



Exact Algorithms

Sommer Term 2020

Lecture 4. Measure & Conquer Based on: [Exact Exponential Algorithms: §6]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

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Lehrstuhl für Informatik I

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- Now: Finer measure \Rightarrow improved timing estimates.
- Many of the fastest algorithms use branching and have their time guarantees established via M&C.

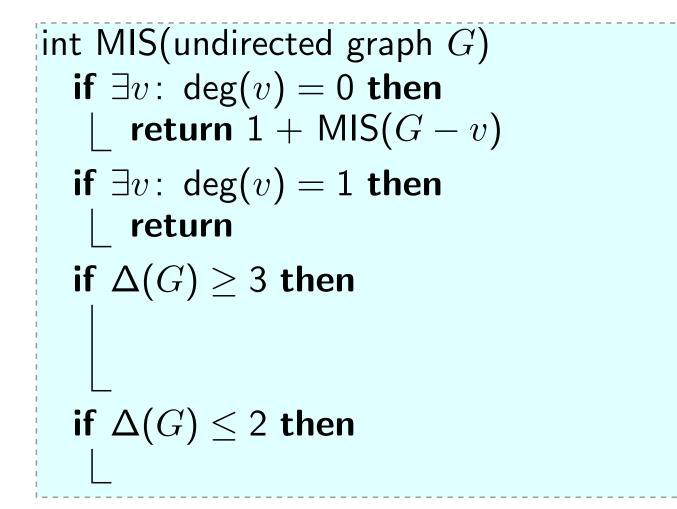
• The measure of an subinstance obtained by a reduction rule or a branching rule is smaller than the measure of the original instance.

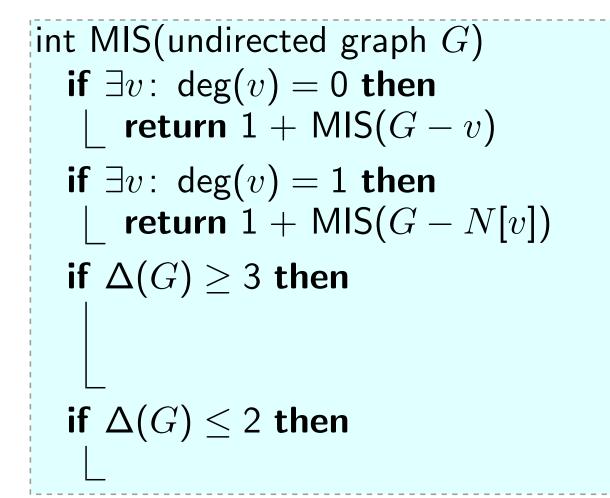
- The measure of an subinstance obtained by a reduction rule or a branching rule is smaller than the measure of the original instance.
- The measure of each instance is nonnegative.

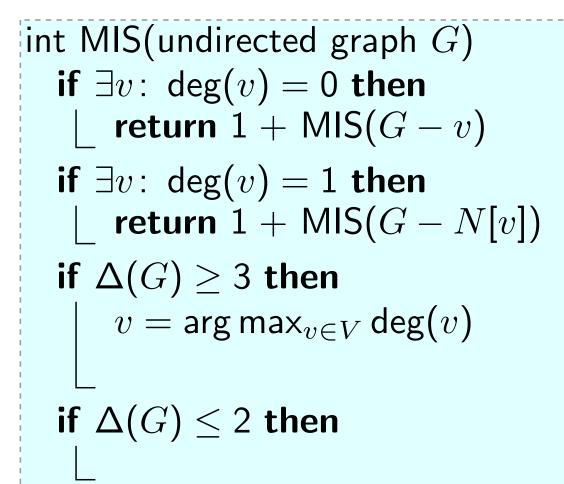
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- Measure of input is upperbounded by function of "natural parameters" of the input, e.g., measure $(G) \leq |V(G)|$.

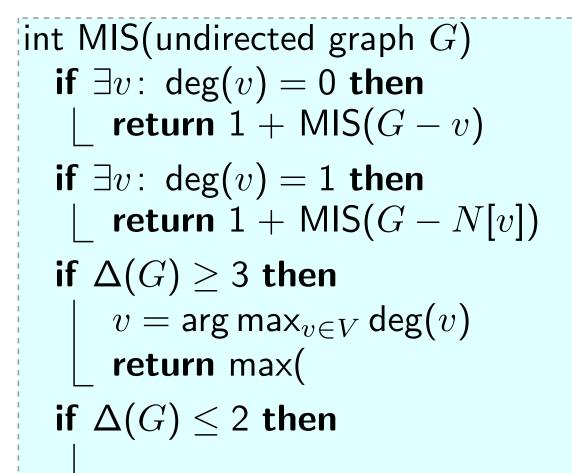
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 ⇒ runtime bounds.

