Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Summer Term 2020

Lecture 7. A General Approach to Inclusion–Exclusion

Based on: [Exact Exponential Algorithms: §3.1.2, §4.3.3]

Further reading: [Parameterized Algorithms: §10.1.3, 10.2]

see also: [J. Nederlof, J.M.M. van Rooij, T.C. van Dijk: Algorithmica (2014),
https://doi.org/10.1007/s00453-013-9759-2]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Thomas van Dijk

Lehrstuhl für Informatik I

Definitions & Notation

Notation: Universe $\mathcal{U}$, Properties $\mathcal{P}$

Def. (as before): Let $S \subseteq \mathcal{P}$.

$$N(S) := |\{e \in \mathcal{U} \mid e \text{ satisfies all properties in } S\}|$$

Def. (as before): Let $S \subseteq \mathcal{P}$.

$$\bar{N}(S) := |\{e \in \mathcal{U} \mid e \text{ satisfies no properties in } S\}|$$

Thm (as before): $N(\mathcal{P}) = \sum_{S \subseteq \mathcal{P}} (-1)^{|S|} \bar{N}(S)$

Idea: Sometimes it is easier to compute $\bar{N}(\cdot)$ than $N(\cdot)$.

"Simplified Problem"

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Example: st-Hamiltonpath

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Example: st-Hamiltonpath

- $\mathcal{U} = \{st\text{-walks of length } n\}$
- $\mathcal{P} = \{P_v \mid v \in V$, and $P_v = \texttt{"walk goes through } v\texttt{"}\}$

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Def.: Let $R \mathbin{\dot\cup} F \mathbin{\dot\cup} O = \mathcal{P}$.

$$N(R, F, O) := |\{e \in \mathcal{U} \mid e \text{ satisfies all properties in } R \text{ and none in } F\}|$$

Required **F**orbidden

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Obs.: For $S \subseteq \mathcal{P}$, $N(S) = N(S, \varnothing, \mathcal{P} \setminus S)$

Obs.: For $S \subseteq \mathcal{P}$, $\bar{N}(S) = N(\varnothing, S, \mathcal{P} \setminus S)$

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Obs.: For $R \mathbin{\dot\cup} F \mathbin{\dot\cup} O \mathbin{\dot\cup} \{p\} = \mathcal{P}$,

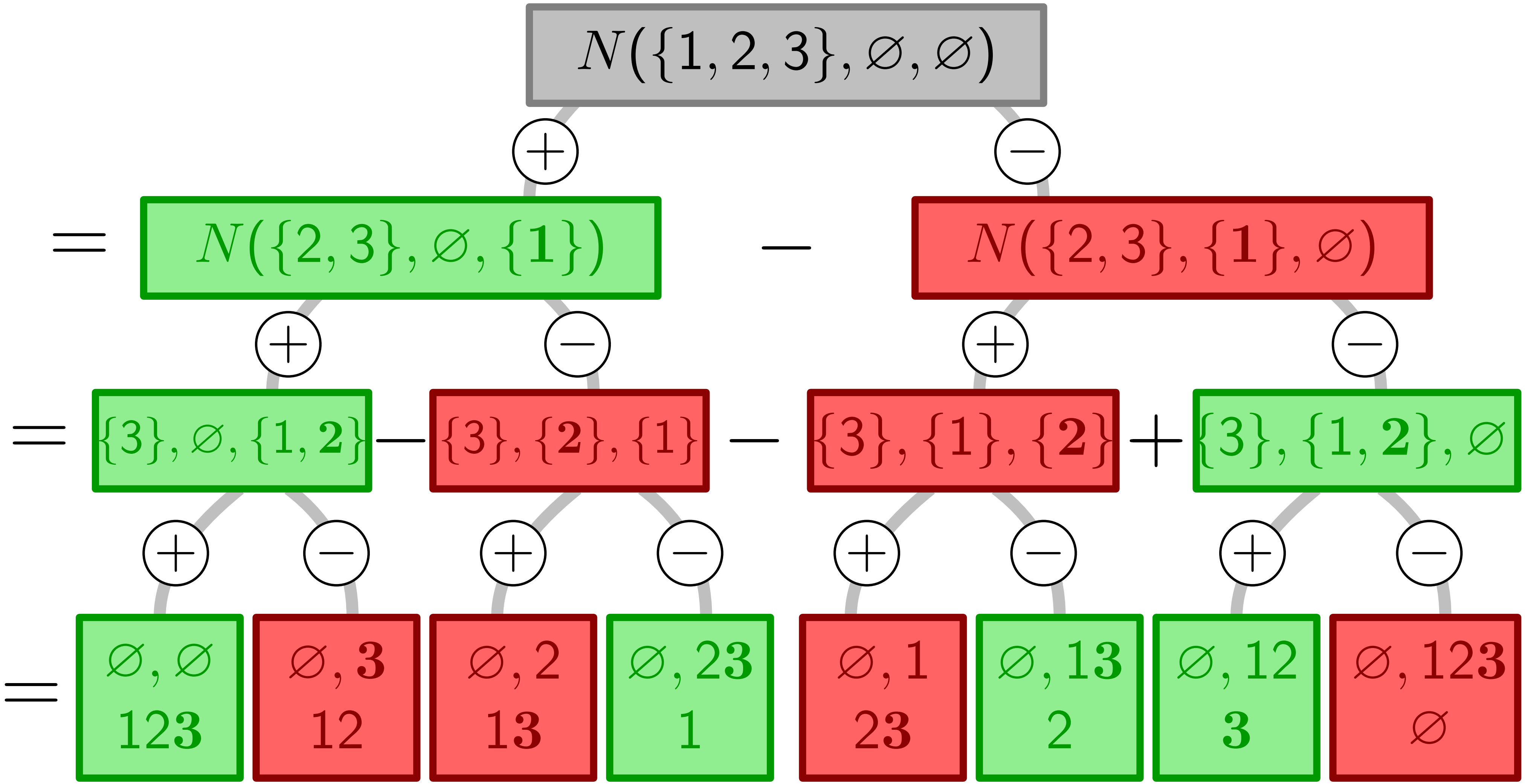
$$N(R, F, O \cup \{p\}) = N(R \cup \{p\}, F, O) + N(R, F \cup \{p\}, O)$$

