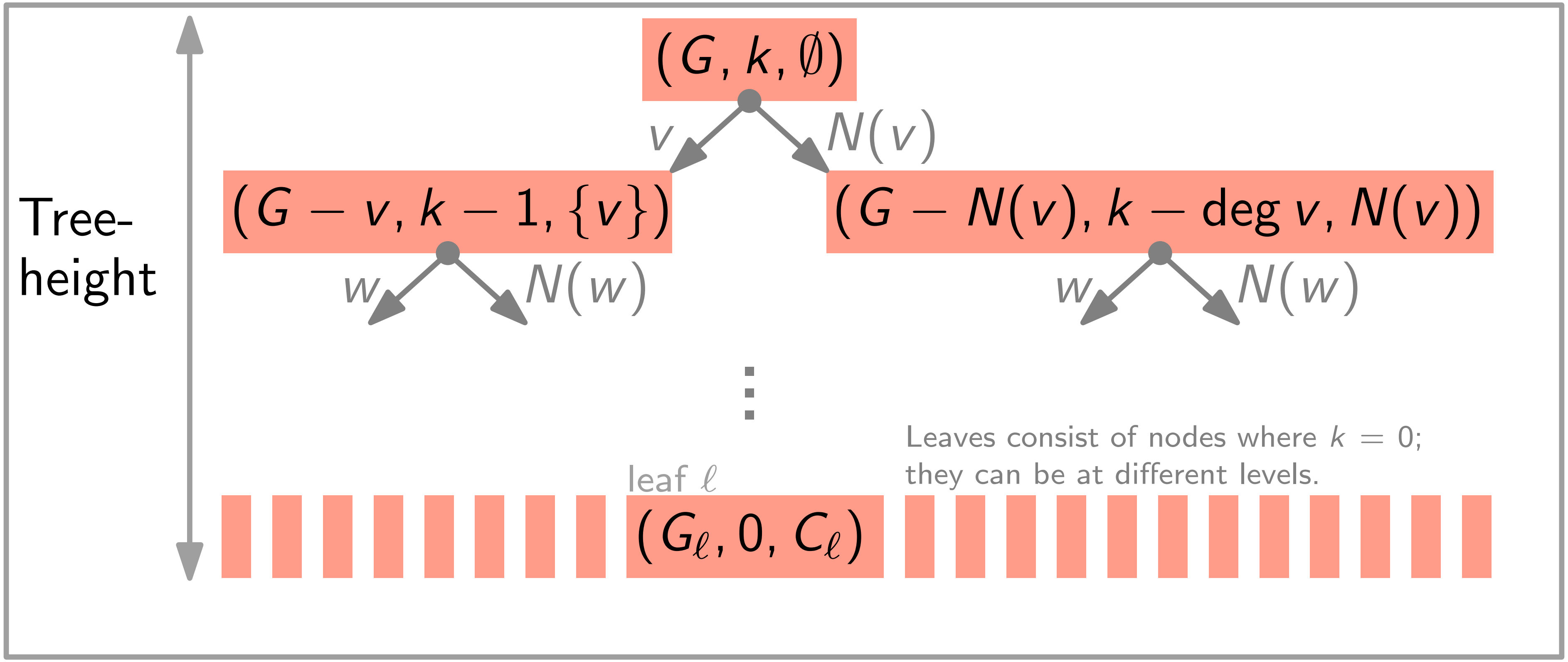

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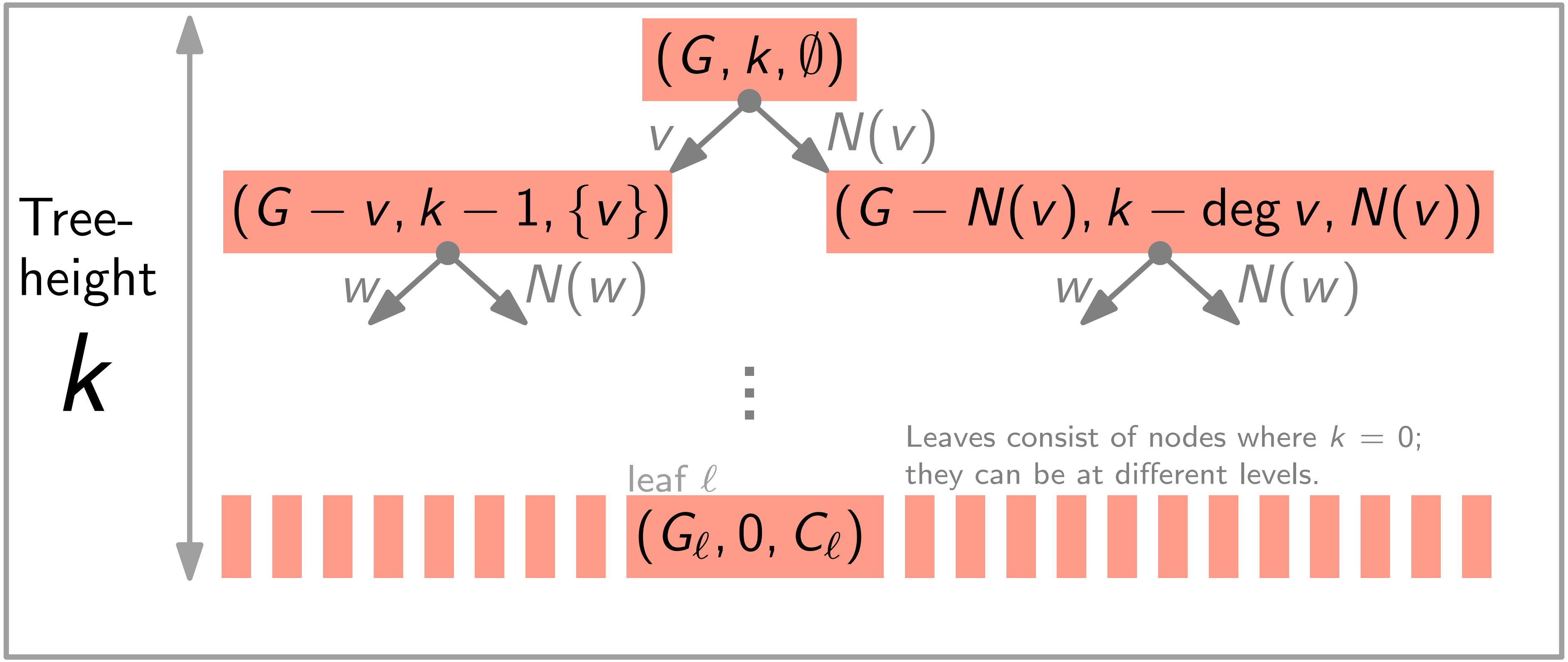

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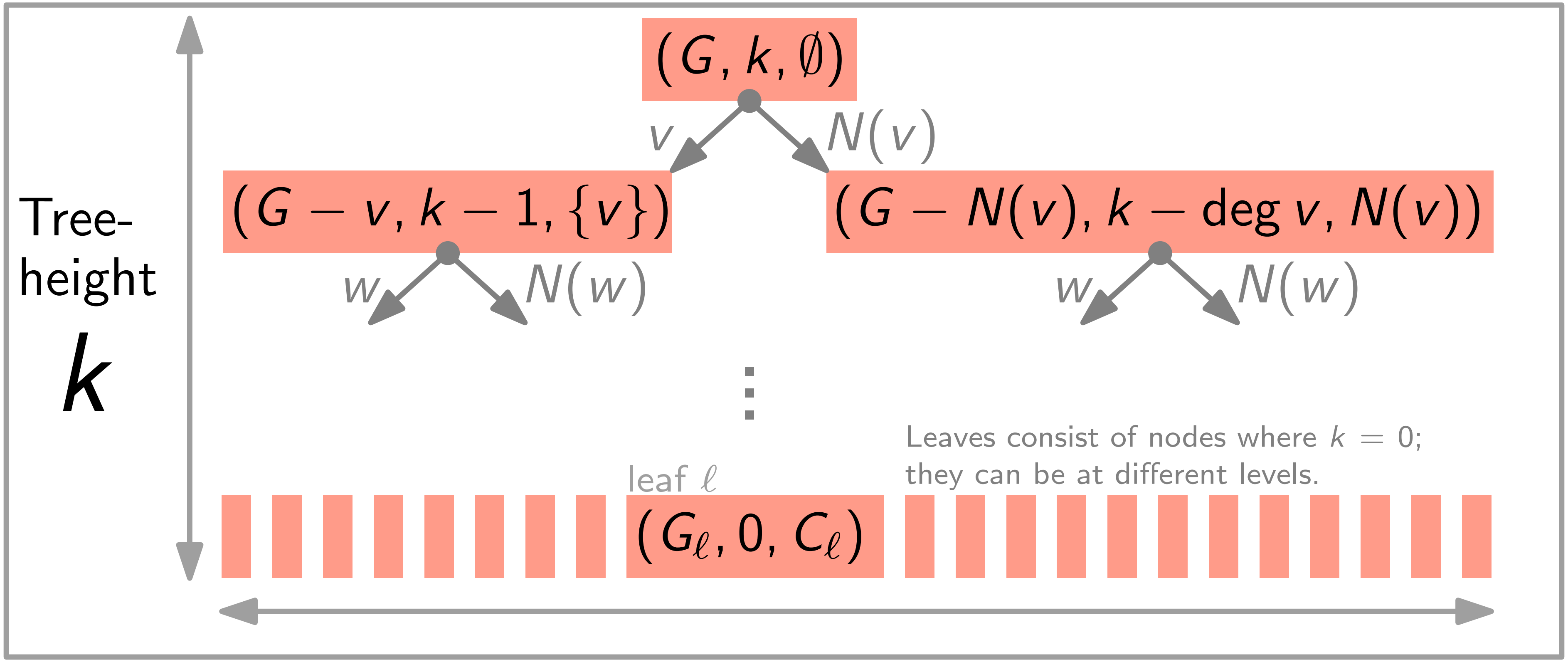

#nodes: $T(k) \leq 2$

YES: If there is a leaf ℓ where $E_\ell = \emptyset$, then C_ℓ is a k-VC of G.
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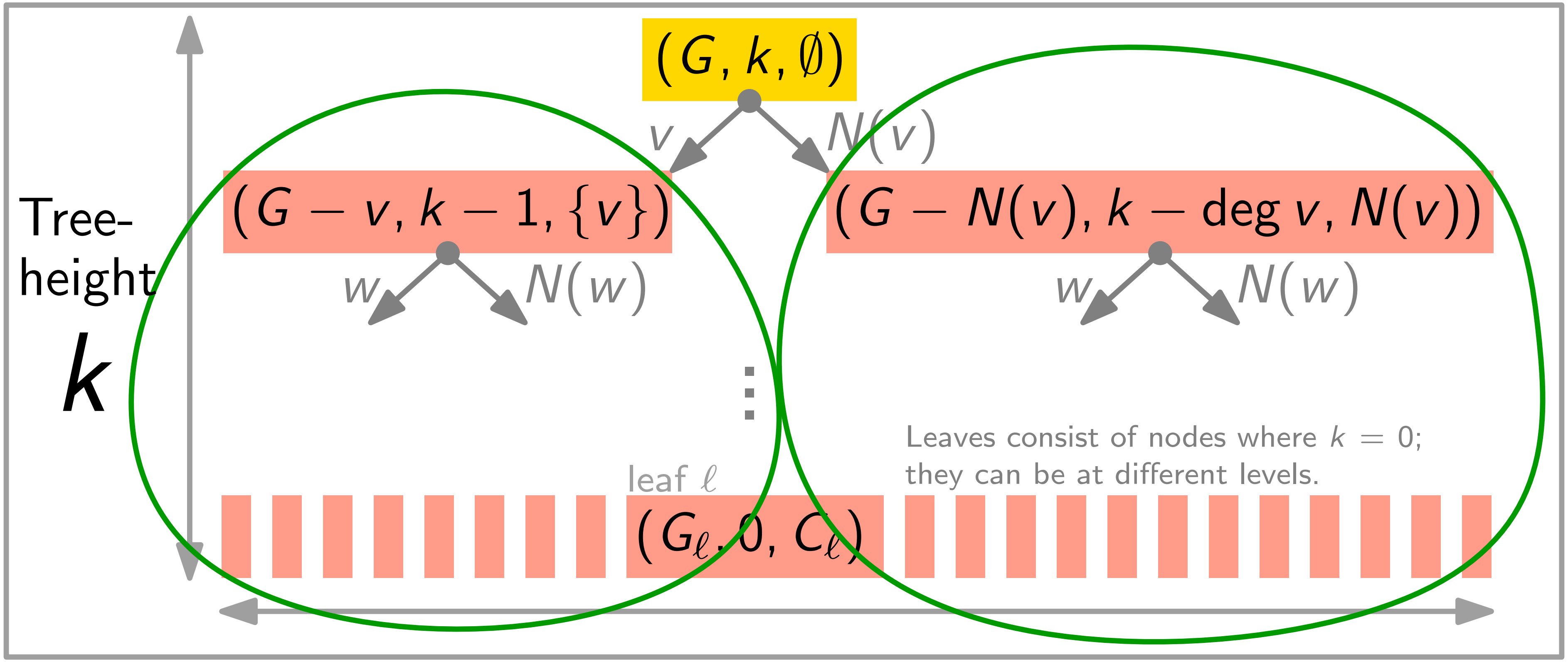

