Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Exact Algorithms

Sommer Term 2020

Lecture 2. Branching Algorithms and Satisfiability

Based on: [Exact Exponential Algorithms: §2]
Further discussions: [Parameterized Algorithms: §3; specifically §3.2]

(slides by J. Spoerhase, Th. van Dijk, S. Chaplick, and A. Wolff)

Alexander Wolff
Lehrstuhl für Informatik I

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- Other names:
  Backtracking, search-tree algorithms, . . .

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  $\Rightarrow$ obtain a solution of the original problem based on the solutions of the subproblems

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- **Specification:**
  Often only calculates the optimal value.

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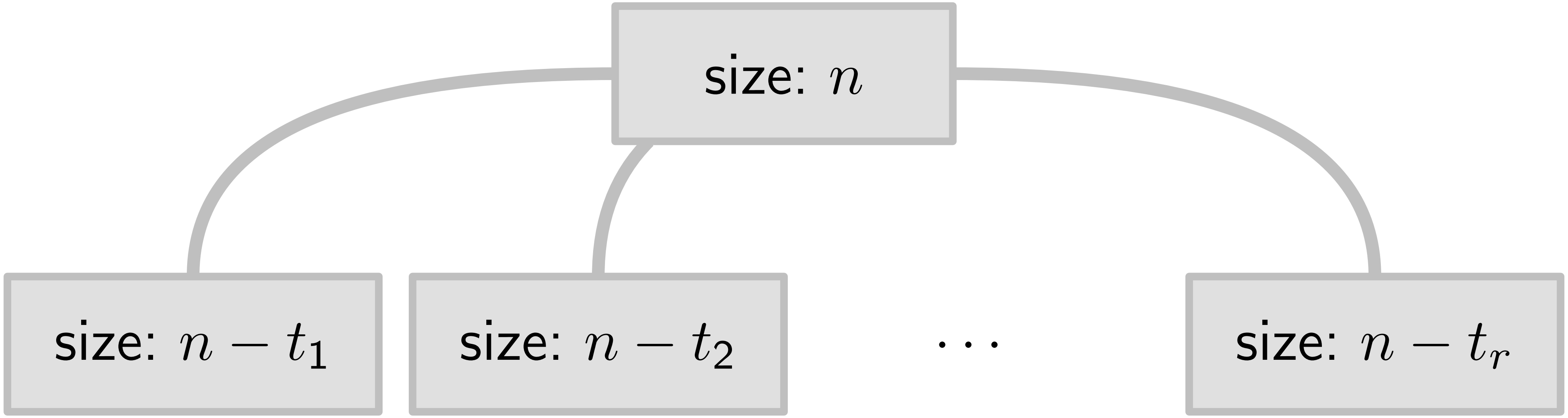
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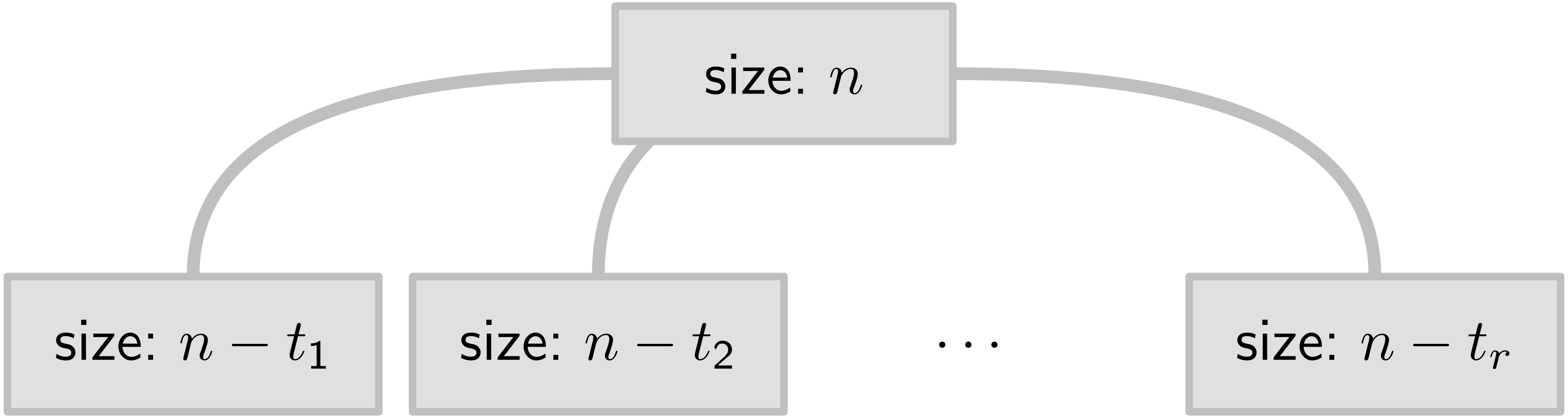