$$N(R \cup \{p\}, F, O) = N(R, F, O \cup \{p\}) - N(R, F \cup \{p\}, O)$$

$N(R, F, O)$ – Required, Forbidden, Optional

Thm: For $R \mathbin{\dot\cup} F \mathbin{\dot\cup} O = \mathcal{P}$ and $e \in R$,

$$N(R, F, O) = N(R \setminus \{p\}, F, O \mathbin{\dot\cup} \{p\}) - N(R \setminus \{p\}, F \mathbin{\dot\cup} \{p\}, O)$$



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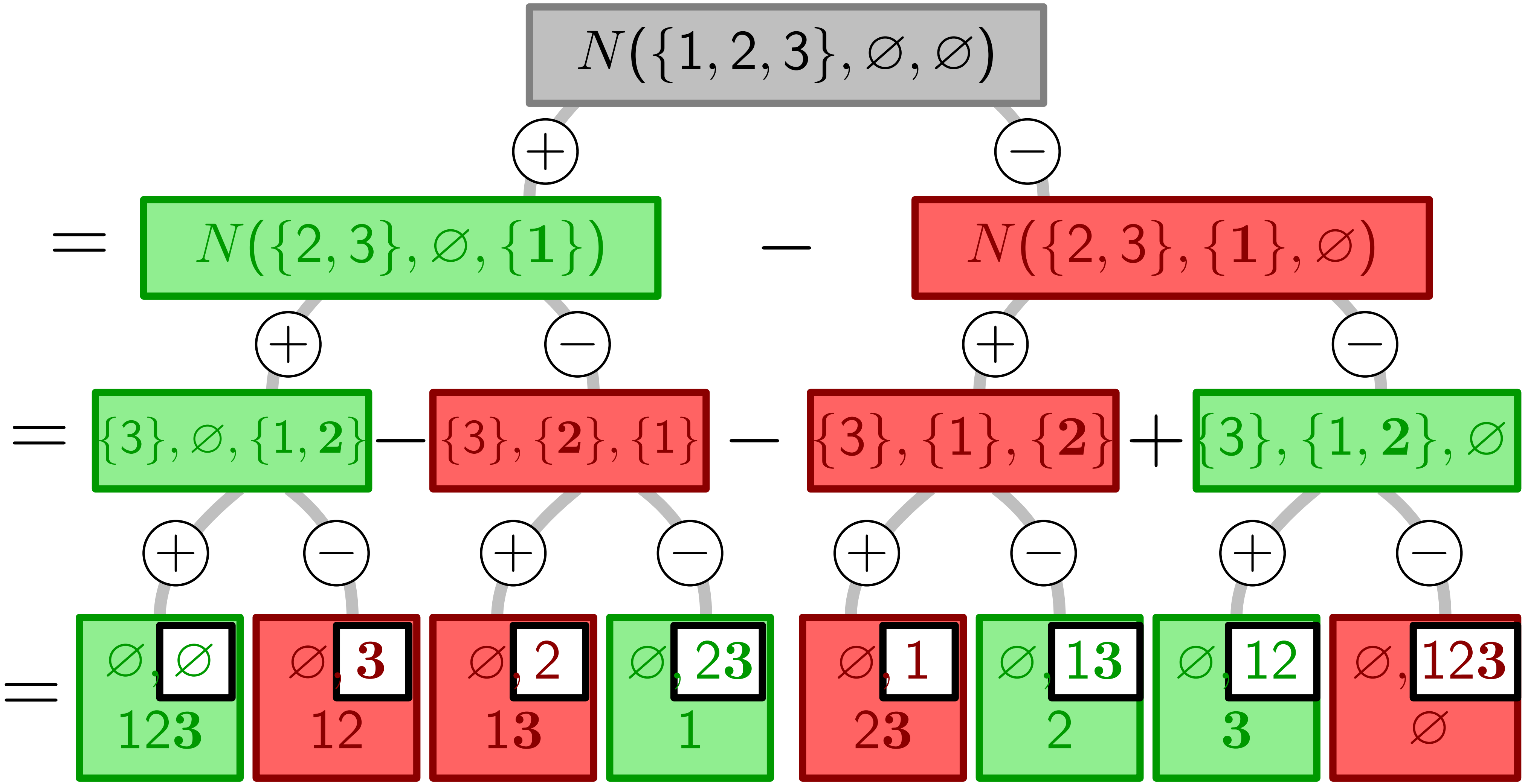

