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Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 4. Measure & Conquer

Based on: [Exact Exponential Algorithms: §6]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

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What is Measure & Conquer?

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- Method to analyse branching algorithms
- So far: Measure progress via instance size ($|V|$, $|E|$, $\ldots$).
- Now: Finer measure $\Rightarrow$ improved timing estimates.
- Many of the fastest algorithms use branching and have their time guarantees established via M & C.

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- Measure of input is upperbounded by function of “natural parameters” of the input, e.g., measure$(G) \leq |V(G)|$.
  $\Rightarrow$ runtime bounds.

Example: Maximum Independent Set

```
int MIS(undirected graph $G$)
    if $\exists v\colon \deg(v) = 0$ then
        return
    if $\exists v\colon \deg(v) = 1$ then
        return
    if $\Delta(G) \geq 3$ then

    if $\Delta(G) \leq 2$ then

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Example: Maximum Independent Set

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int MIS(undirected graph $G$)
  if $\exists v\colon \deg(v) = 0$ then
    return $1 + \mathrm{MIS}(G - v)$
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        return max(                                        )
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    if $\Delta(G) \geq 3$ then
        $v = \arg\max_{v \in V} \deg(v)$
        return $\max(1 + \mathrm{MIS}(G - N[v]), \mathrm{MIS}(G - v))$
    if $\Delta(G) \leq 2$ then
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    solution in polytime
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Measure:

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Measure: $|V(G)|$

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Measure: $|V(G)|$ Branching vector:

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$\rightsquigarrow$ Find positive root of

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Measure: $|V(G)|$ Branching vector: $(1, \deg(v) + 1) \geq (1, 4)$

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i.e., runtime $O^*(\qquad\qquad)$.

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$\rightsquigarrow$ Find positive root of $x^4 - x^3 - 1 = 0 \Rightarrow x \approx 1.3803$.

i.e., runtime $O^*(1.3803^n)$.

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$\Rightarrow$ branching vector: $(1, 1)$ $\Rightarrow$ runtime $O^*(2^n)$:-(

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Idea. Since degree-2-vertices are not removed immediately, they should be considered in our measure!

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(for some weighting $0 < w_2 \leq 1$)

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New measure: $k_2(G) = w_2 n_2 + n_{\geq 3} \leq n$ (!!)

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Lemma. Algorithm MIS runs in $O^*(1.3248^n)$ time.

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New measure: $k_2(G) = w_2 n_2 + n_{\geq 3} \leq n$ (!!)

Lemma. Algorithm MIS runs in $O^*(1.3248^n)$ time.

Proof. Let's try $w_2 = 1/2$ and see . . .

Analysis with M & C

```
return max($1 + \mathsf{MIS}(G - N[v])$, $\mathsf{MIS}(G - v)$)
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Analysis with M & C

return max($1 + \text{MIS}(G - N[v])$, $\text{MIS}(G - v)$)

IN $:=$ decrease of $k_2(G)$ if $N[v]$ is deleted.

Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

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$\Rightarrow$ IN, OUT $\geq$ 1, since v is always deleted and $\deg(v) \geq 3$.

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Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If

2. If

Analysis with M & C

return max($1 + \text{MIS}(G - N[v])$, $\text{MIS}(G - v)$)

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1. If $\deg(w) \geq 3$
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Consider $w \in N(v)$.

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1. If $\deg(w) \geq 3$ $\Rightarrow$ w holds the value
2. If

Analysis with M & C

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Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If $\deg(w) \geq 3$ $\Rightarrow$ w holds the value 1 for IN.
2. If

Analysis with M & C

return max($1 + \text{MIS}(G - N[v])$, $\text{MIS}(G - v)$)

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Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If $\deg(w) \geq 3$ $\Rightarrow$ w holds the value 1 for IN.
2. If $\deg(w) = 2$

Analysis with M & C

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1. If $\deg(w) \geq 3$ $\Rightarrow$ w holds the value 1 for IN.
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Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If $\deg(w) \geq 3$ $\Rightarrow$ w holds the value 1 for IN.
2. If $\deg(w) = 2$ $\frac{1}{2}$ for IN & OUT.

$\overset{\sum}{\Rightarrow}$ IN + OUT $\geq$

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Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If $\text{deg}(w) \geq 3$ $\Rightarrow w$ holds the value 1 for IN.
2. If $\text{deg}(w) = 2$ $\frac{1}{2}$ for IN & OUT.

$\overset{\Sigma}{\Rightarrow}$ IN $+$ OUT $\geq 2 + \text{deg}(v)$.

