Julius-Maximilians-UNIVERSITÄT WÜRZBURG
Lehrstuhl für INFORMATIK I Algorithmen & Komplexität
Institut für Informatik

Exact Algorithms

Summer Term 2020

Lecture 7. A General Approach to Inclusion–Exclusion

Based on: [Exact Exponential Algorithms: §3.1.2, §4.3.3]

Further reading: [Parameterized Algorithms: §10.1.3, 10.2]

see also: [J. Nederlof, J.M.M. van Rooij, T.C. van Dijk: Algorithmica (2014), https://doi.org/10.1007/s00453-013-9759-2]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Thomas van Dijk

Lehrstuhl für Informatik I

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"Simplified Problem"

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Thm: For $R \mathbin{\dot\cup} F \mathbin{\dot\cup} O = \mathcal{P}$ and $e \in R$,
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Suppose: $N(R, F, O)$ is easy when $R = \varnothing$

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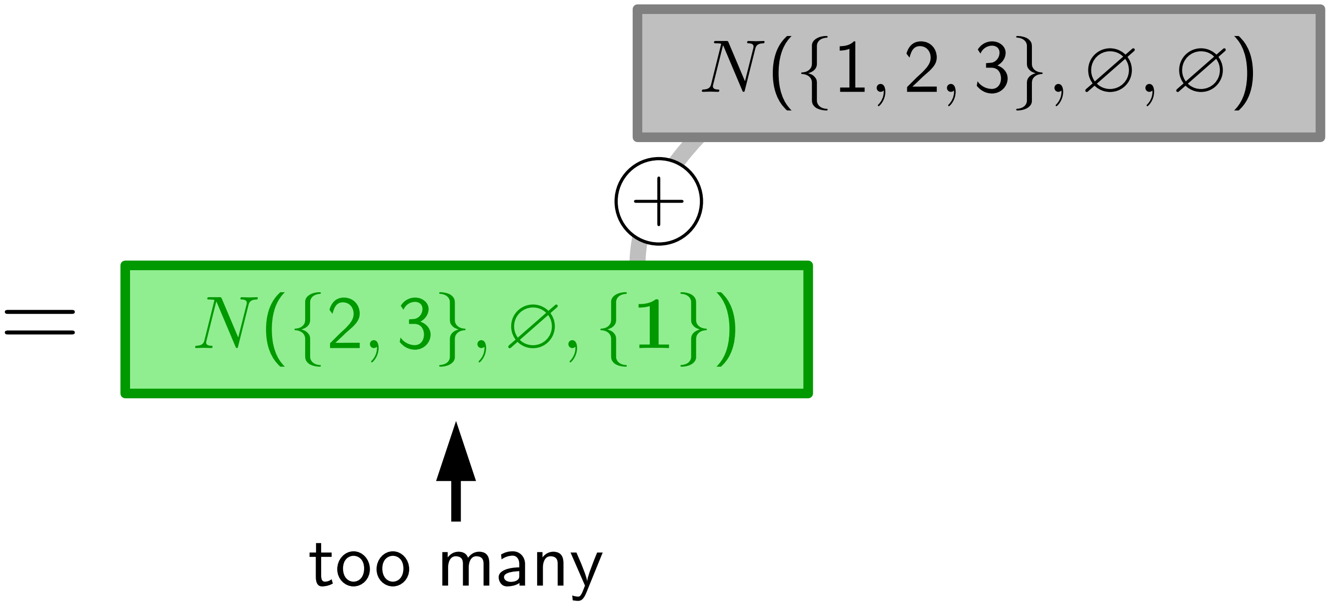

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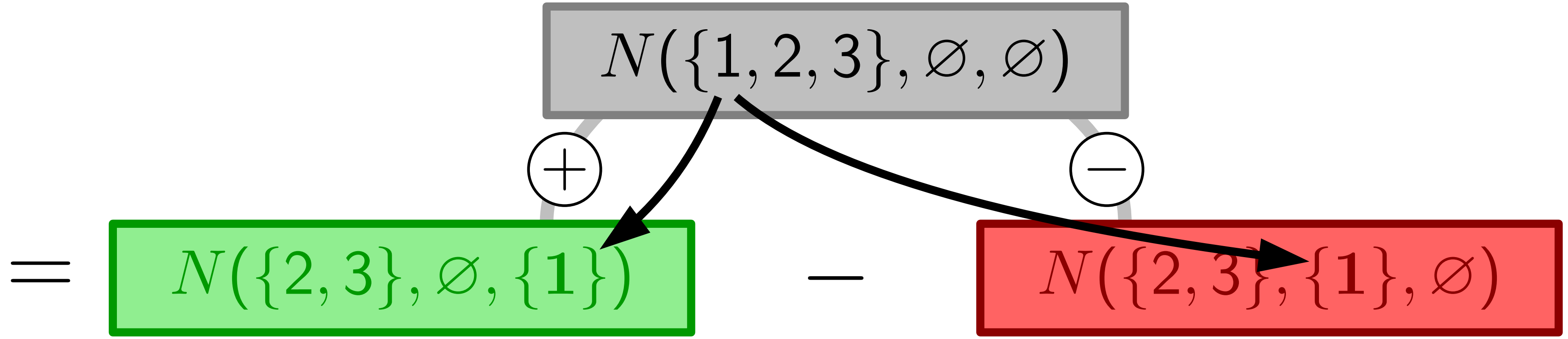

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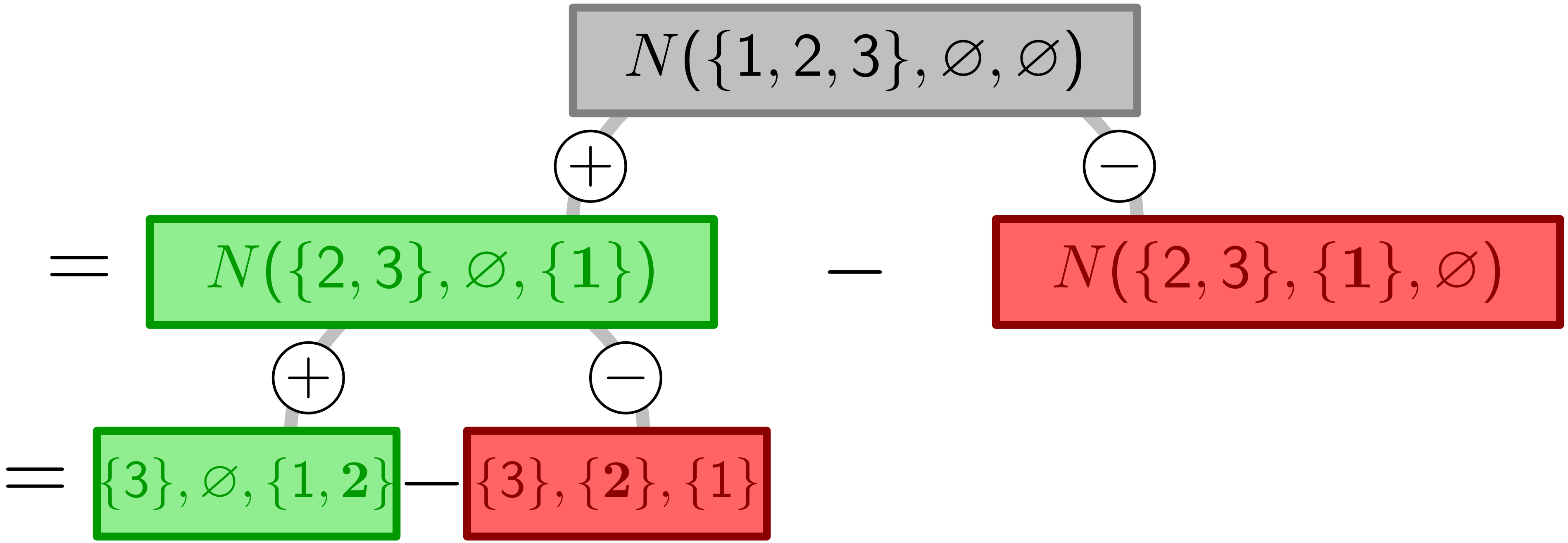

