Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 8. Finding Trees and Partitioning Numbers

Based on: [Exact Exp. Algos: §9.1, Param. Algos: §10.1.2]

Trees: see [J. Nederlof, Algorithmica (2013) 868–884. https://doi.org/10.1007/s00453-012-9630-x]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

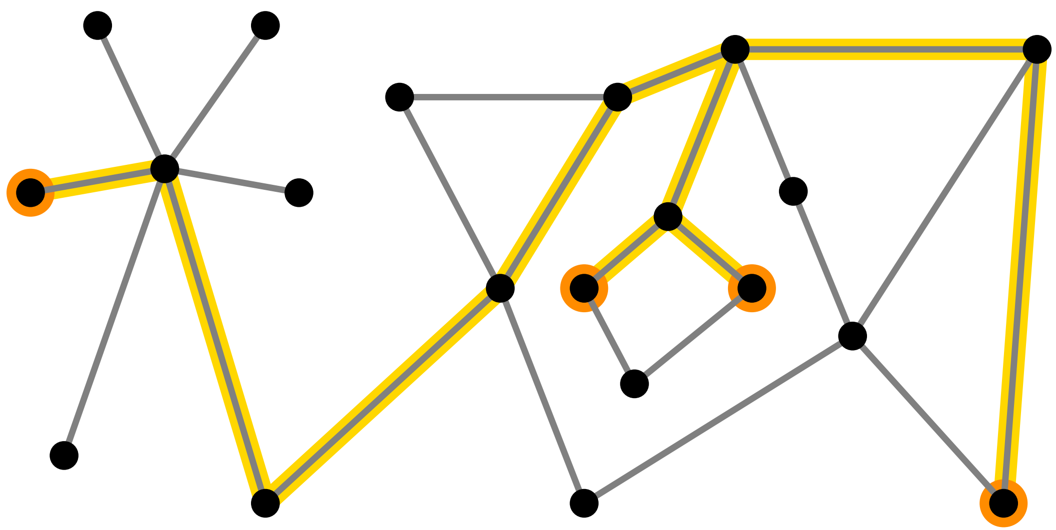
Lehrstuhl für Informatik I

Steiner Tree Problem

Given: Graph $G = (V, E)$, terminals $K \subseteq V$, number c

Question: Does there exist a subtree (V', E') of G such that

- $K \subseteq V'$ and
- $|E'| \leq c$?