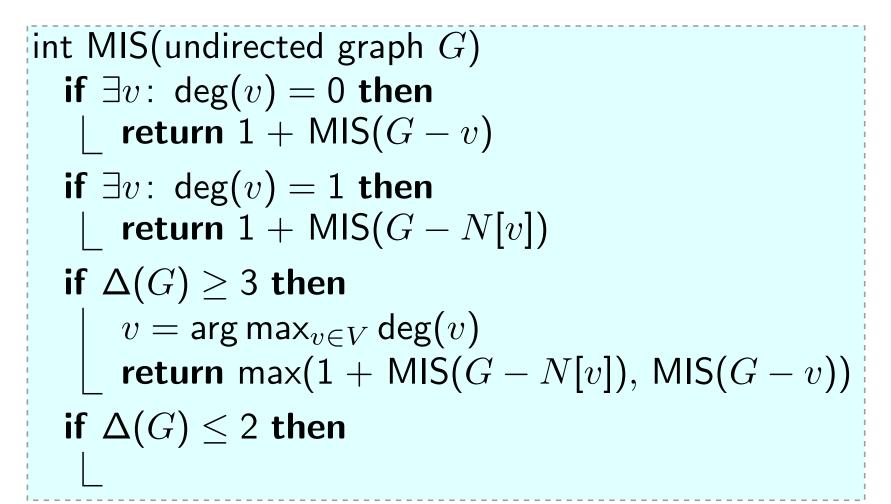
```
int MIS(undirected graph G)
   if \exists v \colon \deg(v) = 0 then
       return
  if \exists v \colon \deg(v) = 1 then
       return
  if \Delta(G) \geq 3 then
  if \Delta(G) \leq 2 then
```

