Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 5. Inclusion–Exclusion

Based on: [Exact Exponential Algorithms: §4]

Further reading: [Parameterized Algorithms: §10.1]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

Lehrstuhl für Informatik I

Puzzle

How many numbers ≤ 1000 are *not* divisible by 2, 3 or 5?

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, $\dots$

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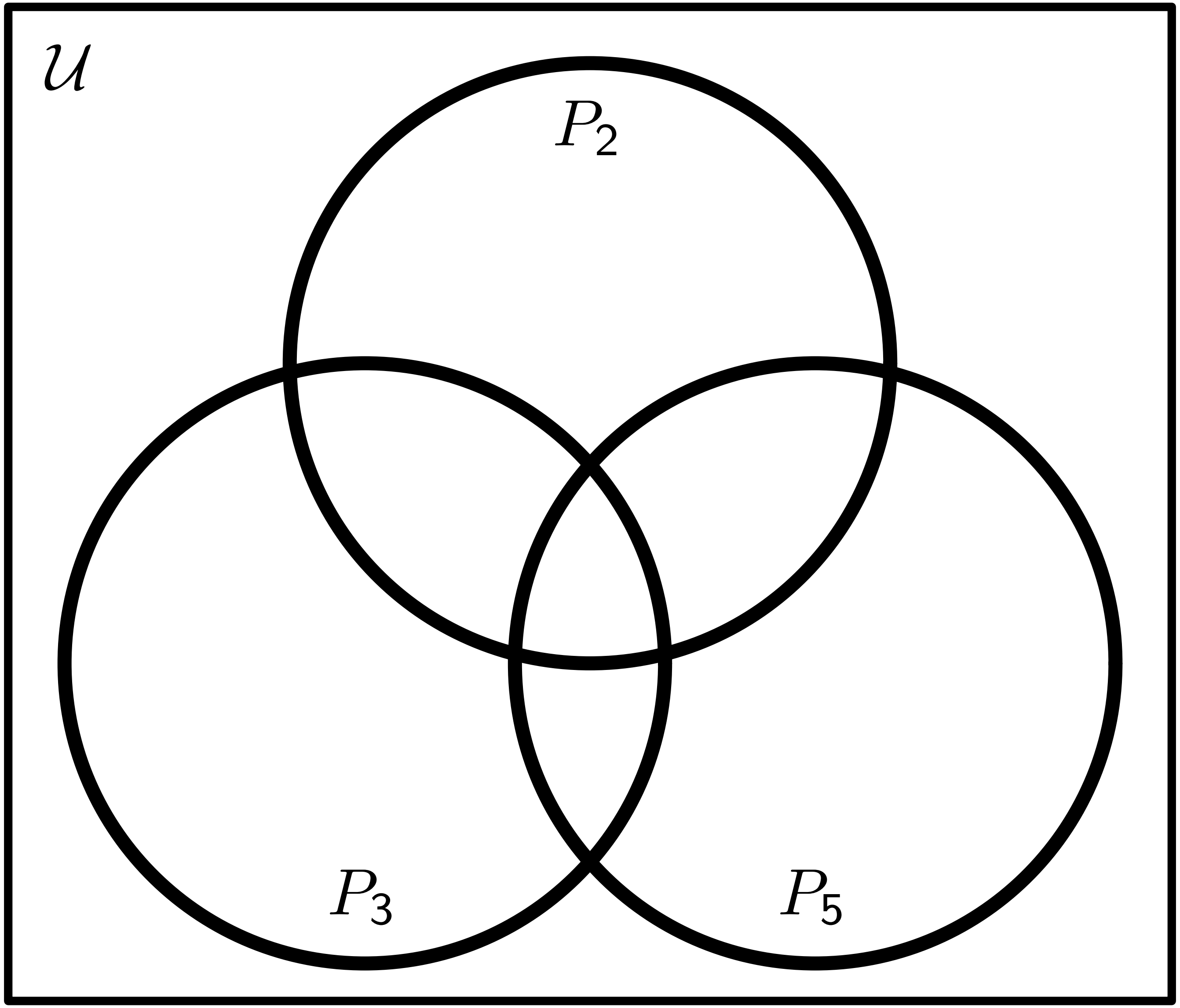

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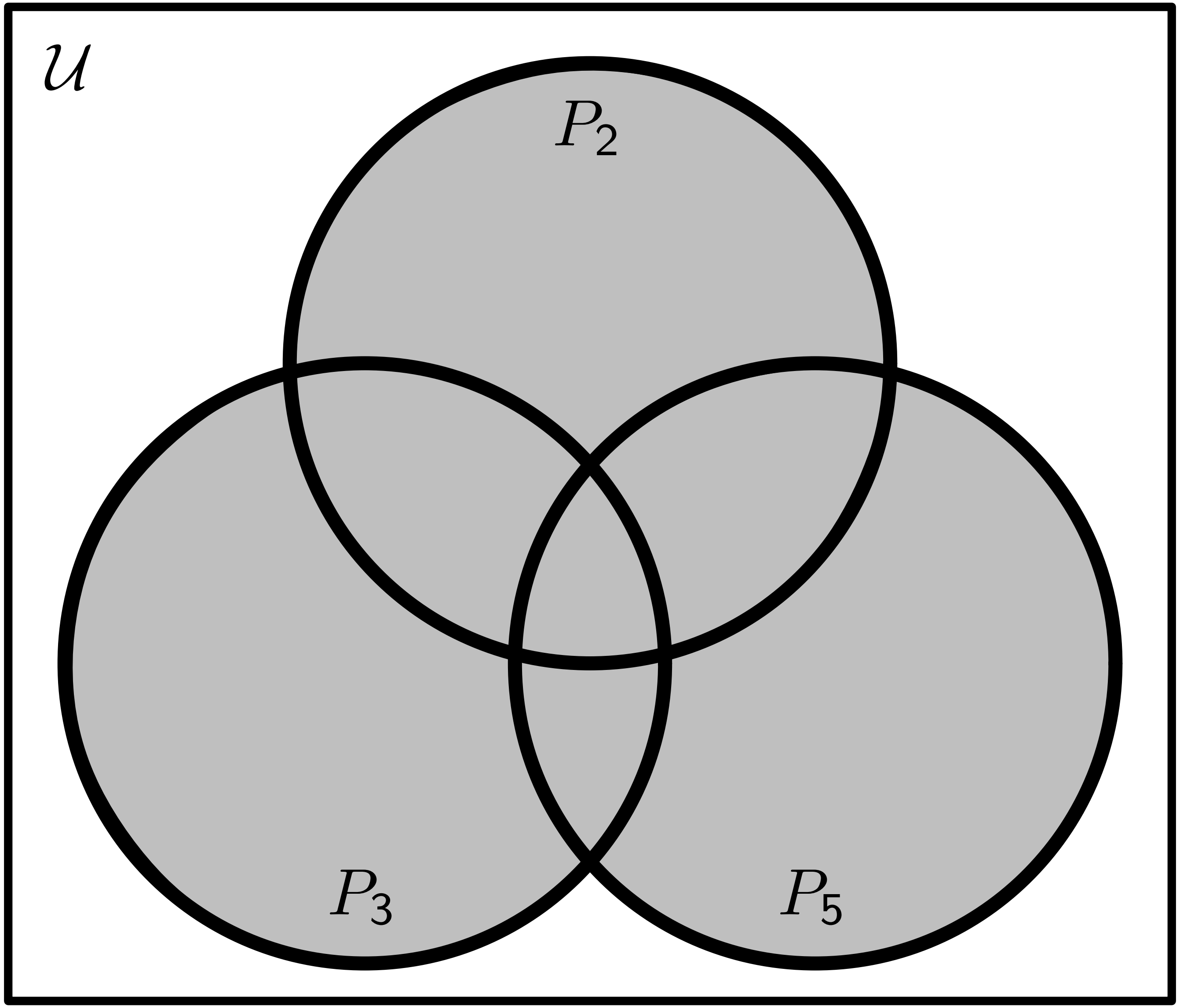