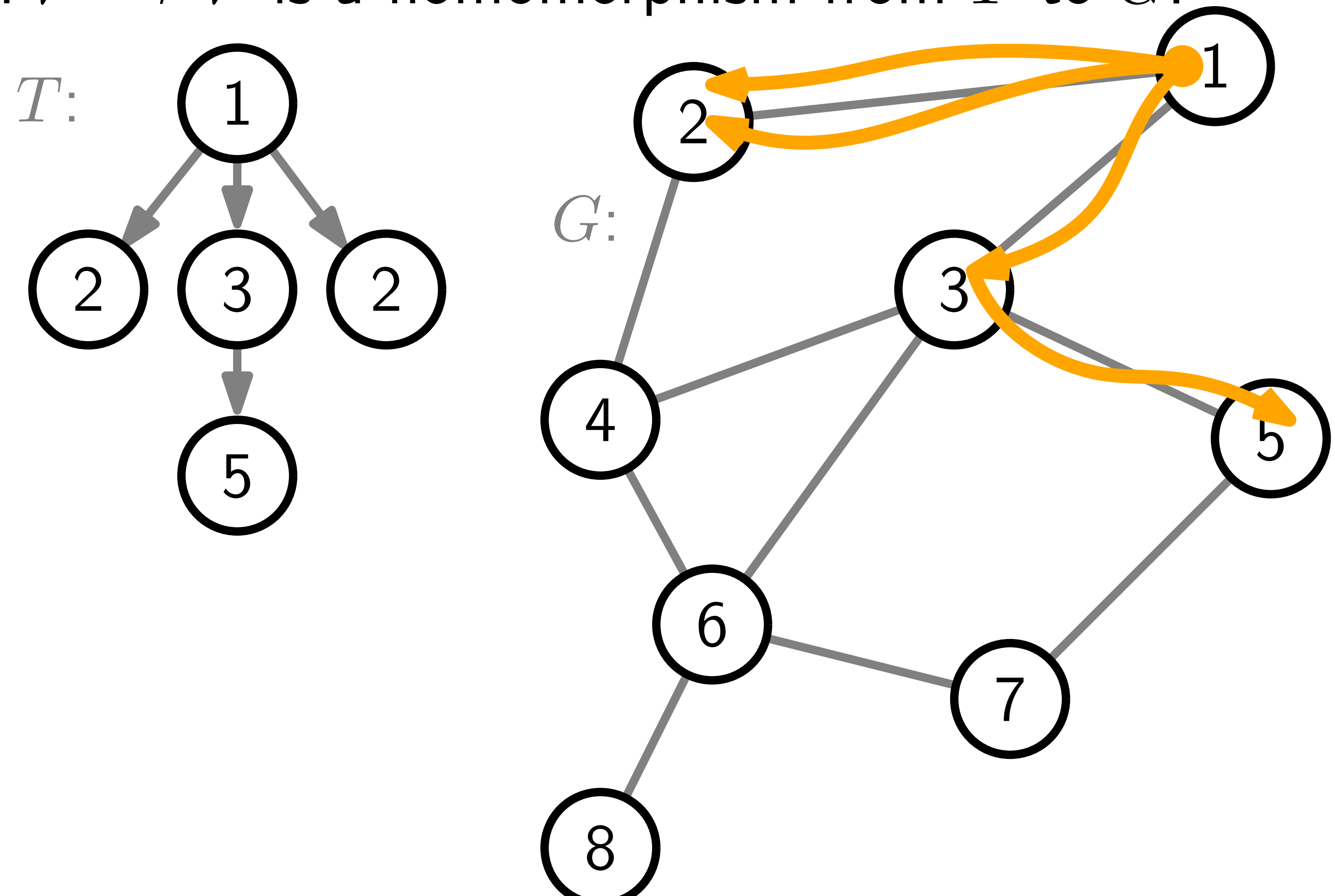


IE-Formulation?

Branching Walks

Def. A *branching walk* in G is a tuple (T, φ) where:

- $T = (V', E')$ is an ordered rooted tree, and
- $\varphi\colon V' \to V$ is a homomorphism from T to G.



