Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 3. Minimum Dominating Set

Based on: [Exact Exponential Algorithms: §3.2]
Further discussions: [Parameterized Algorithms: §6.1]

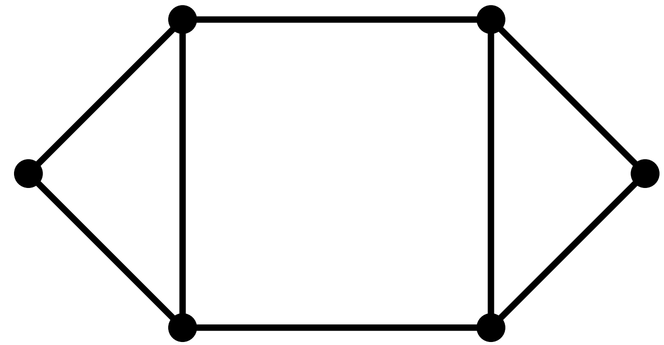
(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff

Lehrstuhl für Informatik I

Dominating Sets

Def. For a graph $G = (V, E)$, a set $D \subseteq V$ *dominates* G if



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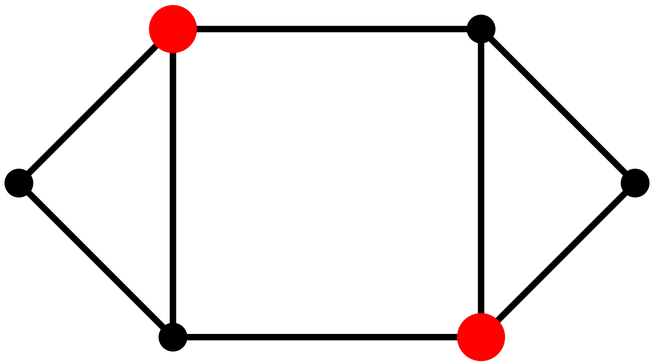

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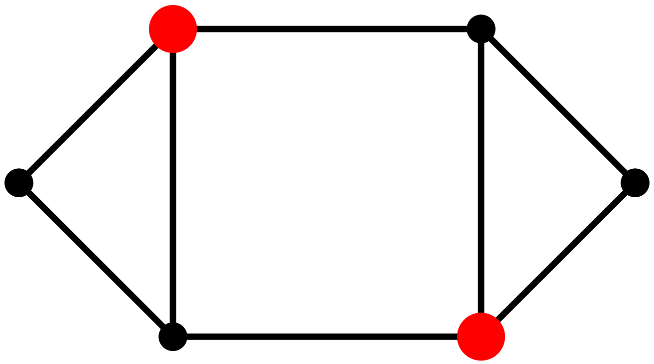

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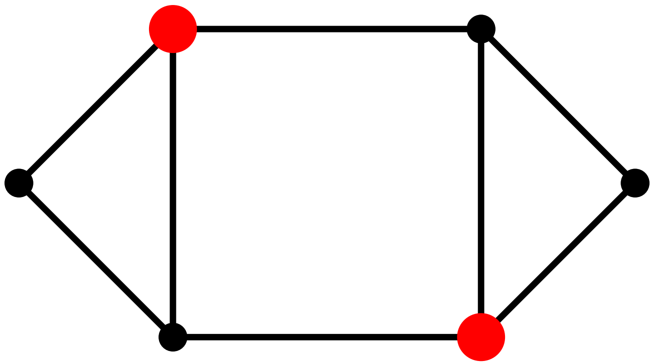

Example Application:

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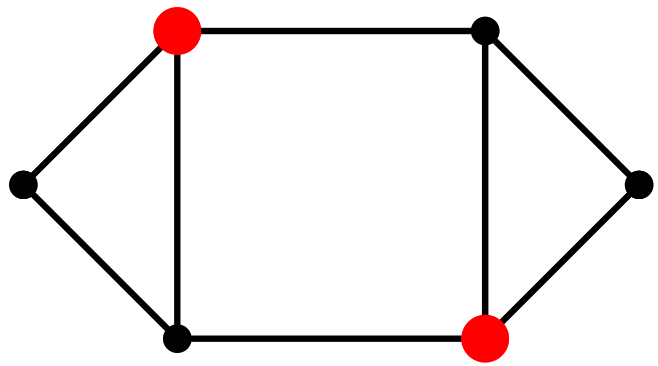

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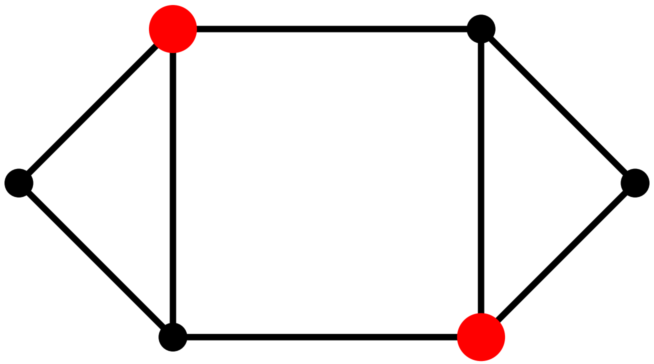

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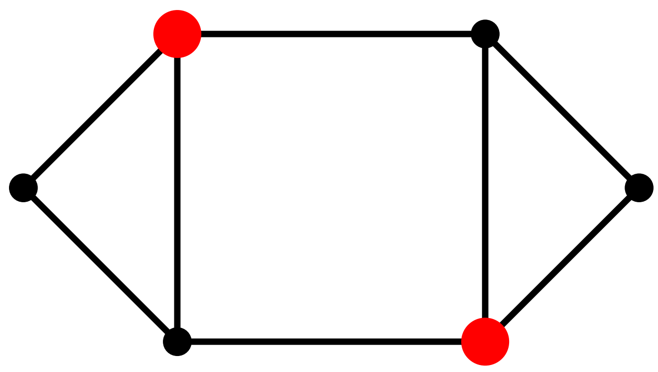

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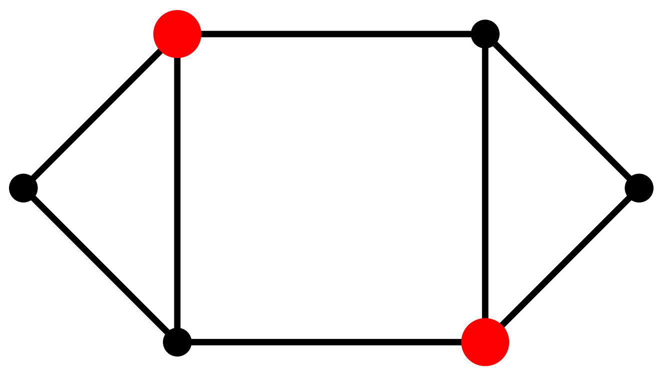

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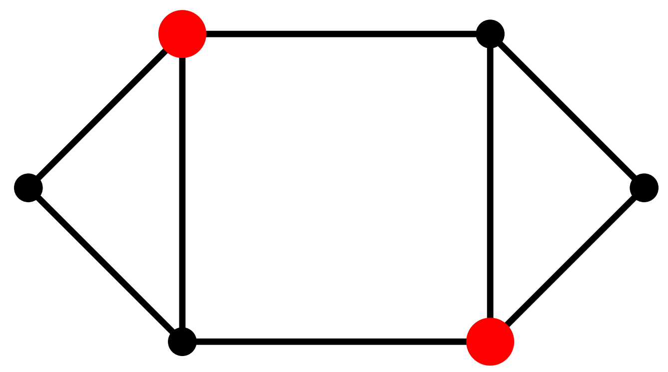

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Given: graph $G = (V, E)$,
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Domination number: $\gamma(G) := |D|$.

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Obs. Every maximal independent set is a dominating set.

Algorithmic Approach for Min. Dom. Set

Brute Force:

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Brute Force: (subset problem!)

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- If I is "small",
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Lemma $\star$. Given a maximal independent set I of G, a minimum dominating set of G can be found in $O^*(2^{n-|I|})$ time.