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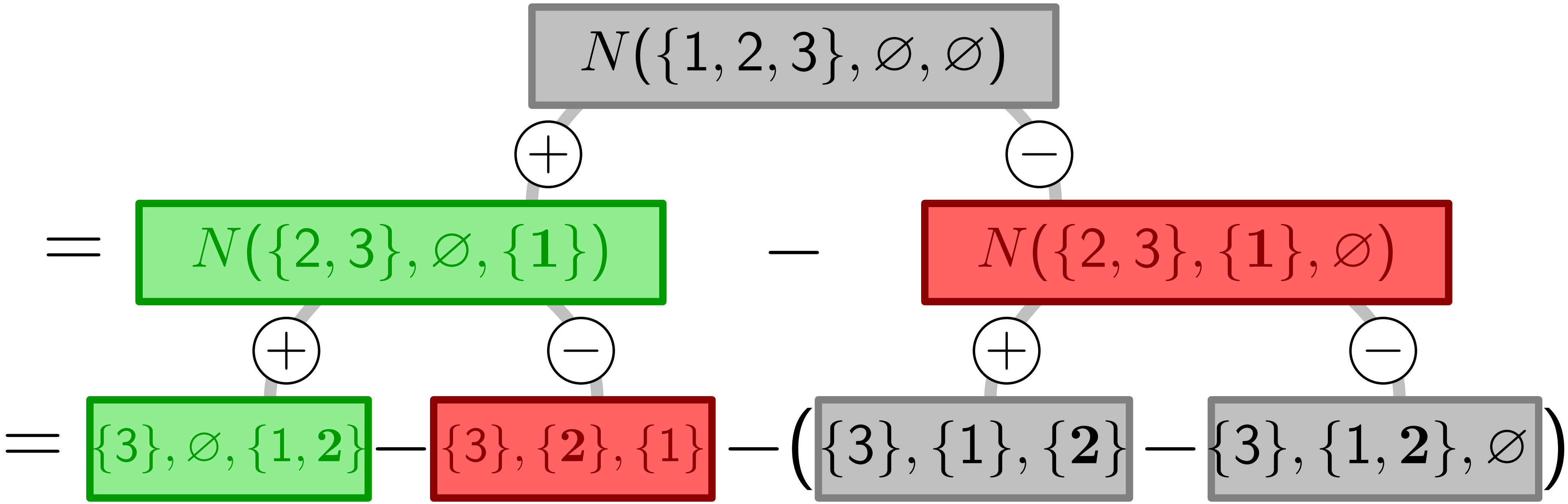

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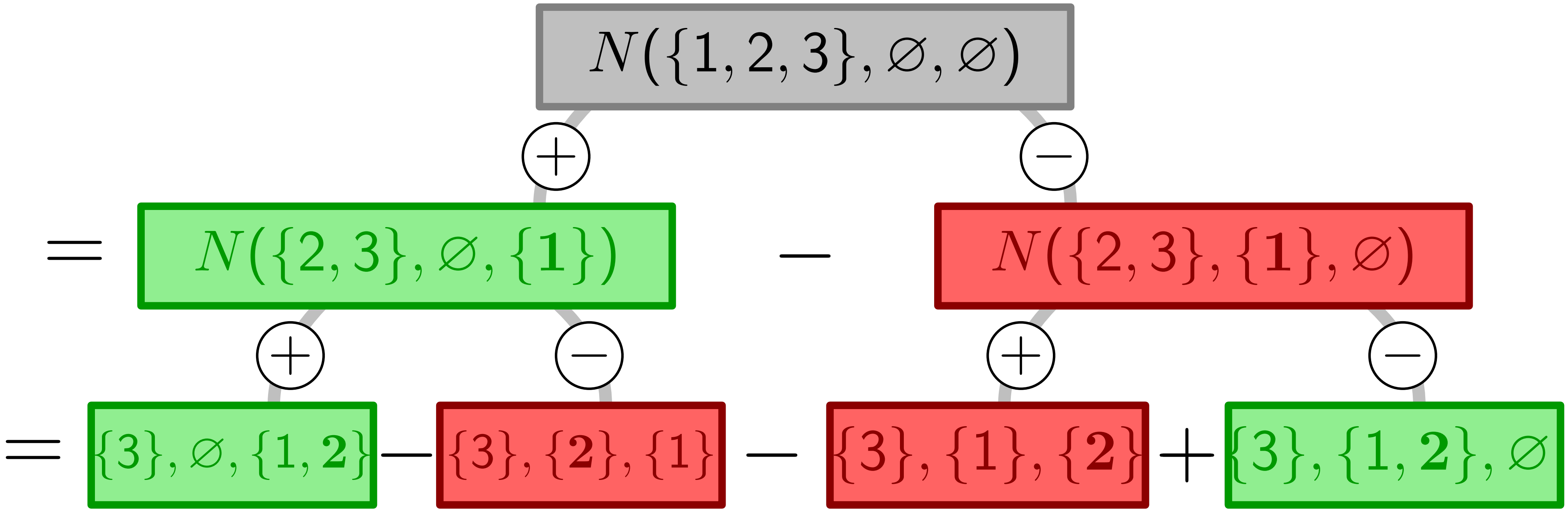