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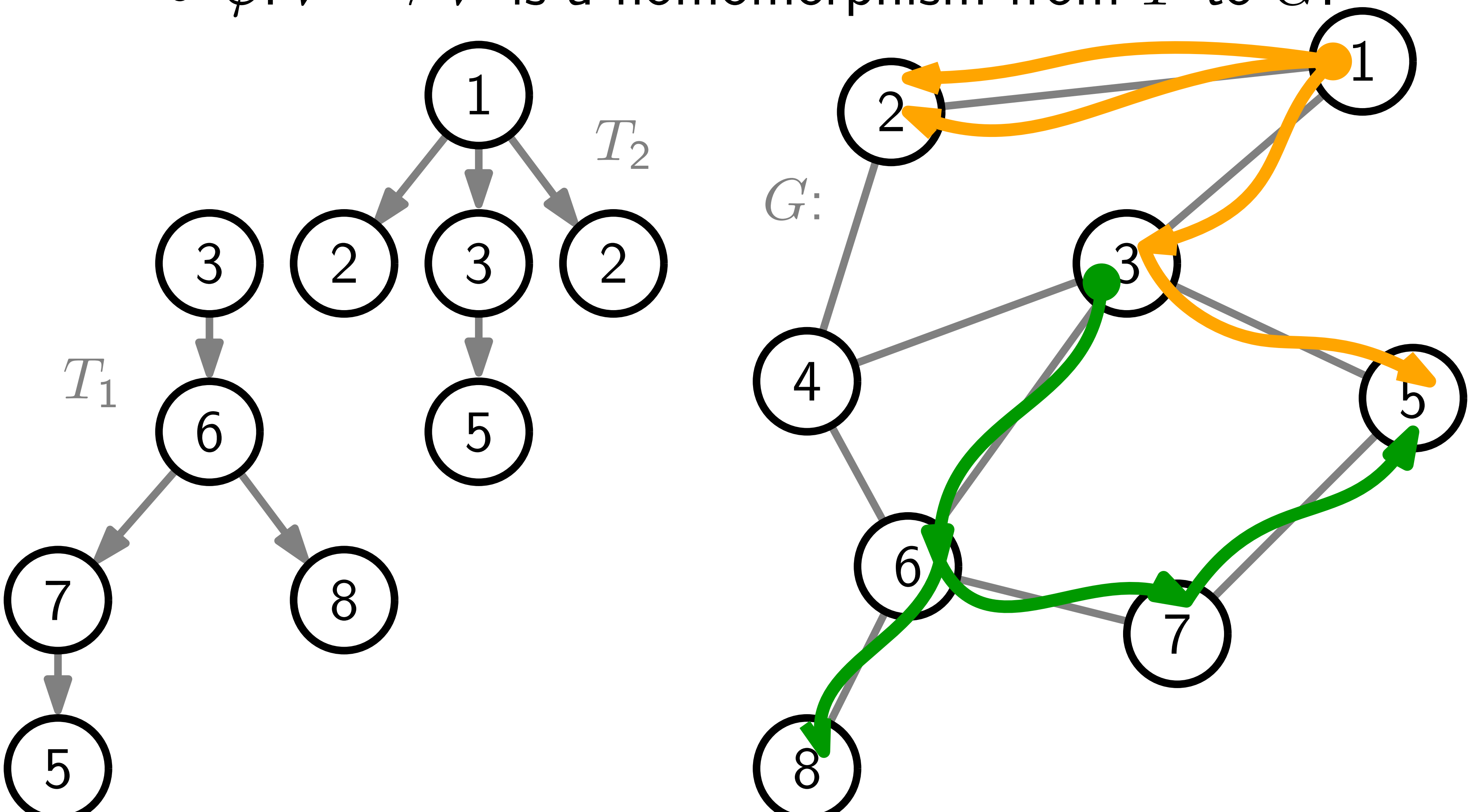