Proof of Lemma $\star$

- Instead of *all* 2^n subsets of V, we consider their $2^{n-|I|}$ *projections* on $J = V \setminus I$

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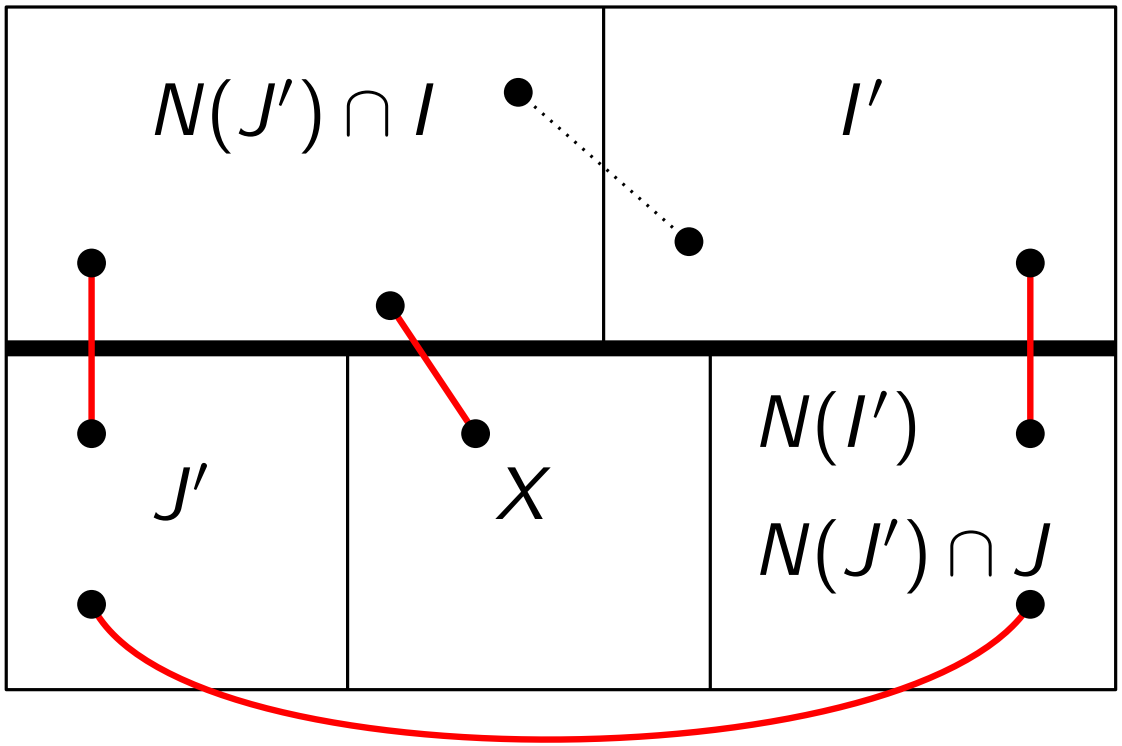
- How can we find a smallest $D_{J'}$ for a given $J' \subseteq J$?

Proof (Lemma $\star$)

independent set: I

J



Def. $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$ for every $S \subseteq V$.

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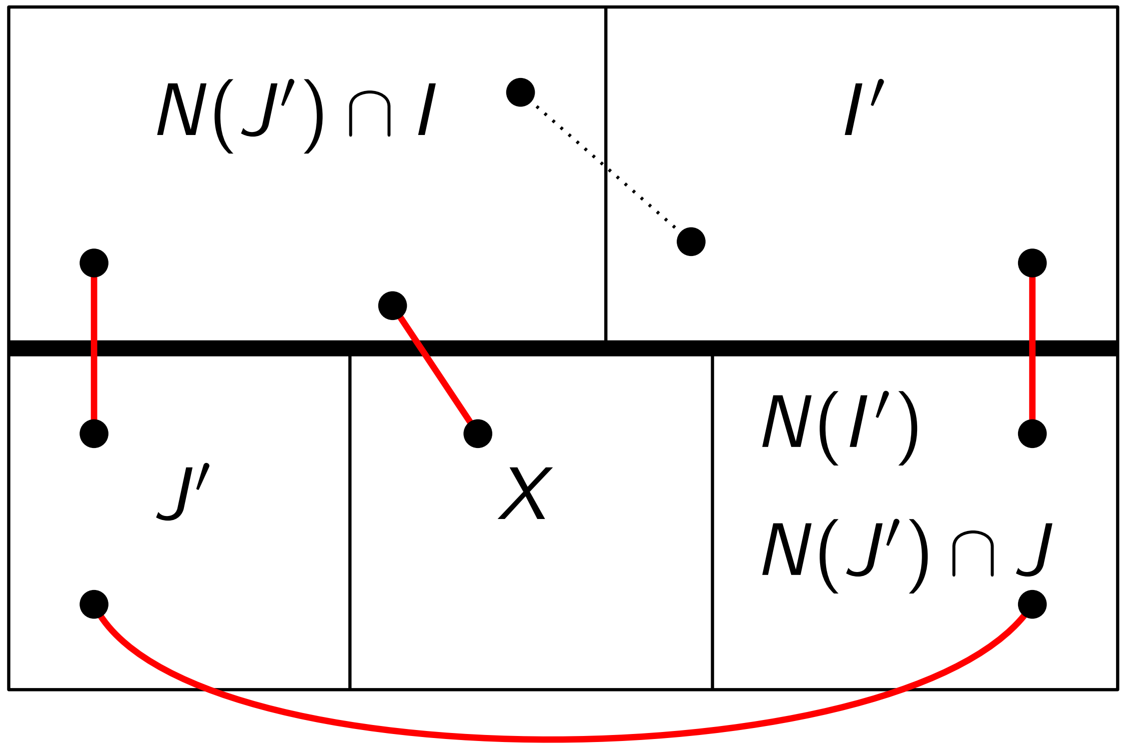

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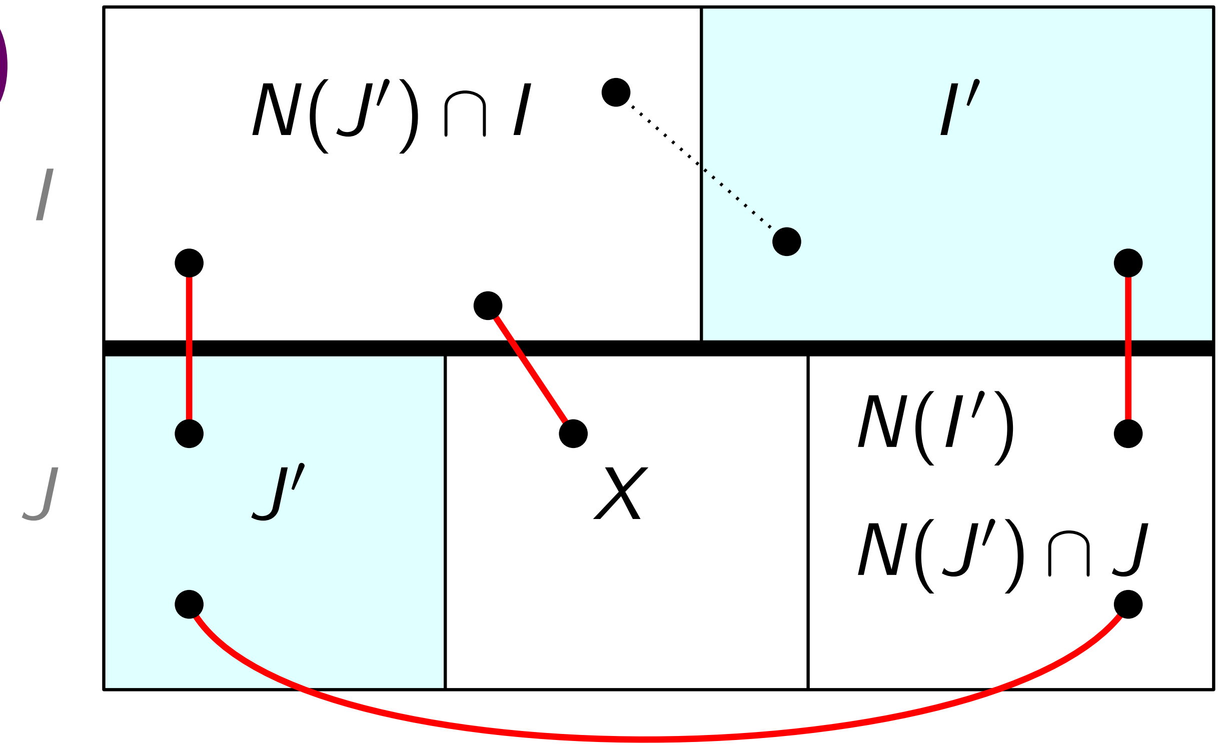

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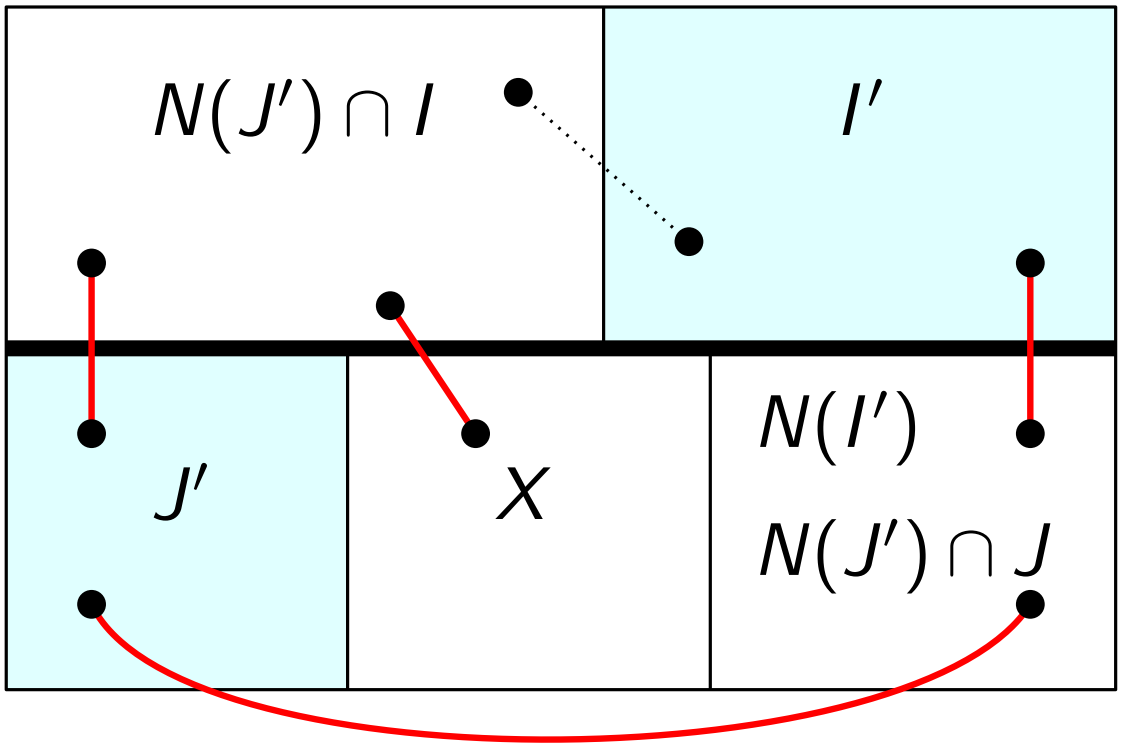