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$$N(R, F, O) = N(R \setminus \{p\}, F, O \uplus \{p\}) - N(R \setminus \{p\}, F \uplus \{p\}, O)$$

$$N(\{1, 2, 3\}, \varnothing, \varnothing)$$

$$N(\mathcal{P}) \stackrel{\text{def}}{=} N(\mathcal{P}, \varnothing, \varnothing)$$
$$= \sum_{F \subseteq \mathcal{P}} (-1)^{|F|} N(\varnothing, F, \mathcal{P} \setminus F)$$
$$\stackrel{\text{def}}{=} \sum_{S \subseteq \mathcal{P}} (-1)^{|S|} \bar{N}(S)$$

$=$

$\varnothing, \varnothing$	$\varnothing, 3$	$\varnothing, 2$	$\varnothing, 23$	$\varnothing, 1$	$\varnothing, 13$	$\varnothing, 12$	$\varnothing, 123$
123	12	13	1	23	2	3	$\varnothing$

Using: Required-Forbidden-Optional

Problem: st-Hamiltonian Path

- $\mathcal{U} = \{st\text{-walks of length } n\}$
- $\mathcal{P} = \{P_v \mid v \in V,\ P_v = \texttt{"walk goes through } v\texttt{"}\ \}$

Solution: $N(V, \varnothing, \varnothing)$

Easier Problem: $N(R, F, O)$ is easy when $R = \varnothing$

Strategy: For $e \in R$.

$$N(R, F, O) = N(R \setminus \{e\}, F, O \mathbin{\dot\cup} \{e\}) - N(R \setminus \{e\}, F \mathbin{\dot\cup} \{e\}, O)$$

"Inclusion-Exclusion"

Using: Required-Forbidden-Optional

Problem: # Independent Sets

- $\mathcal{U} = \{\text{Independent Sets}\}$
- $\mathcal{P} = \{P_v \mid v \in V$, where $P_v =$ "`set contains` v"$\}$

Solution: $N(\varnothing, \varnothing, V)$

Easier Problem: $N(R, F, O)$ is easy when $O = \varnothing$

Strategy: For $v \in O$,
$N(R, F, O) = N(R \cup \{v\}, F, O \setminus \{v\}) + N(R, F \cup \{e\}, O \setminus \{v\})$

standard branching algorithm

Graph Coloring

Given: Graph $G = (V, E)$, number k

Question: $\exists$ proper coloring of V with k colors?

$\equiv \exists$ cover of V by k independent sets?

