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Lehrstuhl für
INFORMATIK I
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Exact Algorithms

Sommer Term 2020

Lecture 2. Branching Algorithms and Satisfiability

Based on: [Exact Exponential Algorithms: §2]

Further discussions: [Parameterized Algorithms: §3; specifically §3.2]

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Problems in $\mathcal{NP}$

- Brute force?
- $\forall$ Yes-instances : $\exists$ certificate
  - polynomial size
  - polynomial-time verifiable
- Runtime?

Problem Types

Subset Problems

- Solutions: subsets of a given ground set
- e.g., independent set in a graph, satisfying assignment to a Boolean formula
- Brute-force in $O^*(2^n)$ time

Permutation Problems

- Solutions: permutations of a given ground set
- e.g., Hamilton path of a graph, or tour as in TSP
- Brute-force in $O^*(n!) = 2^{O(n \log n)}$ time

Partitioning Problems

- Solutions: partitionings of a given ground set
- e.g., graph coloring: partition vertices of a graph into independent sets
- Brute-force in $O^*(n^n) = 2^{O(n \log n)}$ time

Branching Algorithms

- Standard technique
- Typical properties
  - polynomial space
  - in practice, often faster than its worst-case runtime
  - simple speedups (reduction rules, branch-and-bound)
- Other names:
  Backtracking, search-tree algorithms, ...

General Form

- **Branching Rules:**
  Choose a small part of the solution and solve a corresponding subproblem for each choice
  $\Rightarrow$ obtain a solution of the original problem based on the solutions of the subproblems

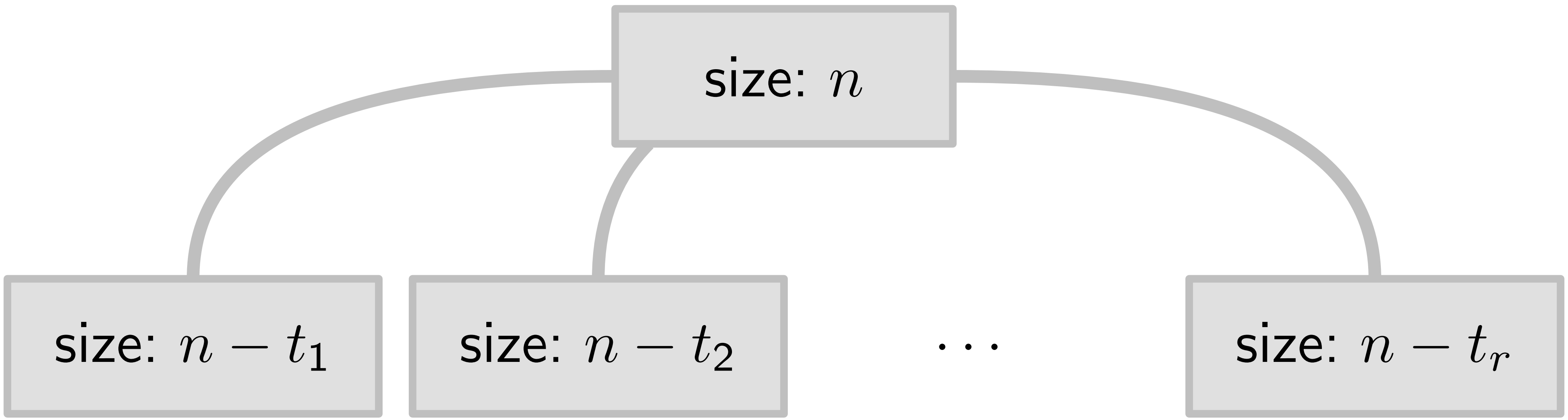
- **Reduction Rules:**
  Reduce the problem size (also: exit/rejection conditions)

- **Correctness:**
  Often immediate/obvious

- **Specification:**
  Often only calculates the optimal value.

Branching Vectors

- Apply a branching rule b to an instance I of size n.
- Suppose that b decomposes I into $r \geq 2$ sub-instances of sizes $n - t_1,\ n - t_2,\ \ldots,\ n - t_r$, where $t_i > 0$ for each i.
- We call $(t_1, t_2, \ldots, t_r)$ *branching vector* of b.