Analysis with M & C

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Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{\geq 3}$

1. If $\deg(w) \geq 3$ $\Rightarrow$ w holds the value 1 for IN.
2. If $\deg(w) = 2$ $\frac{1}{2}$ for IN & OUT.

$\overset{\Sigma}{\Rightarrow}$ IN + OUT $\geq 2 + \deg(v)$. Back to v...

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Analysis with M & C

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Proof. Exercise :-)

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- Define *frequency* $f(v)$ = number of sets that contain v.

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The analysis breaks into two branches: IN and OUT.

Analysis of $\mathcal{S}_{\text{OUT}} := \mathcal{S} \setminus \{S\}$

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$\Delta v_i = 0$ for $i \geq 7$

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(c) If $r_2 > 0$:

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Reduction in $k_v(\mathcal{S})$ from deleting S: $\sum_{i=2}^{6} r_i \cdot \Delta v_i$

$\Delta v_i = 0$ for $i \geq 7$

(c) If $r_2 > 0$:

Let $R_1, \ldots, R_h$ be the sets $\neq S$ that share at least one element of frequency 2 with S.

Analysis of $\mathcal{S}_{\text{OUT}} := \mathcal{S} \setminus \{S\}$

Our new measure $k(\mathcal{S}) = \overset{k_w(\mathcal{S})}{\sum_{i\geq 1} w_i n_i} + \overset{k_v(\mathcal{S})}{\sum_{j\geq 1} v_j m_j}$:

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Removing $S \Rightarrow$ selects $R_1, \ldots, R_h$ *without* branching!

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Selecting $R_i \Rightarrow$ reduction in $k_w(\mathcal{S})$:

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Selecting $R_i \Rightarrow$ reduction in $k_w(\mathcal{S})$: $w_{|R_i|} \geq w_{r_{2,i}+1}$.

At least one element $e_i \in R_i \setminus S$ is covered.

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Selecting $R_i \Rightarrow$ reduction in $k_w(\mathcal{S})$: $w_{|R_i|} \geq w_{r_{2,i}+1}$.

At least one element $e_i \in R_i \setminus S$ is covered.

$\Rightarrow$ Reduction in $k_v(\mathcal{S})$: at least

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(a) Reduction in $k_w(\mathcal{S})$ from deleting S: $w_{|S|}$

(b) $r_i :=$ # elements in S of frequency i $\Rightarrow \sum_{i\geq \cancel{1}2} r_i = |S|$

Reduction in $k_v(\mathcal{S})$ from deleting S: $\sum_{i=2}^{6} r_i \cdot \Delta v_i$

$\Delta v_i = 0$ for $i \geq 7$

(c) If $r_2 > 0$:

Let $R_1, \ldots, R_h$ be the sets $\neq S$ that share at least one element of frequency 2 with S. $\Rightarrow 1 \leq h \leq r_2$.

Removing $S \Rightarrow$ selects $R_1, \ldots, R_h$ *without* branching!

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$R_i \not\subseteq S \Rightarrow |R_i| \geq r_{2,i} + 1$. S largest $\Rightarrow |S| \geq \max_i r_{2,i} + 1$.

Selecting $R_i \Rightarrow$ reduction in $k_w(\mathcal{S})$: $w_{|R_i|} \geq w_{r_{2,i}+1}$.

At least one element $e_i \in R_i \setminus S$ is covered.

$\Rightarrow$ Reduction in $k_v(\mathcal{S})$: at least $v_{f(e_i)} \geq v_2$.

Analysis of $\mathcal{S}_{\mathsf{OUT}}$ (cont'd)

Measure reduced by at least (case distinction)

$$\Delta k' = \begin{cases} 0 & \text{when } r_2 = 0, \\ v_2 + w_2 & r_2 = 1, \\ v_2 + \min\{2w_2, w_3\} & r_2 = 2, \\ v_2 + \min\{3w_2, w_2 + w_3\} & r_2 \geq 3, |S| = 3, \\ v_2 + \min\{3w_2, w_2 + w_3, w_4\} & r_2 \geq 3, |S| \geq 4. \end{cases}$$

Analysis of $\mathcal{S}_{\text{OUT}}$ (cont'd)

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Total reduction for $\mathcal{S}_{\text{OUT}}$ is:

Analysis of $\mathcal{S}_{\text{OUT}}$ (cont'd)

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Total reduction for $\mathcal{S}_{\text{OUT}}$ is:

$$\Delta_{\text{OUT}} = \quad w_{|S|} \quad + \quad \sum_{i=2}^{6} r_i \cdot \Delta v_i \quad + \quad \Delta k'$$

Analysis of $\mathcal{S}_{\mathsf{OUT}}$ (cont'd)