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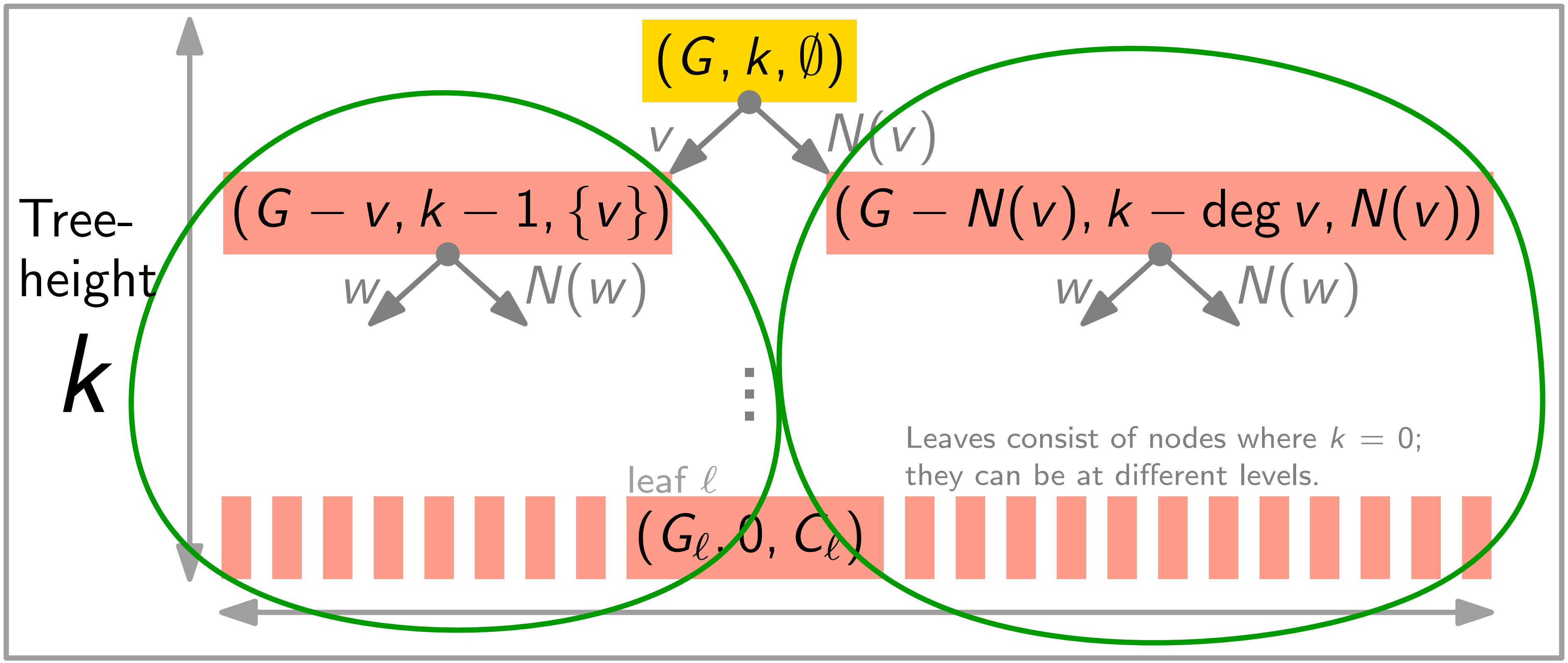

#nodes: $T(k) \leq 2T(k-1) + \boxed{1}$, $T(0) = 1$

YES: If there is a leaf ℓ where $E_\ell = \emptyset$, then C_ℓ is a k-VC of G.

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Search Tree Algorithm

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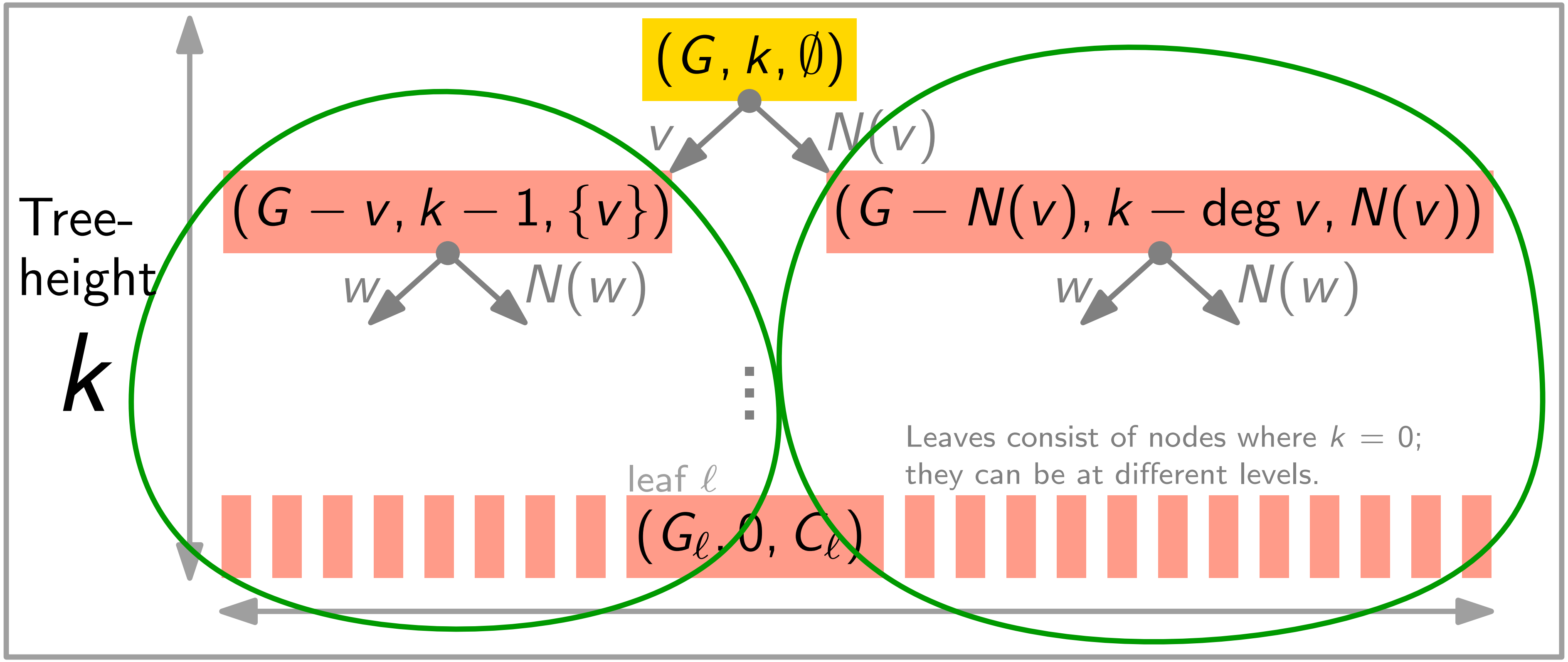

#nodes: $T(k) \leq 2T(k-1) + \boxed{1}$, $T(0) = 1 \quad \Rightarrow T(k) \leq 2^{k+1} - 1 \in O(2^k)$

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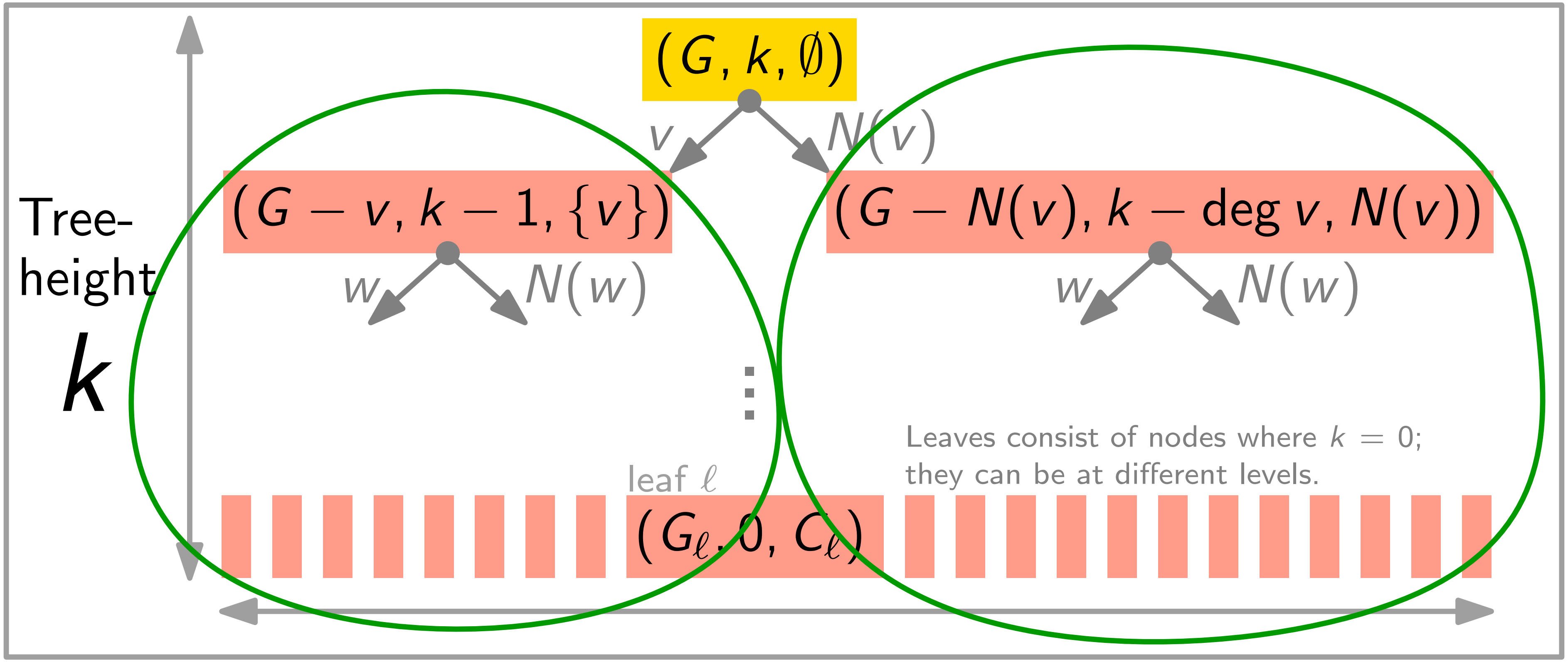

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$\Rightarrow$ **Runtime:**

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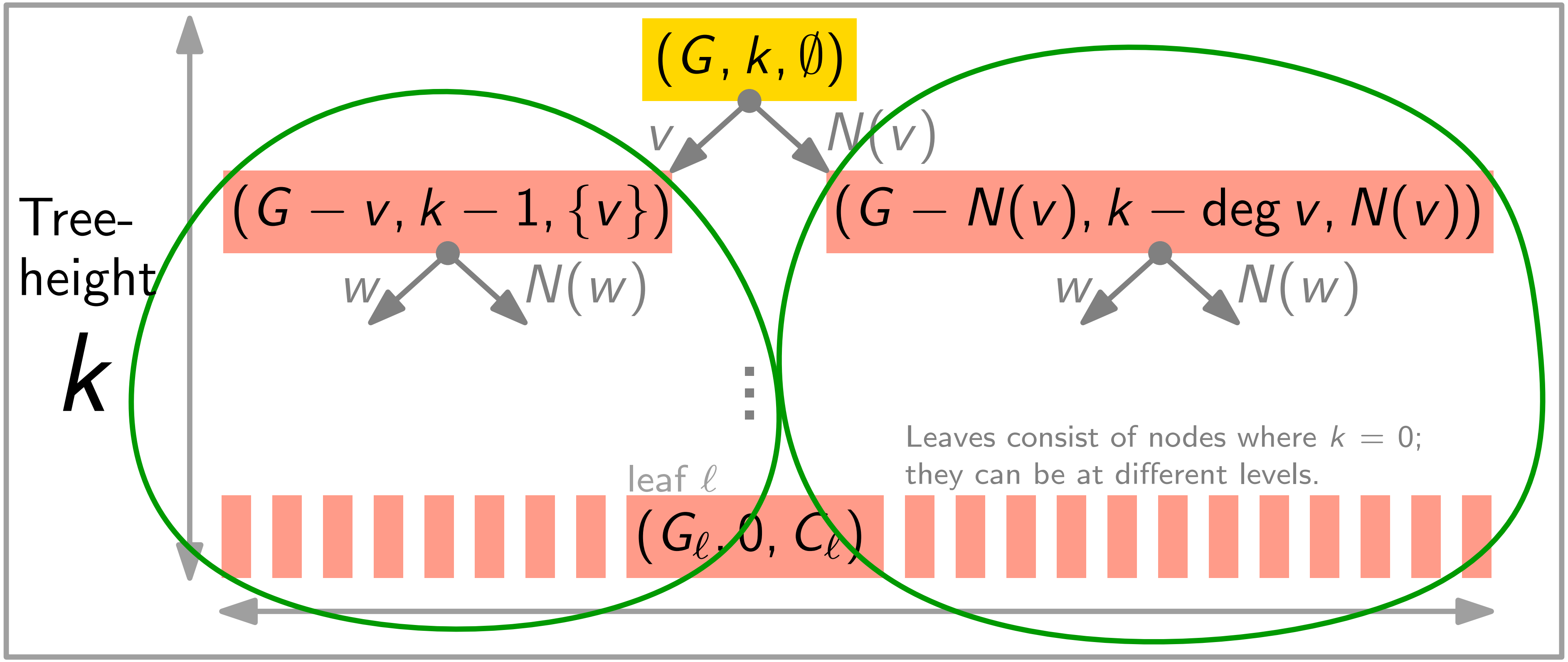

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$\Rightarrow$ **Runtime:** $O^\star(2^k)$

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Degree-4 Algorithm

Idea. Better analysis based on $|N(v)|$.

