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# Exact Algorithms

Sommer Term 2020

## Lecture 5. Inclusion–Exclusion

Based on: [Exact Exponential Algorithms: §4]

Further reading: [Parameterized Algorithms: §10.1]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

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# Puzzle

How many numbers $\leq 1000$ are *not* divisible by 2, 3 or 5?

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, ...

# Puzzle

How many numbers $\leq 1000$ are not divisible by 2, 3 or 5?

Universe: $\mathcal{U} = [1000] := \{n \in \mathbb{N} \colon 1 \leq n \leq 1000\}$

$P_2 = \{n \in \mathcal{U} \mid n \bmod 2 \equiv 0\}$ $\qquad |P_2| = 500$

$P_3 = \{n \in \mathcal{U} \mid n \bmod 3 \equiv 0\}$ $\qquad |P_3| = 333$

$P_5 = \{n \in \mathcal{U} \mid n \bmod 5 \equiv 0\}$ $\qquad |P_5| = 200$

Goal: $|\mathcal{U}| - |P_2 \cup P_3 \cup P_5|$

# Inclusion–Exclusion

$$500 + 333 + 200 - 166 - 100 - 66 + 33 = \mathbf{734}$$

$$|P_2 \cup P_3 \cup P_5| = + |P_2| + |P_3| + |P_5|$$
$$- |P_2 \cap P_3| - |P_2 \cap P_5| - |P_3 \cap P_5|$$
$$+ |P_2 \cap P_3 \cap P_5|$$

$$= \left| \bigcup_{i \in \{2,3,5\}} P_i \right| = \sum_{\varnothing \neq J \subseteq \{2,3,5\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} P_j \right|$$

1 × union

7 × intersection

# An Inclusion–Exclusion Theorem

Given $N$ objects and $n$ properties $\mathcal{P} = \{P_1, \ldots, P_n\}$,
for each $\mathcal{S}$ with $\emptyset \subseteq \mathcal{S} \subseteq \mathcal{P}$, let
$N(\mathcal{S}) := \#$ objects satisfying the properties in $\mathcal{S}$.

($N_0 := \#$ objects with *no* property from $\mathcal{P}$.)

**Thm.** $N_0 = \sum_{\mathcal{S} \subseteq \mathcal{P}} (-1)^{|\mathcal{S}|} N(\mathcal{S})$.

1 union

$2^n$ subsets

Is this useful?

"One of the most useful principles of enumeration in discrete probability and combinatorial theory is the celebrated principle of inclusion–exclusion. When skillfully applied, this principle has yielded the solution to many a combinatorial problem."

Gian-Carlo Rota [1932–1999]

# Proof

**Thm.** $N_0 = \sum_{\mathcal{S} \subseteq \mathcal{P}} (-1)^{|\mathcal{S}|} N(\mathcal{S})$.

*Proof.* Property-less objects are counted exactly once on both sides (on the right side when $\mathcal{S} = \emptyset$).

Consider an object with *precisely* properties $\mathcal{S} = \{P_{i_1}, \ldots, P_{i_s}\}$, $s \geq 1$.

We count this object in $N(\mathcal{S}')$ precisely when $\mathcal{S}' \subseteq \mathcal{S}$.

For an object with properties $\mathcal{S}$, we observe:

$$\sum_{\mathcal{S}' \subseteq \mathcal{S}} (-1)^{|\mathcal{S}'|} = \sum_{i=0}^{s} \binom{s}{i} (-1)^i = \mathbf{0}\text{, because:}$$

– if $s$ is odd: $\binom{s}{0} - \binom{s}{1} + \binom{s}{2} \mp \cdots - \binom{s}{s-2} + \binom{s}{s-1} - \binom{s}{s}$

– if $s$ is even – **exercise!** $\square$

# Corollary

Recall: $N$ objects and $n$ properties $\mathcal{P} = \{P_1, \ldots, P_n\}$.

**Cor.** Let $\bar{N}(\mathcal{S})$ be the number of objects that have *none* of the properties in $\mathcal{S}$. Then

$$N(\mathcal{P}) = \sum_{\mathcal{S} \subseteq \mathcal{P}} (-1)^{|\mathcal{S}|} \bar{N}(\mathcal{S})$$

But is it useful?

*Proof.* Set $P_i{}' :=$ "object does not have property $P_i$".