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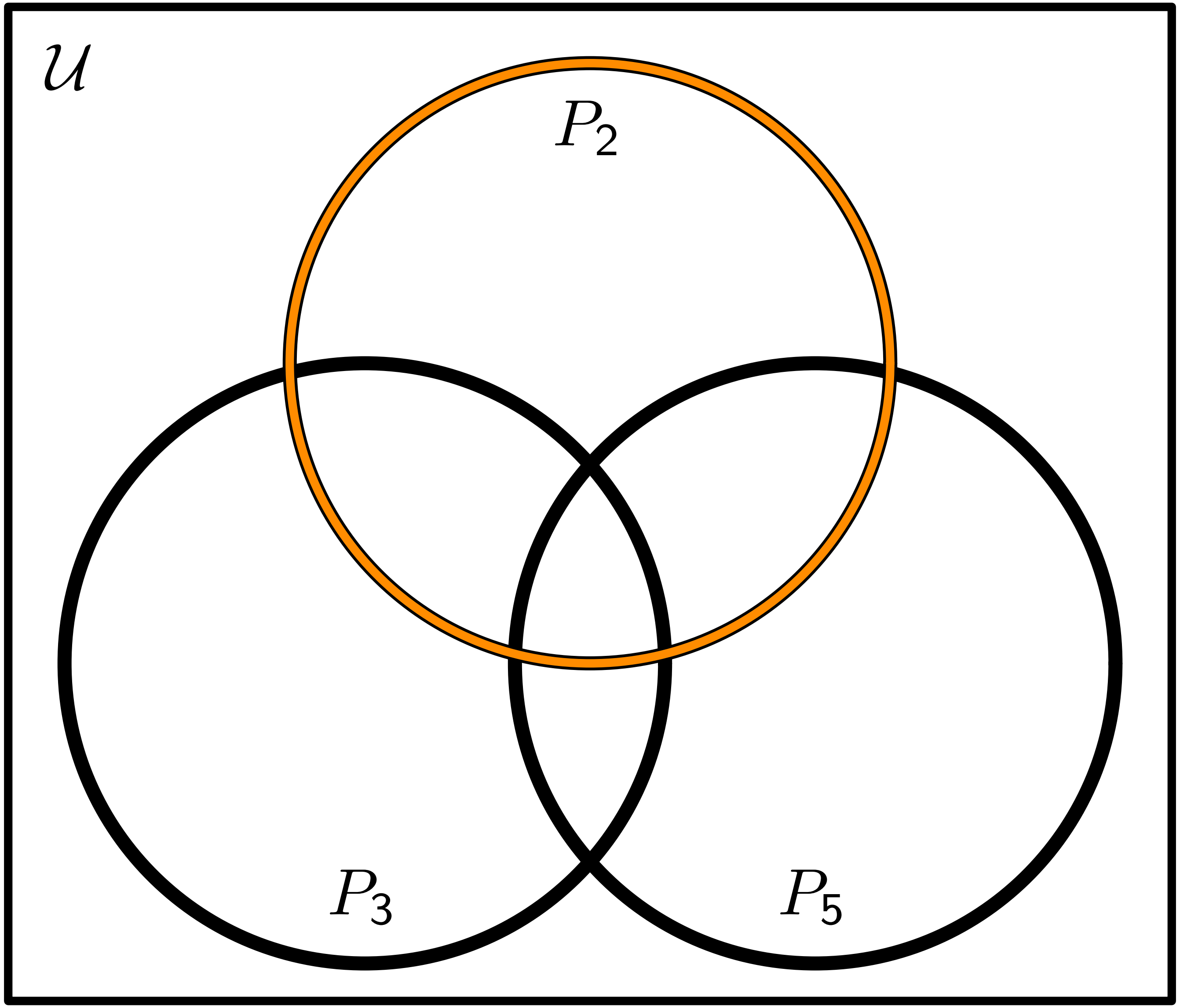

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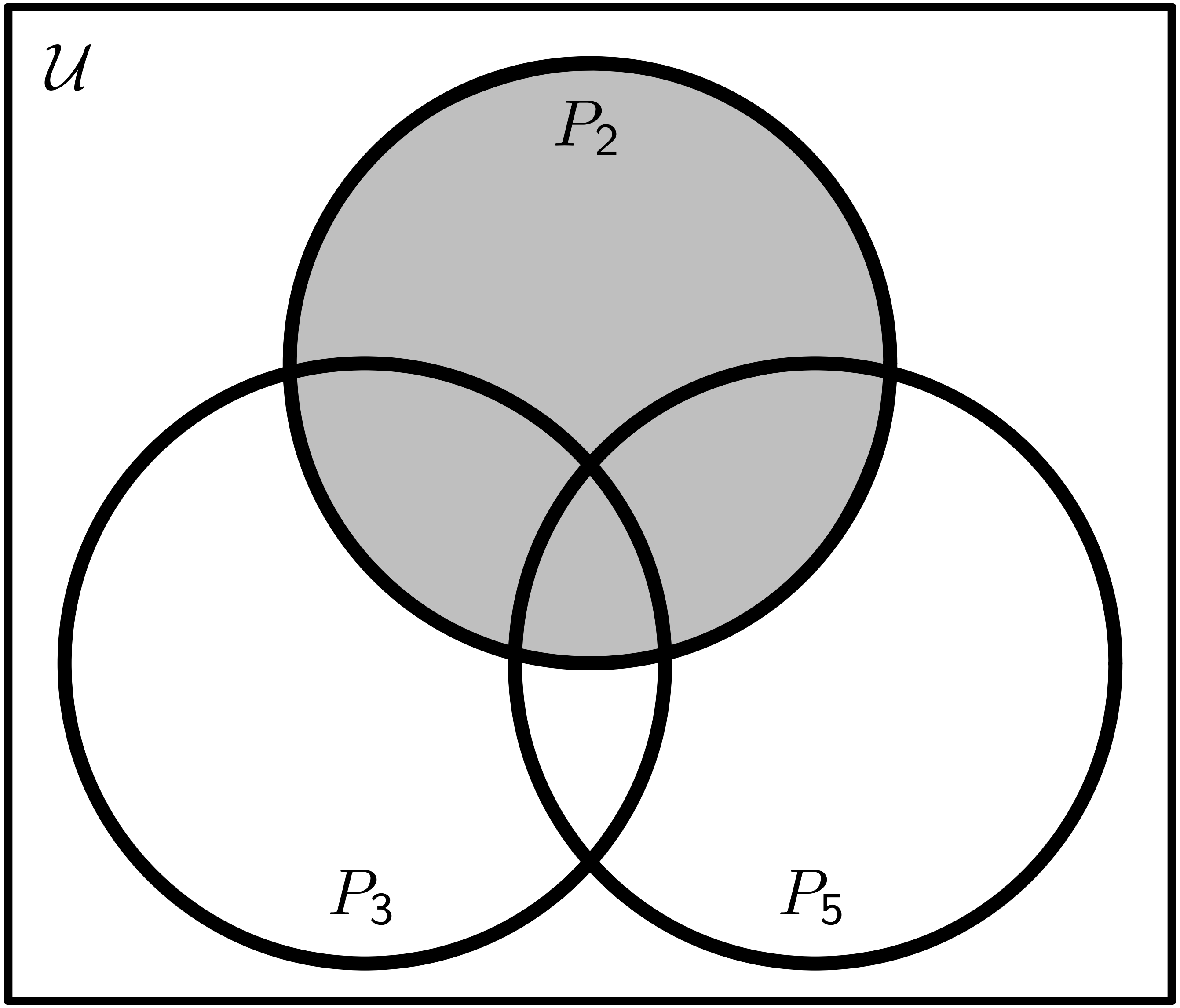

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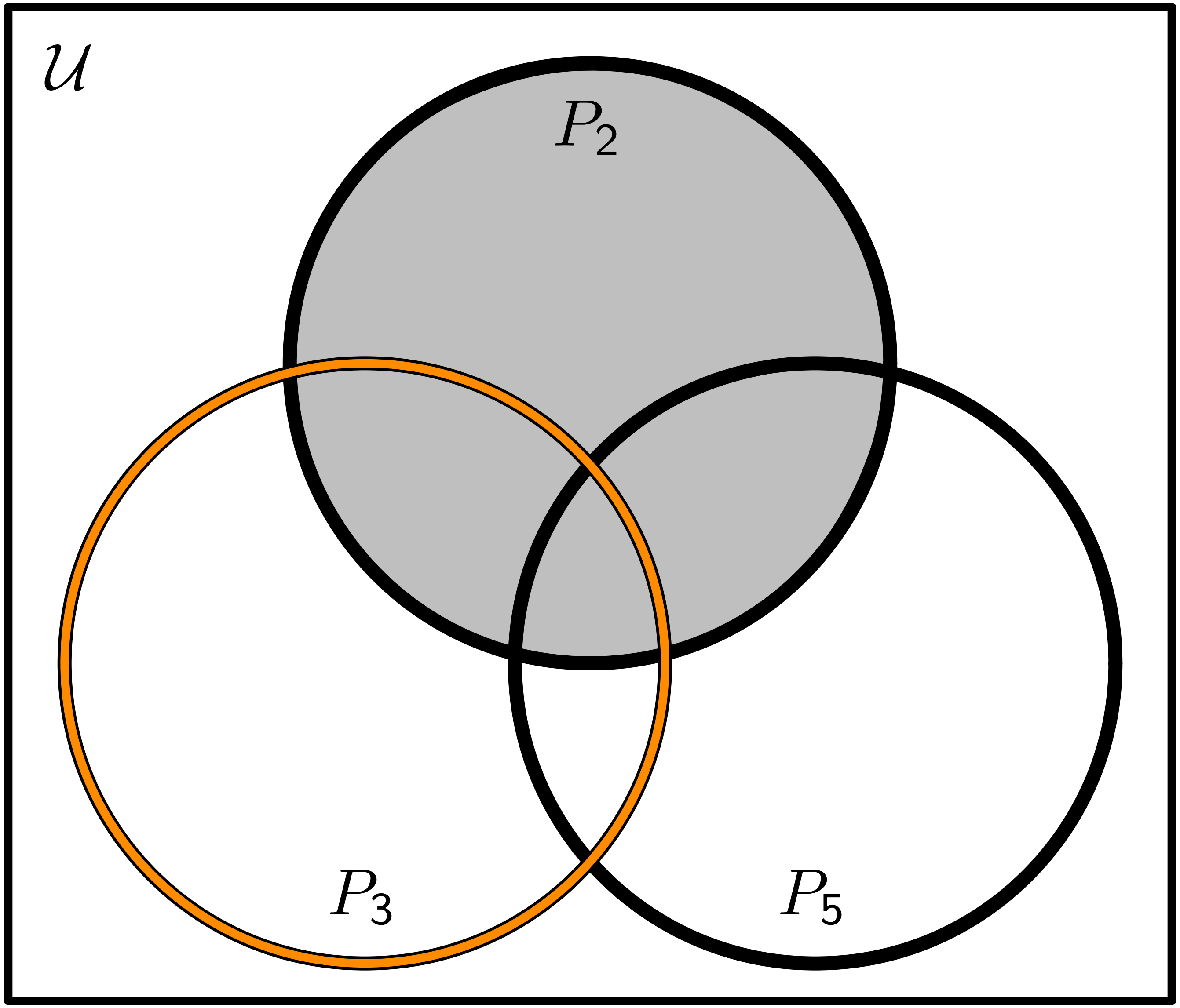