Recurrence: $T(n) \ \leq \ T(n - t_1) + T(n - t_2) + \ldots + T(n - t_r)$

Theorem on Branching Vectors

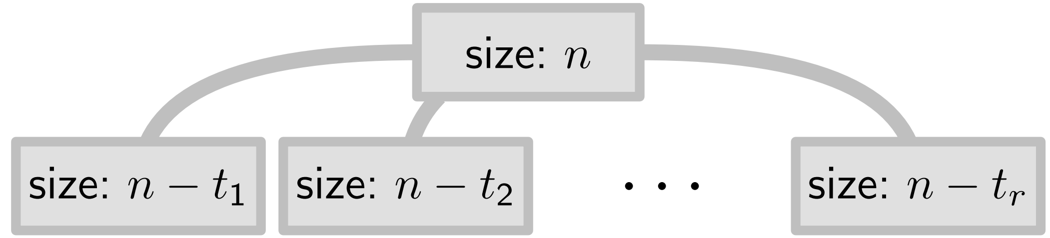
Thm. Let b be a branching rule with vector $(t_1, t_2, \ldots, t_r)$. Then the runtime of an algorithm executing b is $O^*(\alpha^n)$, where α is the unique positive root of

$$\begin{aligned}
\alpha^n &= \alpha^{n-t_1} + \alpha^{n-t_2} + \ldots + \alpha^{n-t_r} \\
\alpha^n - \alpha^{n-t_1} - \alpha^{n-t_2} - \ldots - \alpha^{n-t_r} &= 0 \\
1 - \alpha^{-t_1} - \alpha^{-t_2} - \ldots - \alpha^{-t_r} &= 0 \\
\alpha^m - \alpha^{m-t_1} - \alpha^{m-t_2} - \ldots - \alpha^{m-t_r} &= 0
\end{aligned}$$

where $m := \max_{1 \le i \le r} t_i$.

- Denote solution by $\tau(t_1, t_2, \ldots, t_r)$.
- Often irrational.

Properties of $\tau(\cdot)$



Thm. Let $r \geq 2$ and, for $i = 1, \ldots, r$, let $t_i > 0$. Then:

(i) $\tau(t_1, \ldots, t_r) > 1$,

(ii) $\tau(t_1, \ldots, t_r) = \tau(t_{\pi(1)}, \ldots, t_{\pi(r)})$ for every permutation π,

(iii) $\tau(t_1, t_2 \ldots, t_r) < \tau(t'_1, t_2, \ldots, t_r)$ if $t_1 > t'_1$.

Lemma. For $i, j, k > 0$, the following balancing properties apply:

(i) $\tau(k, k) \leq \tau(i, j)$ if $i + j = 2k$,

(ii) $\tau(i, j) > \tau(i + \varepsilon, j - \varepsilon)$ if $0 < i < j$ and $0 < \varepsilon < \frac{j-i}{2}$.

E.g.:

$\tau(1, 1) = 2$

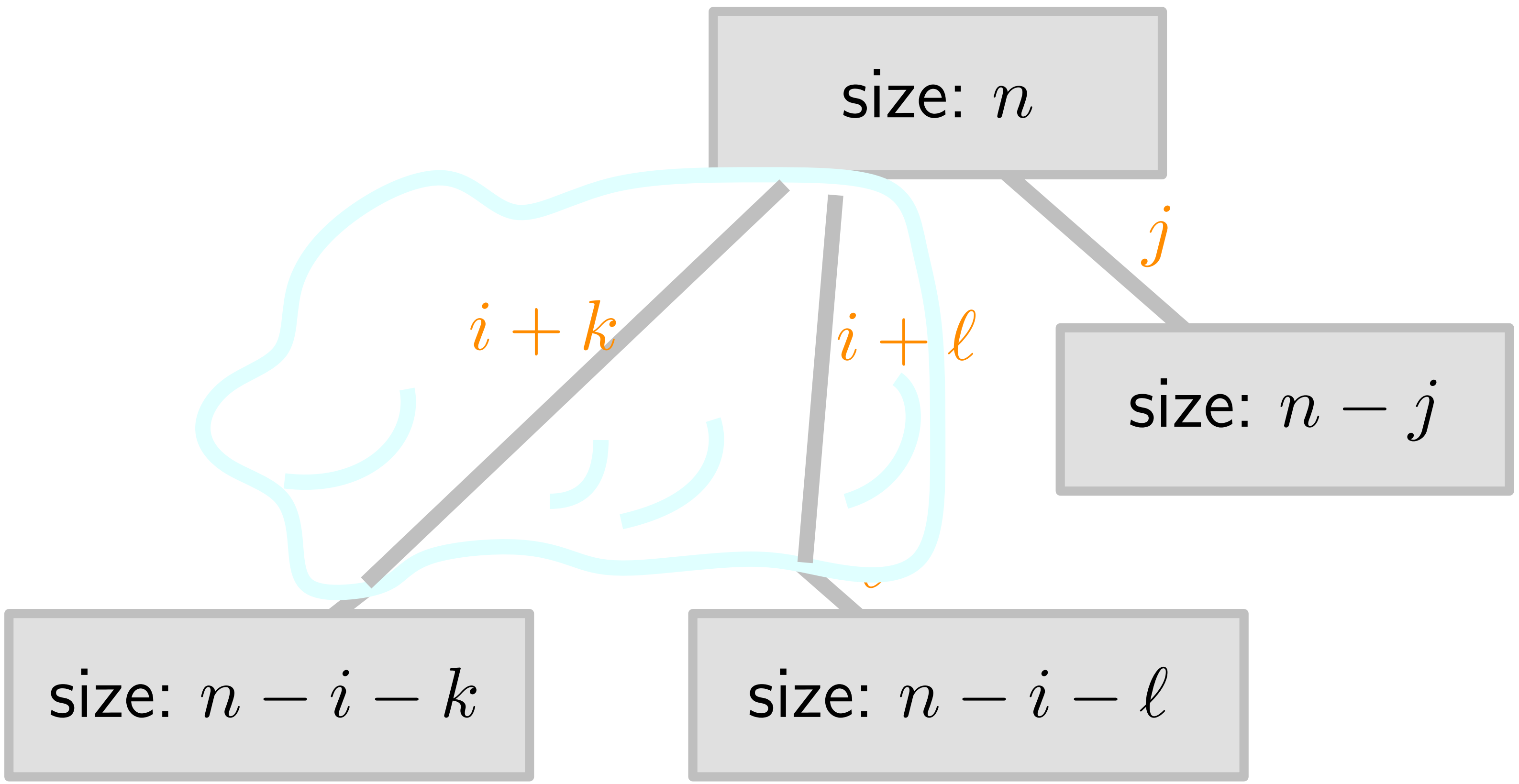
$\tau(1, 2) = \frac{1+\sqrt{5}}{2} < 1.62$

$\tau(3, 3) = \sqrt[3]{2} < 1.26$

$\tau(2, 4) < 1.272$

$\tau(1, 5) < 1.3248$

"Addition" of Branching Vectors



Branching-Vector: $(i+k,\ i+\ell,\ j)$

Satisfiability (SAT)

Input: propositional logic formula F in conjunctive NF

Question: $\exists$ satisfying assignment for F?

E.g., $(x_1 \vee \bar{x}_2 \vee x_3) \wedge \underbrace{(\bar{x}_1 \vee x_4 \vee x_5)}_{\text{clause}}$

variable literal

Brute-Force: Try all 2^n variable assignments.

Runtime: $O(2^n \cdot n \cdot m)$, where $n = \#$variables, $m = \#$clauses.

Strong Exponential Time Hypothesis (SETH) implies that:

$\nexists$ algorithm for SAT in $o(2^n)$ time, i.e., in $O^*((2-\varepsilon)^n), \varepsilon > 0$.

For more on (S)ETH, see [Parameterized Algorithms 14.1].

k-SAT: Each clause has $\leq k$ literals.

A Better Algorithm for k-SAT

- Goal:
  Solve k-SAT in time $O^*({\alpha_k}^n)$, where $\alpha_k < 2$ for every k.

- Idea: Branch on variables.

- For a partial assignment t,
  let $F[t]$ be the *reduced* formula that we get after
  – removing false literals from clauses and
  – removing clauses with true literals.

  Example:

  – $F = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee x_5)$

  – t: $x_1 =$ false

  – $F[t] = (\bar{x}_2 \vee x_3)$

Properties of Reduced Formulas

- $F[t]$ satisfiable $\Rightarrow F$ satisfiable
- $F[t]$ contains an empty clause $\Rightarrow F[t]$ not satisfiable
- $F[t]$ empty $\Rightarrow F[t]$ satisfiable

- $F = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee x_5)$
- $F[x_1 = \mathsf{f}] = (\bar{x}_2 \vee x_3)$ satisfiable
- $F[x_2 = \mathsf{t}, x_3 = x_4 = x_5 = \mathsf{f}] = x_1 \wedge \bar{x}_1$ not satisfiable
- $F[x_1 = x_3 = \mathsf{f}, x_2 = \mathsf{t}] = ()$
- $F[x_3 = x_4 = \mathsf{t}] =$ empty formula