Apply **Thm.** $N_0 = \sum_{\mathcal{S} \subseteq \mathcal{P}} (-1)^{|\mathcal{S}|} N(\mathcal{S})$ to:

$$N_0' = N(\mathcal{P})$$

$$N'(P_{i_1}', \ldots, P_{i_s}') = \bar{N}(P_{i_1}, \ldots, P_{i_s})$$

□

# Directed Hamiltonian Path

Given: Directed Graph $G = (V, E)$,
two special vertices $s, t \in V$.

Question: Is there a *Hamiltonian path* from $s$ to $t$,
i.e., an $s$-$t$-path spanning the vertices of $G$?

Brute Force? Try all permutations of $V \rightsquigarrow \Theta(n!) = 2^{\Theta(n \log n)}$

Another idea: Via TSP $\Rightarrow$ $O^*(2^n)$ time and *space*.

# A Better Algorithm for Hamiltonian Path

**Thm.** The *number* of directed Hamiltonian paths from $s$ to $t$ can be determined in $O^*(2^n)$ time using only *polynomial space*. [Karp '82]

*Proof.* Objects: Directed $s$–$t$ walks of length $n-1$, not necessarily simple. $V' := V - \{s,t\}$

Property $v \in V'$: $s$–$t$ walk visits $v$. For $W \subseteq V'$:

$\bar{N}(W) :=$ # $s$–$t$ walks of length $n-1$, *avoiding* $W$.

$N(V') :=$ # directed Hamiltonian $s$–$t$ paths.

By inclusion–exclusion:

$$N(V') = \sum_{W \subseteq V'} (-1)^{|W|} \bar{N}(W)$$

Compute $\bar{N}(W)$ for each $W \subseteq V'$ separately. How?

# Dynamic Program

For $W \subseteq V'$, $k = 0, \ldots, n-1$, and $u \in V \setminus W$, set:

$$P_W[u,k] := \#\ s\text{–}u \text{ walks of length } k, \text{ avoiding } W$$

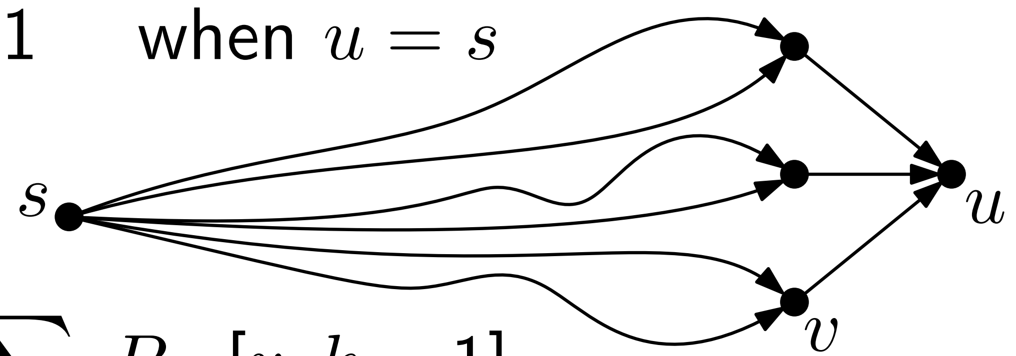
$$\bar{N}(W) := P_W[t, n-1]$$

Recurrence for the DP:

- for $k = 0$

$$P_W[u,0] = \begin{cases} 0 & \text{when } u \neq s \\ 1 & \text{when } u = s \end{cases}$$

- for $k \geq 1$




$$P_W[u,k] = \sum_{\substack{vu \in E \\ v \notin W}} P_W[v, k-1]$$

# Algorithmic Application of the IE Formula

verylongint NumHamPath$(G, s, t)$
  $V' \leftarrow V \setminus \{s, t\}$
  $N(V') \leftarrow 0$
  **foreach** $W \subseteq V'$ **do**
    compute $\bar{N}(W) = P_W[t, n-1]$ using the DP
    $N(V') \leftarrow N(V') + (-1)^{|W|}\bar{N}(W)$
  **return** $N(V')$

$\{O(m^2 n)$ time

[Note that numbers in $P_W$ can be exponential in $m!$]

fix!

$$P_W[u, k] = \sum_{\substack{vu \in E \\ v \notin W}} P_W[v, k-1]$$

Runtime: $O(m^2 n \cdot 2^n)$

Space: $O(mn)$ – if we reuse the memory used by the DP in each iteration of the loop.

How can we use this to *find* a Hamiltonian Path?

$\leadsto$ **Exercise!**

□