Degree-4 Algorithm

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What if we could always branch on a vertex v whose degree is at least 4?

Degree-4 Algorithm

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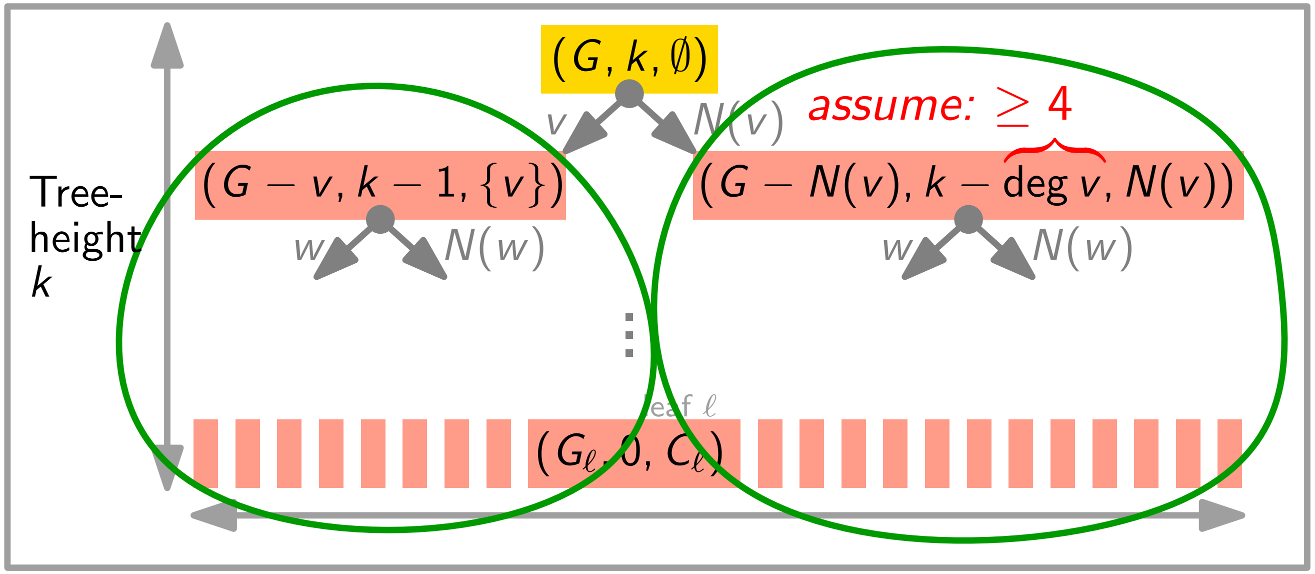

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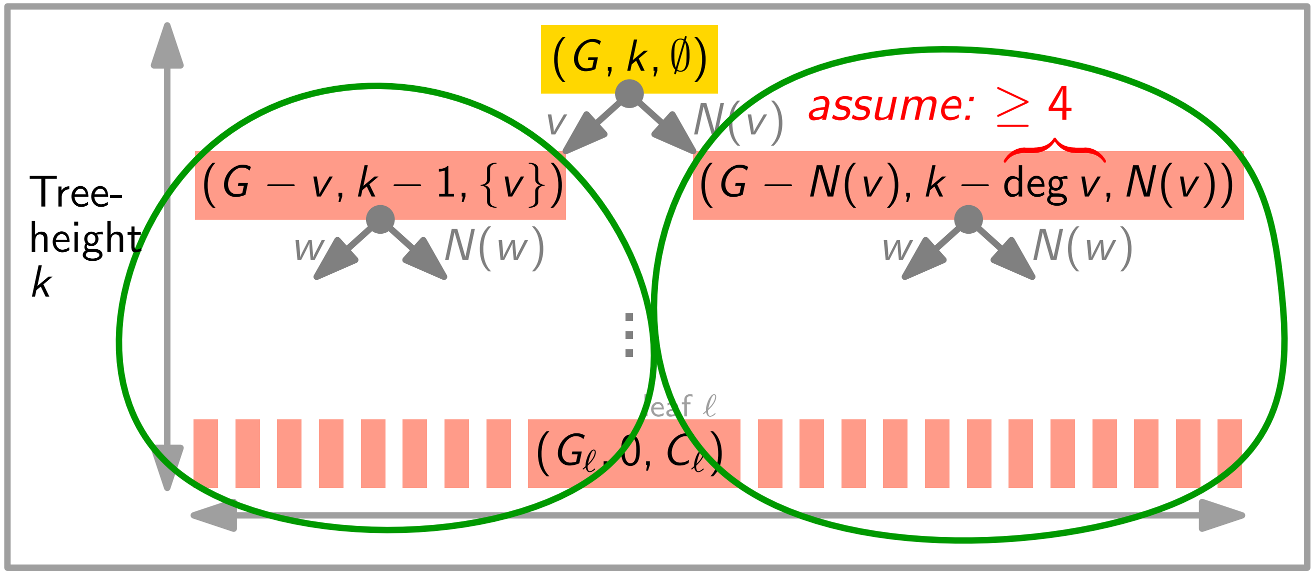

$\Rightarrow \; T(k) =$

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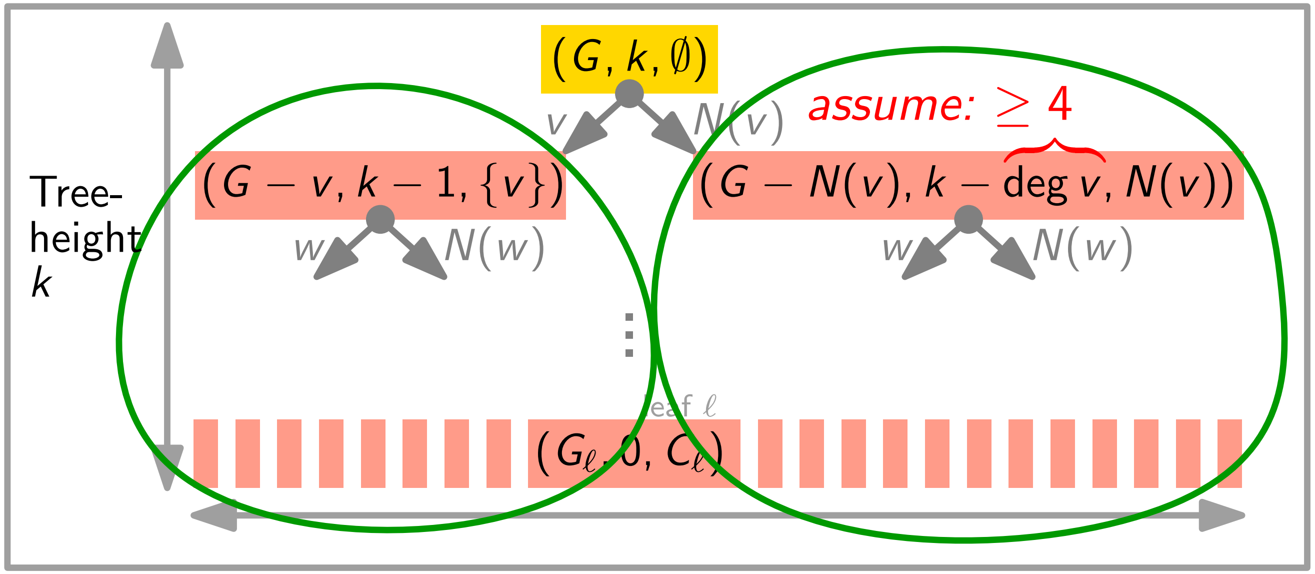

$$\Rightarrow \ T(k) = T(k-4) + T(k-1) + 1, \quad T(\leq 4) = const.$$

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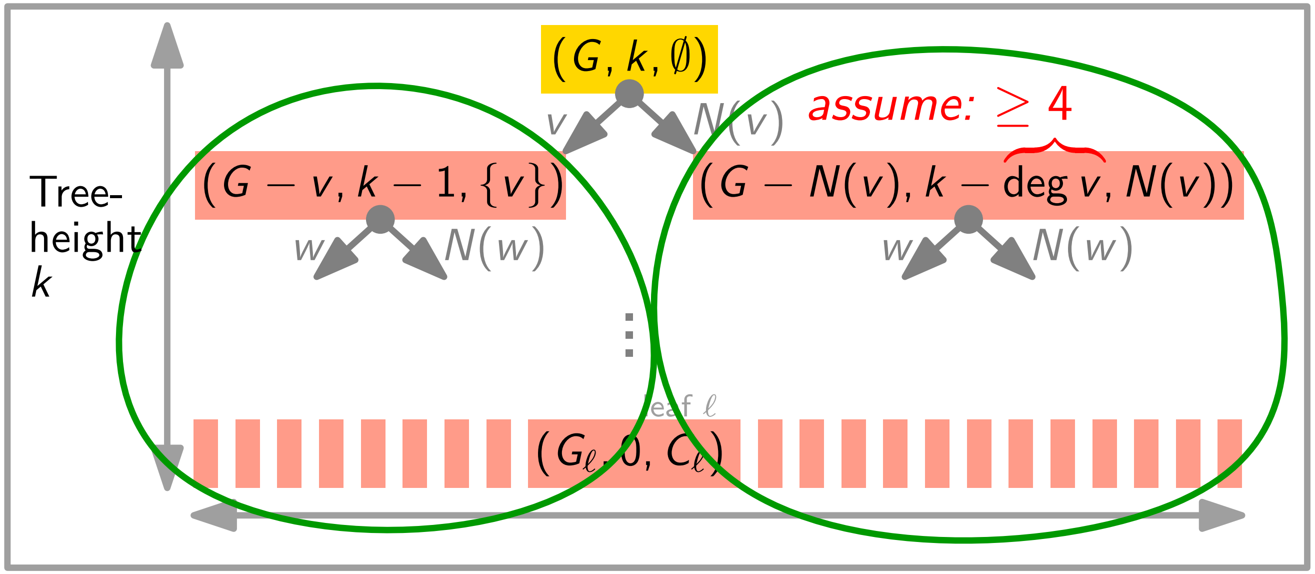