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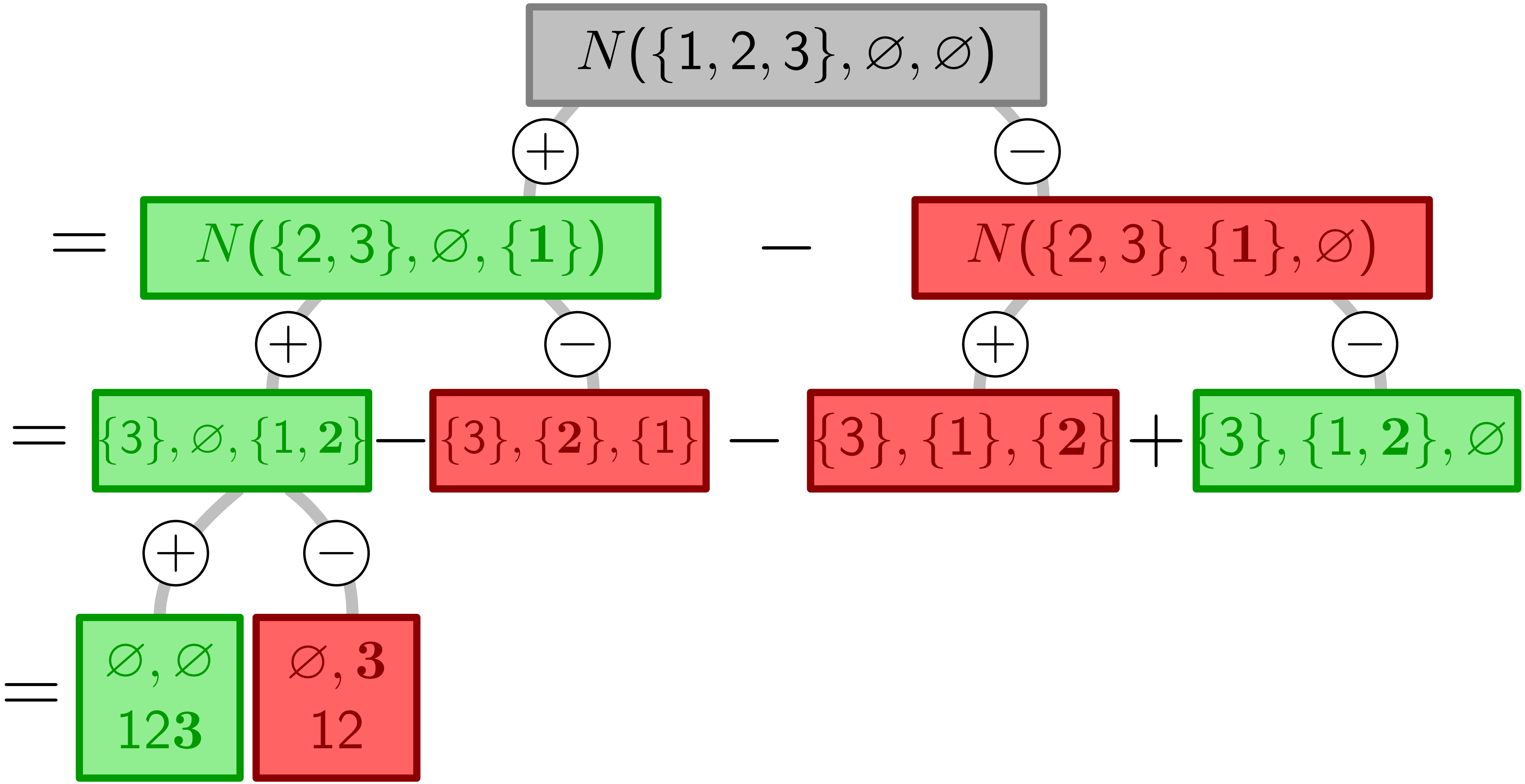

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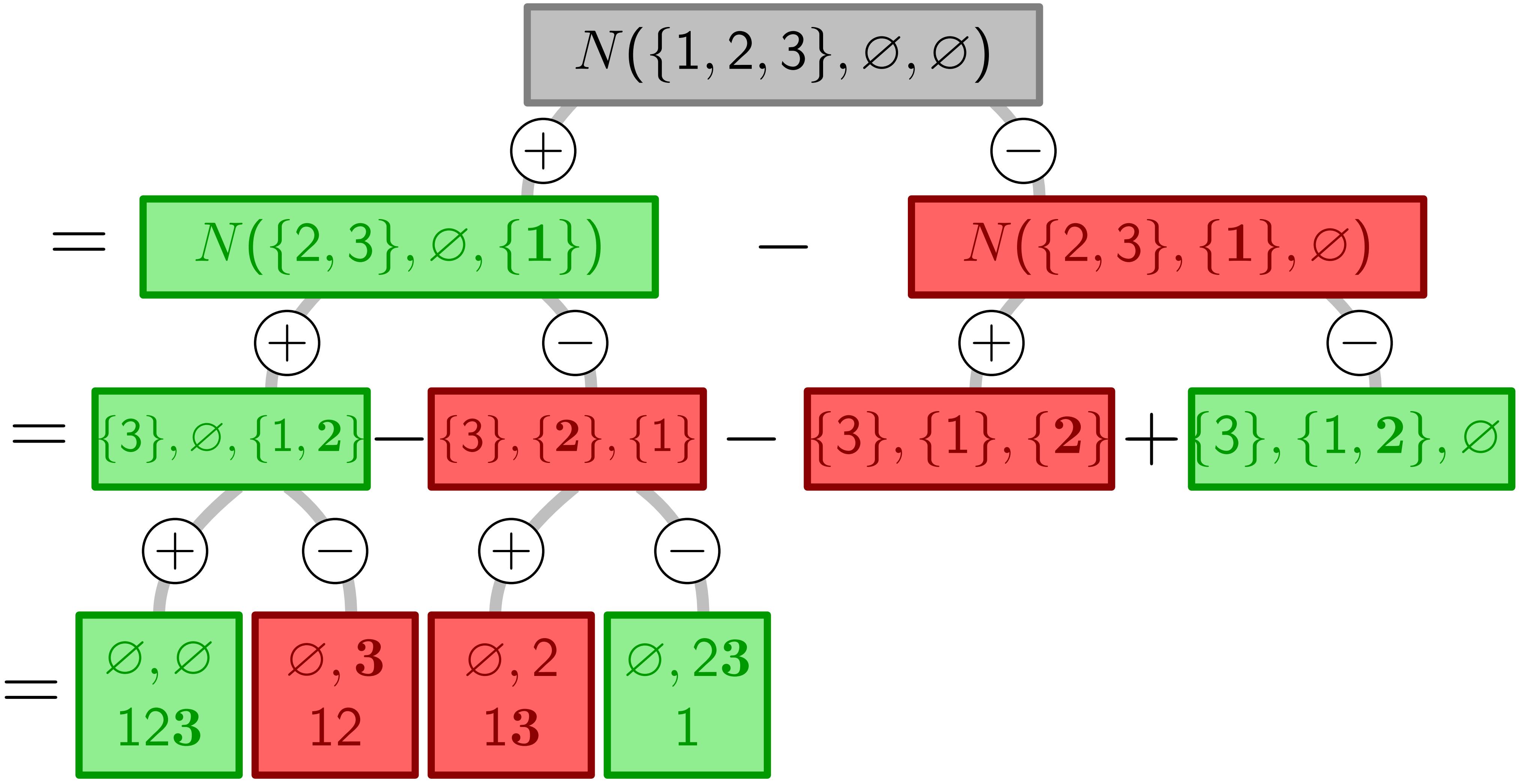

