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Exact Algorithms

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Lecture 4. Measure & Conquer

Based on: [Exact Exponential Algorithms: §6]

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What is Measure & Conquer?

- Method to analyse branching algorithms
- So far: Measure progress via instance size ($|V|$, $|E|$, ...).
- Now: Finer measure $\Rightarrow$ improved timing estimates.
- Many of the fastest algorithms use branching and have their time guarantees established via M & C.

Requirements for such a Measure

- The measure of an subinstance obtained by a reduction rule or a branching rule is smaller than the measure of the original instance.
- The measure of each instance is nonnegative.
- Measure of input is upperbounded by function of "natural parameters" of the input, e.g., measure$(G) \leq |V(G)|$.
  $\Rightarrow$ runtime bounds.

Example: Maximum Independent Set

```
int MIS(undirected graph G)
  if ∃v: deg(v) = 0 then
    return 1 + MIS(G − v)
  if ∃v: deg(v) = 1 then
    return 1 + MIS(G − N[v])
  if Δ(G) ≥ 3 then
    v = arg max_{v∈V} deg(v)
    return max(1 + MIS(G − N[v]), MIS(G − v))
  if Δ(G) ≤ 2 then
    solution in polytime
```

Measure: $|V(G)|$ Branching vector: $(1, \deg(v)+1) \geq (1, 4)$

$\rightsquigarrow$ Find positive root of $x^4 - x^3 - 1 = 0 \Rightarrow x \approx 1.3803$.

i.e., runtime $O^*(1.3803^n)$.

Better Measure than n?

Obs. Only vertices with degree ≥ 3 lead to branching.

Measure: $k_1(G) = n_{\geq 3} = \#$ vertices of degree ≥ 3

– discarding v reduces the size by ≥ 1.

– and (in the worst-case) picking v only removes a single vertex of degree ≥ 3 from G

$\Rightarrow$ branching vector: $(1, 1) \Rightarrow$ runtime $O^*(2^n)$:-(

Idea. Since degree-2-vertices are not removed immediately, they should be considered in our measure!

New measure: $k_2(G) = w_2 n_2 + n_{\geq 3} \leq n$ (!!) (for some weighting $0 < w_2 \leq 1$)

Lemma. Algorithm MIS runs in $O^*(1.3248^n)$ time.

Proof. Let's try $w_2 = 1/2$ and see ...

Analysis with M & C

*) Balancing lemma:
$\tau(i,j) > \tau(i+\epsilon, j-\epsilon)$
for $0 < i < j$ and $0 < \epsilon < \frac{j-i}{2}$

return max($1 + \mathrm{MIS}(G - N[v])$, $\mathrm{MIS}(G - v)$)

IN := decrease of $k_2(G)$ if $N[v]$ is deleted.

OUT := decrease of $k_2(G)$ if v is deleted.

$\Rightarrow$ IN, OUT ≥ 1, since v is always deleted and $\deg(v) \geq 3$.

Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If $\deg(w) \geq 3$ $\Rightarrow w$ holds the value 1 for IN.
2. If $\deg(w) = 2$ $\frac{1}{2}$ for IN & OUT.

$\overset{\sum}{\Rightarrow}$ IN + OUT $\geq 2 + \deg(v)$. Back to v...

1. If $\deg(v) \geq 4$ $\Rightarrow \tau(\mathrm{OUT}, \mathrm{IN}) \overset{(\star)}{\leq} \tau(1,5) < 1.3248$
2. If $\deg(v) = 3$ $\Rightarrow G$ has only vertices of degree 2 or 3.
   $\Rightarrow$ Deleting v or $N[v]$ reduces $k_2(G)$ by $\geq \frac{1}{2}$ per neighbor.
   $\Rightarrow$ IN, OUT $\geq 1 + \frac{3}{2}$ $\Rightarrow \tau(\mathrm{OUT}, \mathrm{IN}) \leq \tau\left(\frac{5}{2}, \frac{5}{2}\right) < 1.3196$.

□

Further Fine Tuning

General measure: $k(G) = \sum_{i=0}^{n-1} w_i n_i \leq n$ if all $w_i \in [0,1]$.

Set $w_0 = w_1 = 0$
because vertices of degree 0 or 1 do not occur when branching.

Pick $0 \leq w_2 \leq w_3 \leq 1$ and set $w_4 = w_5 = \cdots = w_n = 1$.

Best choices: $w_2 = 0.596601$
$w_3 = 0.928643$

Lemma. Algorithm MIS runs in $O^*(1.2905^n)$ time.

Proof. Exercise :-)

Back to Dominating Set...

- Best algorithm from Lecture #3: $O^*(1.7088^n)$.
- Next slide: simple branching algorithm
  - "Convential" analysis: runtime $O^*(1.9052^n)$
  - Measure & Conquer: runtime $O^*(1.5259^n)$
- Model instances of DS as instances $(U, \mathcal{S})$ of Set Cover, where
  - $U = V$ and
  - $\mathcal{S} = \{N[v] \colon v \in V\}$.
- There are faster SC algorithms for large $|\mathcal{S}|$: runtime $O^*(2^n)$.
- Our Goal: runtime $O^*(c^{|U|+|\mathcal{S}|})$ for some $c \ll 2$ for SC.