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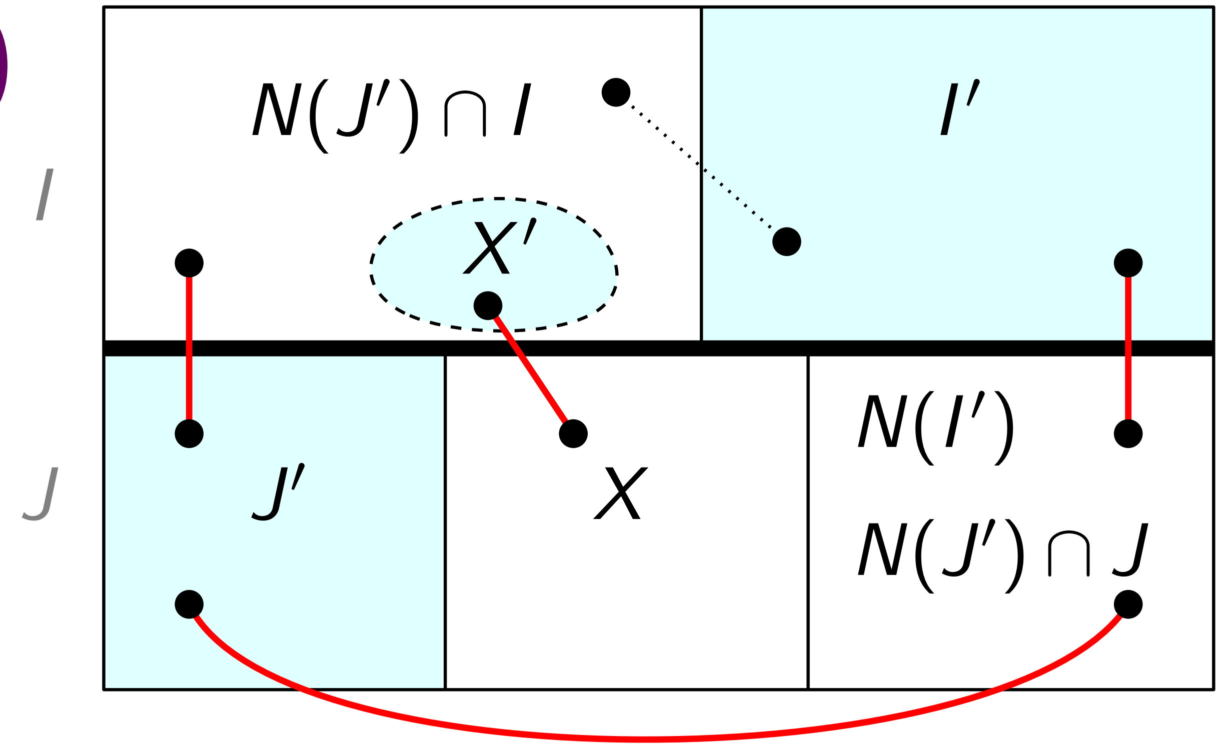

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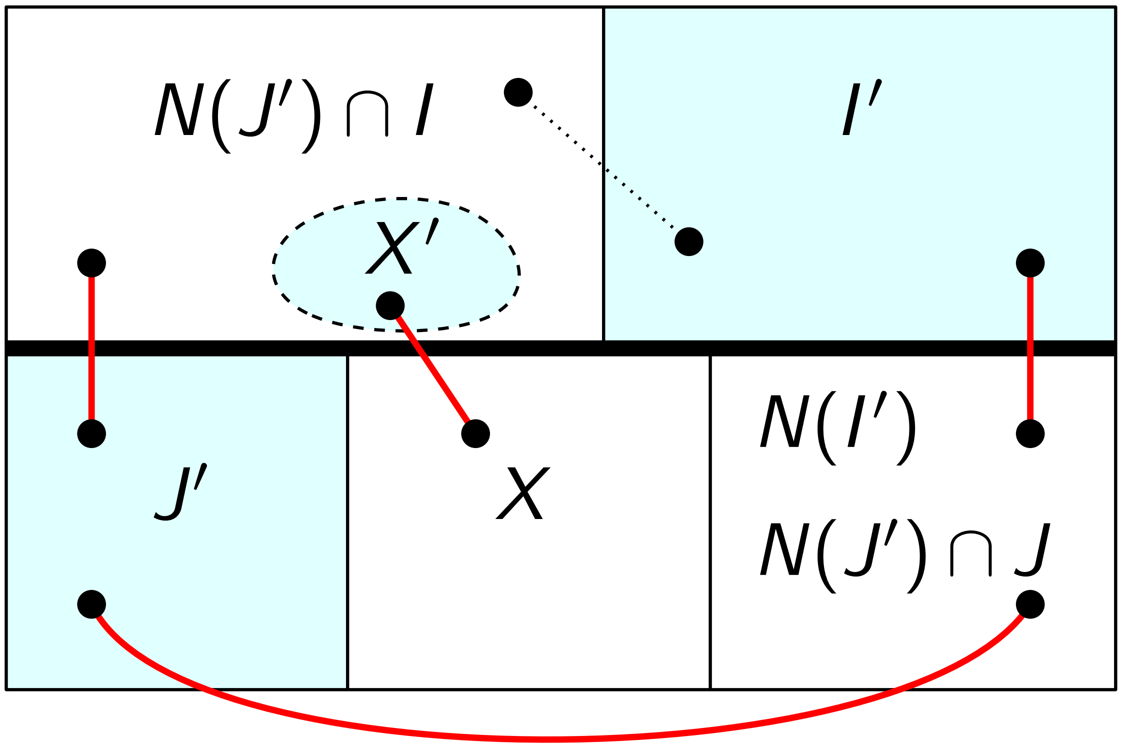

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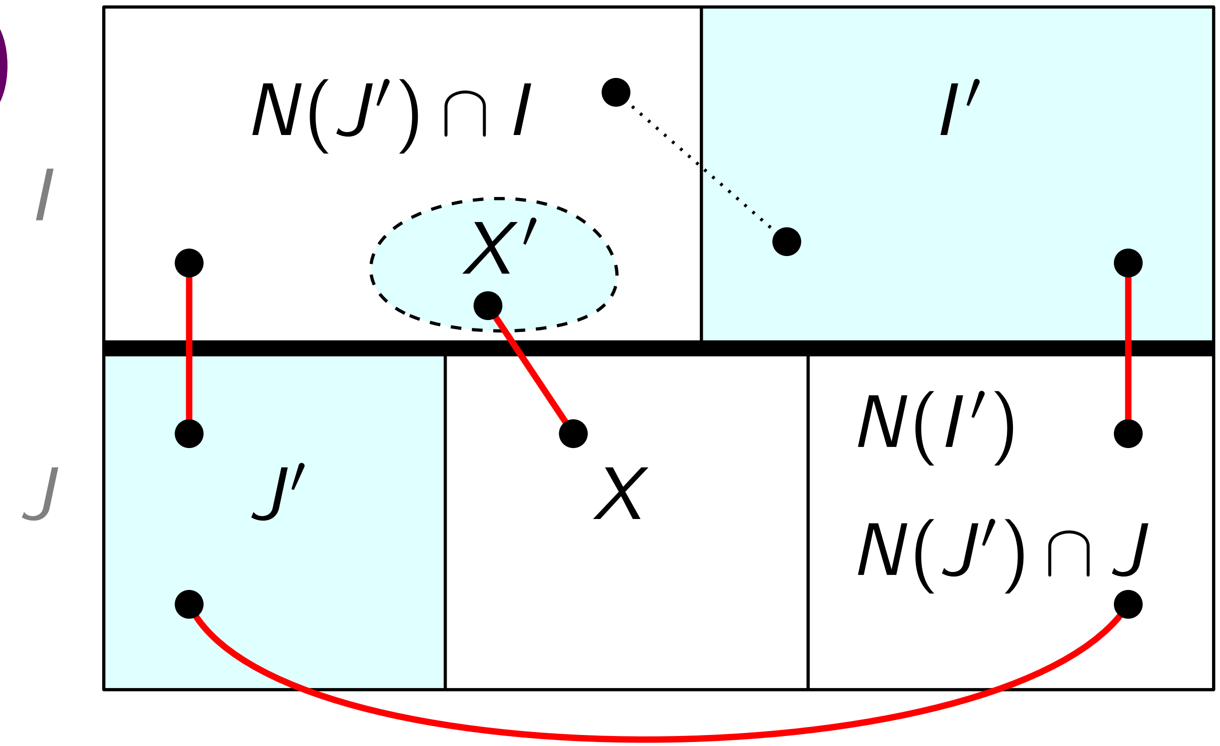