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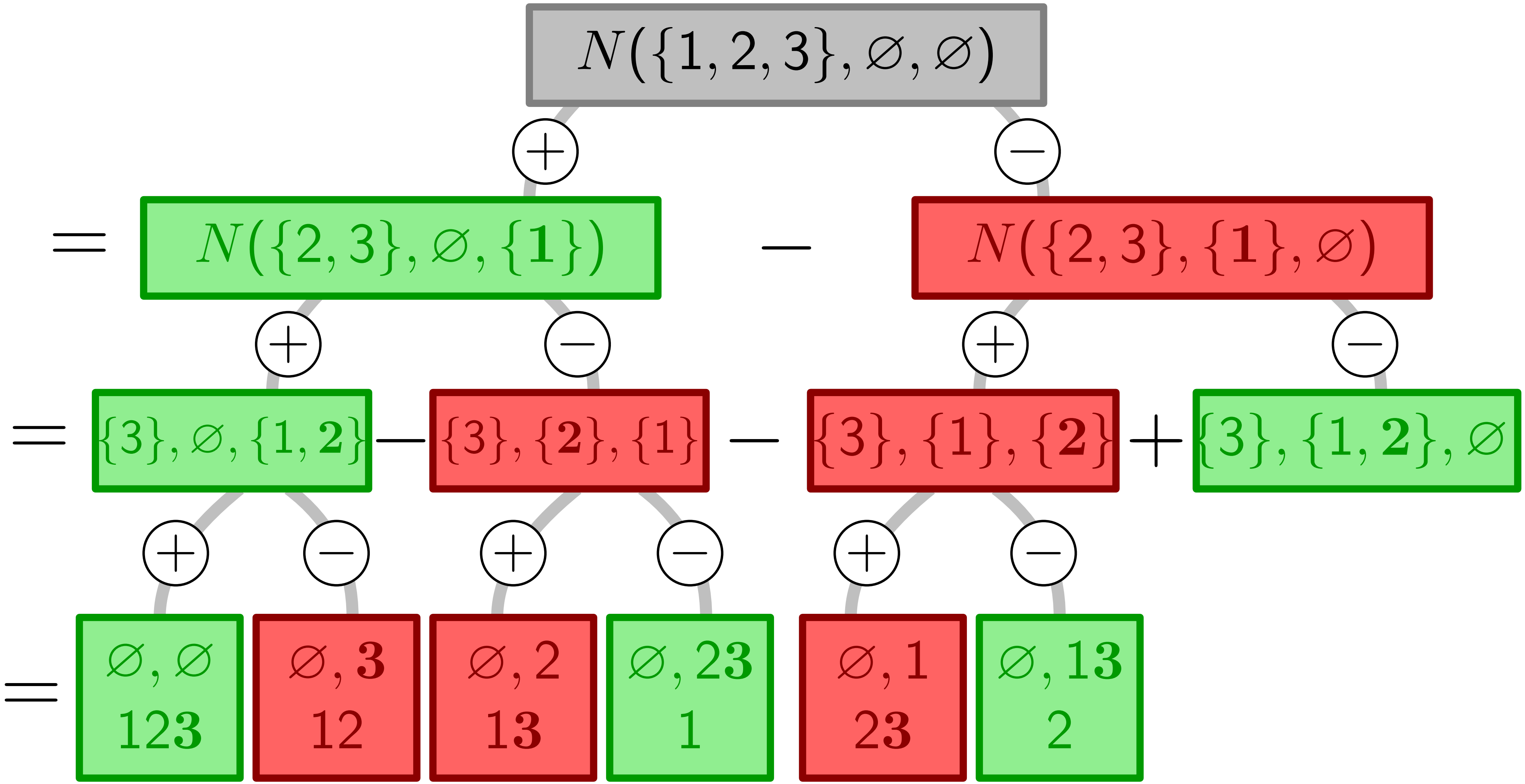

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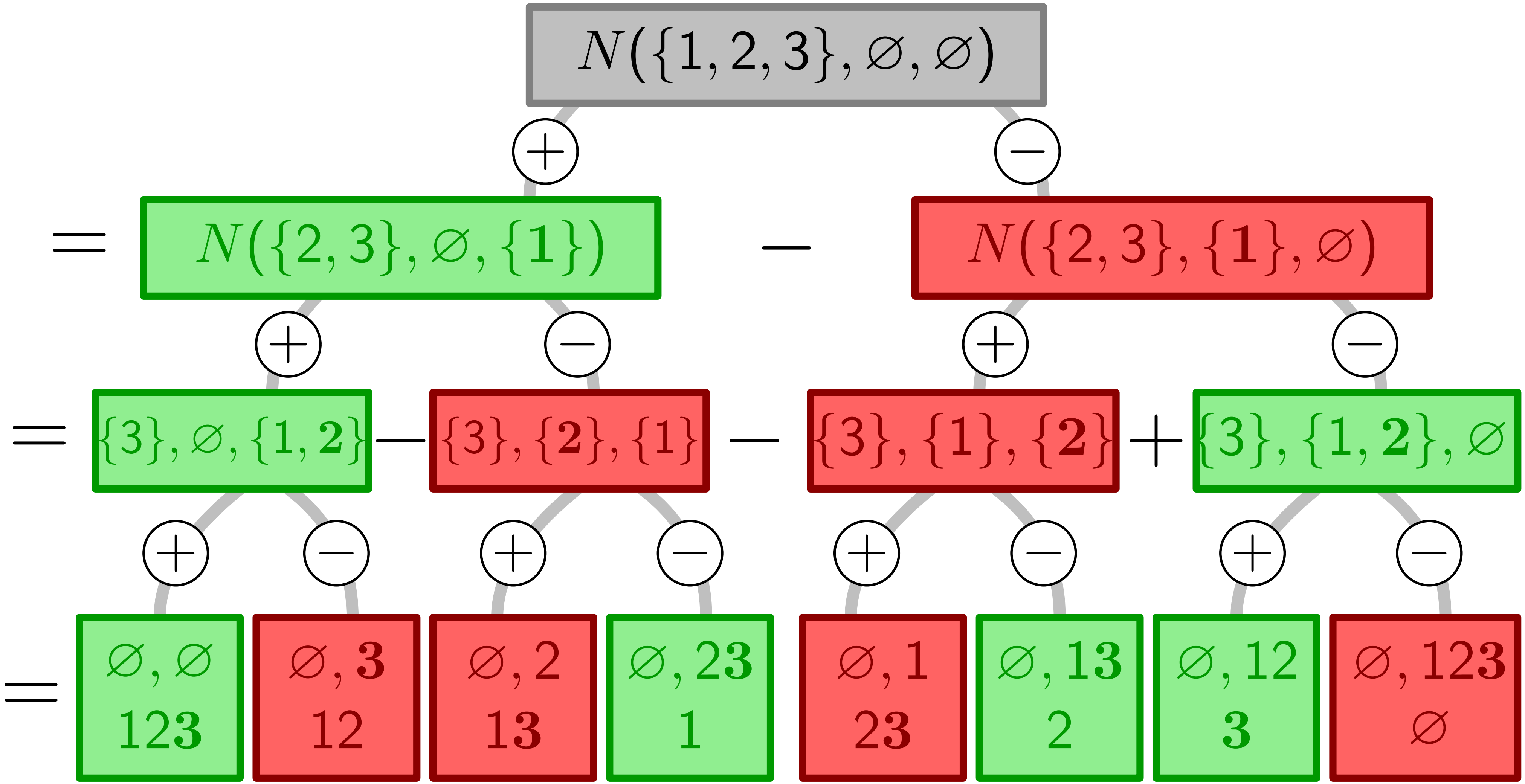