A First Algorithm for k-SAT

Algorithm k-SAT-v1(F)

if F is empty **then**
 └ **return** true
if F contains an empty clause **then**
 └ **return** false
pick clause $c = (\ell_1 \vee \ell_2 \vee \ldots \vee \ell_q)$ from F, where $q \leq k$
t_1: $\ell_1 =$ true
t_2: $\ell_1 =$ false, $\ell_2 =$ true
t_3: $\ell_1 =$ false, $\ell_2 =$ false, $\ell_3 =$ true
$\vdots$
t_q: $\ell_1 =$ false, $\ell_2 =$ false, $\ldots, \ell_{q-1} =$ false, $\ell_q =$ true
return $\bigvee_{i=1}^{q}$ k-SAT-v1($F[t_i]$)

- t_i: first true literal in c is ℓ_i

Runtime

- $F[t_i]$ has $n - i$ variables

  $\Rightarrow$ branching vector $(1, 2, \ldots, k)$

- Runtime: $O^*({\beta_k}^n)$, where $\beta_k = \tau(1, 2, \ldots, k)$.

  Note: $\tau(1, 2, \ldots, k) \;\Rightarrow\; {\beta_k}^k = \sum_{i=0}^{k-1} {\beta_k}^i$. $\quad (*)$

  So, $(*) \cdot \beta_k - (*) \;\Rightarrow\; {\beta_k}^{k+1} - 2{\beta_k}^k + 1 = 0$.

- $\beta_1 = 1$, $\beta_2 < 1.6181$, $\beta_3 < 1.8393$, $\beta_4 < 1.9276$, $\beta_5 < 1.9660$

Speeding Up the Algorithm

- "Branch-on-shortest" rule
- Hope: $\exists$ a clause of length $\leq k-1$

Def. A partial assignment t is an *autark* if every clause with a literal assigned by t also contains a literal assigned true by t.

Expl. $F = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee x_5) \wedge (\bar{x}_3 \vee x_5)$

t: $x_1 = x_4 =$ true *is* an autark

t': $x_1 =$ true, $x_4 =$ false is *not* an autark

Obs.
- If t is an autark, then: F satisfiable $\Leftrightarrow$ $F[t]$ satisfiable.
- If t is *not* an autark, then $F[t]$ contains a clause of length $\leq k-1$.

An Improved k-SAT Algorithm

Algorithm k-SAT-v2(F)

if F is empty **then**
  return true
if F contains an empty clause **then**
  return false
pick a smallest clause $c = (\ell_1 \vee \ell_2 \vee \ldots \vee \ell_q)$ from F, where $q \leq k$
t_1: $\ell_1 =$true
t_2: $\ell_1 =$ false, $\ell_2 =$ true
$\vdots$
t_q: $\ell_1 =$ false, $\ell_2 =$ false, $\ldots, \ell_{q-1} =$ false, $\ell_q =$ true
if t_i is an autark for some $i = 1, \ldots, q$ **then**
  return k-SAT-v2($F[t_i]$) $\leftarrow$ "Reduce"
else
  return $\bigvee_{i=1}^{q} k$-SAT-v2($F[t_i]$) $\leftarrow$ "Branch"

Runtime Analysis

- Claim: The runtime is $O^*(\alpha_k^n)$, where $\alpha_k = \beta_{k-1}$.
- Consider a node v in the search tree (not the root).

  If v is a k-branching, then v's parent is a "Reduce"-node because in a "Branch"-node, for each branch, the formula $F[t_i]$ contains a clause of length $\leq k-1$.

  $$\underbrace{\begin{array}{ll} \text{Now:} & \text{Before:} \\ \Rightarrow \text{ branching vector } (1, 2, \ldots, k-1) & (1, 2, \ldots, k-1, k) \\ \Rightarrow \alpha_k^k - 2\alpha_k^{k-1} + 1 = 0 & \beta_k^{k+1} - 2\beta_k^k + 1 = 0 \end{array}}_{\Rightarrow \; \alpha_k = \beta_{k-1}}$$

- $\beta_1 = 1, \beta_2 < 1.6181, \beta_3 < 1.8393, \beta_4 < 1.9276, \beta_5 < 1.9660$

$\Rightarrow \alpha_2 = 1, \alpha_3 < 1.6181, \alpha_4 < 1.8393, \alpha_5 < 1.9276, \alpha_6 < 1.9660$

$\square$