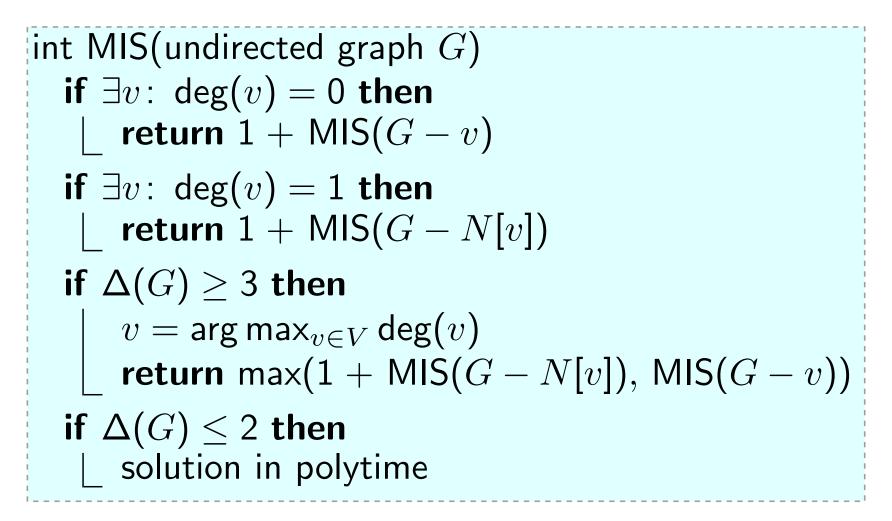


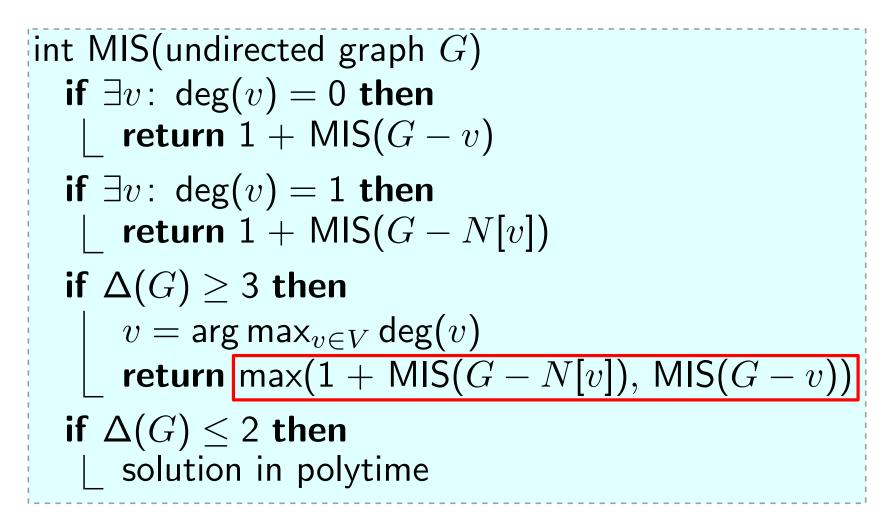


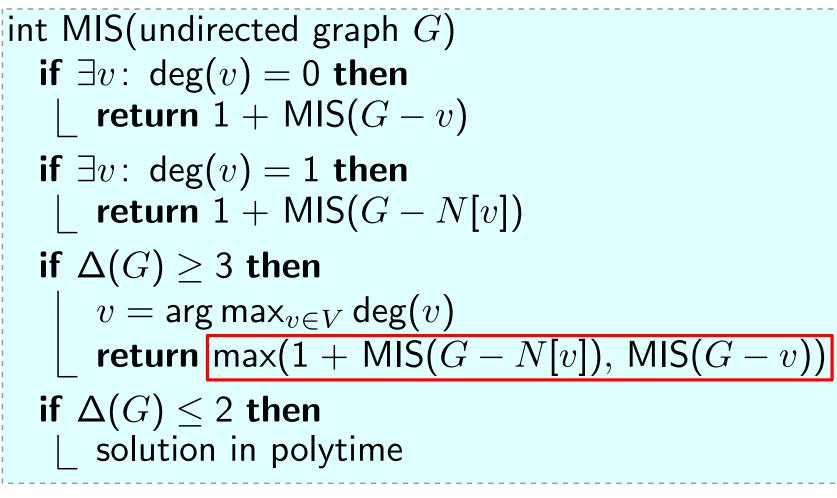