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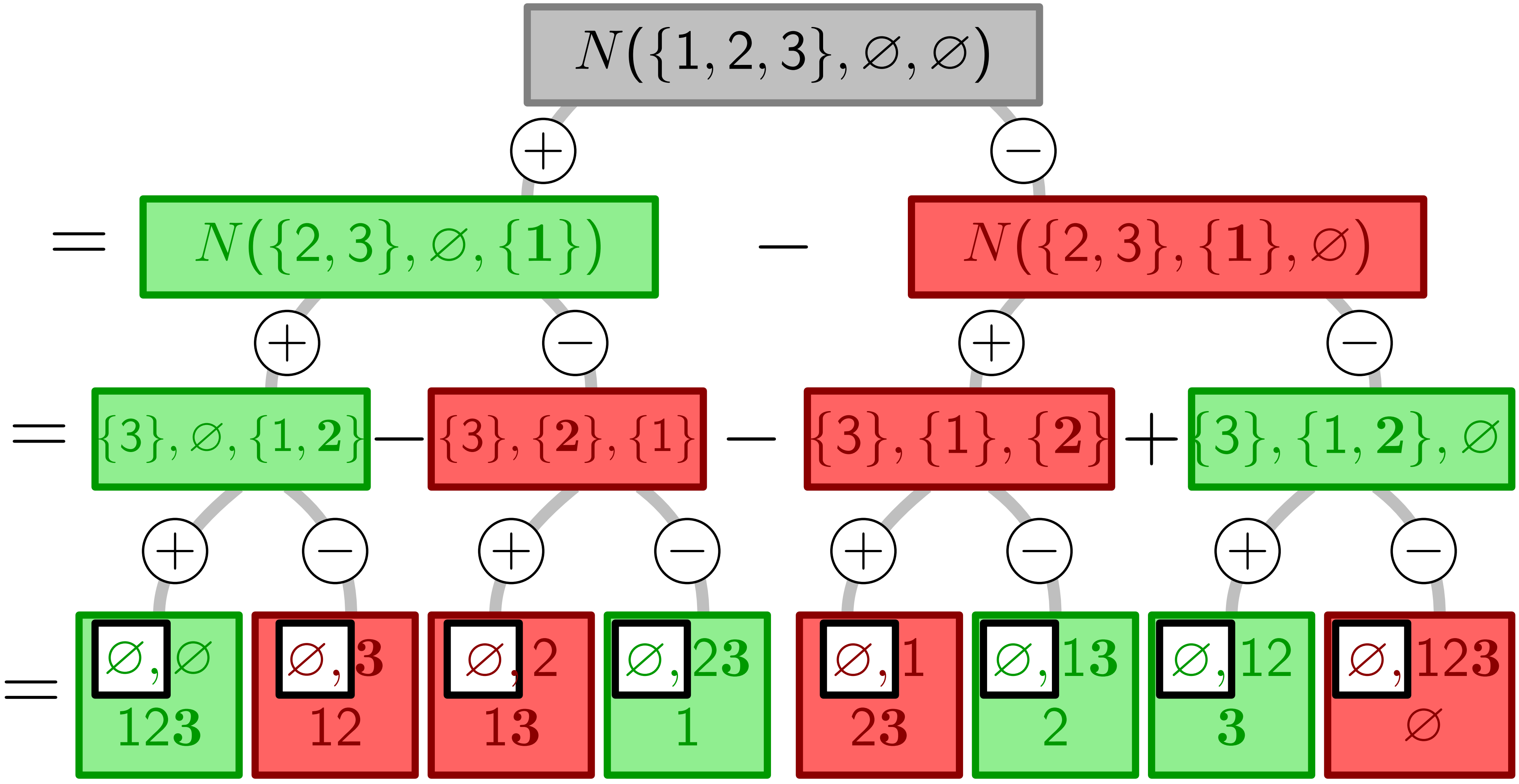

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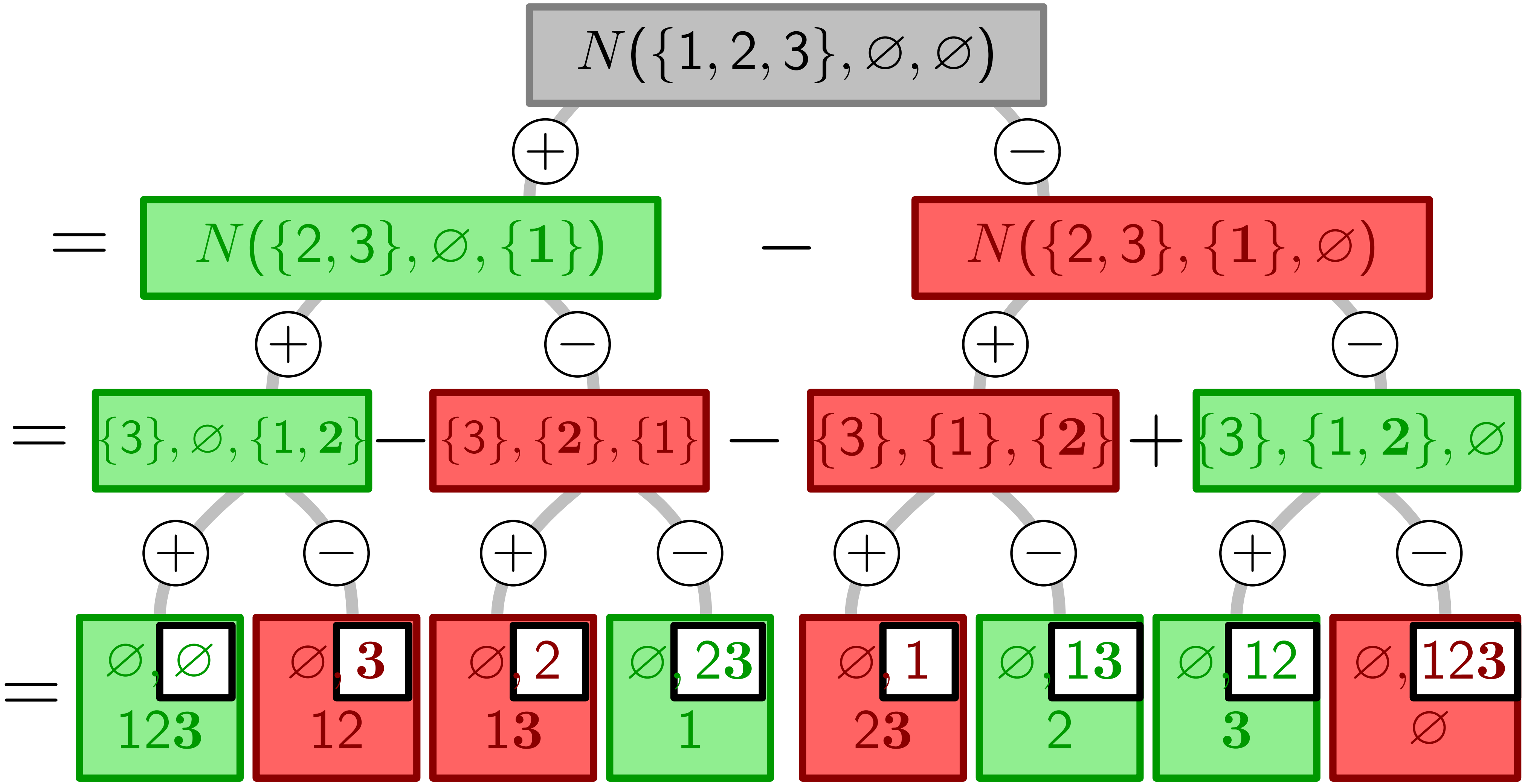