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branching vector $(4, 1)$

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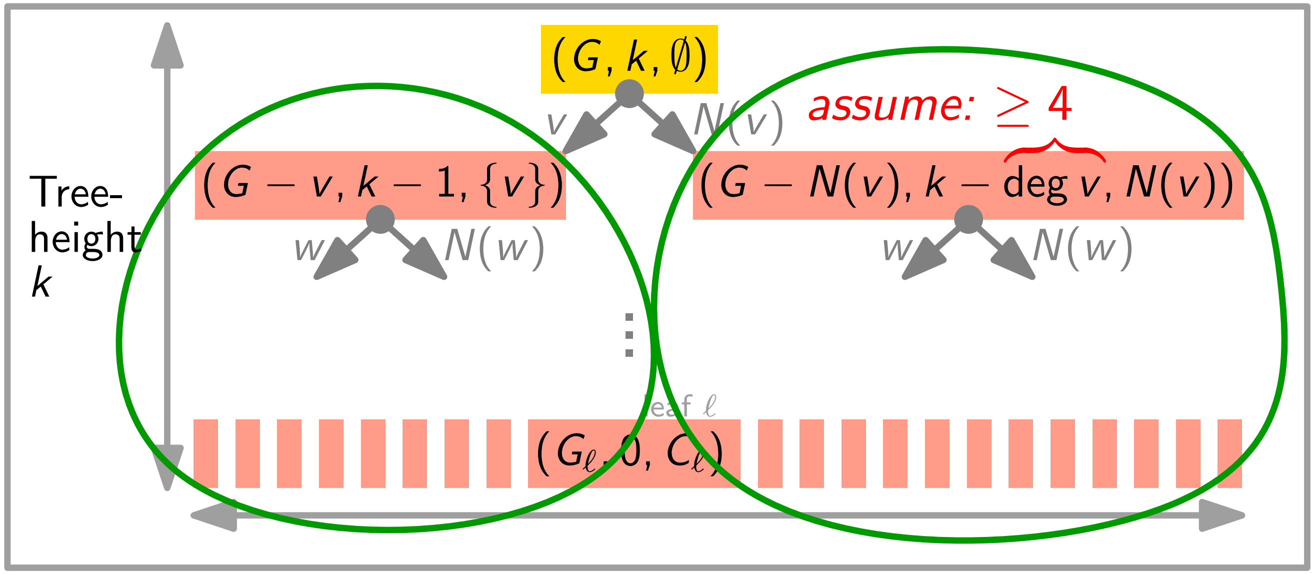

$$\Rightarrow \quad T(k) = T(k - \boxed{4}) + T(k - \boxed{1}) + 1, \quad T(\leq 4) = const.$$

branching vector $(4, 1)$

solve $T(k) = z^k - 1$

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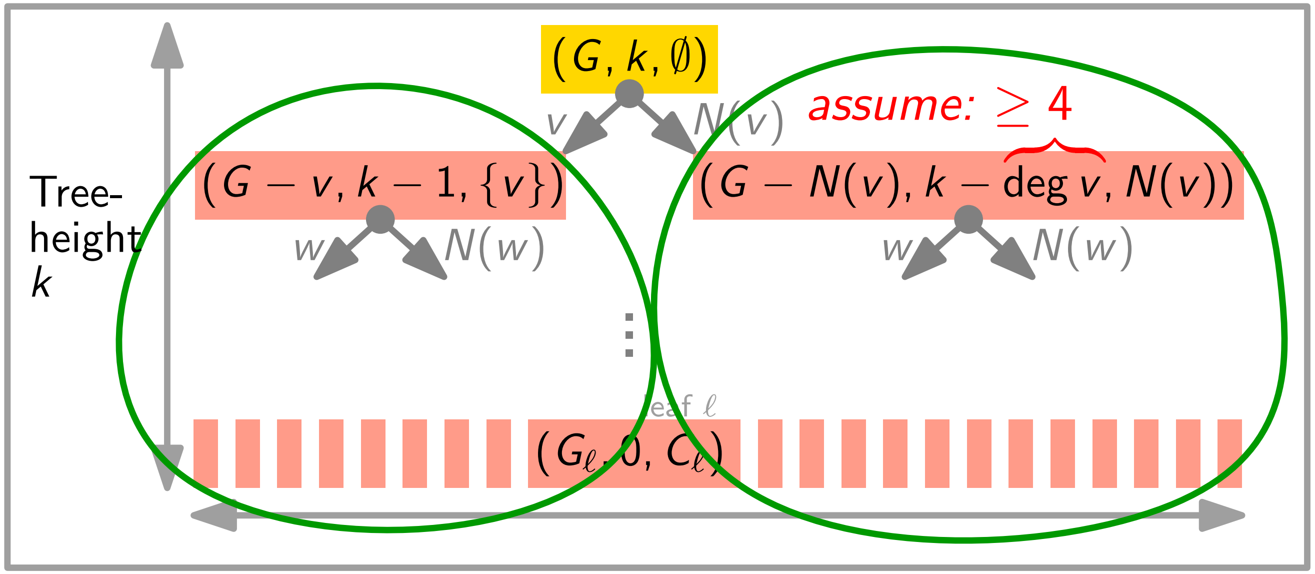

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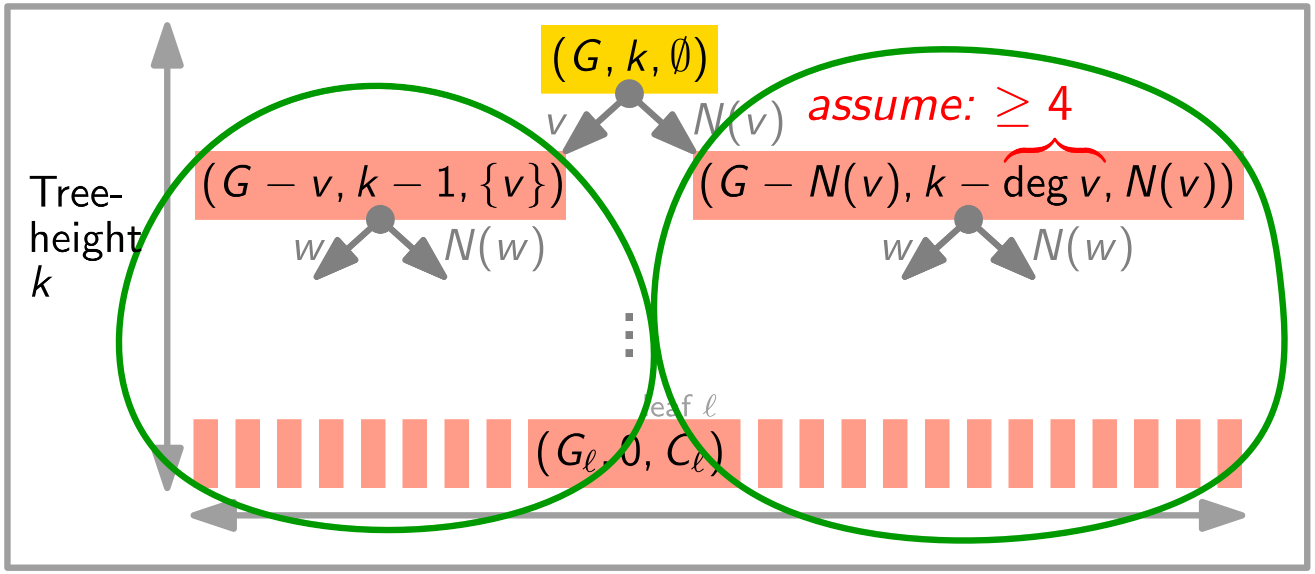

$\Rightarrow \; T(k) = T(k - 4) + T(k - 1) + 1, \; T(\leq 4) = const.$

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solve $T(k) = z^k - 1 \quad \Rightarrow \quad z^k = z^{k-4} + z^{k-1}$

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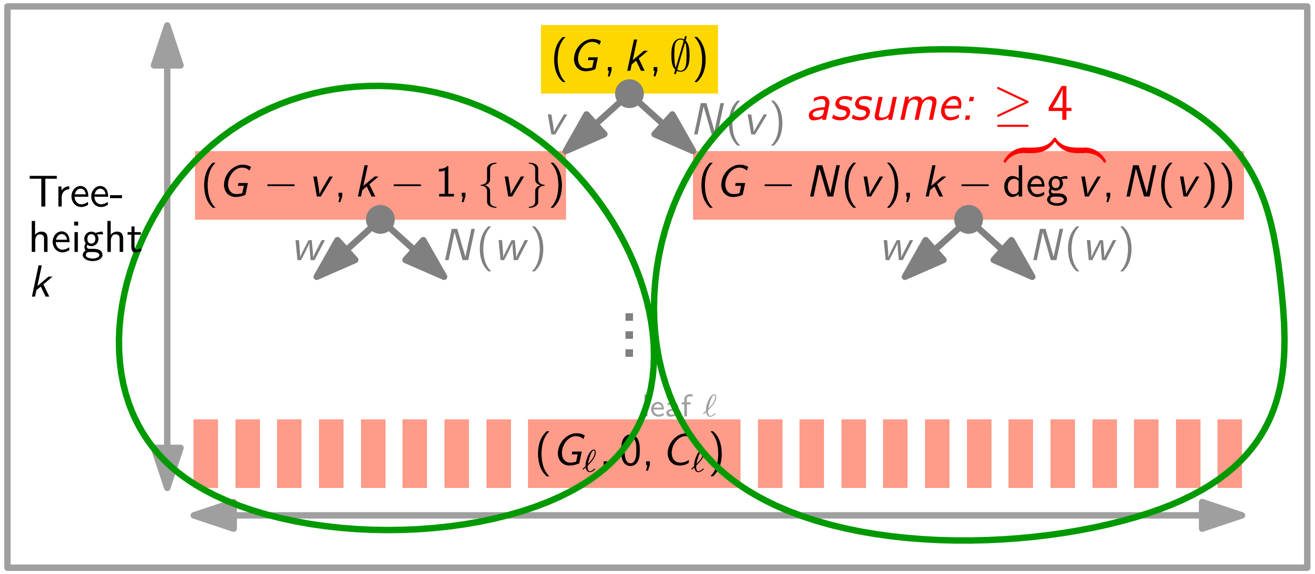

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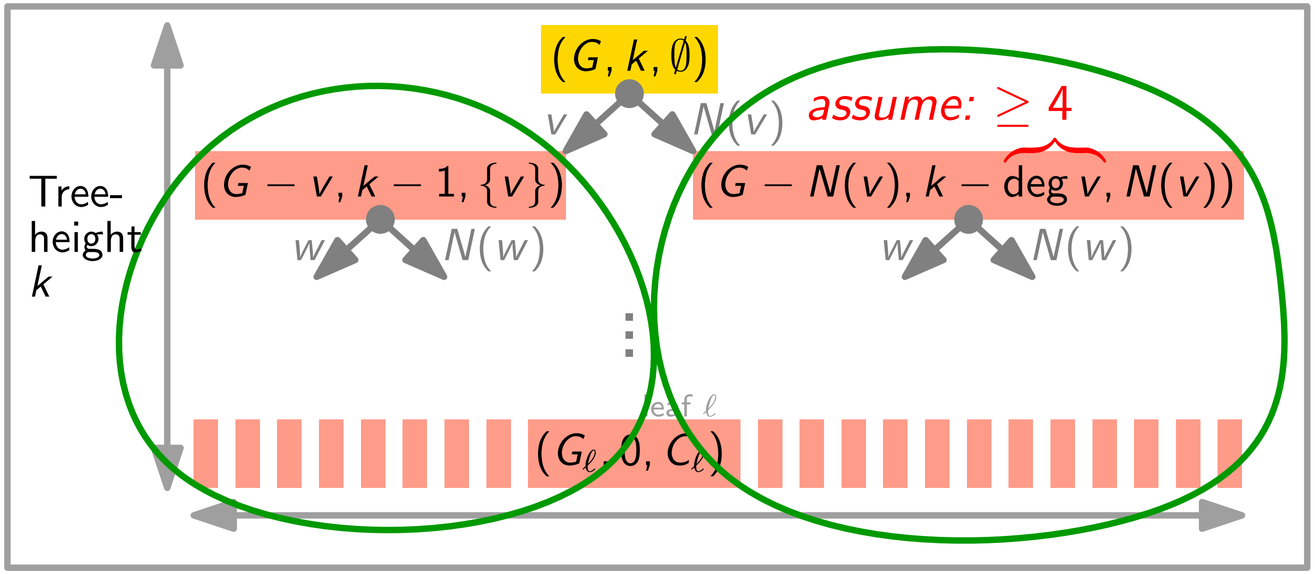

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branching vector $(4, 1)$

solve $T(k) = z^k - 1 \quad \Rightarrow \quad z^k = z^{k-4} + z^{k-1}$ $\cdot \frac{1}{z^{k-4}}$