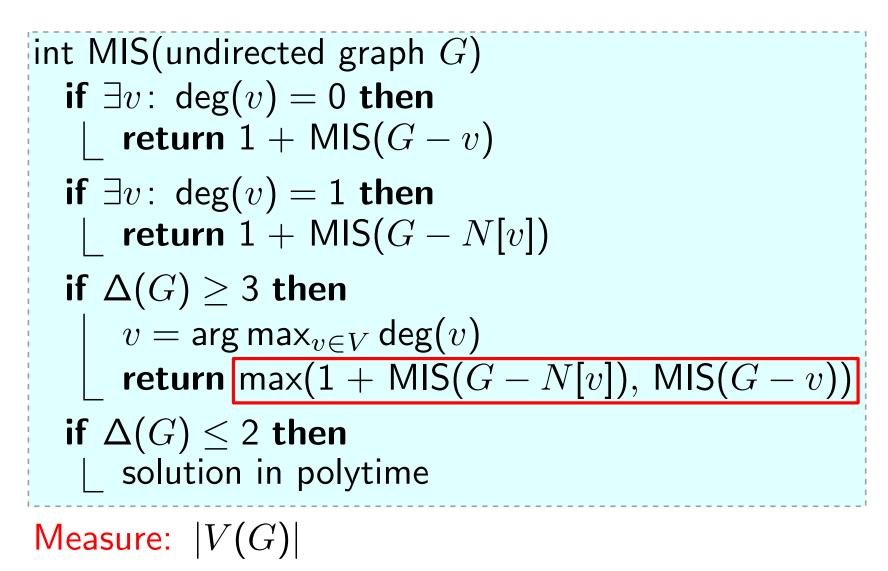


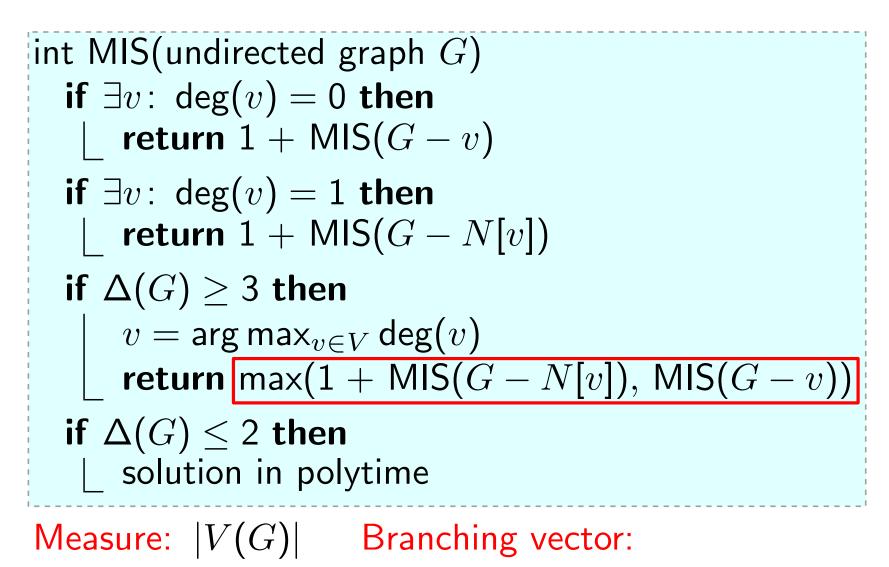