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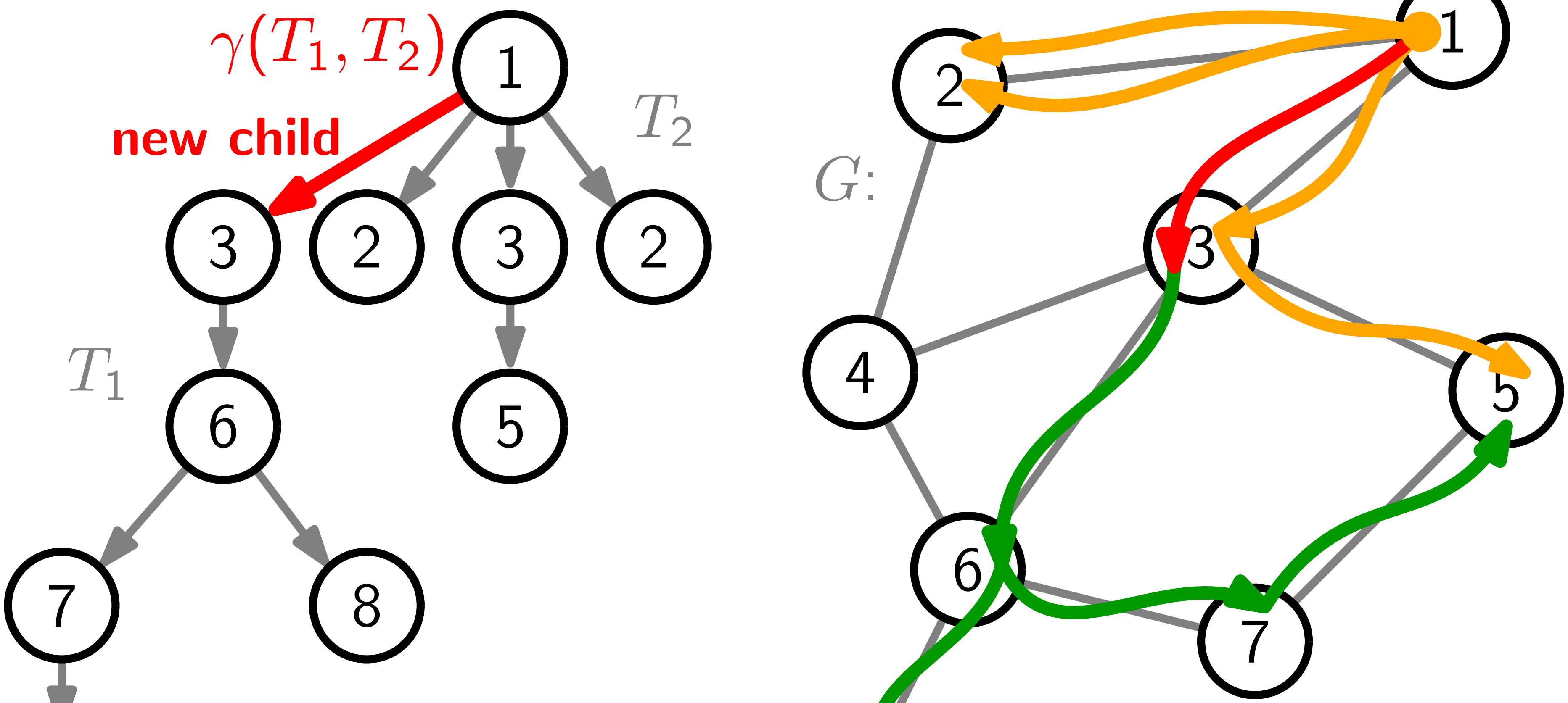