  $\Rightarrow$ runtime $O^*(c^{2n})$ for Dom. Set, where $n = |V|$.
- W.l.o.g. $U = \bigcup \mathcal{S}$ $\Rightarrow$ input given by $\mathcal{S}$.
- Define *frequency* $f(v) =$ number of sets that contain v.

Algorithm for Set Cover

int SC(set family $\mathcal{S}$)
  if $\mathcal{S} = \emptyset$ **then return** 0
  if $\exists S, R \in \mathcal{S}$ and $S \subset R$ **then return** SC($\mathcal{S} \setminus \{S\}$)
  if $\exists v \in \bigcup \mathcal{S}$ with $f(v) = 1$ **then**
    $S \ni v$
    return $1 +$ SC(del$(S, \mathcal{S})$) // del$(S, \mathcal{S}) = \{R \setminus S \neq \emptyset \mid R \in \mathcal{S}\}$
  $S = \arg\max\{|S'| : S' \in \mathcal{S}\}$
  if $|S| = 2$ **then** solve $\mathcal{S}$ in polytime. // Exercise!
  if $|S| \geq 3$ **then return** min(SC($\mathcal{S} \setminus \{S\}$), $1 +$ SC(del$(S, \mathcal{S})$))

Standard analysis: Measure $k(\mathcal{S}) = |\mathcal{S}| + |\bigcup \mathcal{S}|$

$T(k) \leq T(k-1) + T(k-4)$, $T(1,2,3,4) \in O(1)$

$\Rightarrow T(k) \in O^*(1.3803^k)$

$\Rightarrow$ Runtime for DS: $O^*(1.3803^{2n}) = O^*(1.9052^n)$

Idea for a Finer Analysis

- Deleting large sets reduces the frequency of many elements.
- Reducing the frequency of an element can eventually lead to a frequency of 1 $\rightsquigarrow$ selecting a set.
- Deleting a high-frequency element reduces the size of *many* sets.
- Reducing the size of sets is useful since sets contained in other sets are removed.

$n_i = \#$ sets of size i in $\mathcal{S}$

$m_j = \#$ elements of frequency j in $U = \bigcup \mathcal{S}$

New measure: $k(\mathcal{S}) = \sum_{i \geq 1} w_i n_i + \sum_{j \geq 1} v_j m_j \quad w_i, v_j \in [0, 1]$

Note: $k(\mathcal{S}) \leq |\mathcal{S}| + |U| \Rightarrow$ Runtime depends on $|\mathcal{S}| + |U|$ as desired.

Simplifying Observations

Our new measure $k(\mathcal{S}) = \sum_{i \geq 1} w_i n_i + \sum_{j \geq 1} v_j m_j$:

$n_i = \#$ sets of size i in $\mathcal{S}$

$m_j = \#$ elements of frequency j in U

- $w_i \leq w_{i+1}$, and $v_i \leq v_{i+1}$
- $w_1 = v_1 = 0$
- $w_i = v_i = 1$ for every $i \geq 6$
- $\Delta w_i \geq \Delta w_{i+1}$, where $\Delta w_i = w_i - w_{i-1}$ and $\Delta v_i = v_i - v_{i-1}$

The analysis breaks into two branches: IN and OUT.

Analysis of $\mathcal{S}_{\text{OUT}} := \mathcal{S} \setminus \{S\}$

Our new measure $k(\mathcal{S}) = \overbrace{\sum_{i \geq 1} w_i n_i}^{k_w(\mathcal{S})} + \overbrace{\sum_{j \geq 1} v_j m_j}^{k_v(\mathcal{S})}$:

(a) Reduction in $k_w(\mathcal{S})$ from deleting S: $w_{|S|}$

(b) $r_i := \#$ elements in S of frequency i $\Rightarrow \sum_{i \geq 1 \not 2} r_i = |S|$

Reduction in $k_v(\mathcal{S})$ from deleting S: $\sum_{i=2}^{6} r_i \cdot \Delta v_i$

$\Delta v_i = 0$ for $i \geq 7$

(c) If $r_2 > 0$:

Let $R_1, \ldots, R_h$ be the sets $\neq S$ that share at least one element of frequency 2 with S. $\Rightarrow 1 \leq h \leq r_2$.

Removing $S \Rightarrow$ selects $R_1, \ldots, R_h$ *without* branching!

$r_{2,i} := \#$ frequency-2 elements in $R_i \cap S$ $\Rightarrow \sum_{i=1}^{h} r_{2,i} = r_2$

$R_i \not\subseteq S \Rightarrow |R_i| \geq r_{2,i} + 1$. S largest $\Rightarrow |S| \geq \max_i r_{2,i} + 1$.