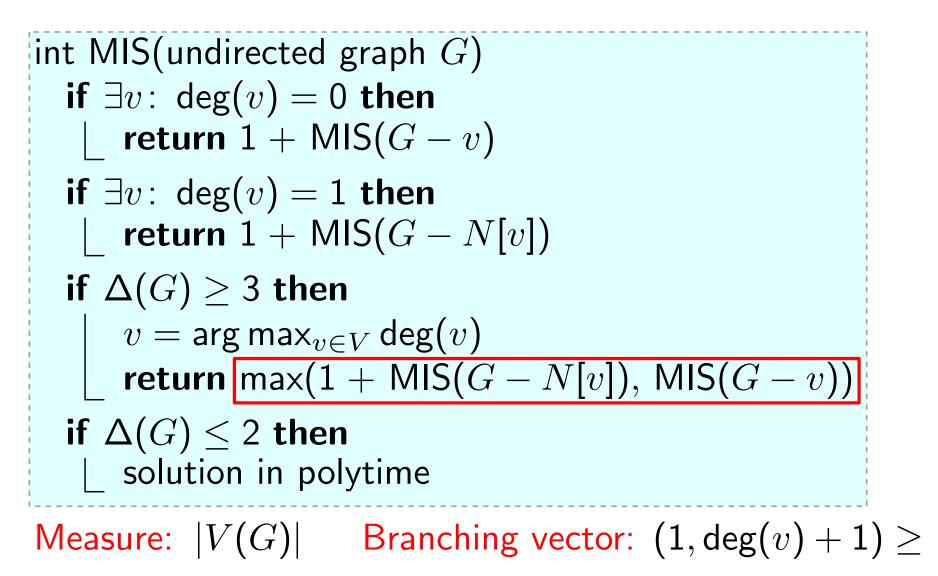


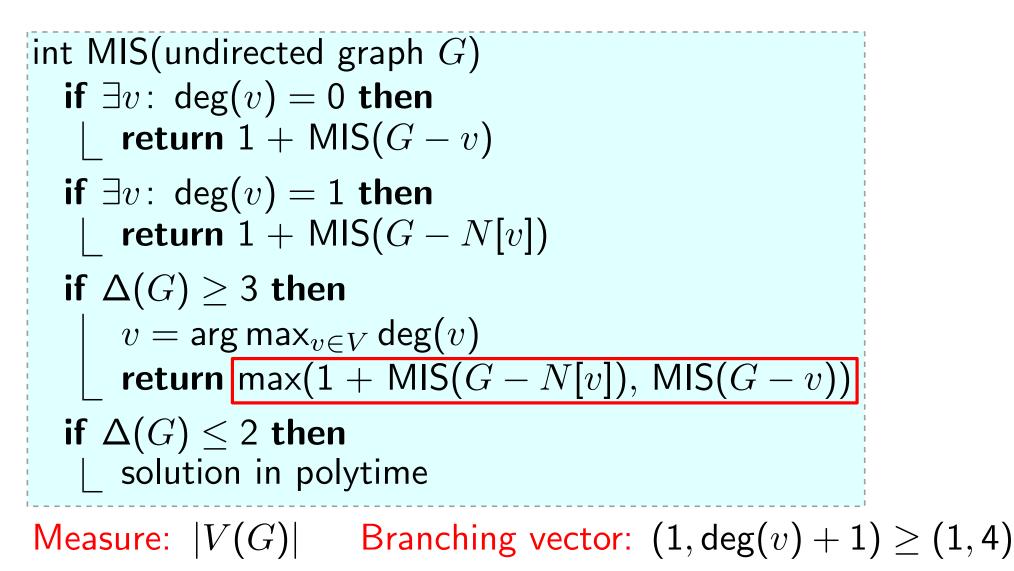


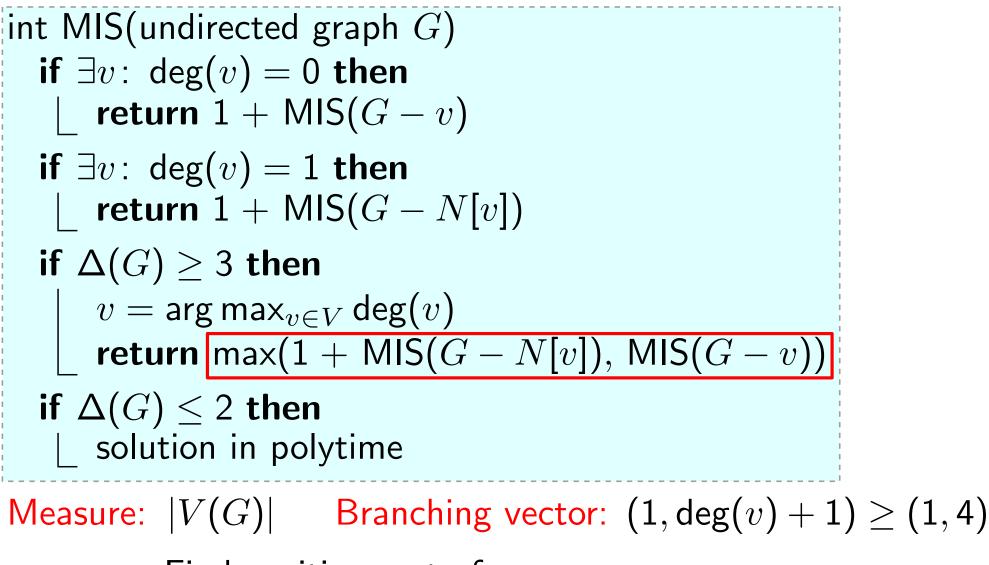
Measure:



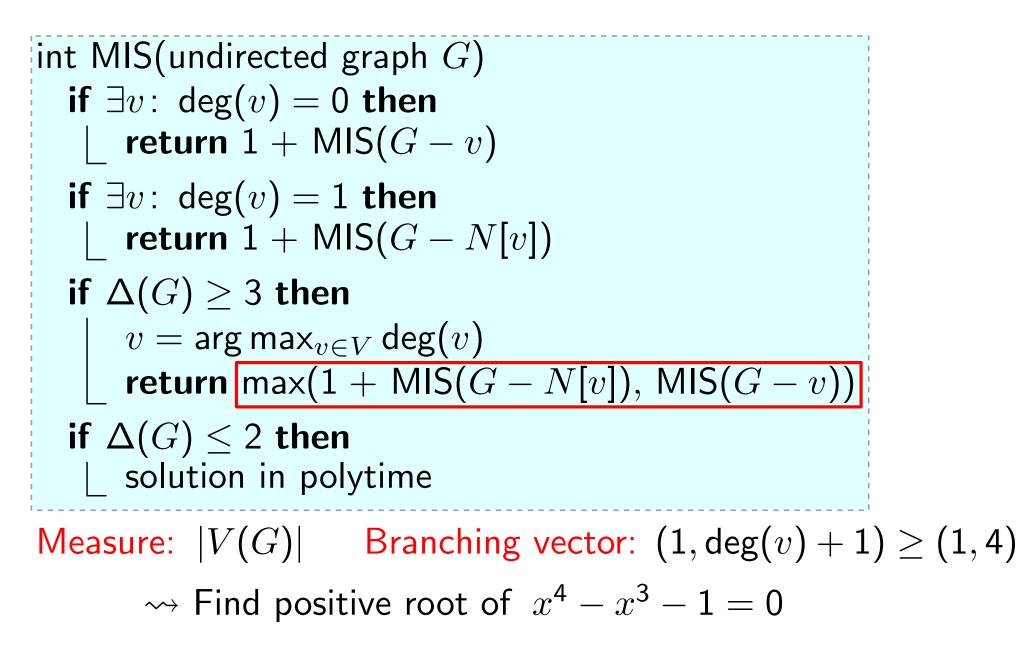


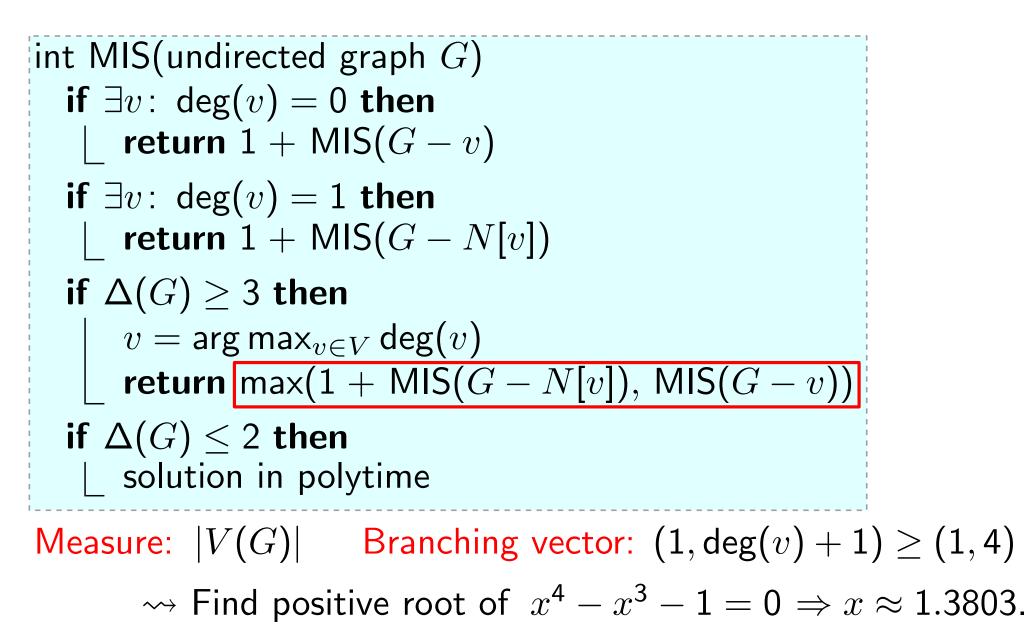


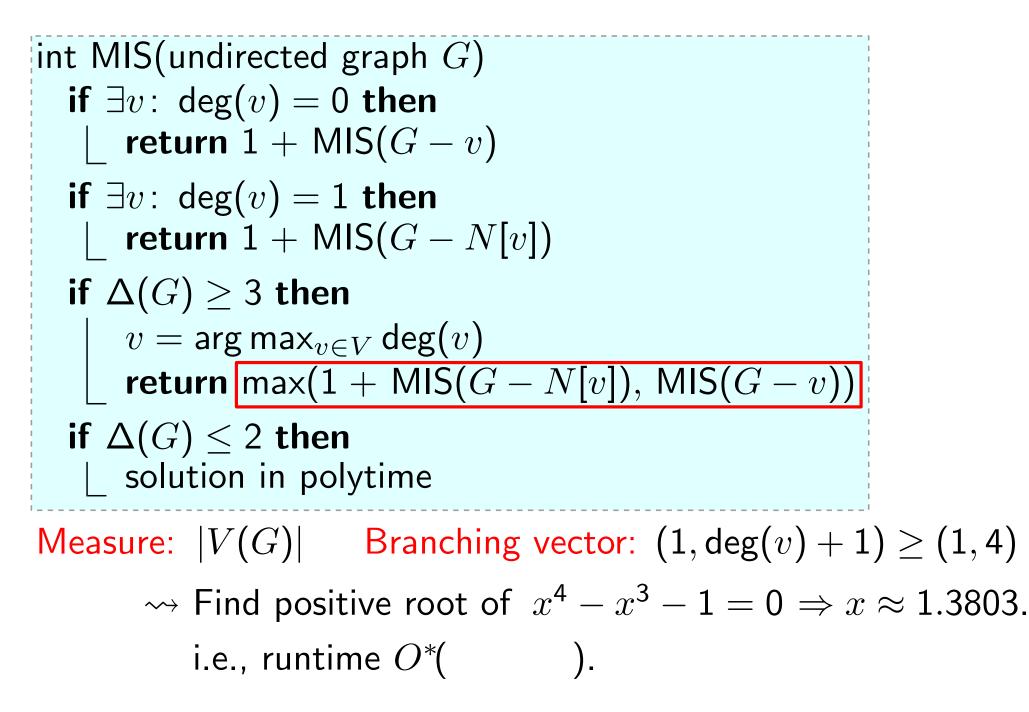


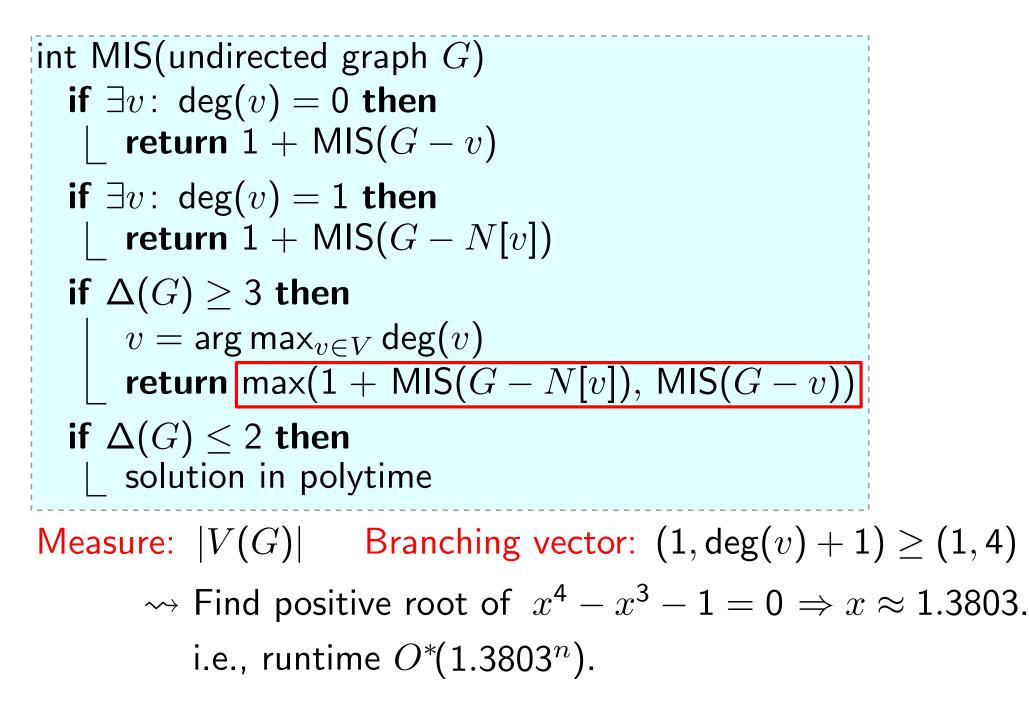


 \rightsquigarrow Find positive root of









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Lemma. Algorithm MIS runs in $O^*(1.3248^n)$ time.

Proof. Let's try $w_2 = 1/2$ and see ...

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IN := decrease of $k_2(G)$ if N[v] is deleted. OUT := decrease of $k_2(G)$ if v is deleted. \Rightarrow IN, OUT \ge 1, since v is always deleted and deg(v) \ge 3. Consider $w \in N(v)$. Recall: $k_2(G) = n_2/2 + n_{>3}$ 1. If $\deg(w) \ge 3 \implies w$ holds the value 1 for IN. $\frac{1}{2}$ for IN & OUT. 2. If deg(w) = 2 \Rightarrow IN + OUT $\ge 2 + \deg(v)$. Back to $v \dots$ 1. If $\deg(v) \ge 4 \implies \tau(\mathsf{OUT},\mathsf{IN}) \stackrel{\scriptscriptstyle (\star)}{\le} \tau(1,5) < 1.3248$ 2. If $deg(v) = 3 \Rightarrow G$ has only vertices of degree 2 or 3. \Rightarrow Deleting v or N[v] reduces $k_2(G)$ by $\geq \frac{1}{2}$ per neighbor. \Rightarrow IN, OUT $\geq 1 + \frac{3}{2} \Rightarrow \tau(OUT, IN) \leq \tau(\frac{5}{2}, \frac{5}{2}) <$

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Further Fine Tuning

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- W.I.o.g. $U = \bigcup S \implies$ input given by S.