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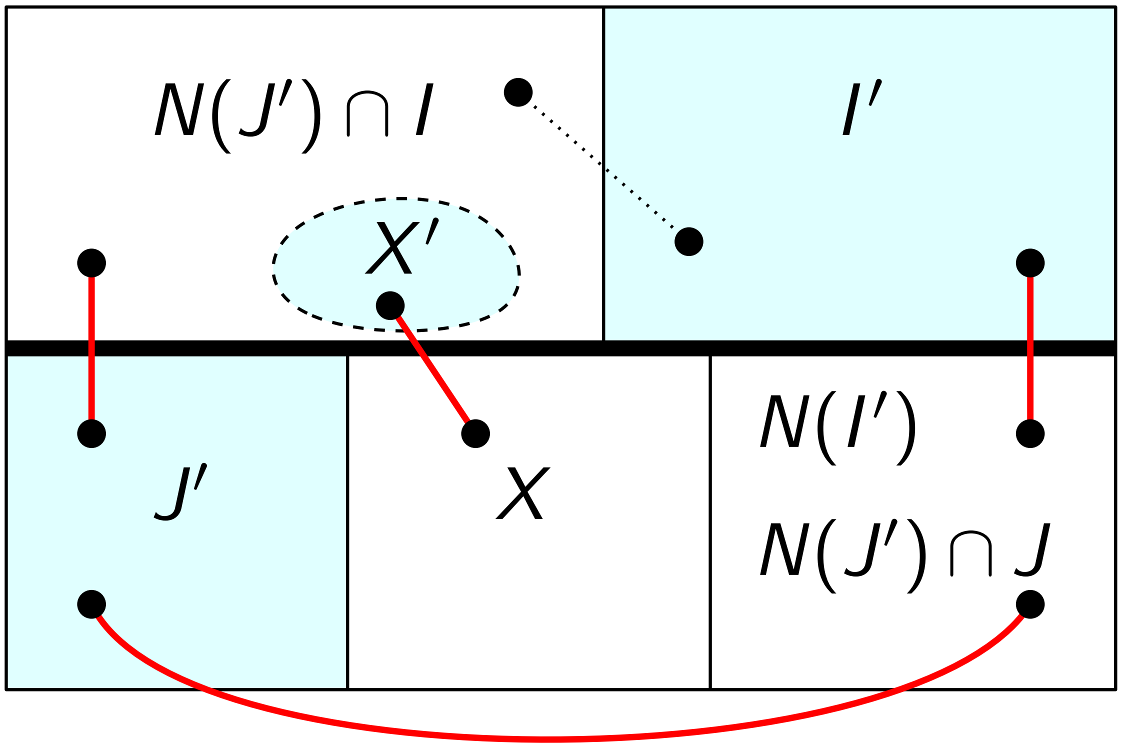

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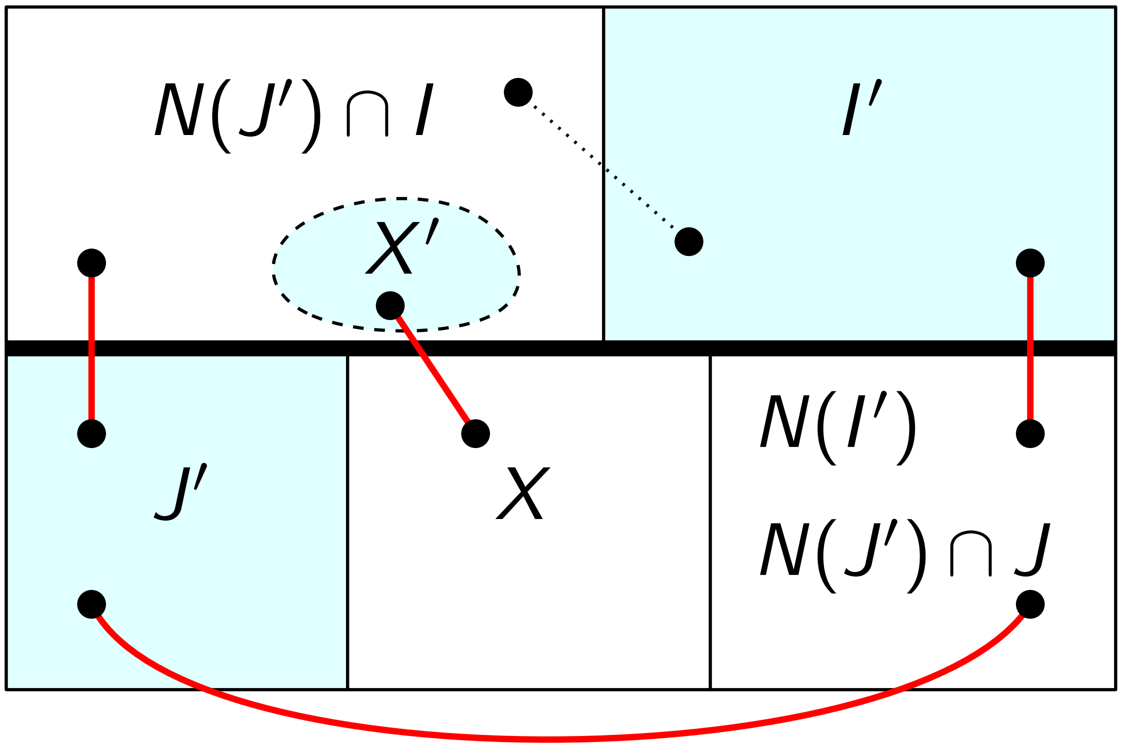

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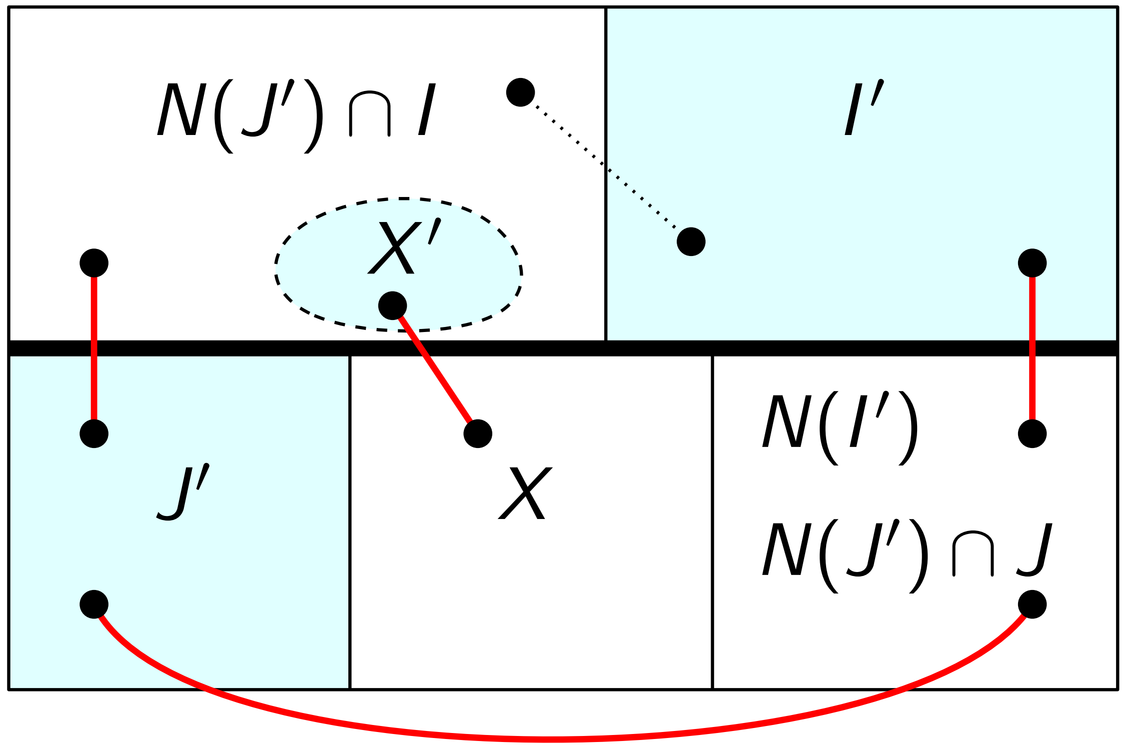