Selecting $R_i \Rightarrow$ reduction in $k_w(\mathcal{S})$: $w_{|R_i|} \geq w_{r_{2,i}+1}$.

At least one element $e_i \in R_i \setminus S$ is covered.

$\Rightarrow$ Reduction in $k_v(\mathcal{S})$: at least $v_{f(e_i)} \geq v_2$.

Analysis of $\mathcal{S}_{\mathsf{OUT}}$ (cont'd)

Measure reduced by at least (case distinction)

$$\Delta k' = \begin{cases} 0 & \text{when } r_2 = 0, \\ v_2 + w_2 & r_2 = 1, \\ v_2 + \min\{2w_2, w_3\} & r_2 = 2, \\ v_2 + \min\{3w_2, w_2 + w_3\} & r_2 \geq 3, |S| = 3, \\ v_2 + \min\{3w_2, w_2 + w_3, w_4\} & r_2 \geq 3, |S| \geq 4. \end{cases}$$

Total reduction for $\mathcal{S}_{\mathsf{OUT}}$ is:

$$\Delta_{\mathsf{OUT}} = \underbrace{w_{|S|}}_{\text{(a) size of } S} + \underbrace{\sum_{i=2}^{6} r_i \cdot \Delta v_i}_{\text{(b) freq. of elements in } S} + \underbrace{\Delta k'}_{\substack{\text{(c) freq. \& size of} \\ \text{frequency-2 elements in } S}}$$

Analysis of $\mathcal{S}_{\text{IN}} := \text{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

(b) Reduction in $k_v(\mathcal{S})$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

(c) Reduction in $k_w(\mathcal{S})$ from shrinking sets that intersect S:

Let R be a set with $S \cap R \neq \emptyset$, and $v \in R \cap S$.

Then v contributes to the reduction: $\Delta w_{|R|} \geq \Delta w_{|S|}$.

i.e. reduction $\geq \quad \Delta k'' := \Delta w_{|S|} \cdot \left(\sum_{i=2}^{6} (i-1) r_i + 6 r_{\geq 7} \right)$

Each element of frequency i belongs to $i - 1$ sets $\neq S$.

$$\Rightarrow \Delta_{\text{IN}} = \quad w_{|S|} \quad + \quad \sum_{i=2}^{6} r_i v_i + r_{\geq 7} \quad + \quad \Delta k''.$$

$\Rightarrow$ Recurrence for fixed weights v and w:

For each $|S| \geq 3$ and $(r_i)_i$ with $|S| = \sum r_i$:

$T(k) \leq T(k - \Delta_{\text{OUT}}) + T(k - \Delta_{\text{IN}})$. What's the worst case?

Optimizing the Branching Vector

Obs. Every branching vector for $|S| \geq 7$ is dominated by some branching vector for $|S| = 7$.

Reason: Consider formulas for Δ_{OUT} and Δ_{IN}:

$$\Delta_{\mathsf{OUT}} = w_{|S|} \quad + \quad \sum_{i=2}^{6} r_i \Delta v_i \quad + \quad \Delta k'$$

$$\Delta_{\mathsf{IN}} \quad = w_{|S|} + \left(\sum_{i=2}^{6} r_i v_i + r_{\geq 7} \right) + \Delta w_{|S|} \cdot (\dots).$$

Shrinking $|S|$ will only reduce the terms.

Important point here is $\Delta w_{|S|}$ in $\Delta_{\mathsf{IN}} \dots$

but $\Delta w_{|S|}$ is 0 when $|S| \geq 7$.

Obs. Hence it is sufficient to consider configurations with $3 \leq |S| \leq 7$, and all possible combinations of $(r_i)_i$'s.

Wrap-Up

For each fixed 8-tuple $(w, v) = (w_2, \ldots, w_5, v_2, \ldots, v_5)$, the runtime is bounded by α^k, where α is the largest root of

$$x^t - x^{t-\Delta_{\mathsf{OUT}}} - x^{t-\Delta_{\mathsf{IN}}} = 0$$

and $t = \max(\Delta_{\mathsf{OUT}}, \Delta_{\mathsf{IN}})$ over all choices of $|S|, r_1, \ldots, r_{|S|}$.

Each (w, v) yields the runtime bound $O^*(\alpha_{(w,v)}^k)$.

Goal: Find (w, v) that minimizes $\alpha_{(w,v)}$!
(Use quasi-convex optimization.)

$$\begin{pmatrix} .377 & .399 \\ .755 & .768 \\ .909 & .930 \\ .976 & .986 \end{pmatrix}$$

The approximate best solution found here is $\alpha_{(w^\star, v^\star)} < 1.2353$.

Thm. SC can be solved in $O^*(1.2353^{|U|+|\mathcal{S}|})$ time.

Corollary. DS can be solved in $O^*(1.2353^{2n}) = O^*(1.5259^n)$ time.