- Define frequency f(v) = number of sets that contain v.

int SC(set family $\mathcal{S})$

9

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 \begin{array}{ll} \text{int SC(set family } \mathcal{S}) \\ \text{if } \mathcal{S} = \emptyset \text{ then return } 0 \\ \text{if } \exists S, R \in \mathcal{S} \text{ and } S \subset R \text{ then return SC}(\mathcal{S} \setminus \{S\}) \\ \text{if } \exists v \in \bigcup \mathcal{S} \text{ with } f(v) = 1 \text{ then} \\ \left\lfloor \begin{array}{c} S \ni v \\ \text{return } 1 + \text{SC}(\text{del}(S, \mathcal{S})) \end{array} \right. // \text{del}(S, \mathcal{S}) = \{R \setminus S \neq \emptyset \mid R \in \mathcal{S}\} \\ \mathcal{S} = \arg \max\{|S'| \colon S' \in \mathcal{S}\} \\ \text{if } |S| = 2 \text{ then solve } \mathcal{S} \text{ in polytime.} \end{array} \right. // \text{Exercise!}
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New measure: $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$ $w_i, v_j \in [0, 1]$ Note: $k(S) \le |S| + |U|$

- Deleting large sets reduces the frequency of many elements.
- Reducing the frequency of an element can eventually lead to a frequency of 1 → selecting a set.
- Deleting a high-frequency element reduces the size of *many* sets.
- Reducing the size of sets is useful since sets contained in other sets are removed.