IE-Formulation:

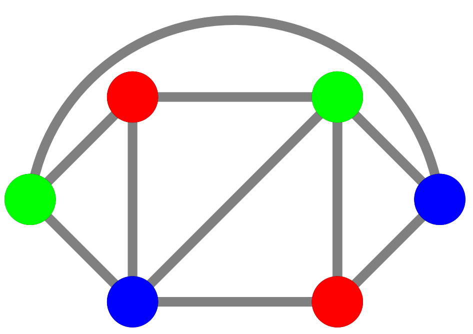
$\mathcal{U} = \{k\text{-tuple of independent sets in } G\}$

$\mathcal{P} = \{P_v \mid v \in V,$

where $P_v =$ `"tuple contains a set with` v `in it"`$\}$

Lemma:

Graph Coloring: G k-colorable $\Leftrightarrow N(\mathcal{P}) > 0$



Easier Problem

Thm.
Graph Coloring can be solved with 2^n queries of $\bar{N}(\cdot)$.

$\mathcal{U} = \{k\text{-tuple of independent sets in } G\}$

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What is the inuitive meaning of $\bar{N}(S)$ for $S \subseteq \mathcal{P}$?

"How many k-tuples of independent sets are there that avoid the vertices in S?"

Def.: $a(S) := \#$ independent sets that avoid S

Lemma: $\bar{N}(S) = a(S)^k$ **Proof:** k sets, each from $a(S)$ (with replacement)

Easier Problem

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Counting Independent Sets

Def.: $a(S) := \#$ independent sets that avoid S

Algorithm 1:
Enumerate all subsets of $V \setminus S$: test independence

Runtime: $O^*(2^{n-|S|})$

Binomial Thm:

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

Thm.
Using Algorithm 1 for Graph Coloring gives us

$$O^*(3^n) \ni \sum_{i=0}^{n} \binom{n}{n-i} 2^{n-i} \mathrm{poly}(n) \quad \text{Time}$$

polynomial Space

Counting Independent Sets

Def.: $a'(S) := \#$ maximal independent sets that avoid S

Algorithm 2:
Enumerate all maximal independent sets of $G[V \setminus S]$

Runtime: $O^*(\sqrt[3]{3}^{n-|S|})$ [as in Lecture 1]

Thm:
Using Algorithm 2 for Graph Coloring gives us

$$O(2.4423^n) \supset \quad O^*((1+\sqrt[3]{3})^n) \quad \text{Time}$$

Runtime from Lawler (1976)

polynomial Space

Counting Independent Sets

Def.: $a(S) :=$ # independent sets that avoid S

Algorithm 3:
Use the algorithm of Fürer & Kasiviswanathan (2007)

Runtime: $O(1.1247^n)$

F&K enumerates satisfying assignments for 2-SAT instances.
$\rightsquigarrow$ enumeration of independent sets :)

Thm:
Using Algorithm 3 for Graph Coloring gives us

5- and 6-coloring from last week are now obsolete

$O(2.1247^n) \supset O((1 + 1.1247)^n)$ Time

Lawler: $O(2.4423^n)$

polynomial Space

Counting Independent Sets

Def.: $a(S) := \#$ independent sets that avoid S

Algorithm 4:
Compute $a(S)$ for each $S \subseteq V$ by DP

Runtime: $O^*(2^n)$ **in total**

Thm:
Using Algorithm 4 for Graph Coloring gives us

$O^*(2^n)$ Time

$O^*(2^n)$ Space

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Runtime: $O^*(2^n)$ **in total**

Thm:
Using Algorithm 4 for Graph Coloring gives us

$O^*(2^n)$ Time

$O^*(2^n)$ Space

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) :=$ # independent sets that avoid S

Def.: $a(R, F, O) :=$ # independent sets that **contain** R and **avoid** F.

independent

Obs.: For $S \subseteq V$, $\quad a(S) = a(\emptyset, S, V \setminus S)$

Obs.: For $R \dot\cup F = V$, $a(R, F, \emptyset) = [R \text{ independent?}] \in \{0, 1\}$

Lemma: For $R \dot\cup F \dot\cup O \dot\cup \{v\} = V$,

$$a(R, F, O \dot\cup \{v\}) = a(R \dot\cup \{v\}, F, O) + a(R, F \dot\cup \{v\}, O)$$

Obs.: For $R \subseteq V$, R not independent $\Rightarrow a(R, \cdot, \cdot) = 0$

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) := \#$ independent sets that avoid S

Def.: $a(R, F, O) := \#$ independent sets that **contain** R and **avoid** F.