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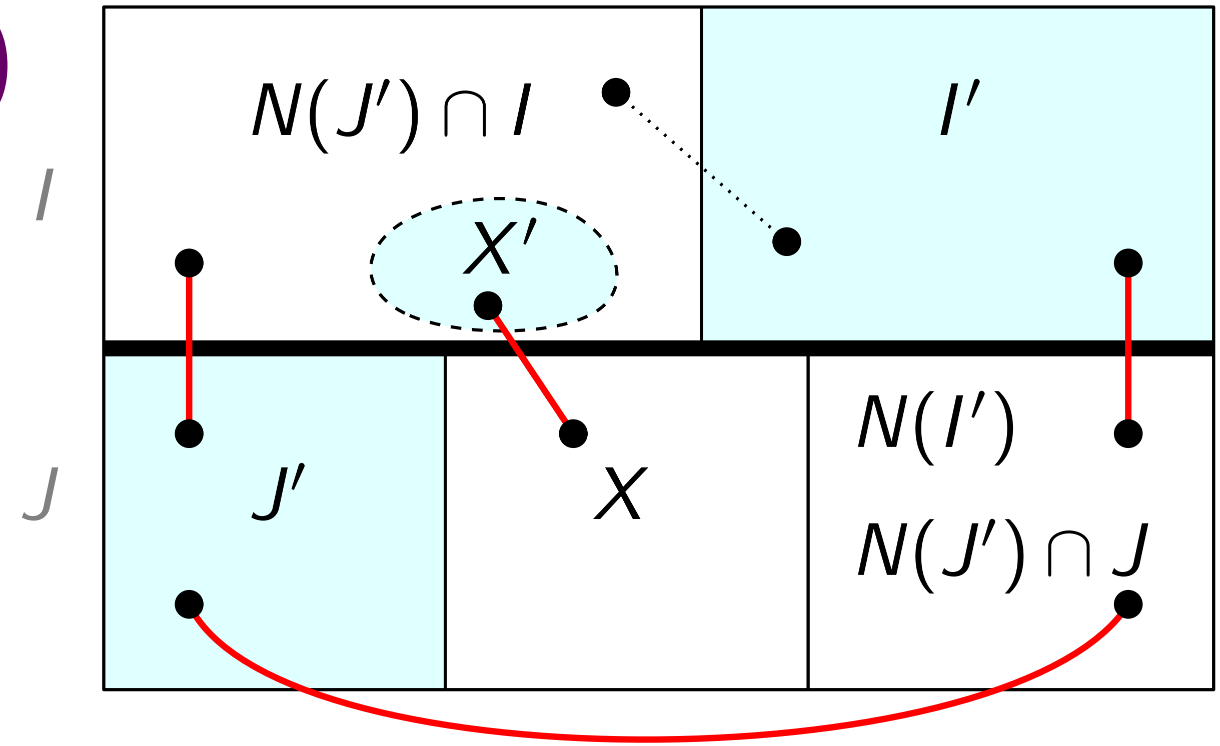

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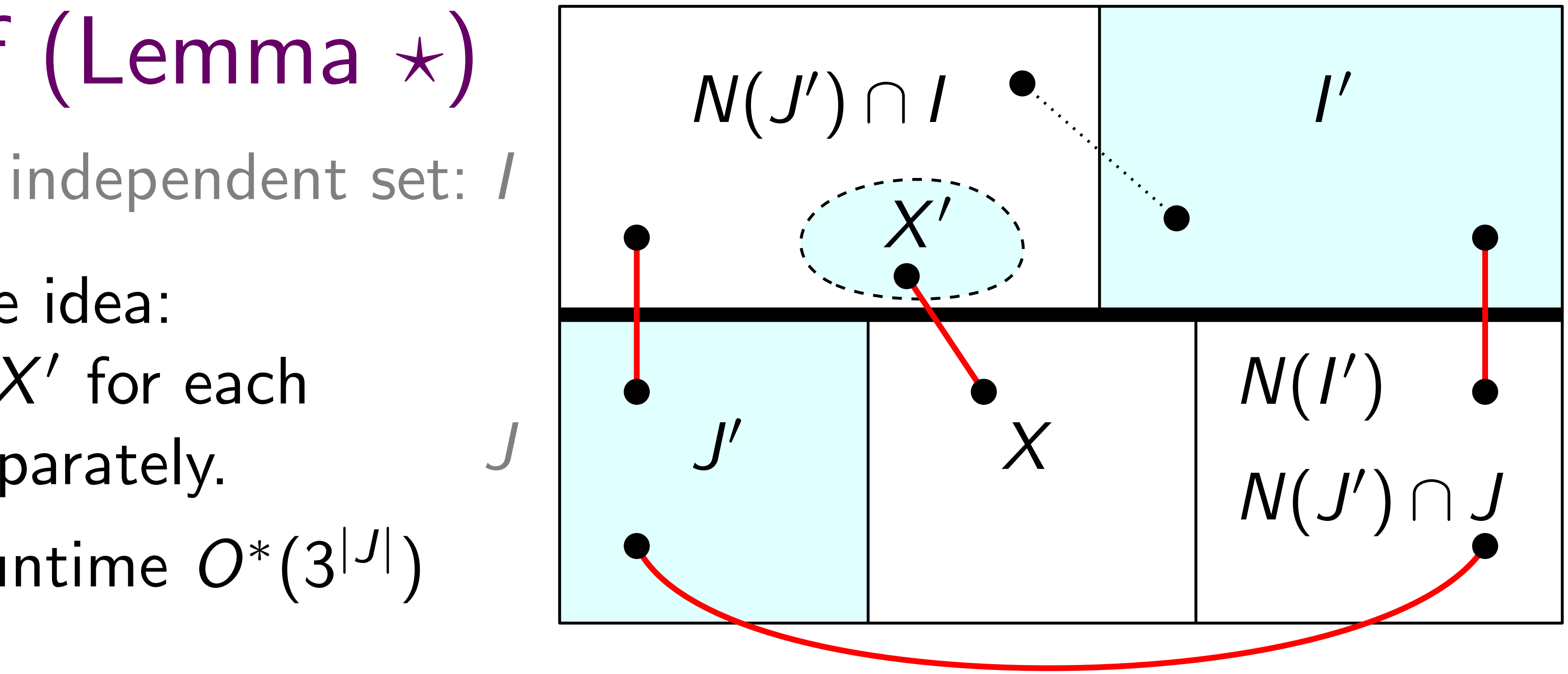

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$\Rightarrow$ total runtime of the algorithm: $O^*(2^{|J|}) = O^*(2^{n-|I|})$ $\square$

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Proof. Compute a maximal independent set I.

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$\Rightarrow \gamma(G) \leq \alpha n$

2. If $|I| > \alpha n$:

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Thm. A minimum dominating set of a given graph can be found in $O(\beta^n)$ time, for some $\beta < 2$.

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Helper Lemma

Lemma. For $\alpha \in (0, \frac{1}{2}]$, we have

$$\sum_{i=1}^{\alpha n} \binom{n}{i} \in O^*\Big(2^{h(\alpha)n}\Big),$$

where $h(\alpha) = -\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha)$.

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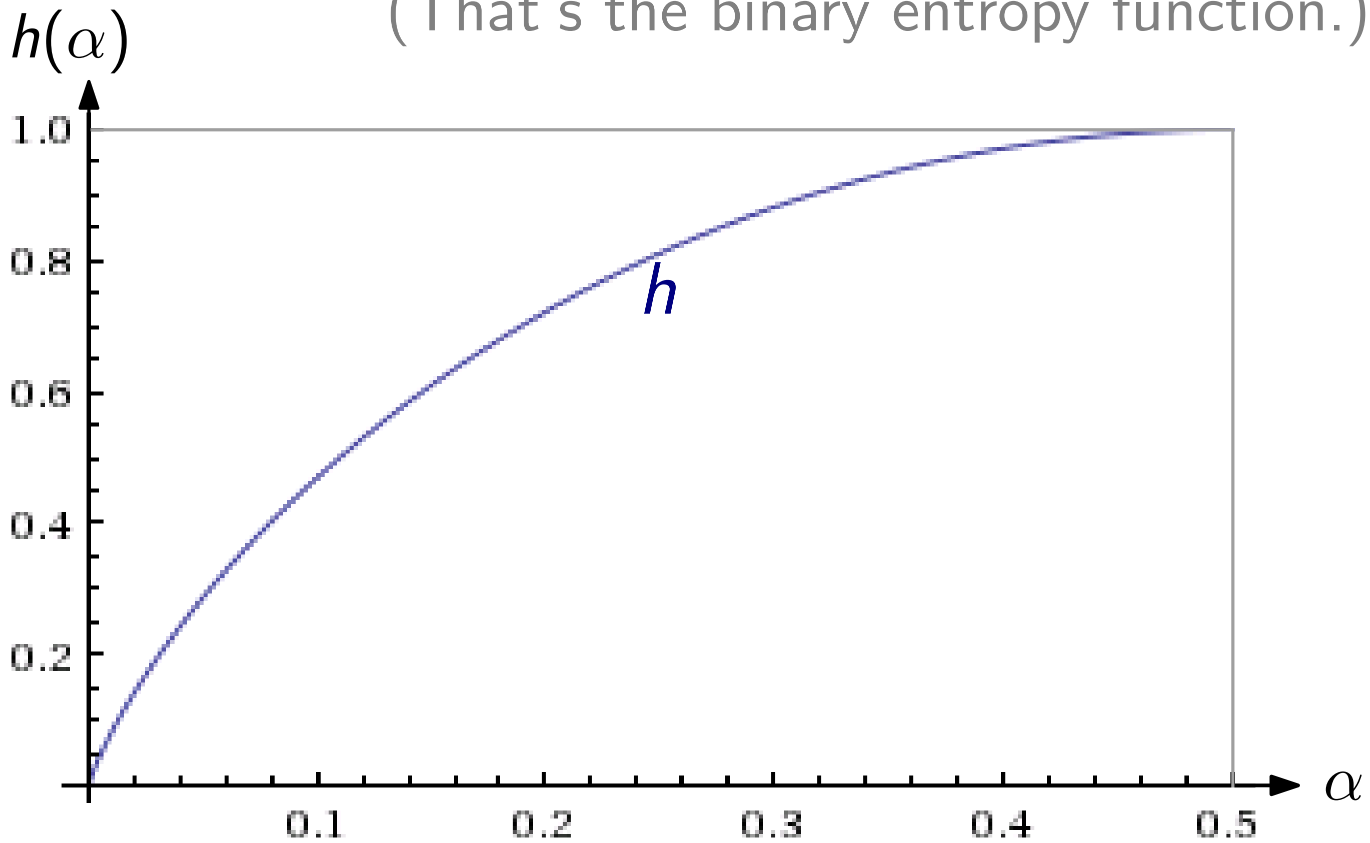