Degree-4 Algorithm

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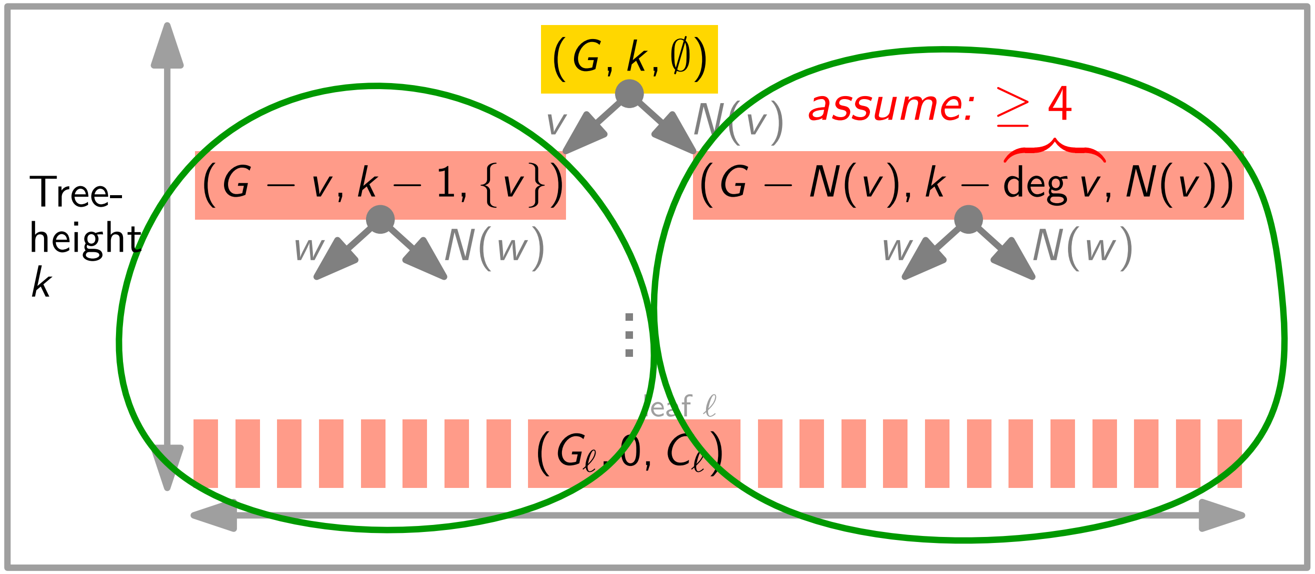

$\Rightarrow \; T(k) = T(k - 4) + T(k - 1) + 1, \; T(\leq 4) = const.$

branching vector $(4, 1)$

solve $T(k) = z^k - 1 \quad \Rightarrow \quad z^k = z^{k-4} + z^{k-1}$ $\quad \cdot \frac{1}{z^{k-4}}$

$\Rightarrow$ Characteristic polynomial: $z^4 = 1 + z^3$

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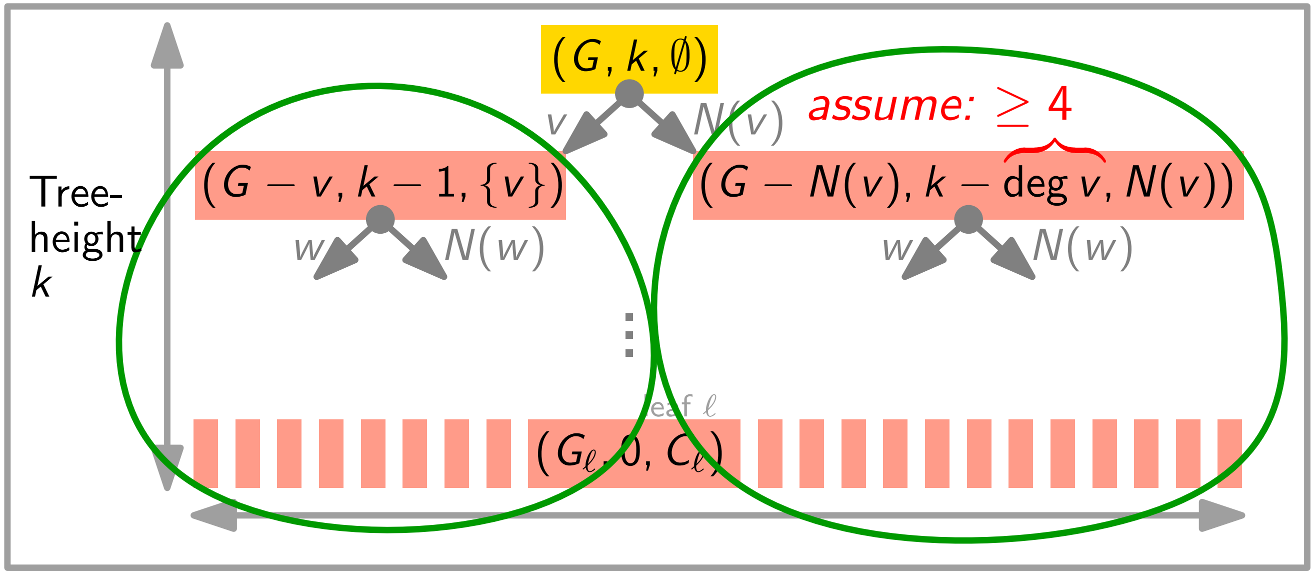

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branching vector $(4, 1)$

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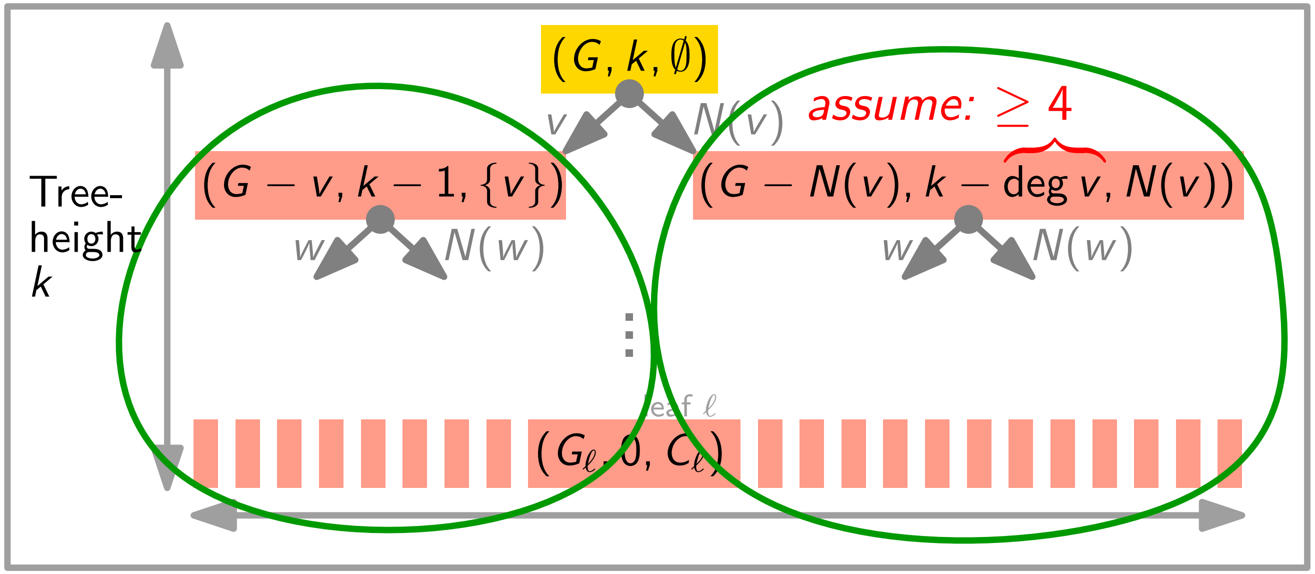
$\Rightarrow$ Characteristic polynomial: $z^4 = 1 + z^3$

$\Rightarrow$ largest positive solution:

Degree-4 Algorithm

Idea. Better analysis based on $|N(v)|$.



$\Rightarrow \; T(k) = \; T(k - \boxed{4}) + T(k - \boxed{1}) + 1, \; T(\leq 4) = const.$

branching vector $(4, 1)$

solve $T(k) = z^k - 1 \quad \Rightarrow \quad z^k = z^{k-4} + z^{k-1}$ $\quad \cdot \frac{1}{z^{k-4}}$

$\Rightarrow$ Characteristic polynomial: $z^4 = 1 + z^3$

$\Rightarrow$ largest positive solution: $z \approx 1.38 \quad$ (*branching value*)

Degree-4 Algorithm

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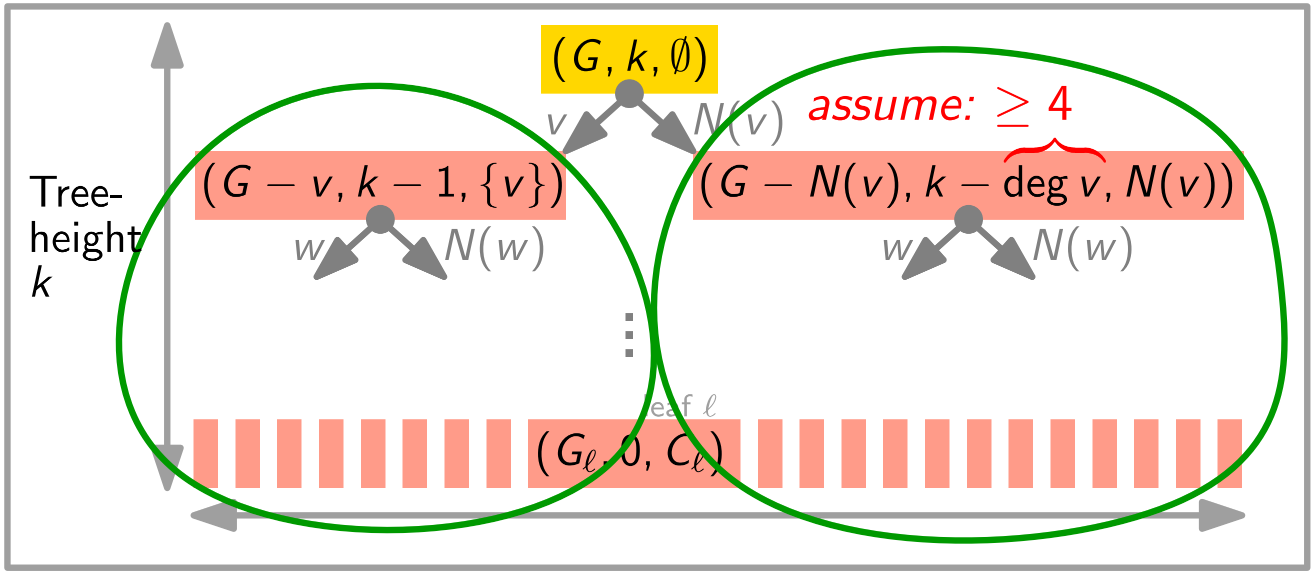

$\Rightarrow$ $T(k) = T(k - \boxed{4}) + T(k - \boxed{1}) + 1$, $T(\leq 4) = const.$

branching vector $(4, 1)$

solve $T(k) = z^k - 1$ $\Rightarrow$ $z^k = z^{k-4} + z^{k-1}$ $\quad \cdot \frac{1}{z^{k-4}}$

$\Rightarrow$ Characteristic polynomial: $z^4 = 1 + z^3$

$\Rightarrow$ largest positive solution: $z \approx 1.38$ (*branching value*)

$\Rightarrow$ $T(k) \in O(1.38^k)$.

Degree-4 Algorithm

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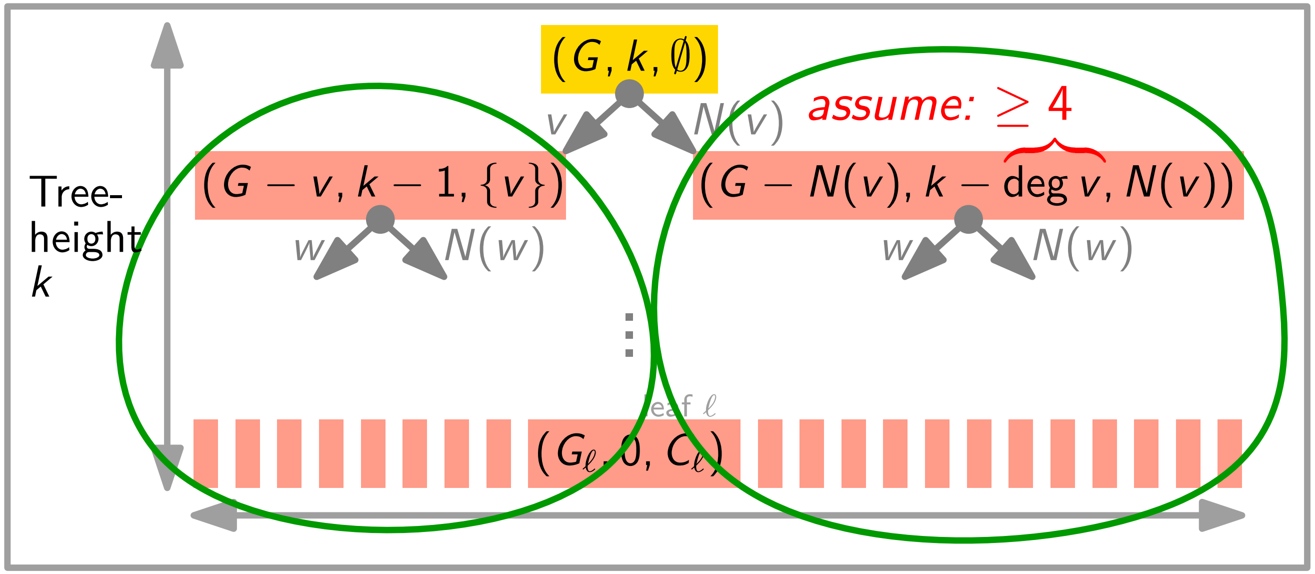

$\Rightarrow \quad T(k) = T(k - \boxed{4}) + T(k - \boxed{1}) + 1, \quad T(\leq 4) = const.$

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$\Rightarrow \quad T(k) \in O(1.38^k)$. How can we ensure $\deg v \geq 4$?