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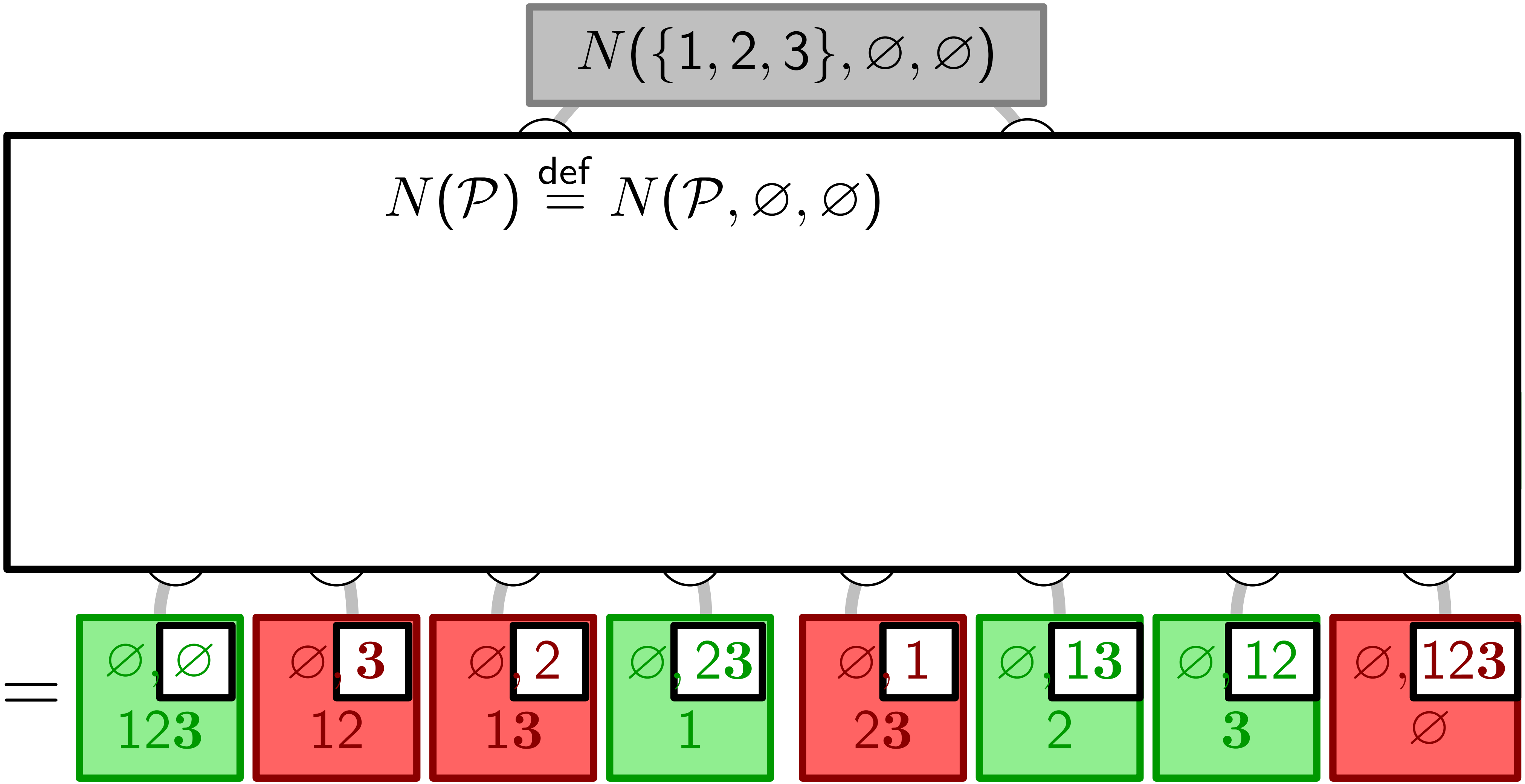

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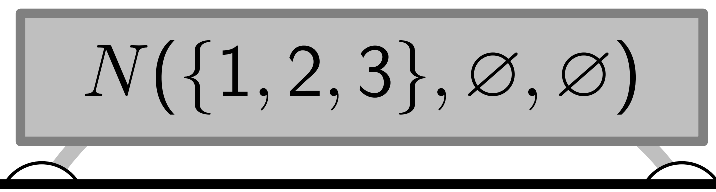

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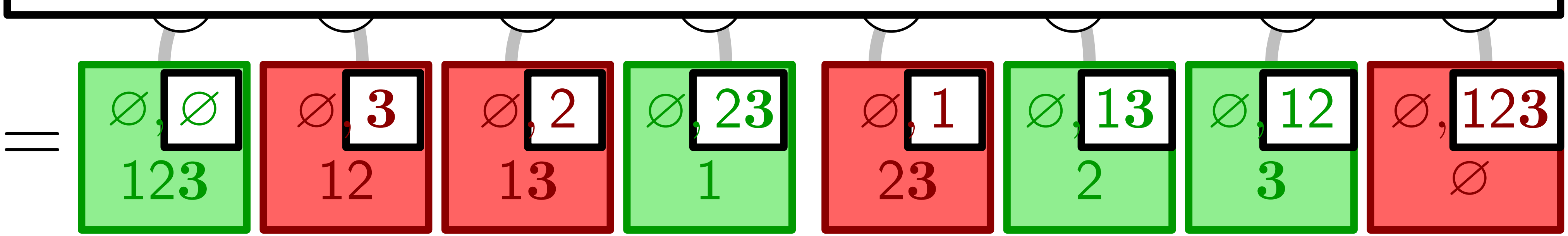

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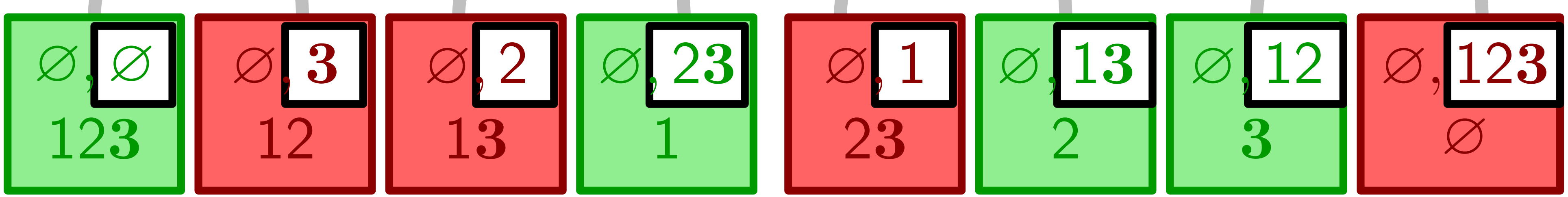
$=$ 