Def. $\gamma(A, B) :=$ result of making $root(A)$ first child of $root(B)$

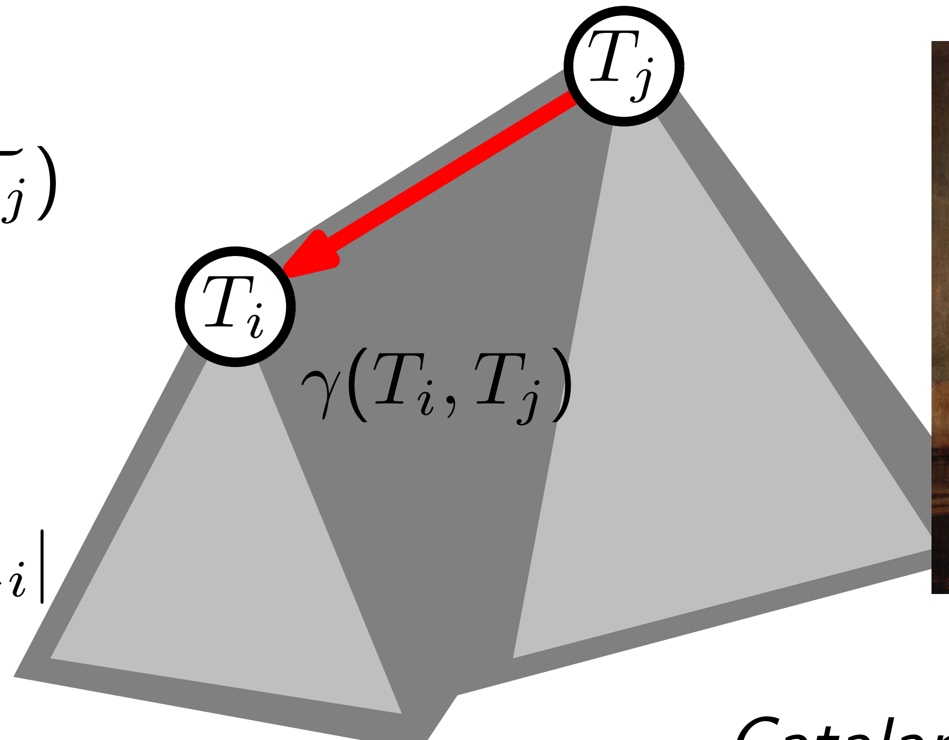
Tree Counting

Def. $\mathcal{T}_n :=$ set of different ordered rooted trees with n edges.

Obs. $\forall T \in \mathcal{T}_n \colon \exists!\ i, j \in \mathbb{N},\ T_i \in \mathcal{T}_i,\ T_j \in \mathcal{T}_j$
such that $T = \gamma(T_i, T_j) \ldots i + j = ?$

Bijective mapping:

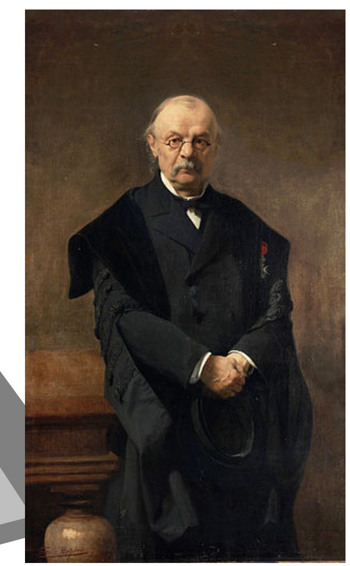
$$\mathcal{T}_n \leftrightarrow \bigcup_{i+j=n-1} (\mathcal{T}_i \times \mathcal{T}_j)$$



Theorem.

$$|\mathcal{T}_n| = \sum_{i=0}^{n-1} |\mathcal{T}_i| \cdot |\mathcal{T}_{n-1-i}|$$

$$|\mathcal{T}_0| = ?$$



Recall:
Catalan numbers

$C_0 = 1; C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$

Counting Branching Walks

$|\mathcal{T}_n|$ works for counting rooted ordered trees... still need φ.

Def. $\mathcal{B}_F(x,c)$:= all branching walks of type (T,φ)
- contained in $G[V \setminus F]$
- starting at $x \in V \setminus F$
- ~~with $\leq c$ edges~~ total edge weight $\leq c$

Let $b_F(x,c) := |\mathcal{B}_F(x,c)|$.

[and with edge weights $\omega\colon E \to \mathbb{Z}_{\geq 0}$?]

Thm. For $F \subseteq K$ and $x \in V \setminus F$:

$$b_F(x,c) = \begin{cases} 1 & \text{if } c = 0, \\ \displaystyle\sum_{t \in N(x) \setminus F} \sum_{c_1 + c_2 \leq \cancel{c-1}\; c - \omega(xt)} b_F(t,c_1) \cdot b_F(x,c_2) & \text{otherwise.} \end{cases}$$

Runtime: $O(n^2 \cdot n^3) = O(n^5)$ – unweighted case
$O(nc \cdot nc^2) = O(n^2c^3)$ – weighted case

Steiner Tree: Summary

Given: Graph $G = (V, E)$, terminals $K \subseteq V$, number c
and edge weights $\omega \colon E \to \mathbb{Z}_{\geq 0}$,

Question: Does there exist a subtree (V', E') of G such that
- $K \subseteq V'$ and
- $\cancel{|E'| \leq c}$? $\omega(E') \leq c$