Kernel Construction II

Previous version:

Rule K: Reduce vertices of degree $> k$

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New rules:

Rule 1:

Rule 2:

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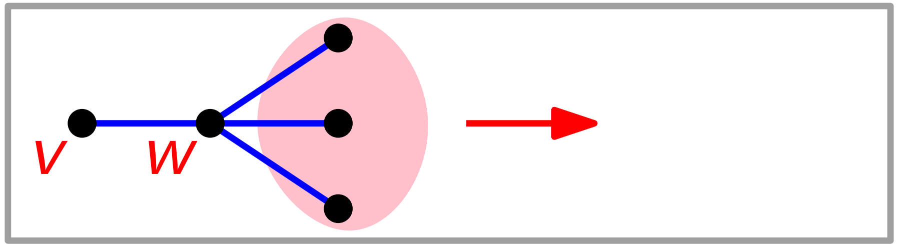

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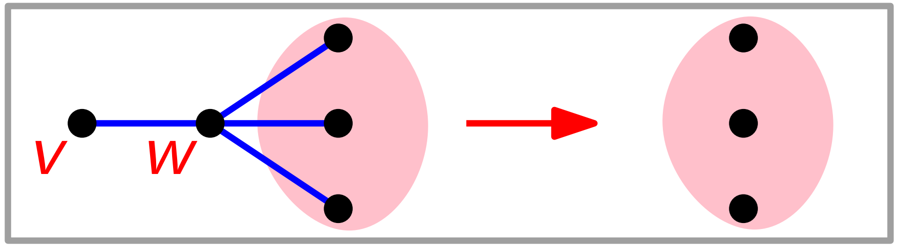

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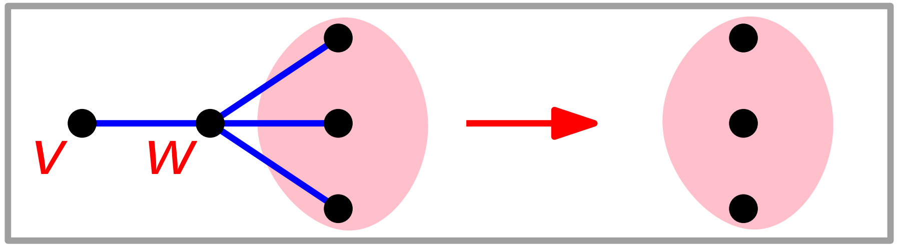

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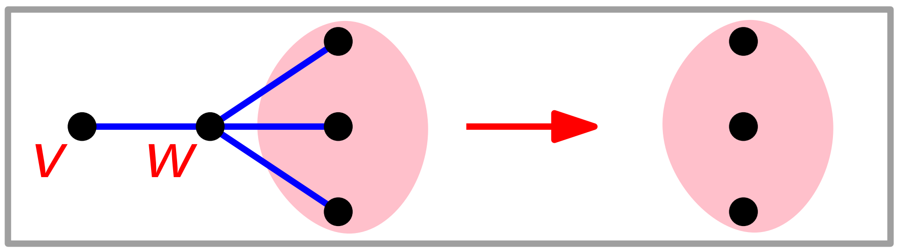

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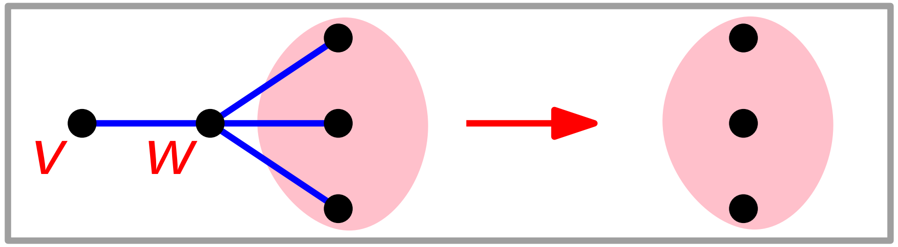

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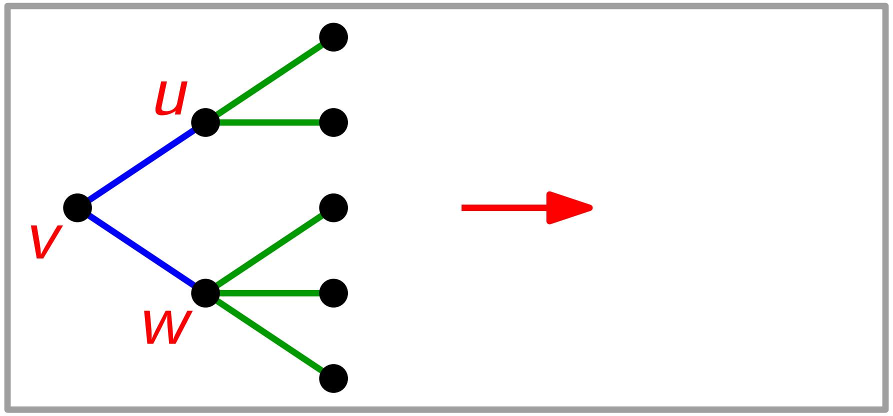

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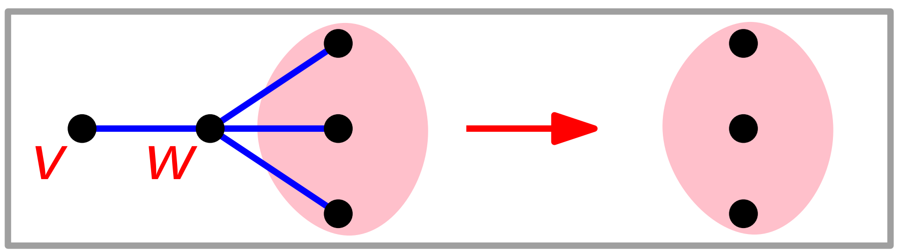

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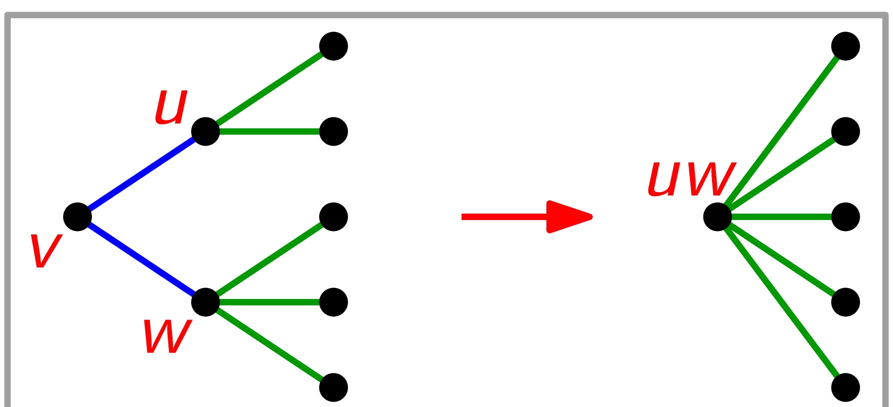

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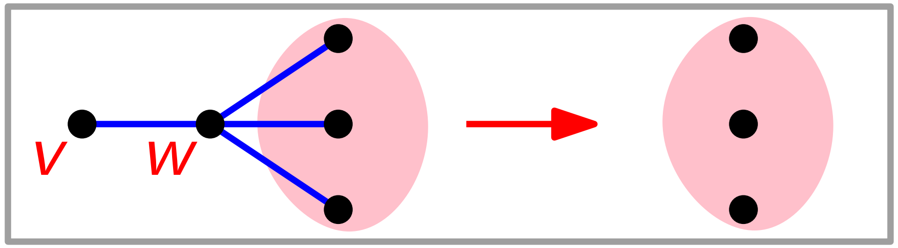

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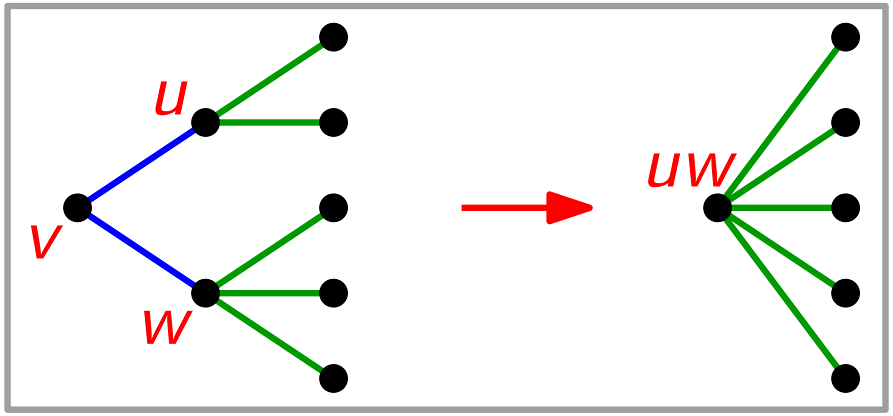

when uw is an edge, put u and w in C, otherwise $v \in C$, and merge u, w.

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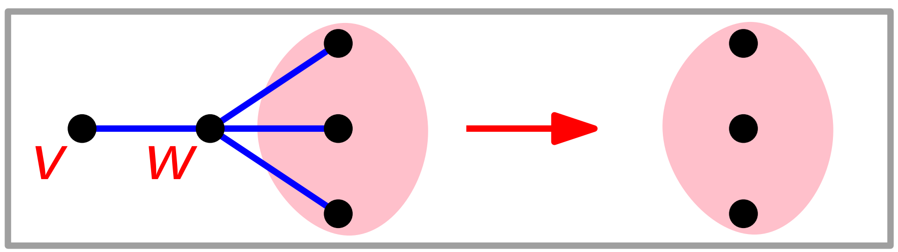

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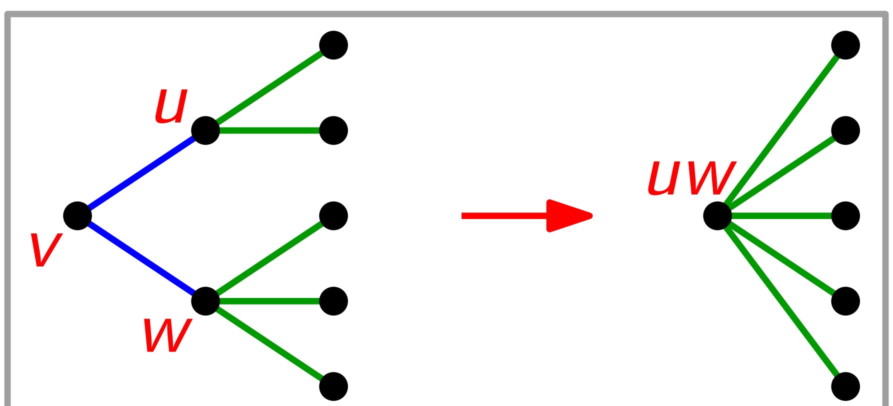