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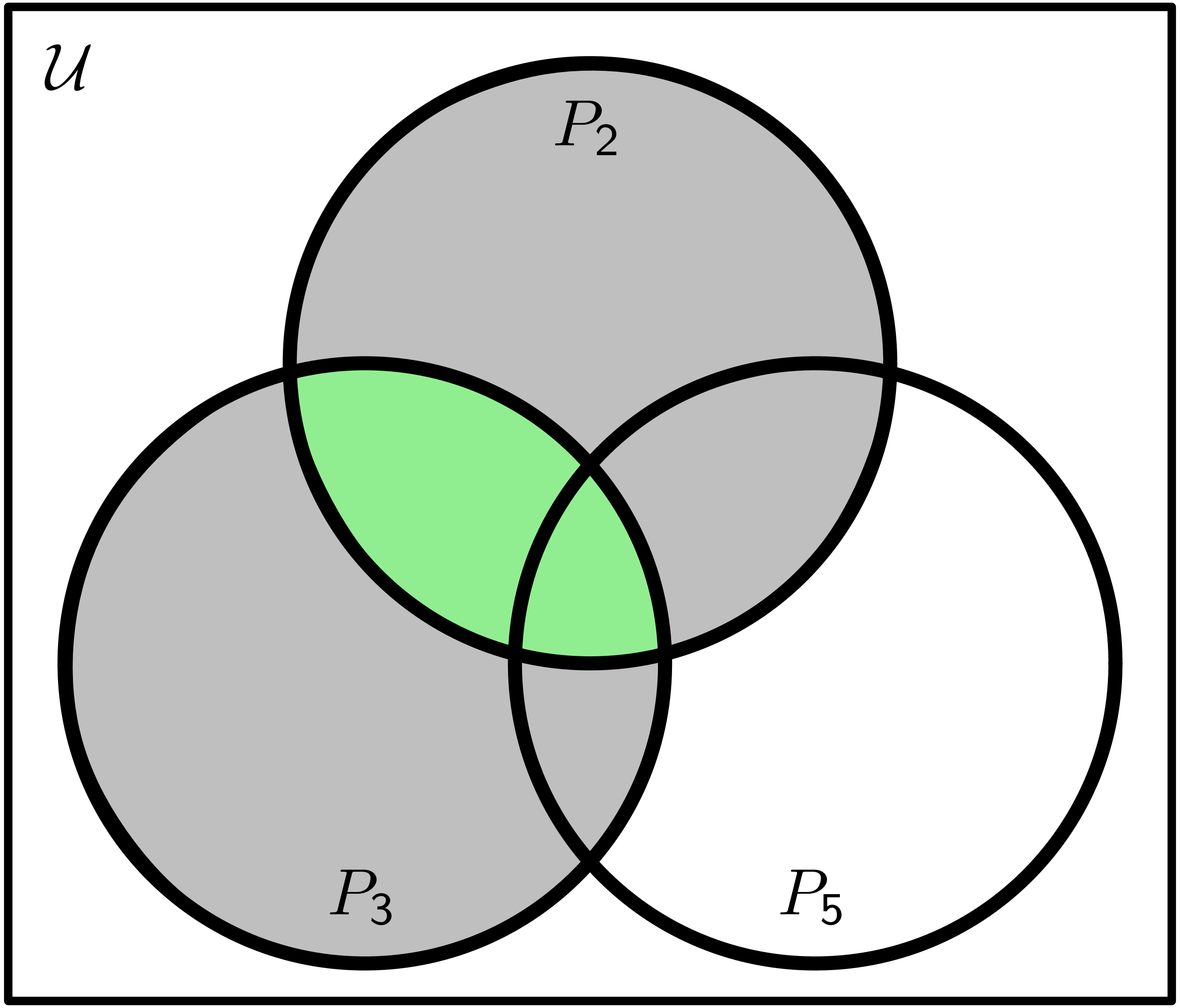
$500 + 333$



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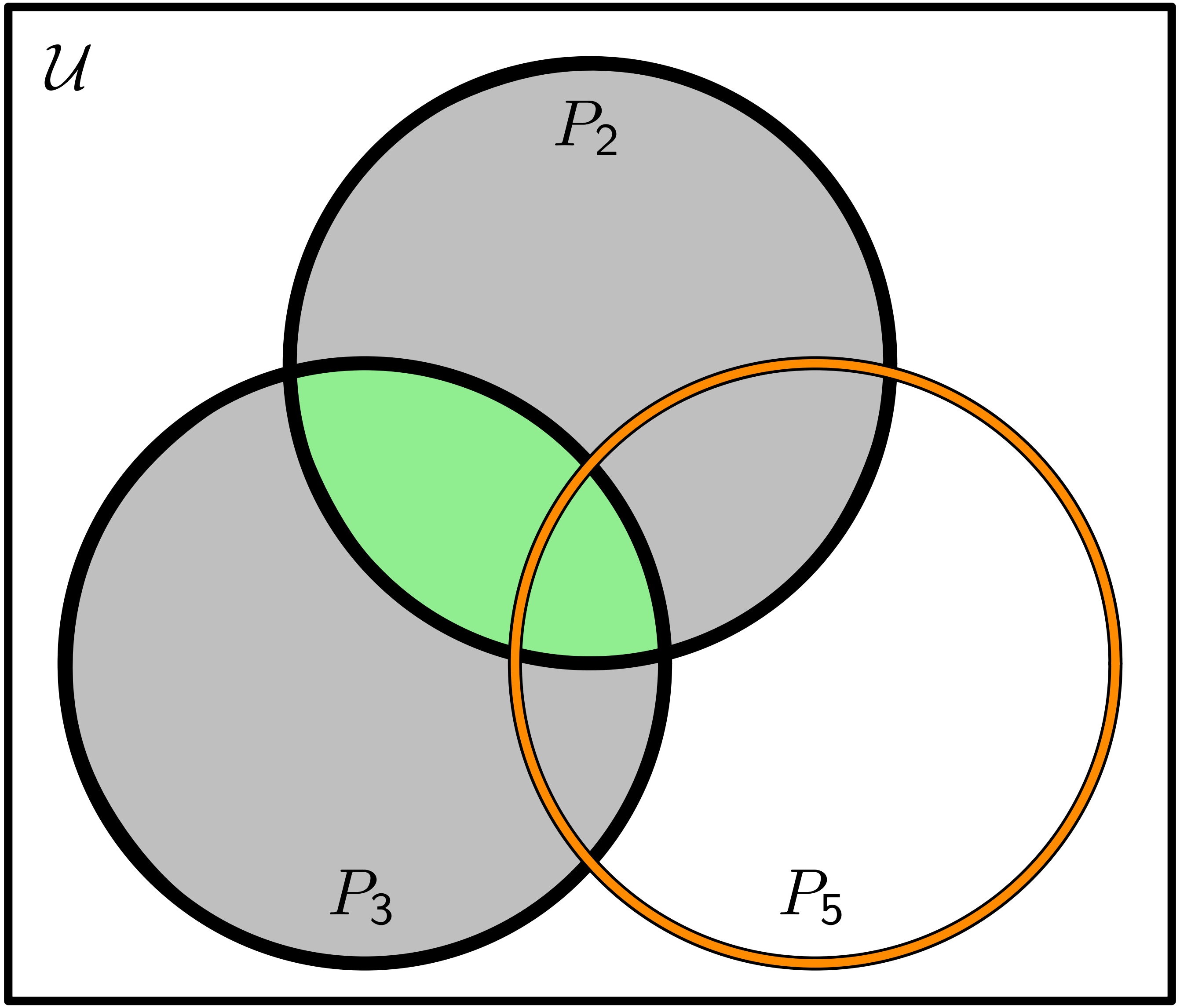