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"Inclusion-Exclusion"

Using: Required-Forbidden-Optional

Problem: # Independent Sets

- $\mathcal{U} = \{\text{Independent Sets}\}$
- $\mathcal{P} = \{P_v \mid v \in V$, where $P_v =$ "`set contains` v"$\}$

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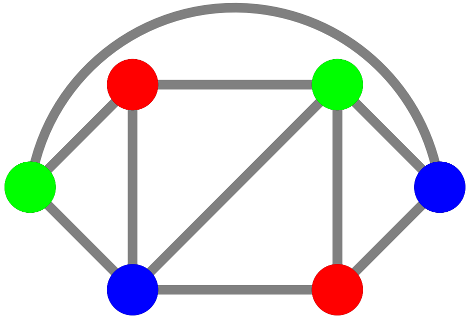
standard branching algorithm

Graph Coloring

Given: Graph $G = (V, E)$, number k

Question: $\exists$ proper coloring of V with k colors?



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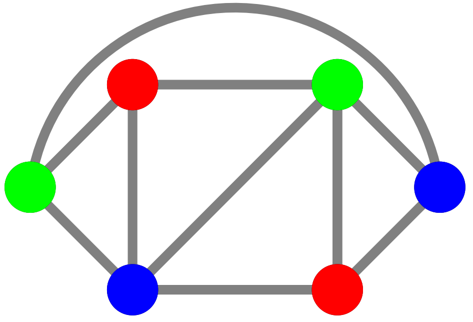

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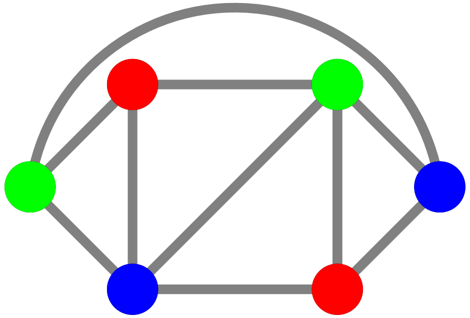

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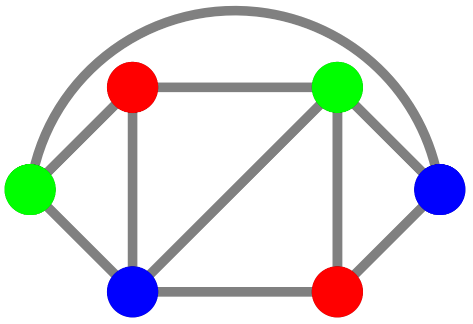

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ordered! (different from the others!)

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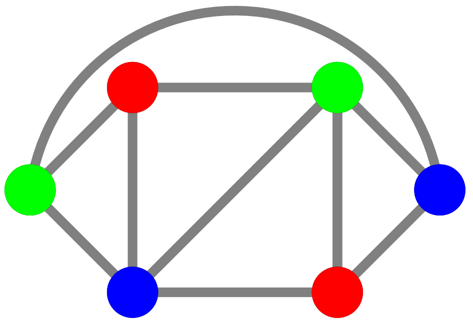

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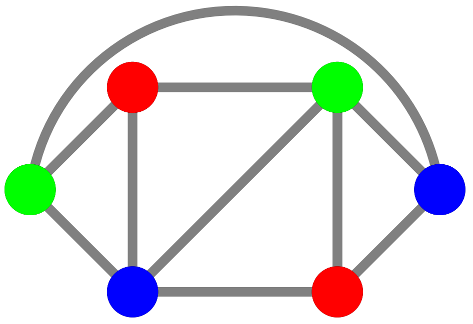

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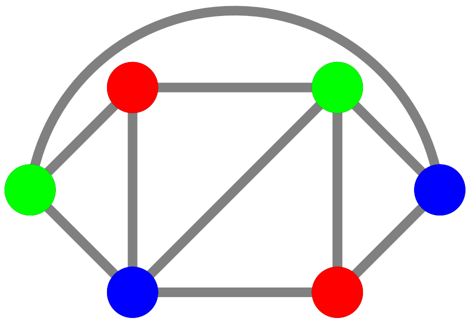
Lemma:

Graph Coloring: G k-colorable $\Leftrightarrow N(\mathcal{P}) > 0$



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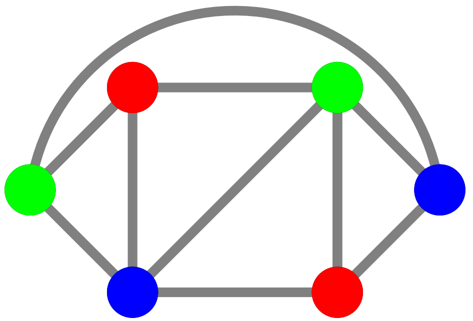
Graph Coloring can be decided by $2^{|\mathcal{P}|} = 2^n$ queries of $\bar{N}(\cdot)$



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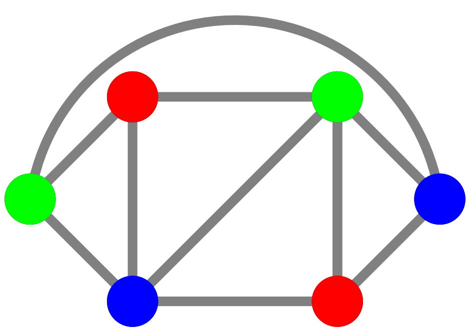

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Using Algorithm 1 for Graph Coloring gives us

Time

Space

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polynomial Space

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Time

polynomial Space

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polynomial Space

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Thm:
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$O^*((1+\sqrt[3]{3})^n)$ Time

polynomial Space

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Thm:
Using Algorithm 2 for Graph Coloring gives us

$$O(2.4423^n) \supset \quad O^*((1+\sqrt[3]{3})^n) \quad \text{Time}$$

Runtime from Lawler (1976)

polynomial Space

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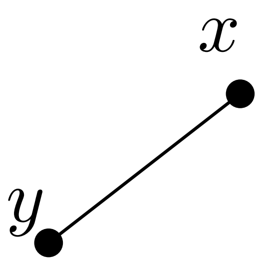

$\Leftrightarrow$

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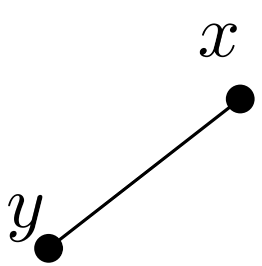

$$\Leftrightarrow \quad (\neg x \vee \neg y)$$

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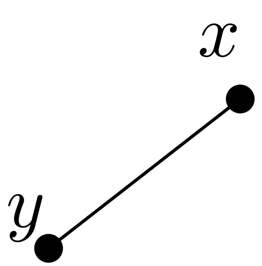
$\leadsto$ enumeration of independent sets :)



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Thm:
Using Algorithm 3 for Graph Coloring gives us

$$O(2.1247^n) \supset O((1+1.1247)^n) \quad \text{Time}$$

Lawler: $O(2.4423^n)$

polynomial Space

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Using Algorithm 3 for Graph Coloring gives us