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(Proof at the end!)

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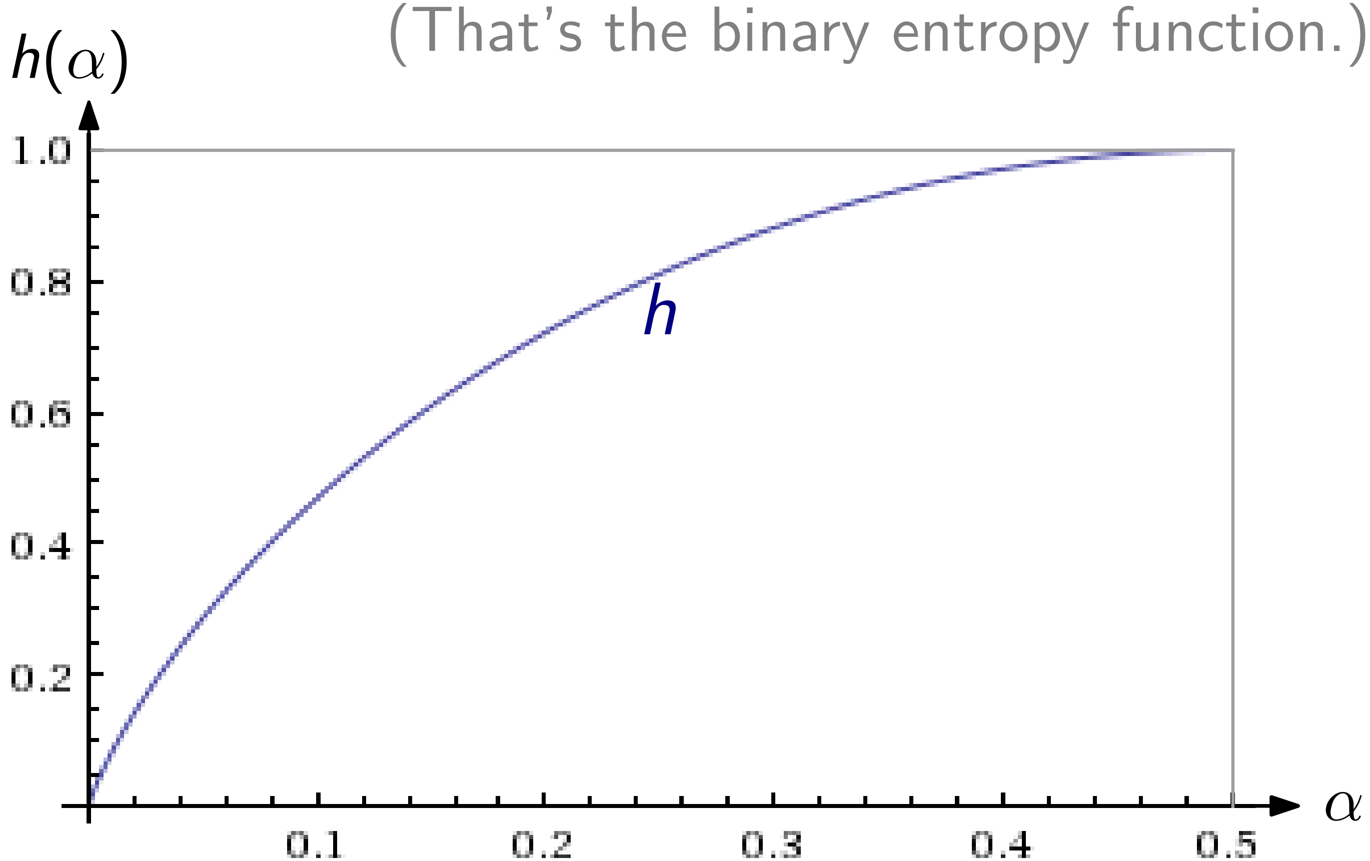

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TO DO: Determine the value $\alpha^\star$ for α, to balance 1. and 2.

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Thm. A minimum dominating set of a given graph can be found in $O(1.7088^n)$ time.

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Finding $\alpha^\star$ and the Base

For $\alpha^\star =$ we have a total runtime of
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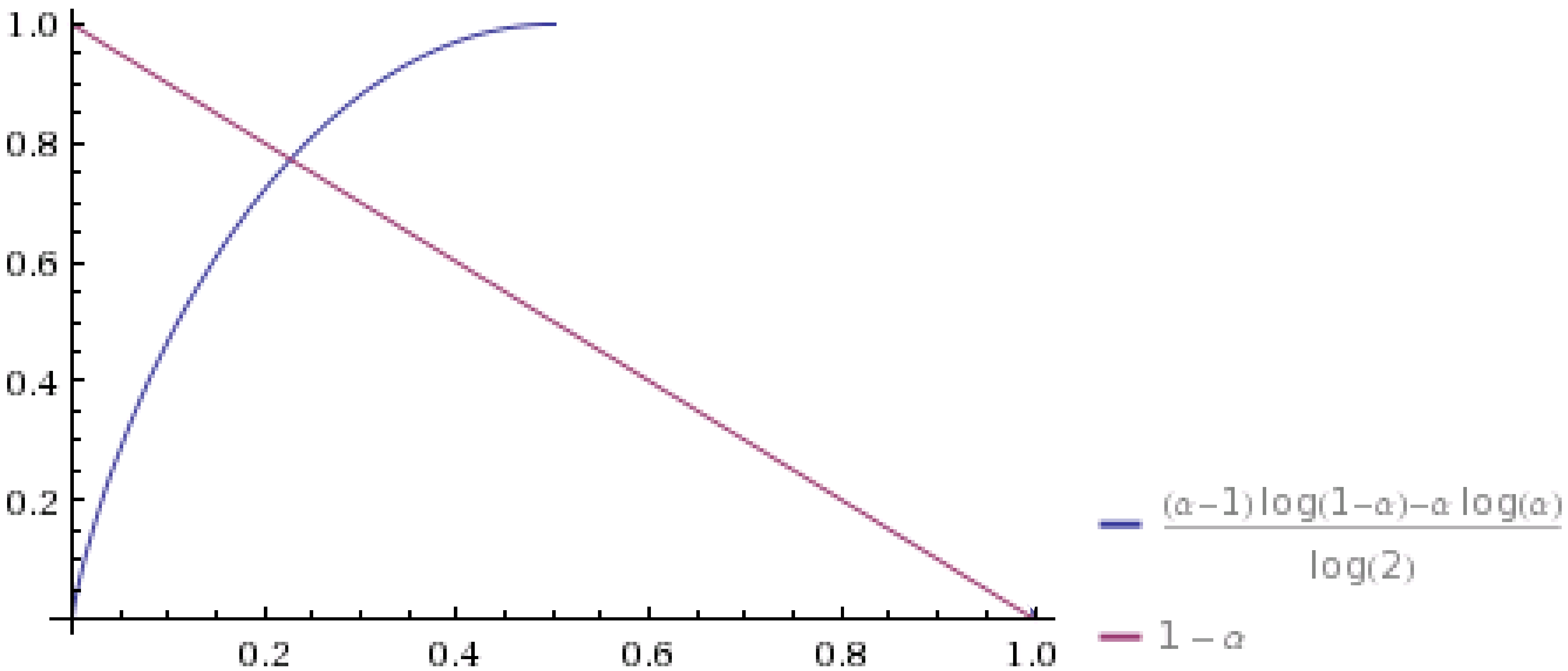