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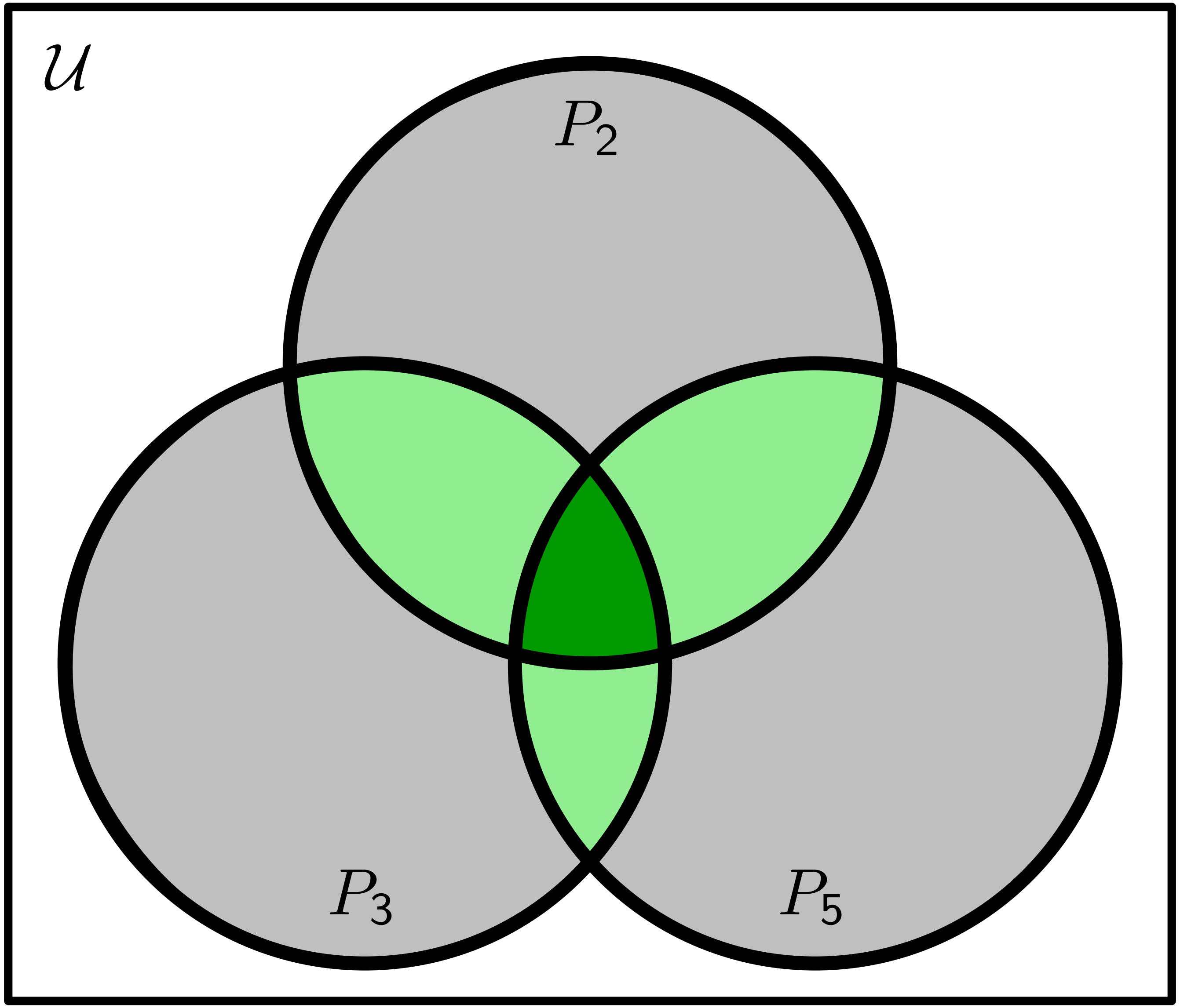

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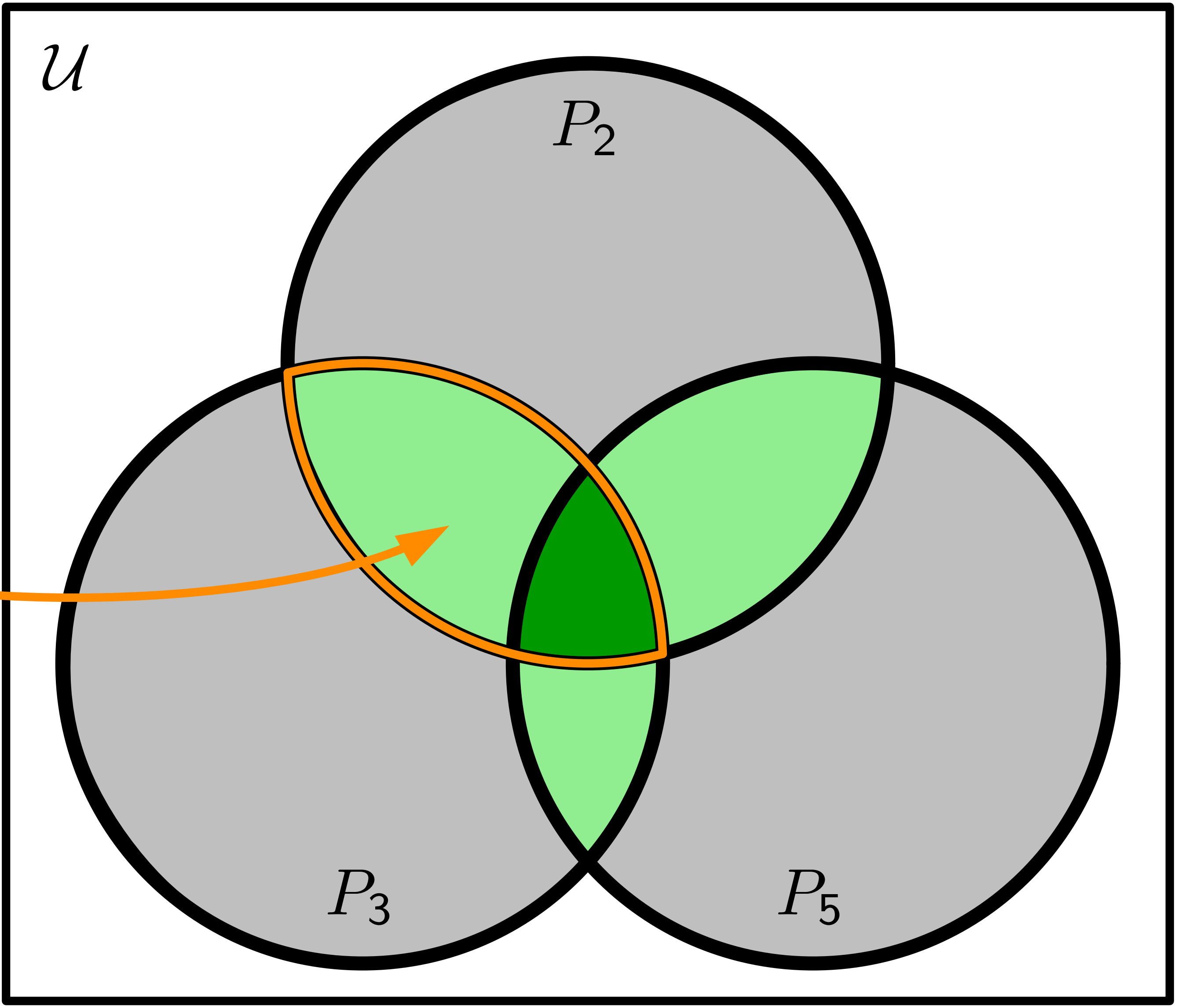
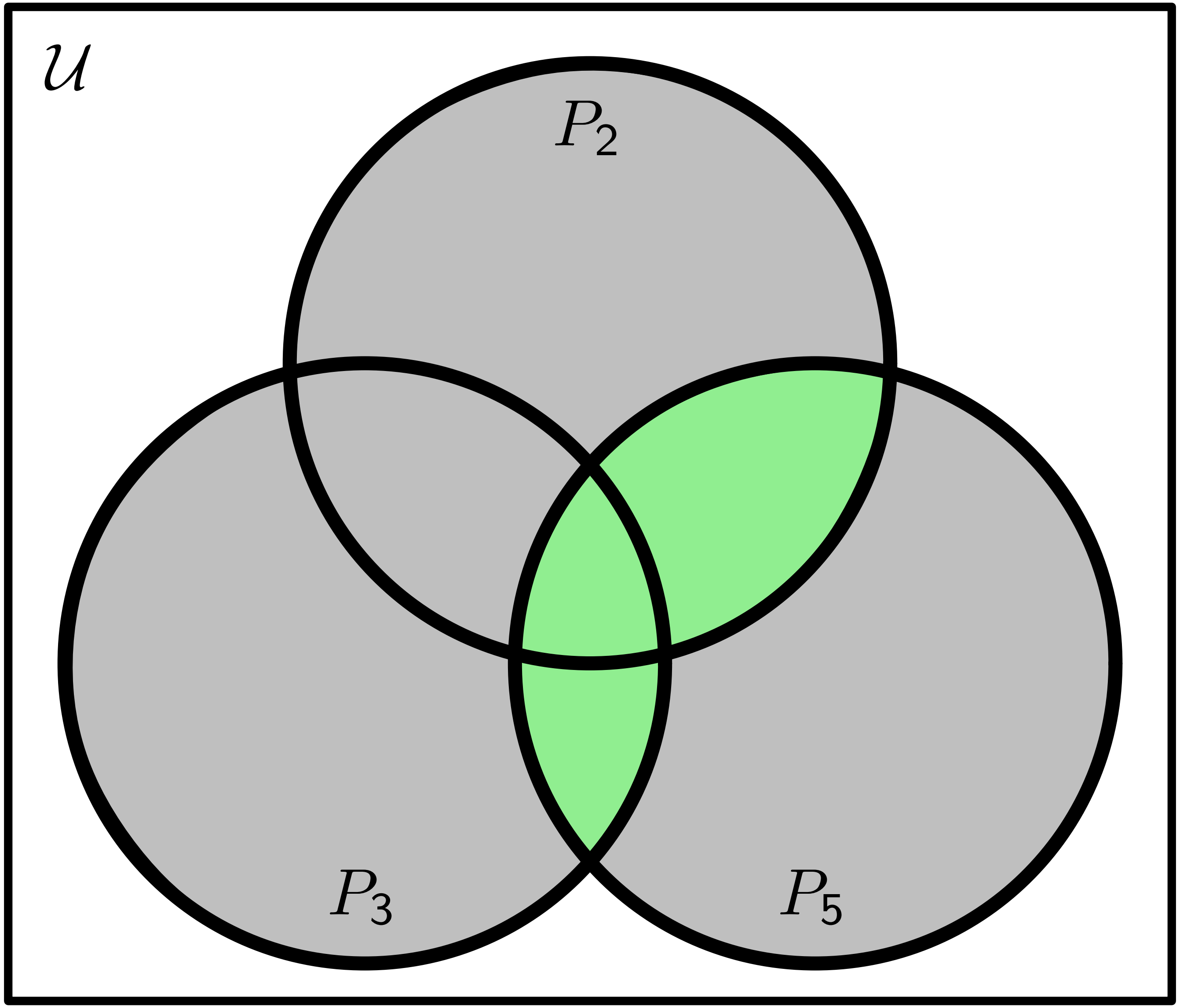