Obs.: For $S \subseteq V$, $\quad a(S) = a(\varnothing, S, V \setminus S)$

Obs.: For $R \cup F = V$, $a(R, F, \varnothing) = [R \text{ independent?}] \in \{0, 1\}$

Lemma: For $R \cup F \cup O \cup \{v\} = V$,

$$a(R, F, O \cup \{v\}) = a(R \cup \{v\}, F, O) + a(R, F \cup \{v\}, O)$$

Obs.: For $R_1 \cup F_1 \cup O = R_2 \cup F_2 \cup O = V$, and R_i's indep.
$\nexists$ edge between R_i and $O \Rightarrow a(R_1, F_1, O) = a(R_2, F_2, O)$.

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) :=$ # independent sets that avoid S

Def.: $a(R, F, O) :=$ # independent sets that **contain** R and **avoid** F.

Obs.: For $S \subseteq V$, $\quad a(S) = a(\varnothing, S, V \setminus S)$

Obs.: For $R \dot{\cup} F = V$ and R indep.. $\quad a(R, F, \varnothing) = 1$

Lemma: For $R \dot{\cup} F \dot{\cup} O = V$ and $v \in O$, $\qquad U :=$ neighborhood

$$a(R, F, O) = a(R, F \dot{\cup} \{v\}, O \setminus \{v\}) + a(R \dot{\cup} \{v\}, F \dot{\cup} U(v), O \setminus U[v])$$

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) :=$ # independent sets that avoid S

Def.: $a(R, F, O) :=$ # independent sets that **contain** R and **avoid** F.

Obs.: For $S \subseteq V$, $\quad a(S) = a(\varnothing, S, V \setminus S)$

Obs.: For $R \,\dot\cup\, F = V$ and R indep.. $\quad a(R, F, \varnothing) = 1$

Lemma: For $R \,\dot\cup\, F \,\dot\cup\, O = V$ and $v \in O$, and R independent, and $\nexists$ edge between R and O

$$a(R, F, O) = a(R, F \,\dot\cup\, \{v\}, O \setminus \{v\})$$
$$+ \, a(R \,\dot\cup\, \{v\}, F \,\dot\cup\, U(v), O \setminus U[v])$$

independent

no edges between $R \,\dot\cup\, \{v\}$ and $O \setminus U[v]$

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) :=$ # independent sets that avoid S

Def.: $a(R, F, O) :=$ # independent sets that **contain** R and **avoid** F.

Obs.: For $S \subseteq V$, $\quad a(S) = a(\emptyset, S, V \setminus S)$

Obs.: For $R \,\dot\cup\, F = V$ and R indep.. $\quad a(R, F, \emptyset) = 1$

Lemma: For $R \,\dot\cup\, F \,\dot\cup\, O = V$ and $v \in O$, and R independent, and $\nexists$ edge between R and O

$$b(O) = b(O \setminus \{v\}) + b(O \setminus U[v])$$

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) :=$ # independent sets that avoid S

Def.: $a(R, F, O) :=$ # independent sets that **contain** R and **avoid** F.

Obs.: For $S \subseteq V$, $a(S) = a(\emptyset, S, V \setminus S) = b(V \setminus S)$

Obs.: For $R \cup F = V$ and R indep.. $a(R, F, \emptyset) = 1$

Lemma: For $R \cup F \cup O = V$ and $v \in O$, and R independent, and $\nexists$ edge between R and O

$$b(O) = b(O \setminus \{v\}) + b(O \setminus U[v])$$

$$b(\emptyset) = 1$$

Thm: Table with $a(S)$ for each $S \subseteq V$ can be computed in $O^*(2^n)$ time. □

Graph Coloring: Summary

Given: Graph $G = (V, E)$, number k

Question: $\exists$ proper k-coloring of V?

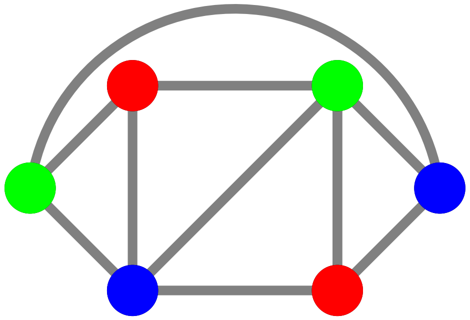
IE-Formulation:

$\mathcal{U} = \{k\text{-tuple of independent sets from } G\}$

$\mathcal{P} = \{P_v \mid v \in V$, where $P_v =$ `"tuple contains a set with` v `in it"`$\}$

Algorithm:

- Compute $\bar{N}(S) = a(S)^k$ for each $S \subseteq V$
- Apply Inclusion-Exclusion



Thm: Graph Coloring can be decided using $O^*(2^n)$ time and space