IE-Formulation:

$s_0 \in K$

$\mathcal{U} = \{\text{branching walks with root } s_0 \text{ and weight} \leq c \text{ in } G\}$

$\mathcal{P} = \{P_v \mid v \in K \setminus \{s_0\}$,
where $P_v =$ "branching walk contains v" $\}$

Runtime: $O(2^k \cdot \text{poly}(n))$ unweighted
$O(2^k \cdot \text{poly}(n, c))$ weighted

Recall: $N(\mathcal{P}) = \sum_{F \subseteq \mathcal{P}} (-1)^{|F|} \underbrace{N(\emptyset, F, \mathcal{P} \setminus F)}_{\beta_F(s_0, c)}$

Degree-Constrained Spanning Tree

Given: Graph $G = (V, E)$, number $1 \leq c \leq n$

Question: $\exists$ spanning tree of G of maximum degree $\leq c$?

IE-Formulation:

$\mathcal{U} = \{$branching walks of length $n-1$ and of degree $\leq c$ in $G\}$

$\mathcal{P} = \{P_v \mid v \in V,$

where $P_v =$ "`branching walk contains` v" $\}$

Easier Problem: (find and solve it yourself!)

Runtime: $O^*(2^n)$, improving over $O^*(5.92^n)$
[Amini et al., ICALP'09]

Maximum Internal Spanning Tree

Given: Graph $G = (V, E)$, number $1 \leq c \leq n$

Question: $\exists$ spanning tree of G with $\geq c$ internal vertices?

IE-Formulation:

$\mathcal{U} = \{$branching walks of length $n - 1$ with $\geq c$ internal vtc.$\}$

$\mathcal{P} = \{P_v \mid v \in V,$

where $P_v =$ "`branching walk contains` v" $\}$

Easier Problem: (find and solve it yourself!)

Runtime: $O^*(2^n)$, improving over $O^*(3^n)$
[Fernau et al., WG'09]

Partitioning Numbers

PARTITION

Given: Set S of integers.

Question: $\exists$ partition of S into two sets with the same sum?

SUBSETSUM

Given: Set S of integers and an integer t.

Question: $\exists$ subset of S that sums to t?

3-PARTITION

Given: Set S of integers

Question: $\exists$ partition of S into 3-tuples with the same sum?

Subset Sum – A Question

Thm. SubSetSum is *weakly* NP-hard.

Obs. Standard DP needs $O^*(n \cdot t)$ time and space

Question: What is possible without depending on $\sum S$?

Obs. For S partitioned into $S_1 \mathbin{\dot\cup} S_2$, note:

$$S \text{ has a subset of sum } t$$

$$\Updownarrow$$

$$\exists t_1, t_2 \text{ such that } t_1 + t_2 = t \text{ and } S_1 \text{ has a subset of sum } t_1$$
$$\text{and } S_2 \text{ has a subset of sum } t_2$$

Subset Sum – A Solution

$$S = \{\underbrace{\ldots,\ldots,\ldots,\ldots,\ldots,\ldots}_{\text{sum } t_1},\underbrace{\ldots,\ldots,\ldots,\ldots,\ldots,\ldots}_{\text{sum } t_2}\}$$

Def. For S partitioned into $S_1 \cup S_2$:
$\sum_1 :=$ all possible subset sums of S_1
$\sum_2 :=$ all possible subset sums of S_2

Algo: Compute $\sum_1$ and $\sum_2$.

sort $\sum_1$ and $\sum_2$

Binary seach for each t_1!

Test: $\exists t_1 \in \sum_1, t_2 \in \sum_2$ with $t_1 + t_2 = t$

Runtime: $O(\sqrt{2}^n \cdot \log_2(\sqrt{2}^n)) \subseteq O(\sqrt{2}^n \cdot n) \subseteq O^*(\sqrt{2}^n)$

Can we get rid of this factor?