$|P_2 \cap P_3|$

Inclusion–Exclusion

$500 + 333 + 200 - 166$



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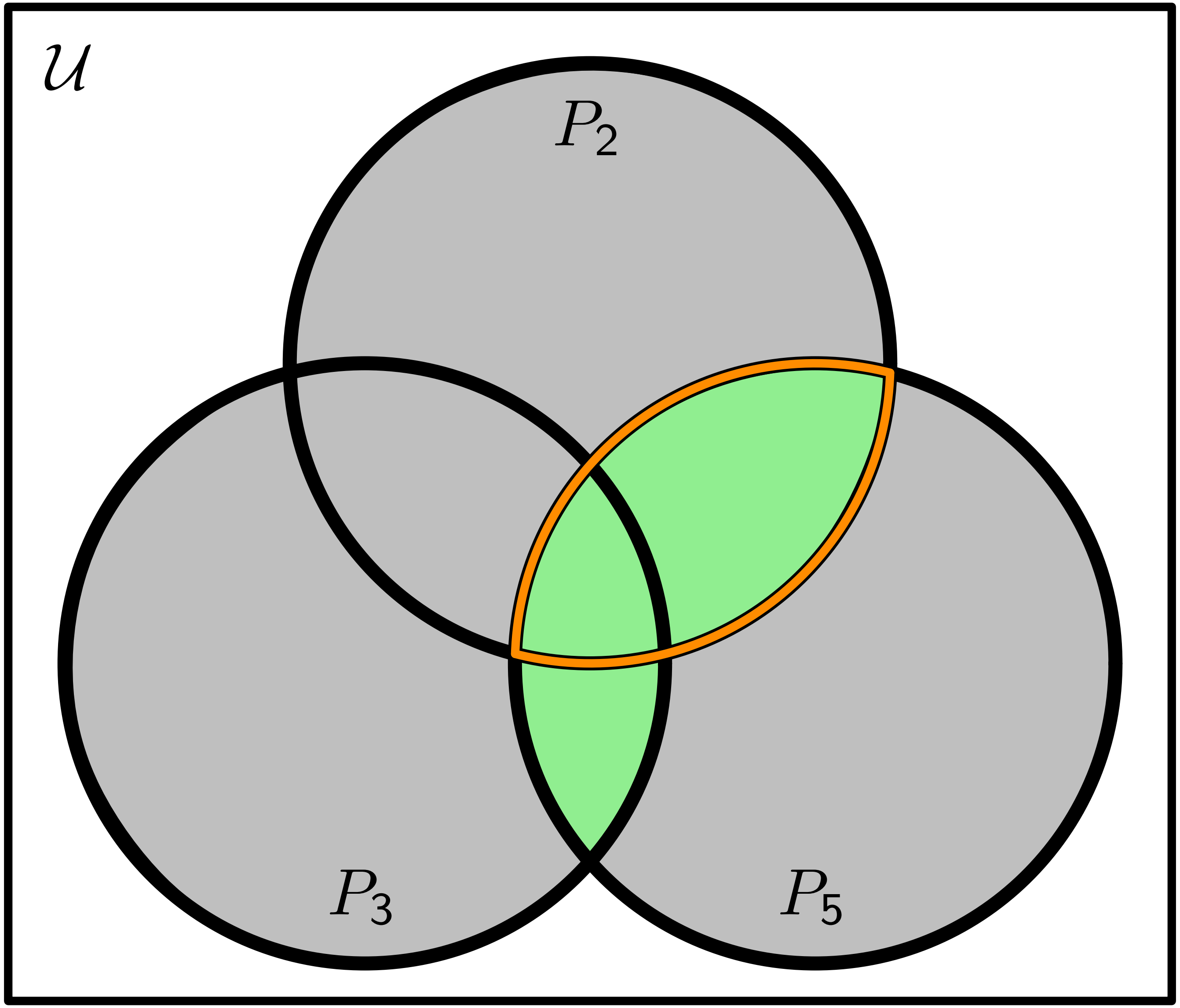

Inclusion–Exclusion

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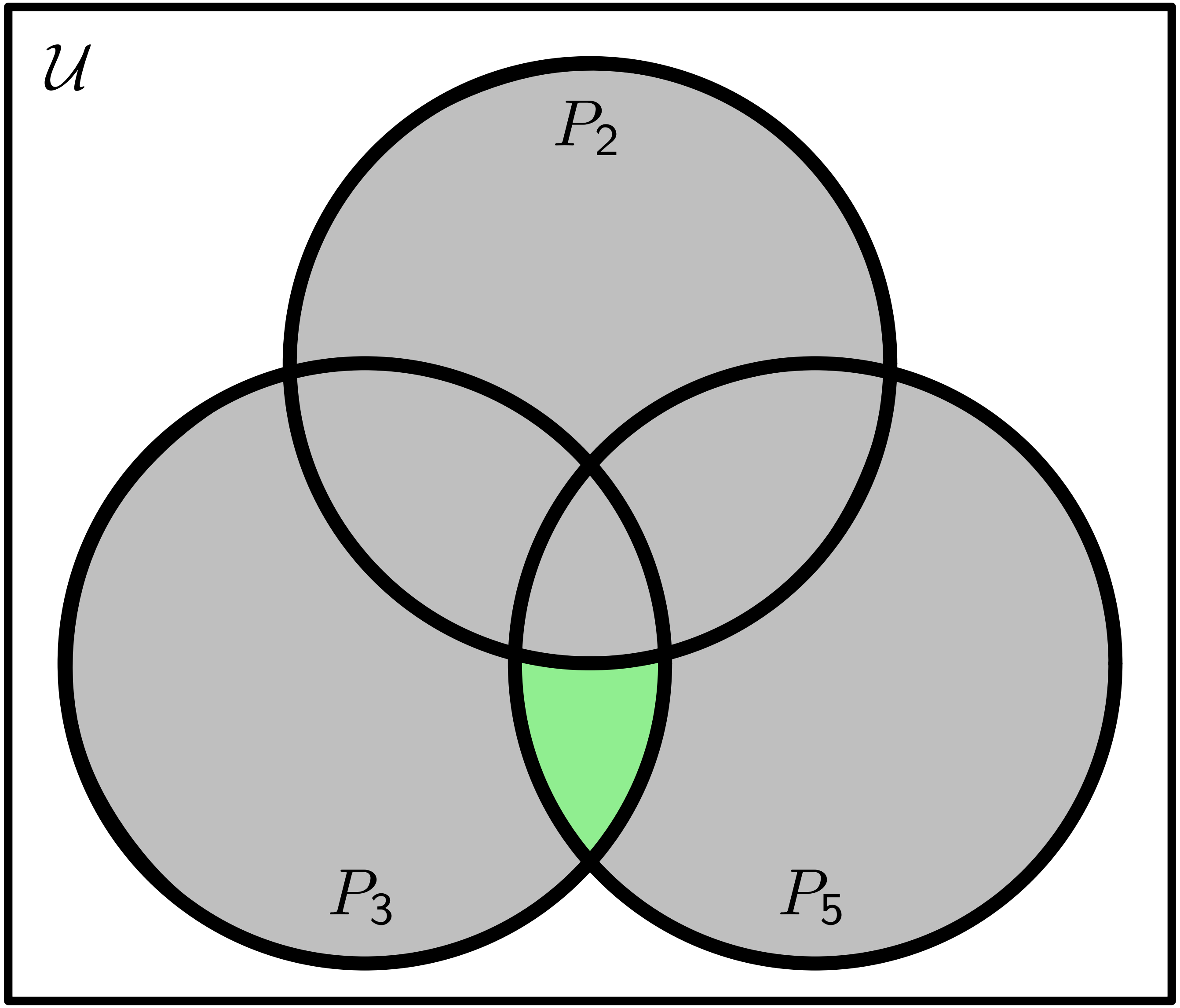