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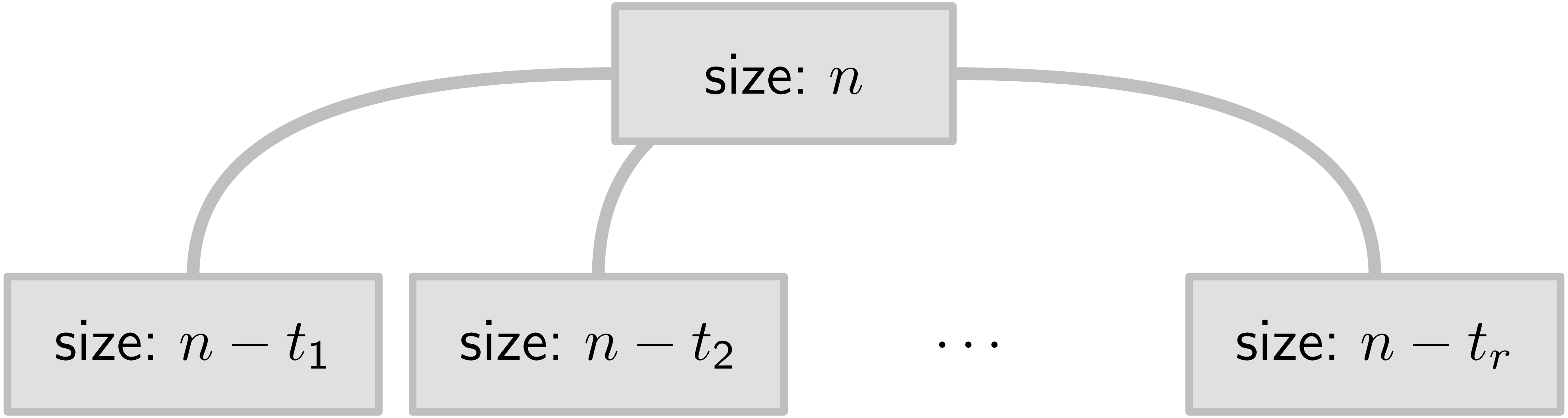

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Recurrence: $T(n) \ \leq \ T(n - t_1) + T(n - t_2) + \ldots + T(n - t_r)$

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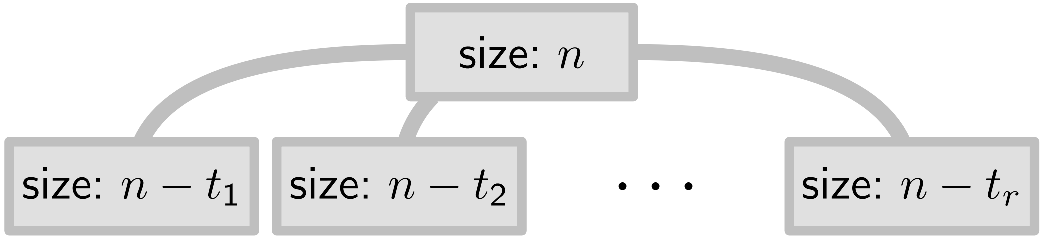
- Denote solution by $\tau(t_1, t_2, \ldots, t_r)$.
- Often irrational.

Properties of $\tau(\cdot)$



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Thm. Let $r \geq 2$ and, for $i = 1, \ldots, r$, let $t_i > 0$. Then:

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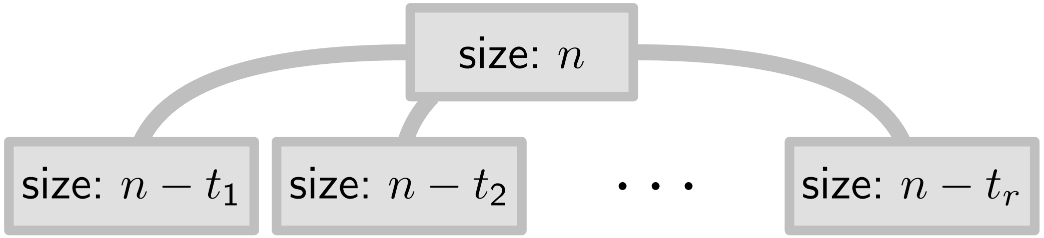

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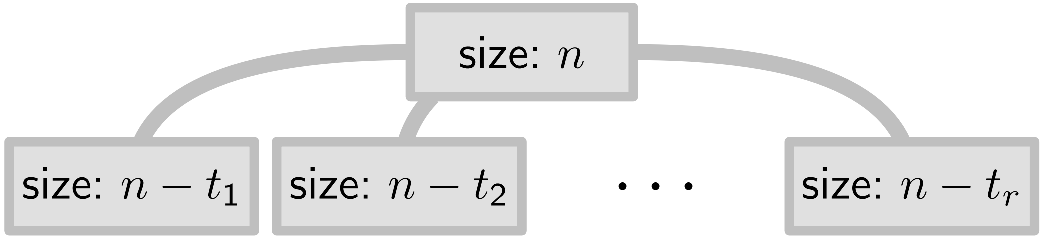