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$$\Delta k' = \begin{cases} 0 & \text{when } r_2 = 0, \\ v_2 + w_2 & r_2 = 1, \\ v_2 + \min\{2w_2, w_3\} & r_2 = 2, \\ v_2 + \min\{3w_2, w_2 + w_3\} & r_2 \geq 3, |S| = 3, \\ v_2 + \min\{3w_2, w_2 + w_3, w_4\} & r_2 \geq 3, |S| \geq 4. \end{cases}$$

Total reduction for $\mathcal{S}_{\mathsf{OUT}}$ is:

$$\Delta_{\mathsf{OUT}} = \quad \underbrace{w_{|S|}}_{\text{(a) size of } S} \quad + \quad \sum_{i=2}^{6} r_i \cdot \Delta v_i \quad + \quad \Delta k'$$

Analysis of $\mathcal{S}_{\mathsf{OUT}}$ (cont'd)

Measure reduced by at least (case distinction)

$$\Delta k' = \begin{cases} 0 & \text{when } r_2 = 0, \\ v_2 + w_2 & r_2 = 1, \\ v_2 + \min\{2w_2, w_3\} & r_2 = 2, \\ v_2 + \min\{3w_2, w_2 + w_3\} & r_2 \geq 3, |S| = 3, \\ v_2 + \min\{3w_2, w_2 + w_3, w_4\} & r_2 \geq 3, |S| \geq 4. \end{cases}$$

Total reduction for $\mathcal{S}_{\mathsf{OUT}}$ is:

$$\Delta_{\mathsf{OUT}} = \underset{\text{(a) size of } S}{w_{|S|}} + \underset{\text{(b) freq. of elements in } S}{\sum_{i=2}^{6} r_i \cdot \Delta v_i} + \Delta k'$$

Analysis of $\mathcal{S}_{\text{OUT}}$ (cont'd)

Measure reduced by at least (case distinction)

$$\Delta k' = \begin{cases} 0 & \text{when } r_2 = 0, \\ v_2 + w_2 & r_2 = 1, \\ v_2 + \min\{2w_2, w_3\} & r_2 = 2, \\ v_2 + \min\{3w_2, w_2 + w_3\} & r_2 \geq 3, |S| = 3, \\ v_2 + \min\{3w_2, w_2 + w_3, w_4\} & r_2 \geq 3, |S| \geq 4. \end{cases}$$

Total reduction for $\mathcal{S}_{\text{OUT}}$ is:

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Analysis of $\mathcal{S}_{\mathsf{IN}} := \mathsf{del}(S, \mathcal{S})$

Analysis of $\mathcal{S}_{\mathsf{IN}} := \mathsf{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S:

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(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

Analysis of $\mathcal{S}_{\mathsf{IN}} := \mathsf{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

(b) Reduction in $k_v(\mathcal{S})$ by dropping S:

Analysis of $\mathcal{S}_{\mathsf{IN}} := \mathsf{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

(b) Reduction in $k_v(\mathcal{S})$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

Analysis of $\mathcal{S}_{\mathsf{IN}} := \mathsf{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

(b) Reduction in $k_v(\mathcal{S})$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

(c) Reduction in $k_w(\mathcal{S})$ from shrinking sets that intersect S:

Analysis of $\mathcal{S}_{\text{IN}} := \text{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

(b) Reduction in $k_v(\mathcal{S})$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

(c) Reduction in $k_w(\mathcal{S})$ from shrinking sets that intersect S:

Let R be a set with $S \cap R \neq \emptyset$, and $v \in R \cap S$.

Analysis of $\mathcal{S}_{\text{IN}} := \text{del}(S, \mathcal{S})$

(a) Reduction in $k_w(\mathcal{S})$ by dropping S: $w_{|S|}$

(b) Reduction in $k_v(\mathcal{S})$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

(c) Reduction in $k_w(\mathcal{S})$ from shrinking sets that intersect S:

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Then v contributes to the reduction:

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(b) Reduction in $k_v(\mathcal{S})$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

(c) Reduction in $k_w(\mathcal{S})$ from shrinking sets that intersect S:

Let R be a set with $S \cap R \neq \emptyset$, and $v \in R \cap S$.

Then v contributes to the reduction: $\Delta w_{|R|} \geq \Delta w_{|S|}$.

Analysis of $\mathcal{S}_{\text{IN}} := \text{del}(S, \mathcal{S})$

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Obs. Hence it is sufficient to consider configurations with $3 \leq |S| \leq 7$, and all possible combinations of $(r_i)_i$'s.

Wrap-Up

For each fixed 8-tuple $(w, v) = (w_2, \ldots, w_5, v_2, \ldots, v_5)$, the runtime is bounded by α^k, where α is the largest root of

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Corollary. DS can be solved in $O^*(1.2353^{2n}) = O^*(1.5259^n)$ time.