Inclusion–Exclusion

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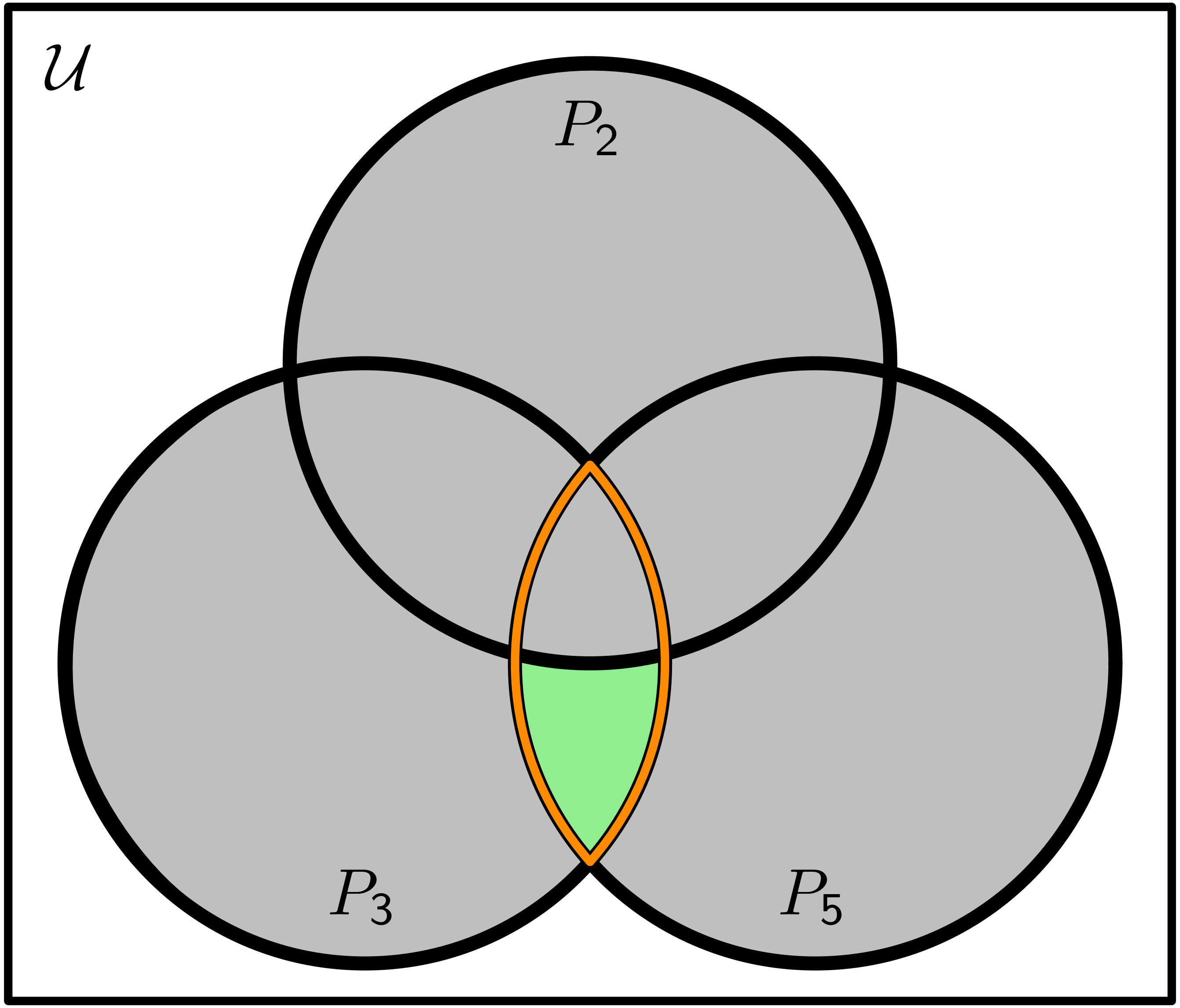

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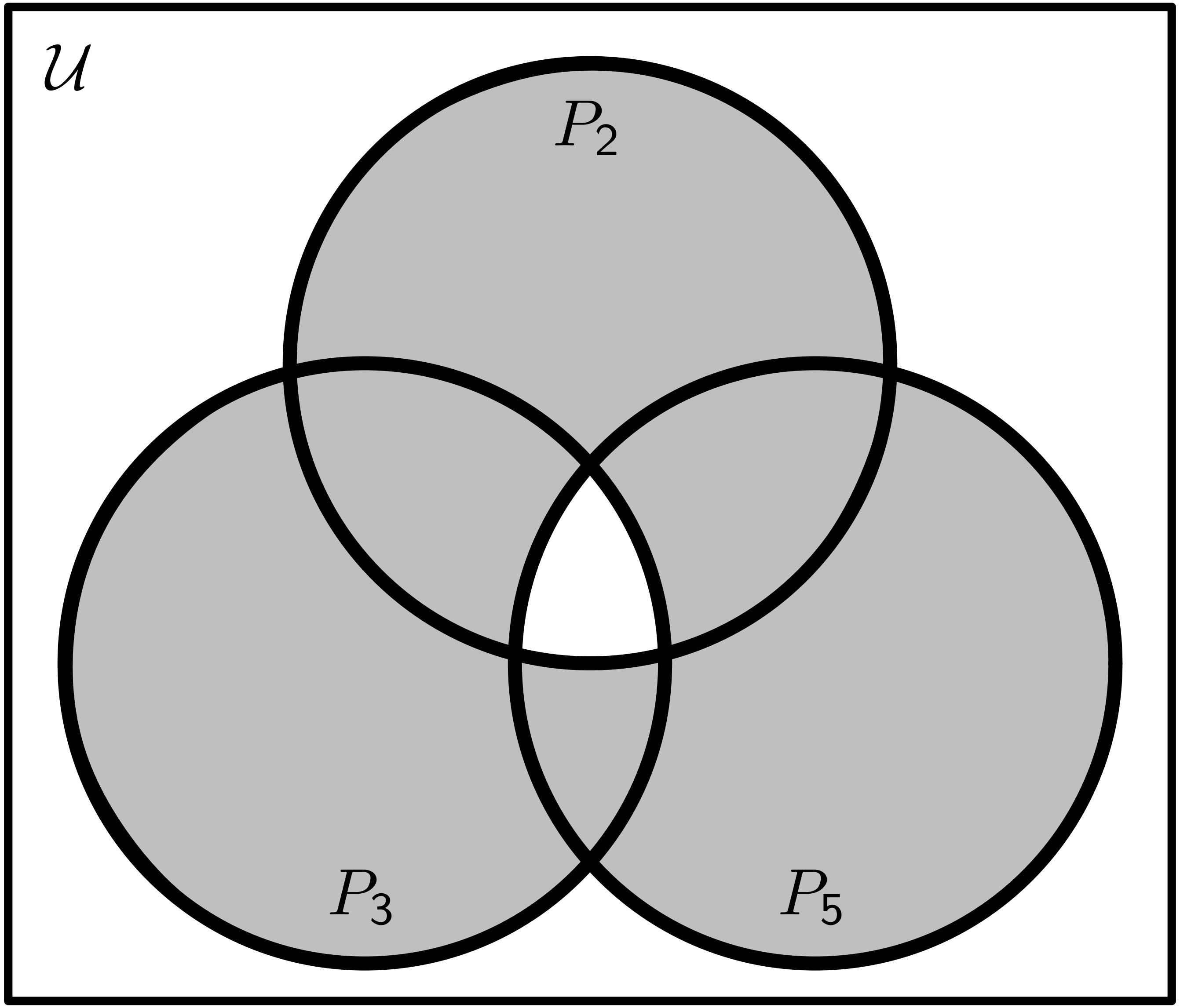