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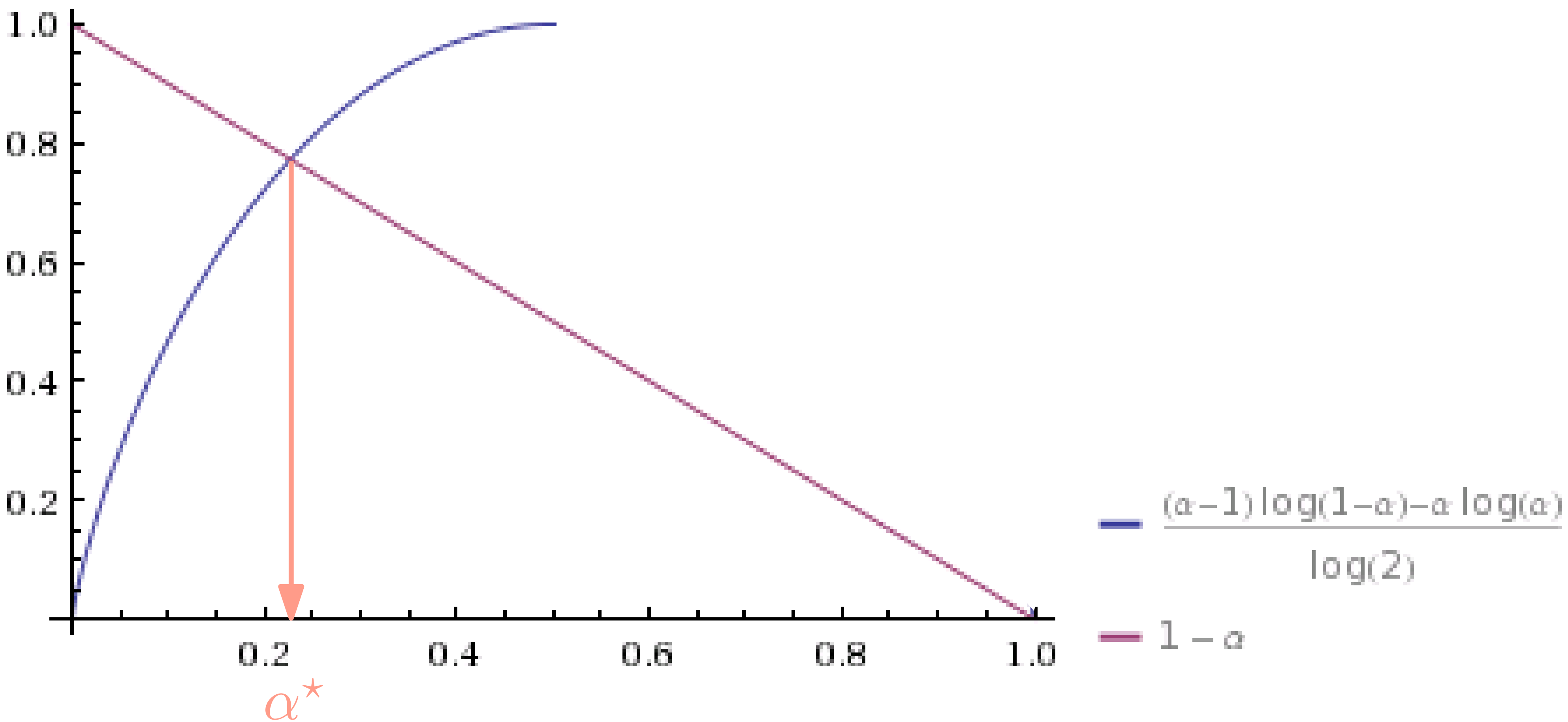

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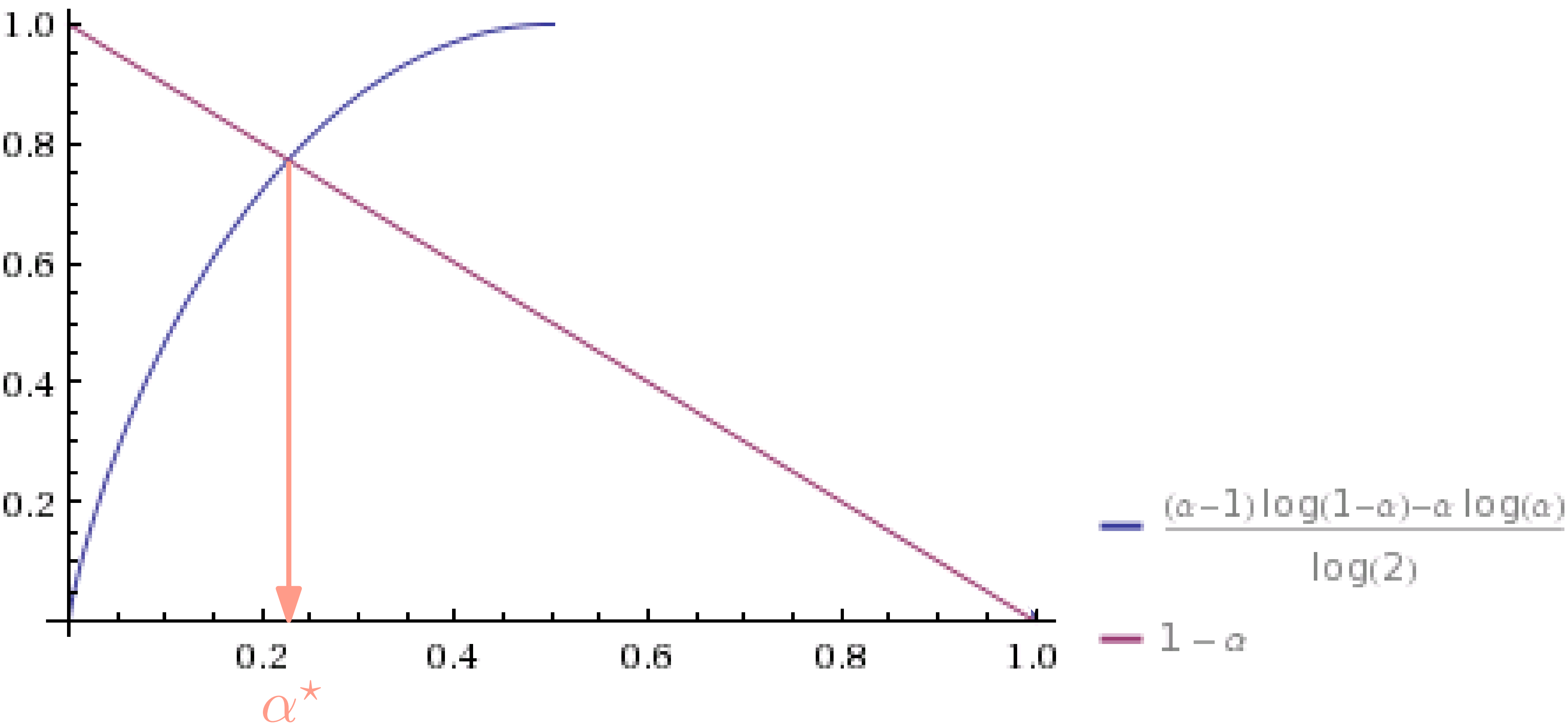