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New measure: $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$ $w_i, v_j \in [0, 1]$ Note: $k(S) \le |S| + |U| \Rightarrow$ Runtime depends on |S| + |U|as desired.

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The analysis breaks into two branches: IN and OUT.

Analysis of $S_{OUT} := S \setminus \{S\}$ Our new measure $k(S) = \sum_{i \ge 1}^{k_w(S)} w_i n_i + \sum_{j \ge 1}^{k_v(S)} v_j m_j$: Analysis of $S_{OUT} := S \setminus \{S\}$ Our new measure $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: Analysis of $S_{OUT} := S \setminus \{S\}$ $k_w(S) = \sum_{k_w(S)} k_v(S)$ Our new measure $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: $w_{|S|}$ Analysis of $S_{OUT} := S \setminus \{S\}$ $k_w(S) = \sum_{k_w(S)} k_v(S)$ Our new measure $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: $w_{|S|}$ (b) $r_i := \#$ elements in S of frequency i Analysis of $S_{OUT} := S \setminus \{S\}$ Our new measure $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: $w_{|S|}$ (b) $r_i := \#$ elements in S of frequency $i \implies \sum_{i \ge 1} r_i =$ Analysis of $S_{OUT} := S \setminus \{S\}$ Our new measure $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: $w_{|S|}$ (b) $r_i := \#$ elements in S of frequency $i \Rightarrow \sum_{i \ge 1} r_i = |S|$ Analysis of $S_{OUT} := S \setminus \{S\}$ Our new measure $k(S) = \sum_{i \ge 1} w_i n_i + \sum_{j \ge 1} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: $w_{|S|}$ (b) $r_i := \#$ elements in S of frequency $i \implies \sum_{i \ge 1} r_i = |S|$ Analysis of $S_{OUT} := S \setminus \{S\}$ Our new measure $k(S) = \sum_{i \ge 1}^{k_w(S)} w_i n_i + \sum_{j \ge 1}^{k_v(S)} v_j m_j$: (a) Reduction in $k_w(S)$ from deleting S: $w_{|S|}$ (b) $r_i := \#$ elements in S of frequency $i \Rightarrow \sum_{i \ge 1}^{k_v(S)} r_i = |S|$ Reduction in $k_v(S)$ from deleting S:

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 $\Delta v_i = 0$ for i > 7

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 $\Delta k' = \begin{cases} 0 & \text{when } r_2 = 0, \\ v_2 + w_2 & r_2 = 1, \\ v_2 + \min\{2w_2, w_3\} & r_2 = 2, \\ v_2 + \min\{3w_2, w_2 + w_3\} & r_2 \ge 3, |S| = 3, \\ v_2 + \min\{3w_2, w_2 + w_3, w_4\} & r_2 \ge 3, |S| \ge 4. \end{cases}$

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Total reduction for \mathcal{S}_{OUT} is:

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(b) freq. of elements in S
(c) freq. & size of frequency-2 elements in S

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(b) Reduction in $k_v(S)$ by dropping S: $\sum_{i=2}^{6} r_i v_i + r_{\geq 7}$

(c) Reduction in $k_w(S)$ from shrinking sets that intersect S:

(a) Reduction in k_w(S) by dropping S: w_{|S|}
(b) Reduction in k_v(S) by dropping S: ∑⁶_{i=2} r_iv_i + r_{≥7}
(c) Reduction in k_w(S) from shrinking sets that intersect S: Let R be a set with S ∩ R ≠ Ø, and v ∈ R ∩ S.

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 $\begin{array}{lll} \Delta_{\mathsf{OUT}} = w_{|S|} &+& \sum_{i=2}^{6} r_i \Delta v_i &+& \Delta k' \\ \Delta_{\mathsf{IN}} &= w_{|S|} + \left(\sum_{i=2}^{6} r_i v_i + r_{\geq 7}\right) + \Delta w_{|S|} \cdot (\dots). \\ \text{Shrinking } |S| \text{ will only reduce the terms.} \\ \text{Important point here is } \Delta w_{|S|} \text{ in } \Delta_{\mathsf{IN}} \dots \\ \text{but } \Delta w_{|S|} \text{ is 0 when } |S| \geq 7. \end{array}$

Obs. Hence it is sufficient to consider configurations with $3 \le |S| \le 7$, and all possible combinations of $(r_i)_i$'s.

For each fixed 8-tuple $(w, v) = (w_2, \ldots, w_5, v_2, \ldots, v_5)$, the runtime is bounded by α^k , where α is the largest root of

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Thm. SC can be solved in $O^*(1.2353^{|U|+|S|})$ time. Corollary. DS can be solved in $O^*(1.2353^{2n}) = O^*(1.5259^n)$ time.