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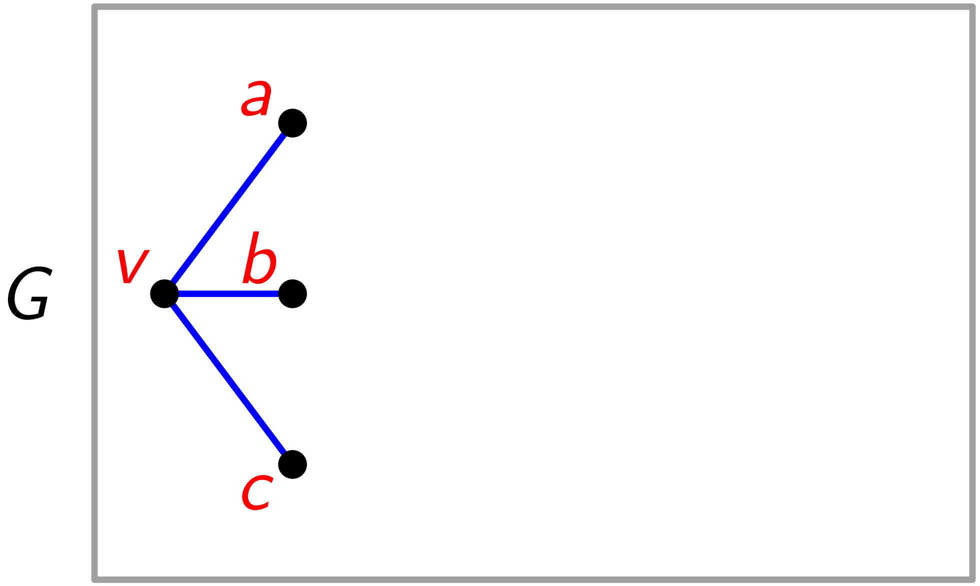
Rule 3: Reduce Degree-3 Vertices

Rule 3.1: $G[N(v)]$ has no edges



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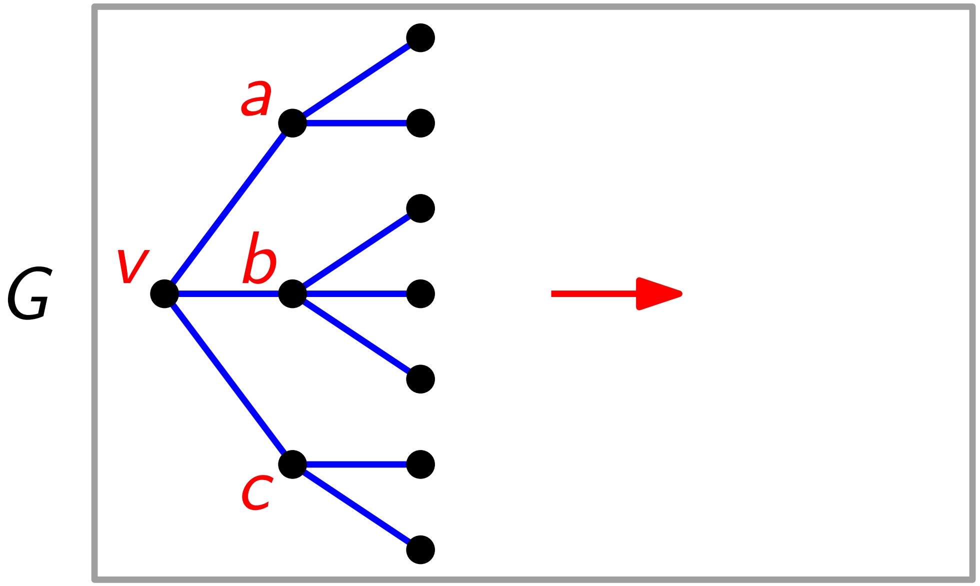

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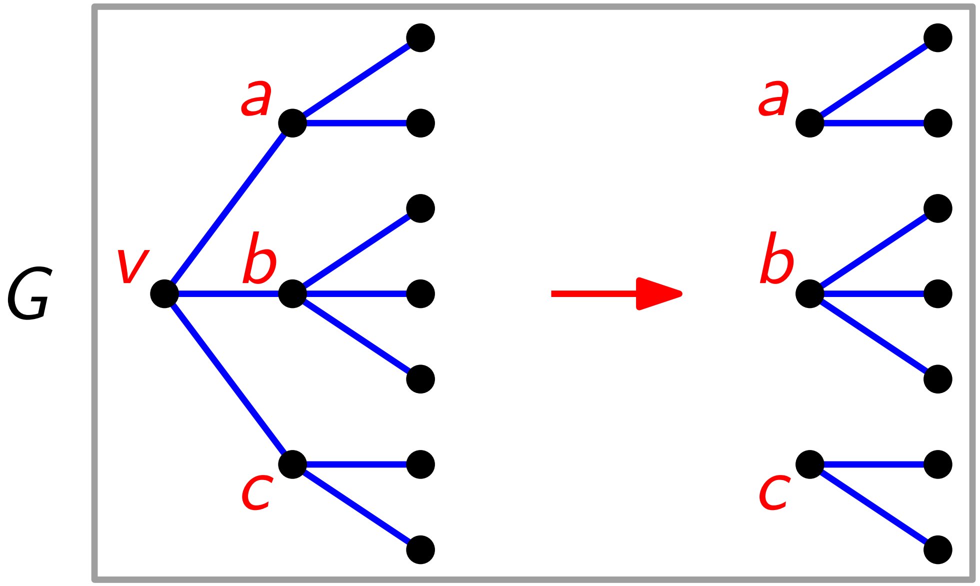

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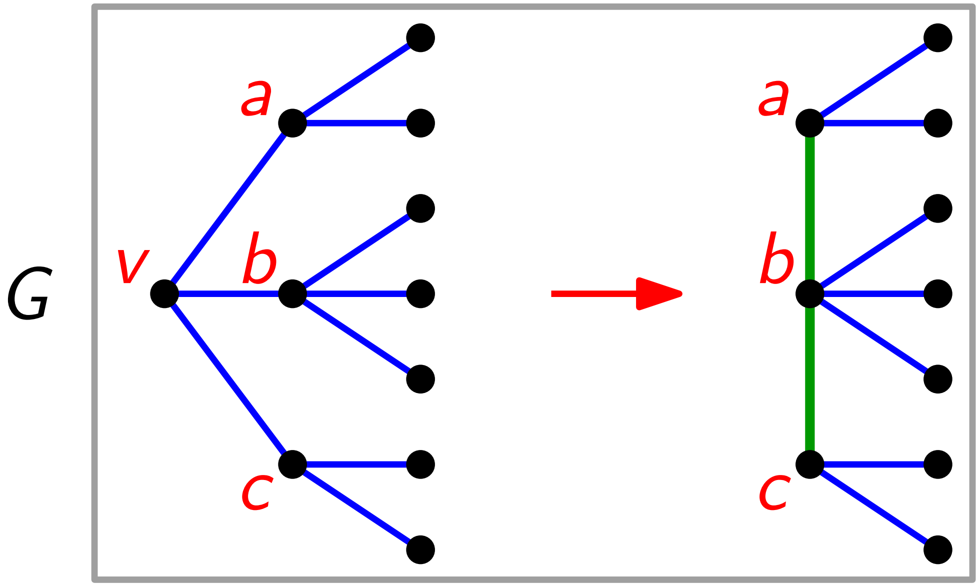

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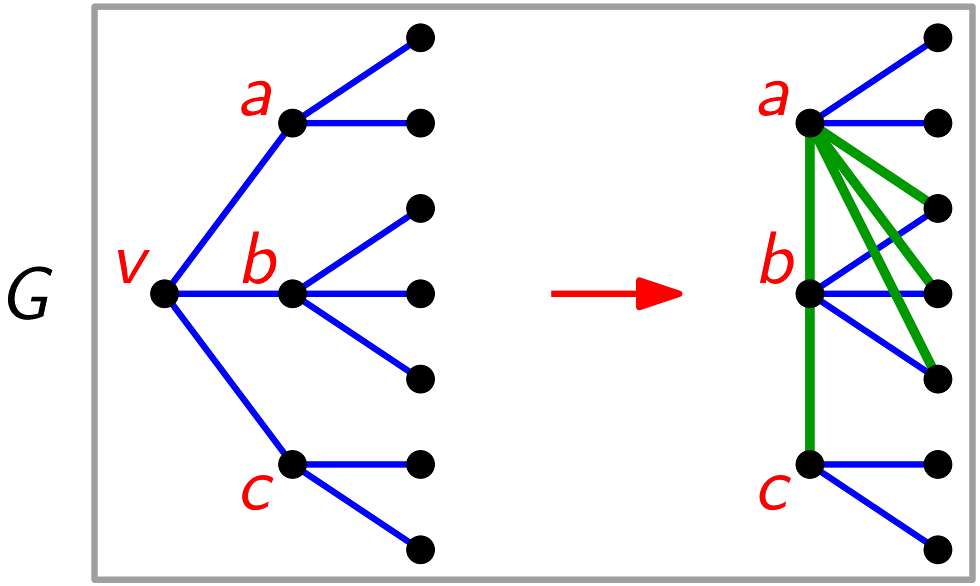

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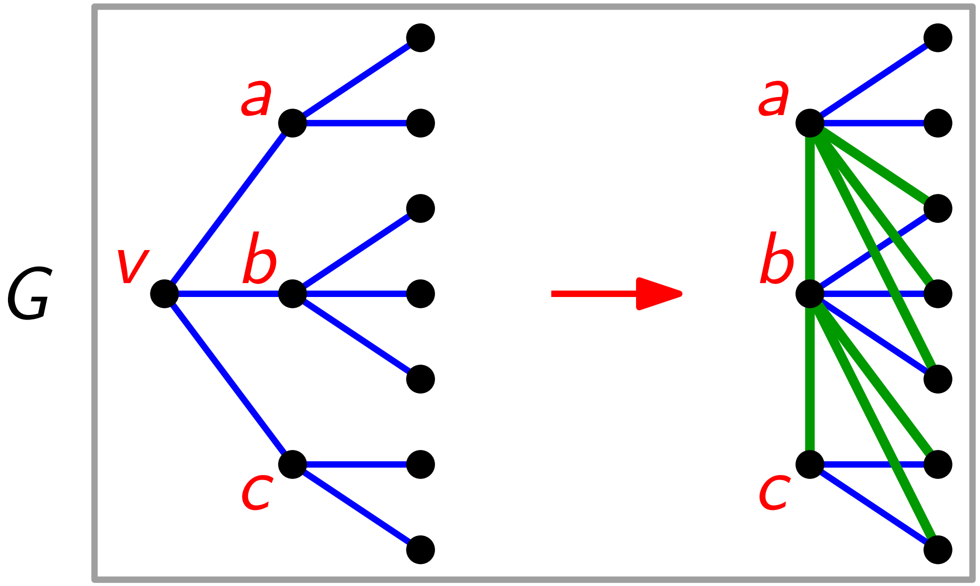

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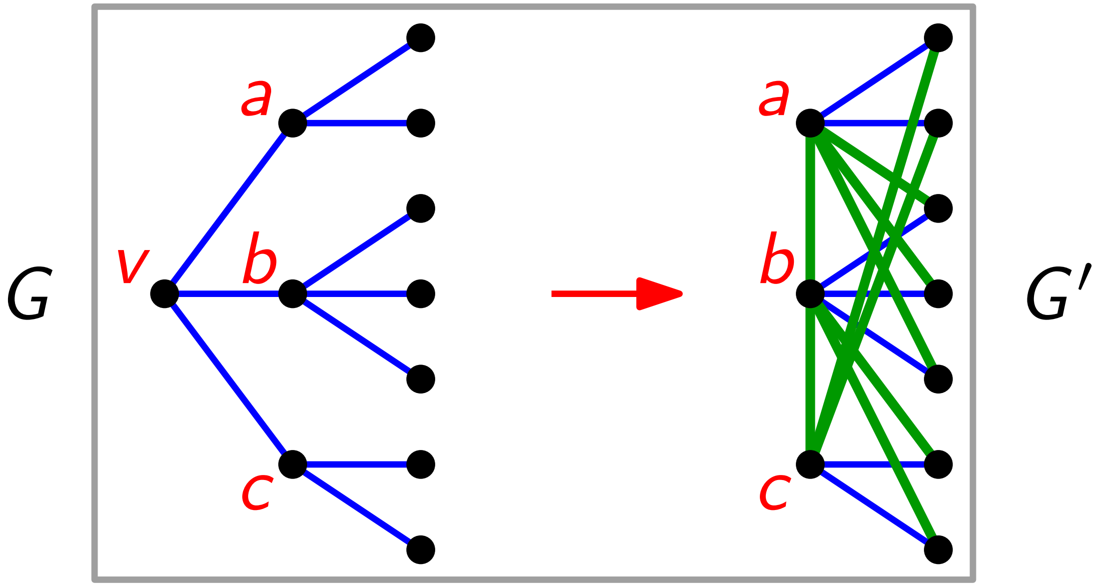

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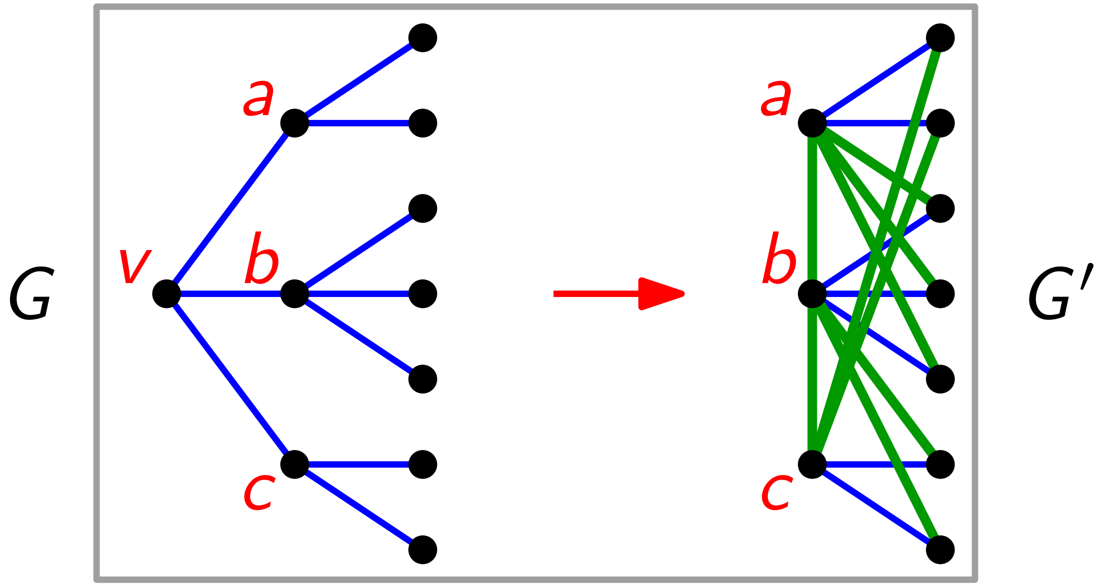

Note. k-VC in $G \Leftrightarrow k$-VC in G'.