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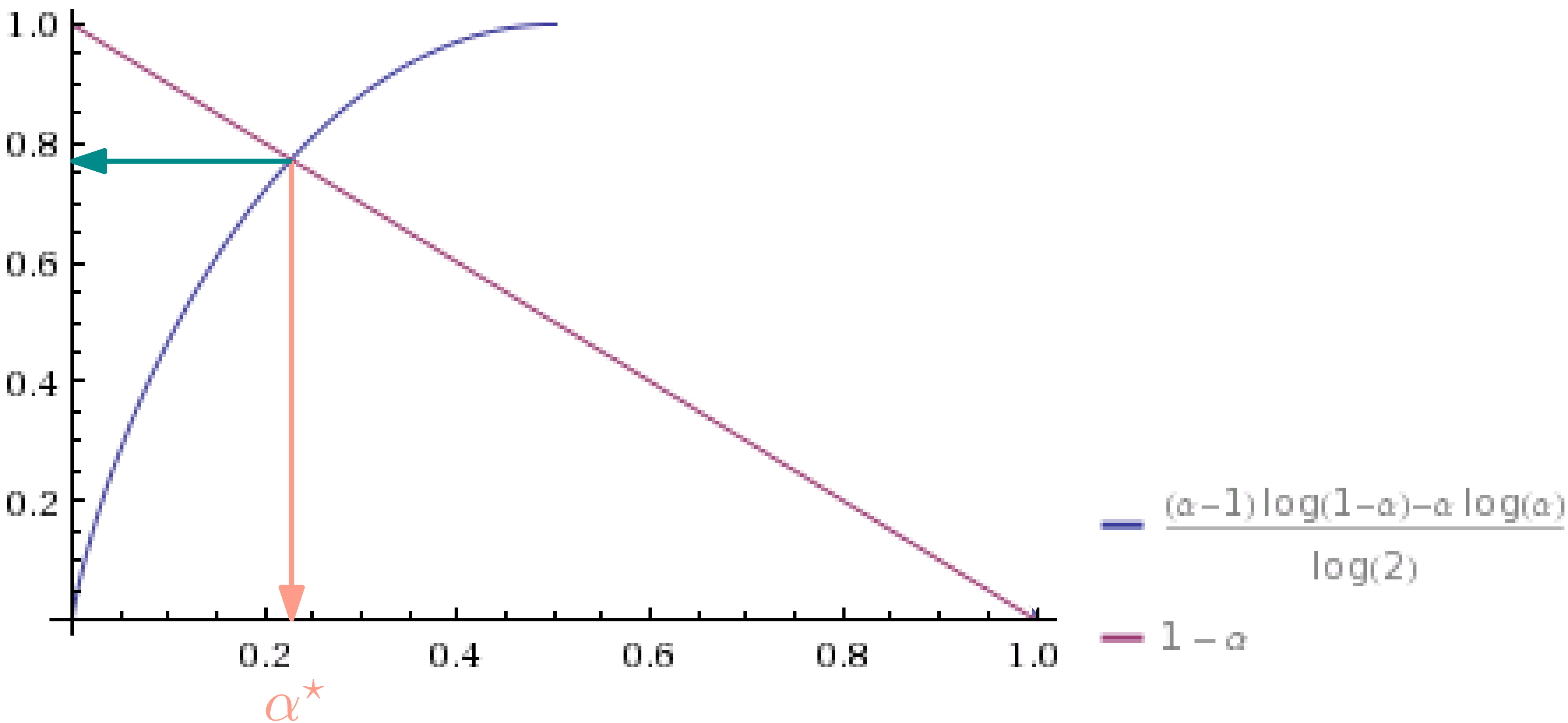
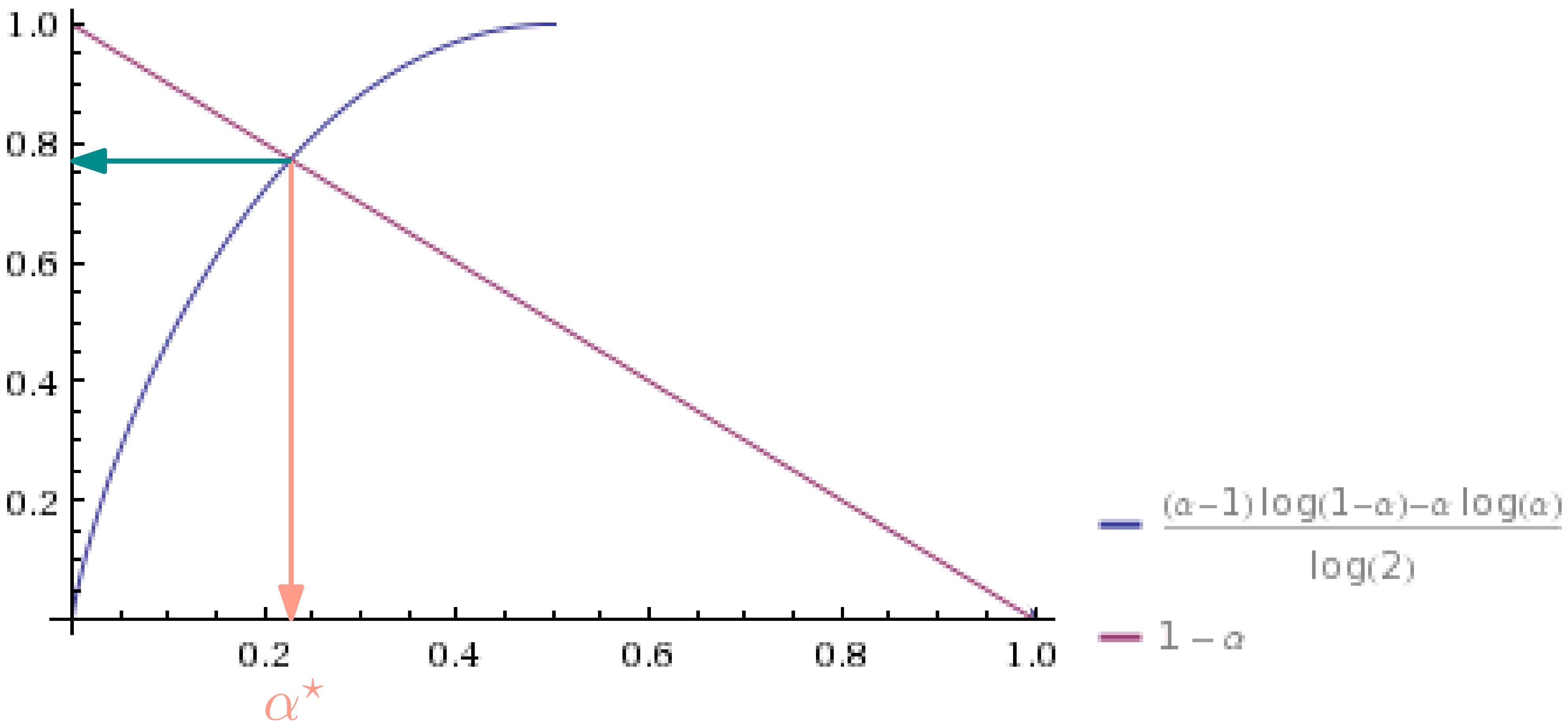


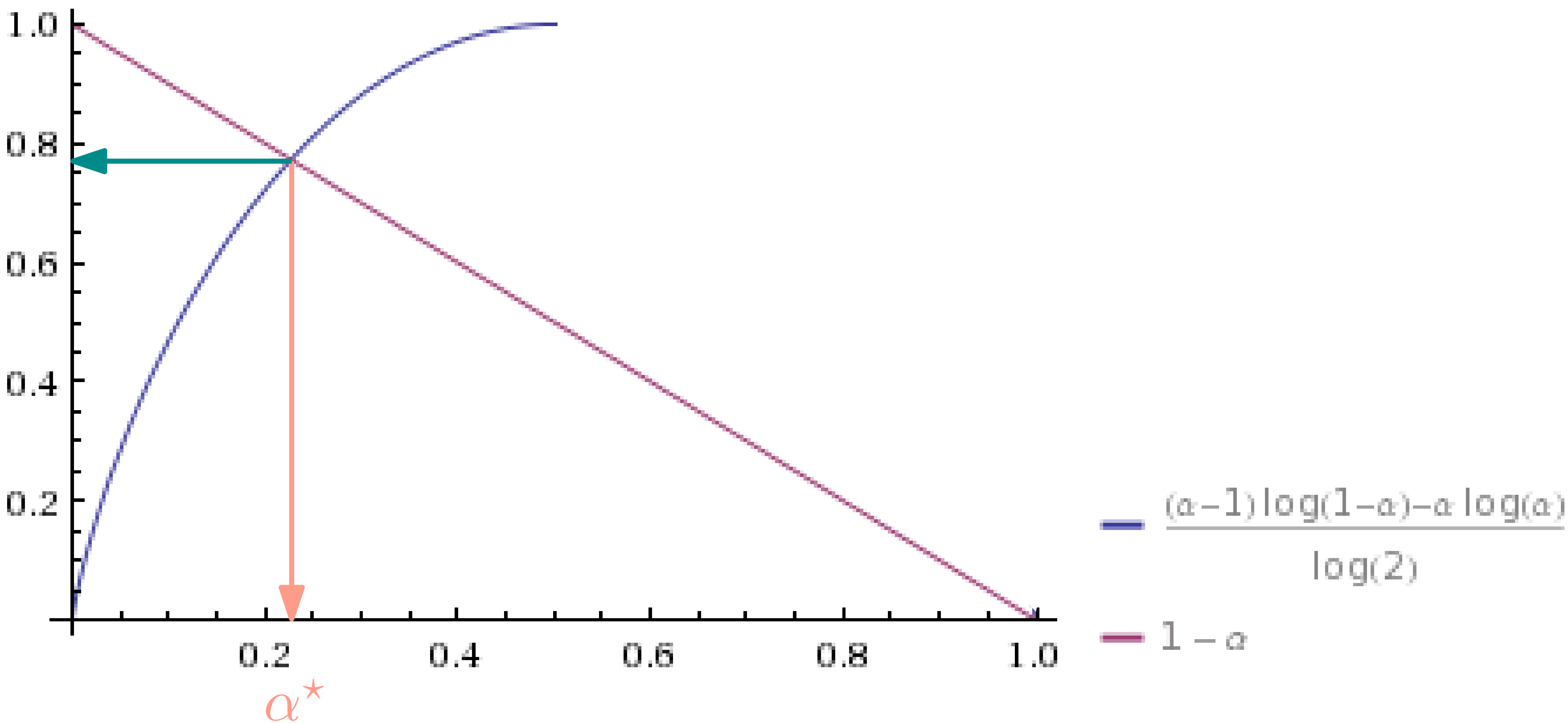
Finding $\alpha^\star$ and the Base

For $\alpha^\star = 0.22711$, we have a total runtime of $O^*(2^{0.7729n})$



Finding $\alpha^\star$ and the Base

For $\alpha^\star = 0.22711$, we have a total runtime of $O^*(2^{0.7729n}) = O(1.7088^n)$.



Proof of the Helper Lemma

Recall the statement: For $\alpha \in (0, \frac{1}{2}]$, we have $\sum_{i=1}^{\alpha n} \binom{n}{i} \in O^*\big(2^{h(\alpha)n}\big)$, where $h(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)$.

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Proof of the Helper Lemma (cont'd)

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\begin{aligned}
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&= O^*\left(2^{h(\alpha)\cdot n}\right) \qquad \square
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