5- and 6-coloring from last week are now obsolete

$O(2.1247^n) \supset O((1+1.1247)^n)$ Time

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Dynamic Program for $a(S)$, $S \subseteq V$

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Obs.: For $S \subseteq V$, $\quad a(S) = a(\varnothing, S, V \setminus S)$

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Obs.: For $R \mathbin{\dot\cup} F = V$, $a(R, F, \emptyset) =$

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Obs.: For $R \mathbin{\dot\cup} F = V$, $a(R, F, \emptyset) = [R \text{ independent?}] \in \{0, 1\}$

Lemma: For $R \mathbin{\dot\cup} F \mathbin{\dot\cup} O \mathbin{\dot\cup} \{v\} = V$,

$$a(R, F, O \mathbin{\dot\cup} \{v\}) = a(R \mathbin{\dot\cup} \{v\}, F, O) + a(R, F \mathbin{\dot\cup} \{v\}, O)$$

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DP with parameters $R \cup F \cup O$ seems like it would use $O^*(3^n)$ time and space... but does it?

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Obs.: For $R \subseteq V$, R not independent $\Rightarrow a(R, \cdot, \cdot) = 0$

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independent

Obs.: For $S \subseteq V$, $a(S) = a(\varnothing, S, V \setminus S)$

Obs.: For $R \cup F = V$, $a(R, F, \varnothing) = [R$ independent?$] \in \{0, 1\}$

Lemma: For $R \cup F \cup O \cup \{v\} = V$,

$$a(R, F, O \cup \{v\}) = a(R \cup \{v\}, F, O) + a(R, F \cup \{v\}, O)$$

Obs.: For $R \subseteq V$, R not independent $\Rightarrow a(R, \cdot, \cdot) = 0$

Dynamic Program for $a(S)$, $S \subseteq V$

Def.: $a(S) := \#$ independent sets that avoid S

Def.: $a(R, F, O) := \#$ independent sets that **contain** R and **avoid** F.

Obs.: For $S \subseteq V$, $\quad a(S) = a(\varnothing, S, V \setminus S)$

Obs.: For $R \mathbin{\dot\cup} F = V$, $a(R, F, \varnothing) = [R \text{ independent?}] \in \{0, 1\}$

Lemma: For $R \mathbin{\dot\cup} F \mathbin{\dot\cup} O \mathbin{\dot\cup} \{v\} = V$,

$$a(R, F, O \mathbin{\dot\cup} \{v\}) = a(R \mathbin{\dot\cup} \{v\}, F, O) + a(R, F \mathbin{\dot\cup} \{v\}, O)$$

Obs.: For $R_1 \mathbin{\dot\cup} F_1 \mathbin{\dot\cup} O = R_2 \mathbin{\dot\cup} F_2 \mathbin{\dot\cup} O = V$, and R_i's indep.

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Obs.: For $R_1 \cup F_1 \cup O = R_2 \cup F_2 \cup O = V$, and R_i's indep. $\nexists$ edge between R_i and $O \Rightarrow a(R_1, F_1, O) = a(R_2, F_2, O)$.

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Obs.: For $R \,\dot\cup\, F = V$ and R indep.. $\quad a(R, F, \emptyset) = 1$

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Lemma: For $R \dot\cup F \dot\cup O = V$ and $v \in O$,

U := neighborhood

$$a(R, F, O) = a(R, F \dot\cup \{v\}, O \setminus \{v\}) + a(R \dot\cup \{v\}, F \dot\cup U(v), O \setminus U[v])$$

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independent

no edges between $R \mathbin{\dot\cup} \{v\}$ and $O \setminus U[v]$

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Lemma: For $R \dot\cup F \dot\cup O = V$ and $v \in O$, and R independent, and $\nexists$ edge between R and O

$$a(R, F, O) = a(R, F \dot\cup \{v\}, O \setminus \{v\}) + a(R \dot\cup \{v\}, F \dot\cup U(v), O \setminus U[v])$$

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Lemma: For $R \cup F \cup O = V$ and $v \in O$, and R independent, and $\nexists$ edge between R and O

$$b(O) = b(O \setminus \{v\}) + b(O \setminus U[v])$$

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Obs.: For $S \subseteq V$, $\quad a(S) = a(\varnothing, S, V \setminus S) \quad = b(V \setminus S)$

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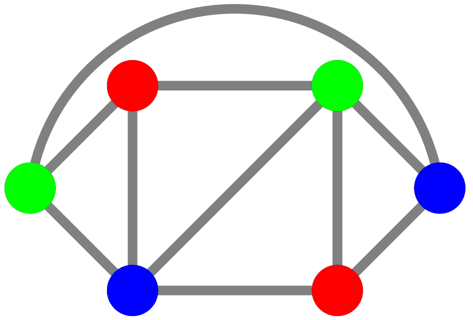
$$b(\emptyset) = 1$$

Thm: Table with $a(S)$ for each $S \subseteq V$ can be computed in $O^*(2^n)$ time. $\square$

Graph Coloring: Summary

Given: Graph $G = (V, E)$, number k

Question: $\exists$ proper k-coloring of V?



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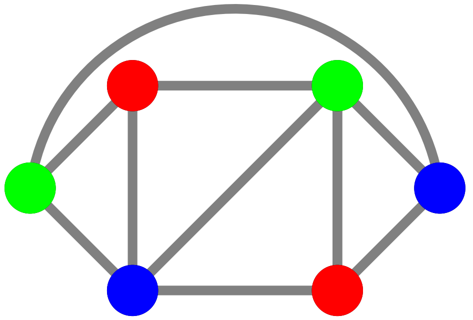
IE-Formulation:

$\mathcal{U} = \{k\text{-tuple of independent sets from } G\}$

$\mathcal{P} = \{P_v \mid v \in V$, where $P_v =$ `"tuple contains a set with` v `in it"`$\}$



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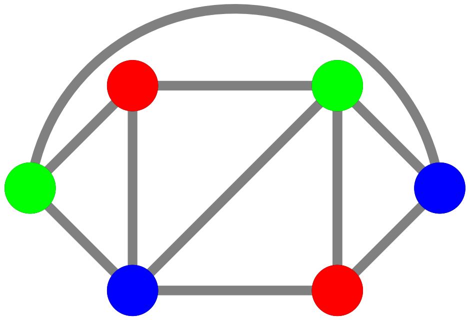
Algorithm:

- Compute $\bar{N}(S) = a(S)^k$ for each $S \subseteq V$
- Apply Inclusion-Exclusion



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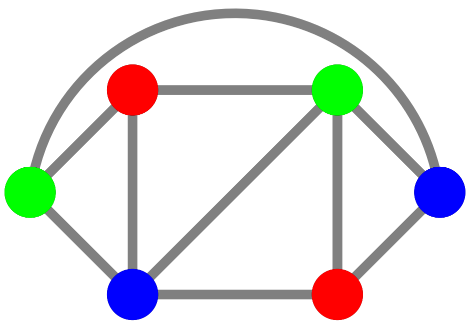

Thm: Graph Coloring can be decided using $O^*(2^n)$ time and space