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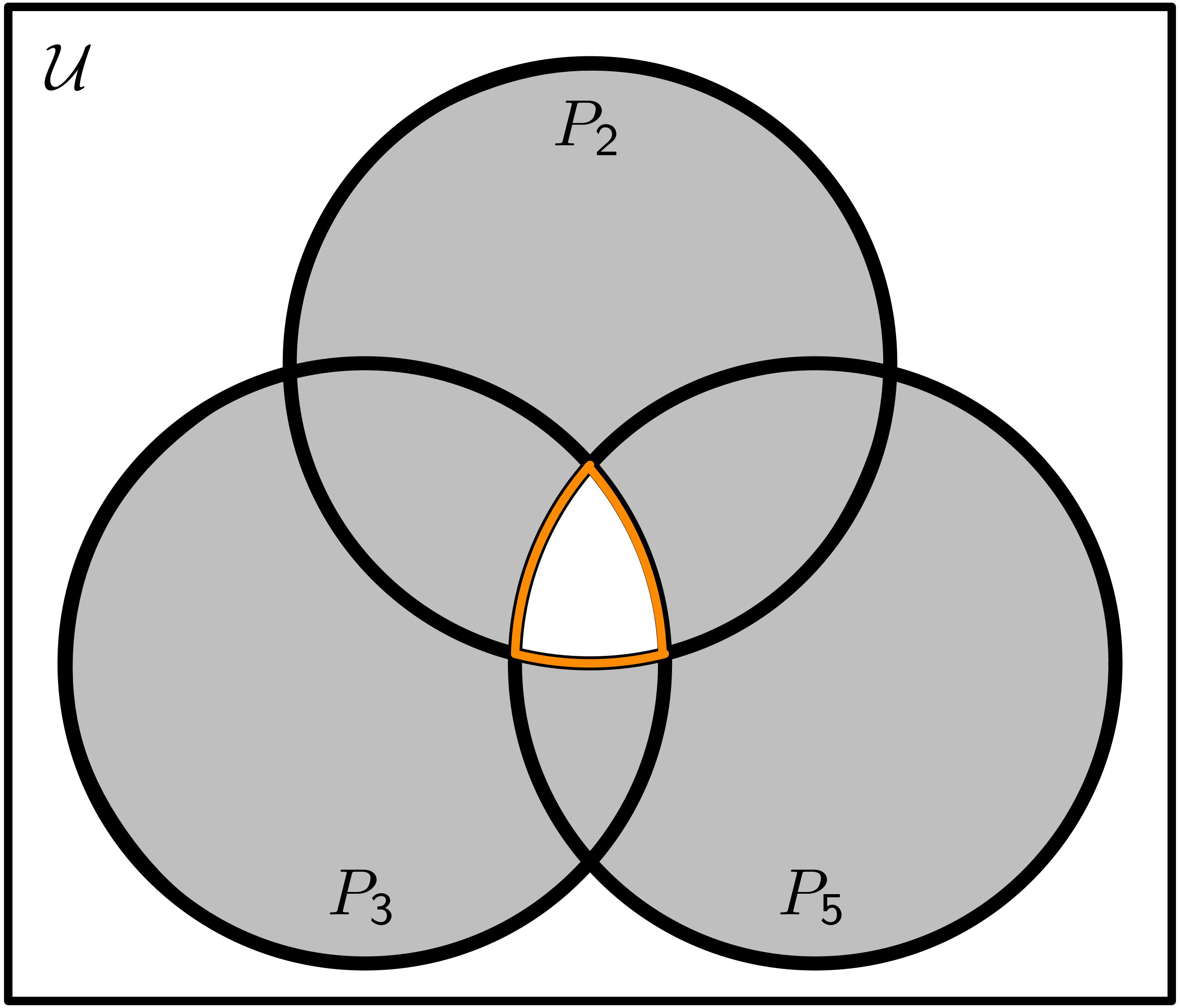

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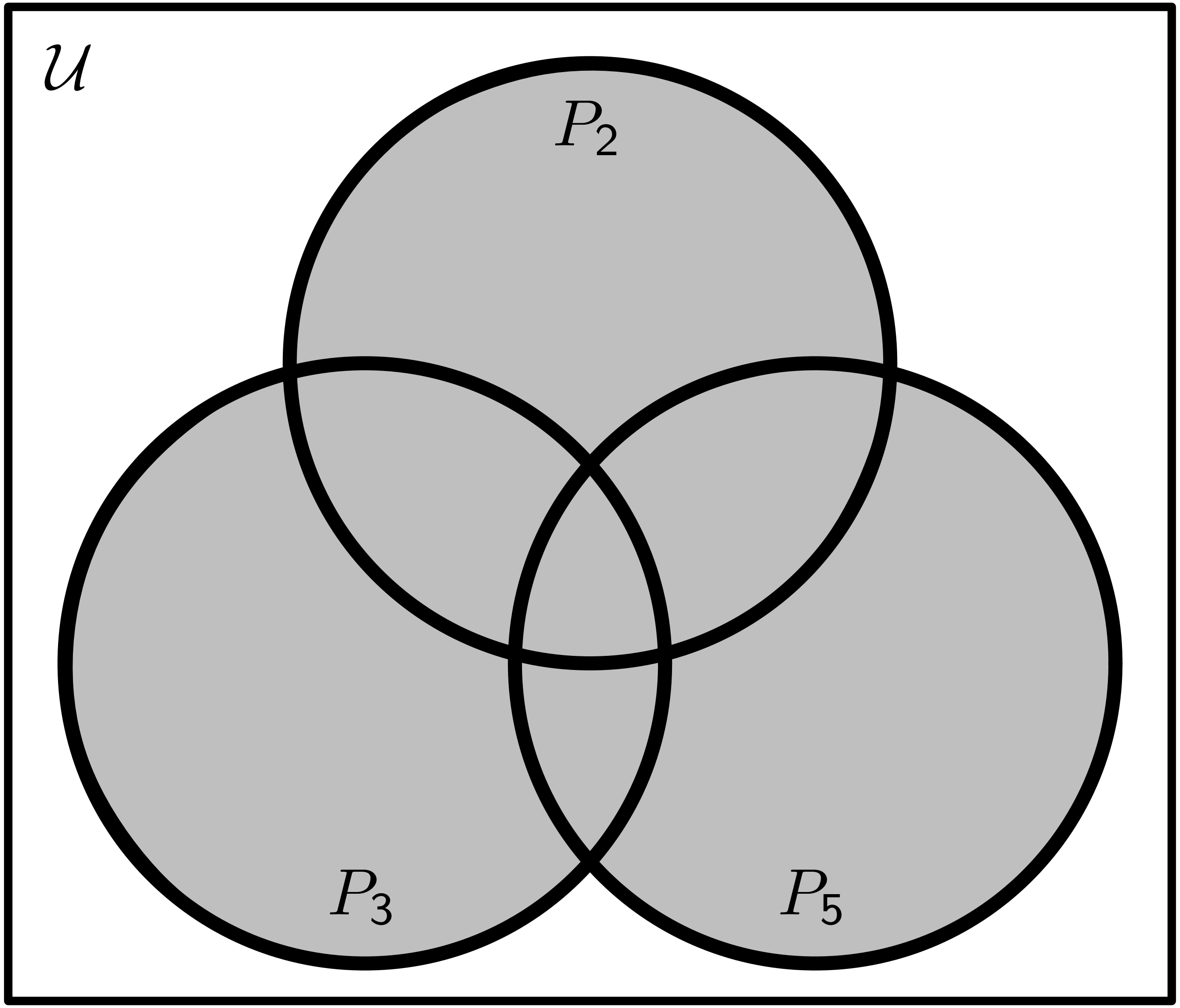

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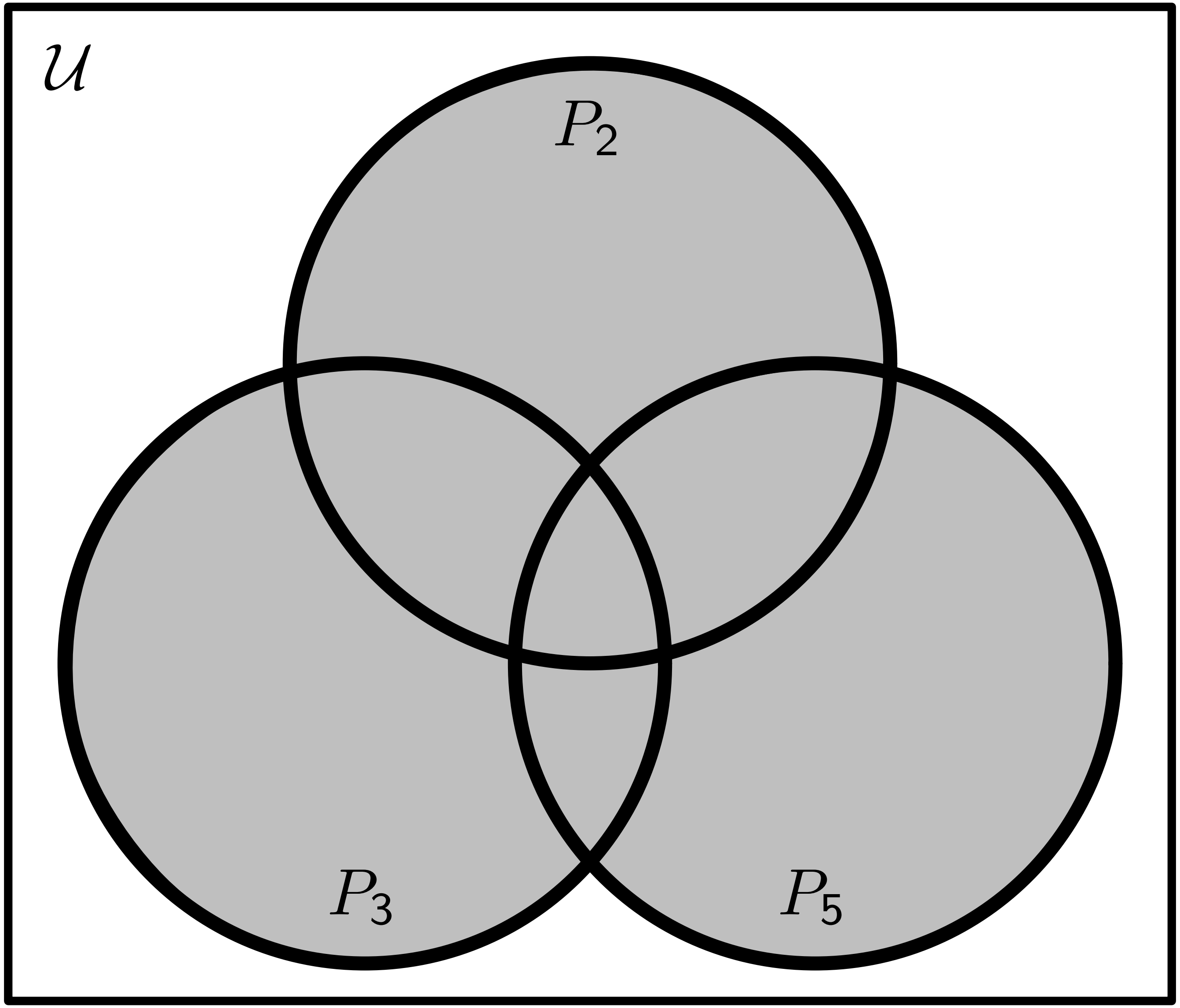
$$500 + 333 + 200 - 166 - 100 - 66 + 33 \; = \mathbf{734}$$