[proof omitted]

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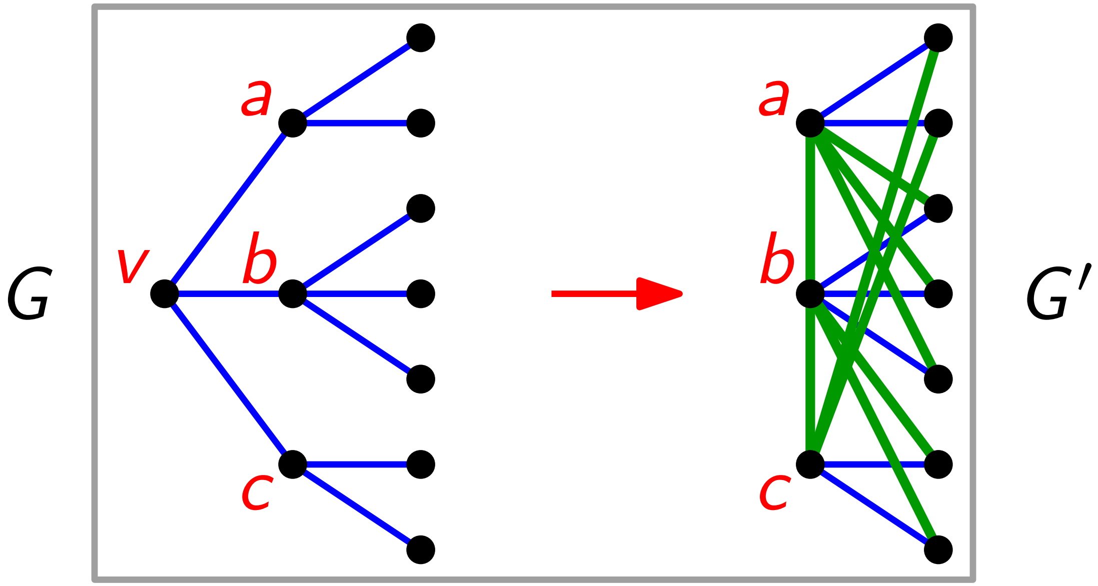

Note. k-VC in G $\Leftrightarrow$ k-VC in G'.

[proof omitted]

Rule 3.2: $G[N(v)]$ contains an edge

$\dots$

Degree-4 Algorithm

Idea: Apply the improved kernelization approach at each node of the search tree.

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Preprocessing   Kernelization in each node

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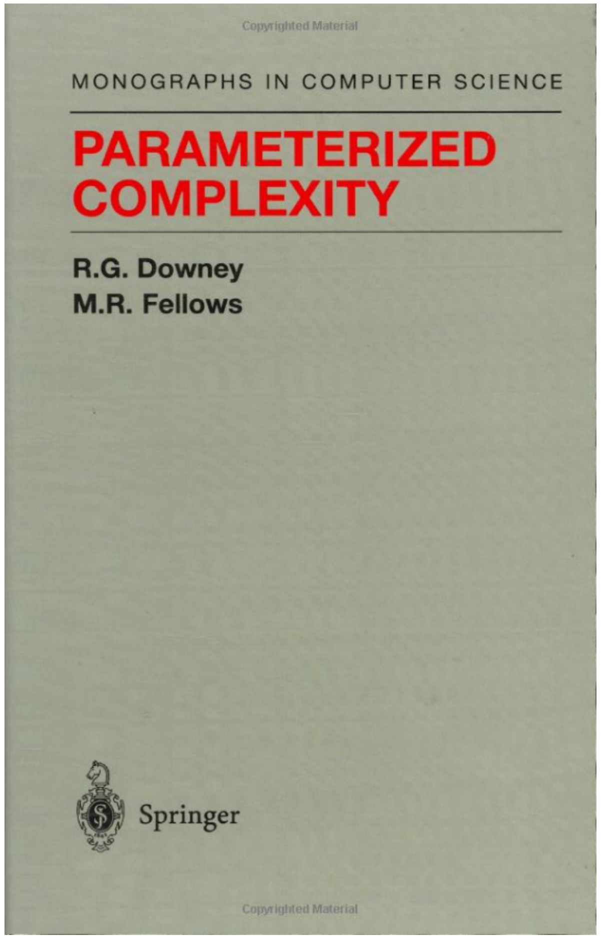
$\Rightarrow$ **Runtime**: $O(\underbrace{nk}_{\text{Preprocessing}} + \underbrace{k^2}_{\text{Kernelization in each node}} \cdot 1.38^k) \subseteq O^\star(1.38^k)$

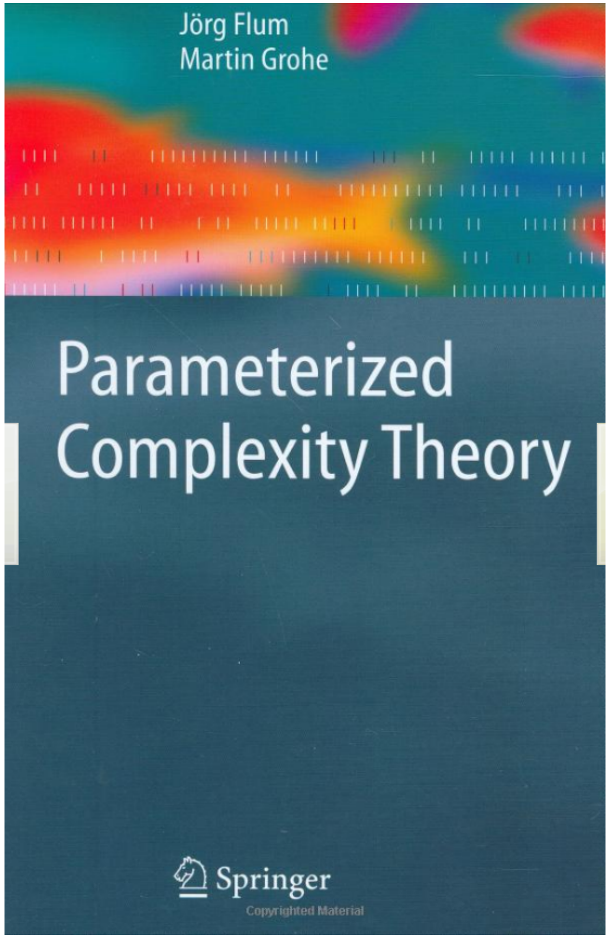
Summary

- k-VC can be solved in $O(nk + 1.38^k k^2)$ time.
- parameterized complexity =
  new approach to hard problems: kernelization, search trees,
  ...
- always a good idea look for parameterized analysis
  as in FPT !
- Ideally:
  "natural" problem $P \in \mathcal{FPT} \Rightarrow$ reasonable $f(k)$.

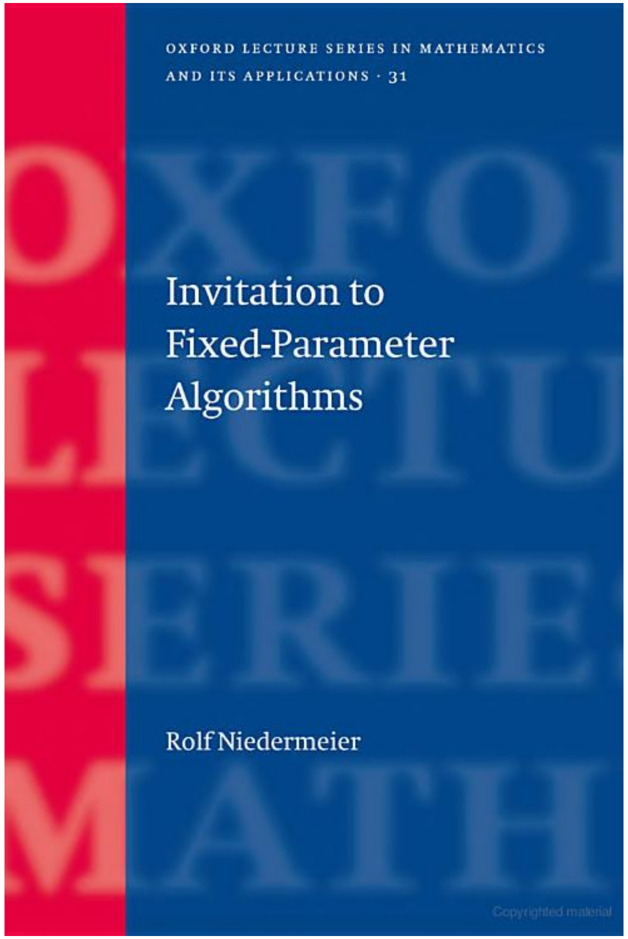
Books on the Topic



1999



2006

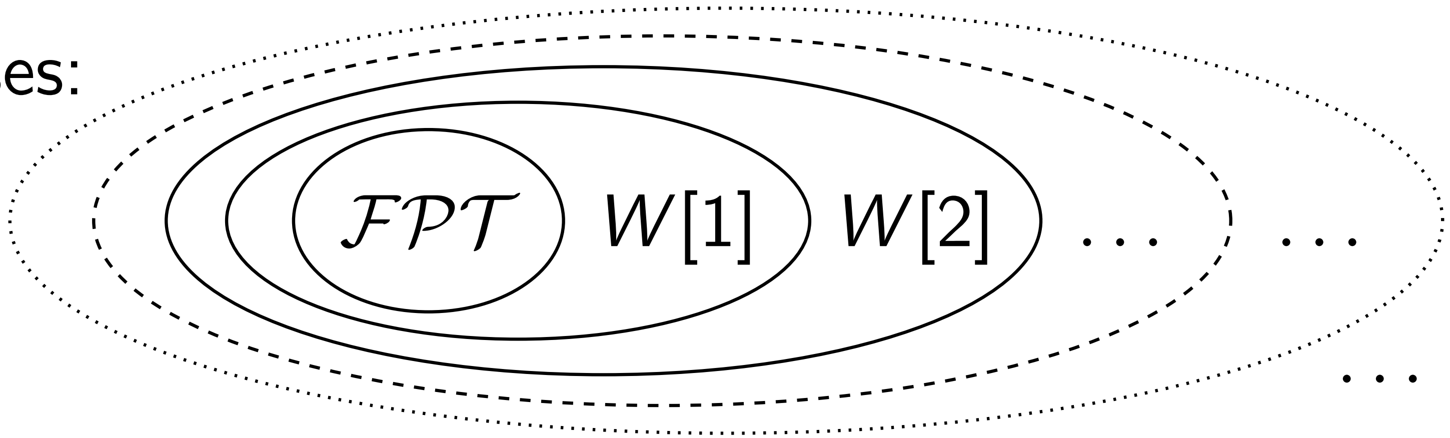


2006

Also, the textbook we are using:
Parameterized Algorithms

Computational Complexity

- FPT-reduction
- Decision circuits: weft and depth
- Problem Classes:



- Example $W[1]$-complete problems
  - k-IndependentSet
  - k-Clique
- Example of a $W[2]$-complete problem:
  - k-DominatingSet

Exercise: Show that these problems are in W[1]/W[2]