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7 × intersection

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“One of the most useful principles of enumeration in discrete probability and combinatorial theory is the celebrated principle of inclusion–exclusion. When skillfully applied, this principle has yielded the solution to many a combinatorial problem.”

Gian-Carlo Rota [1932–1999]

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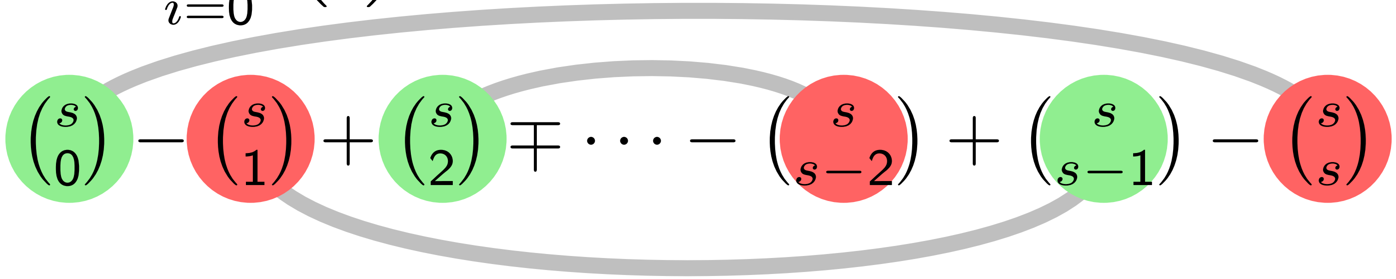

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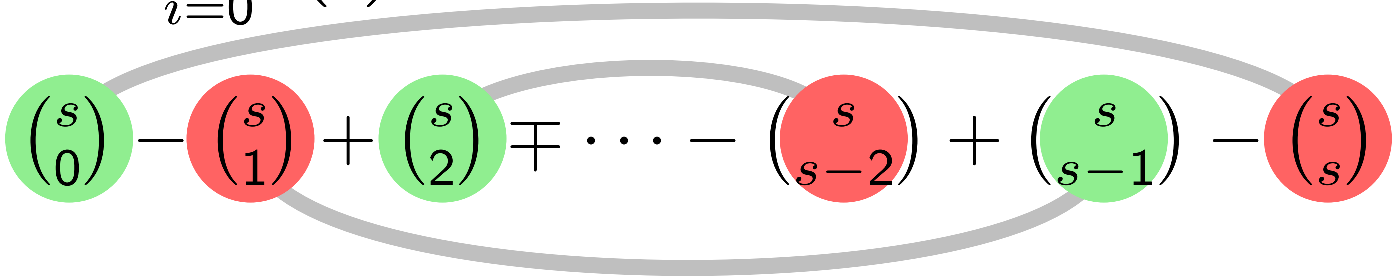

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– if s is even – **exercise!** □

Corollary

Recall: N objects and n properties $\mathcal{P} = \{P_1, \ldots, P_n\}$.

Cor. Let $\bar{N}(\mathcal{S})$ be the number of objects that have *none* of the properties in $\mathcal{S}$. Then

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But is it useful?

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Given: Directed Graph $G = (V, E)$,
two special vertices $s, t \in V$.

Question: Is there a *Hamiltonian path* from s to t,
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Another idea: Via TSP $\Rightarrow$ $O^*(2^n)$ time and *space*.

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Dynamic Program

For $W \subseteq V'$, $k = 0, \ldots, n-1$, and $u \in V \setminus W$, set:

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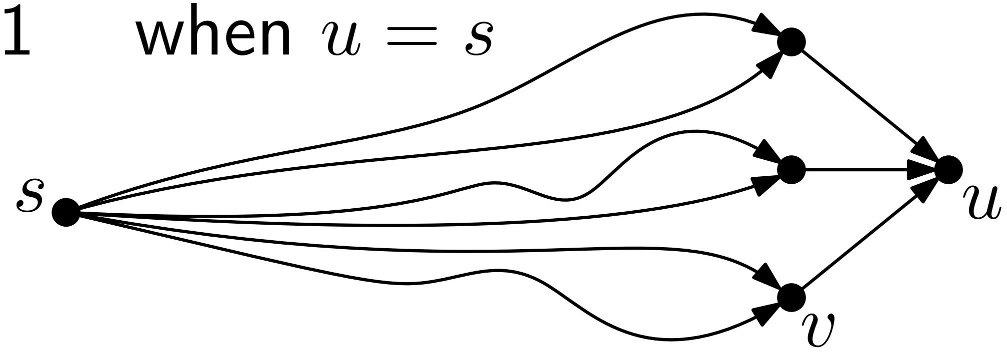

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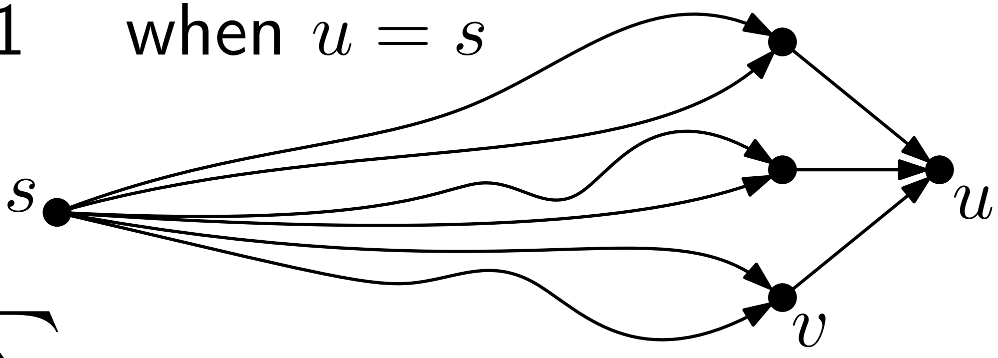

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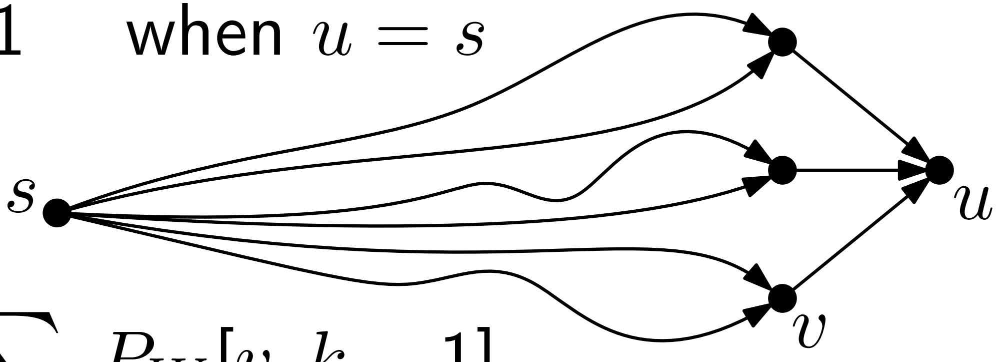

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Algorithmic Application of the IE Formula

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verylongint NumHamPath($G, s, t$)
    $V' \leftarrow V \setminus \{s, t\}$
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[Note that numbers in P_W can be exponential in m!]

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Runtime: $O(m^2 n \cdot 2^n)$

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$\rightsquigarrow$ **Exercise!**

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