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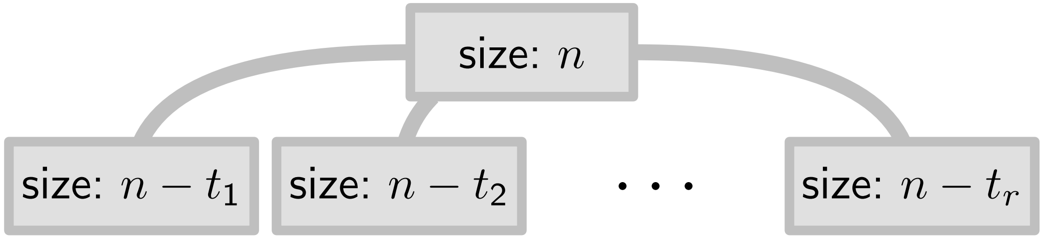

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Lemma. For $i, j, k > 0$, the following balancing properties apply:

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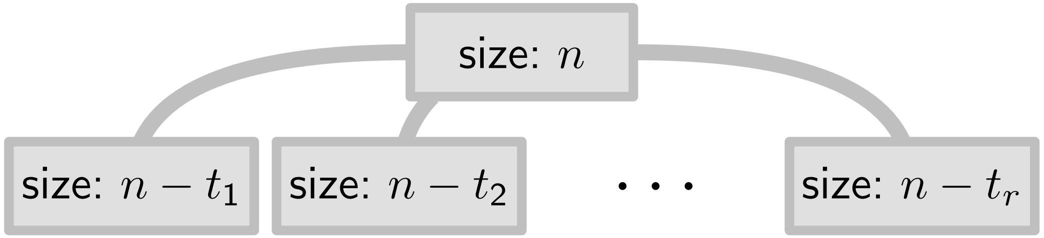

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$$\tau(1, 1) = 2$$

$$\tau(1, 2) = \frac{1+\sqrt{5}}{2} < 1.62$$

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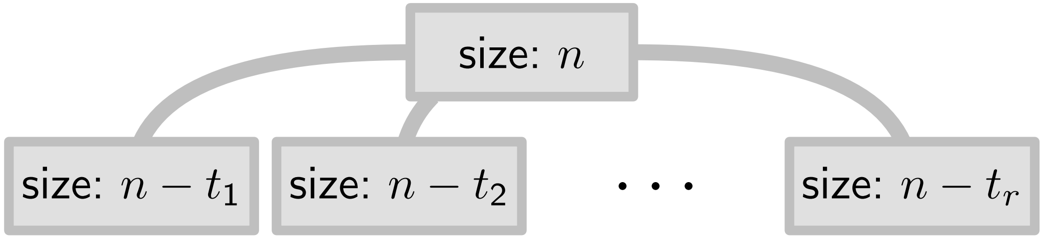

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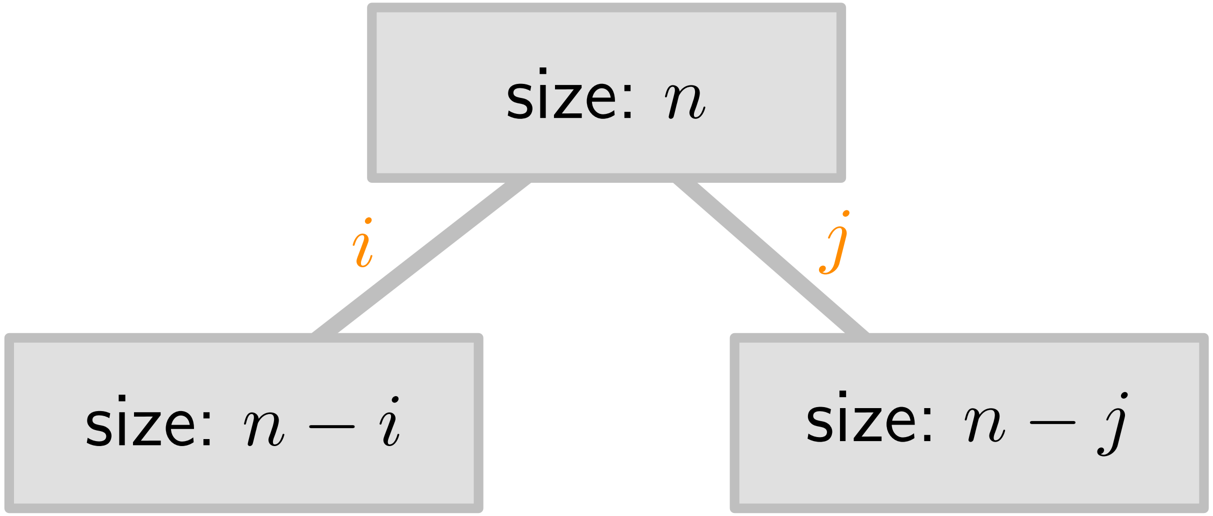
$\tau(3,3) = \sqrt[3]{2} < 1.26$

$\tau(2,4) < 1.272$

$\tau(1,5) < 1.3248$

“Addition” of Branching Vectors



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j

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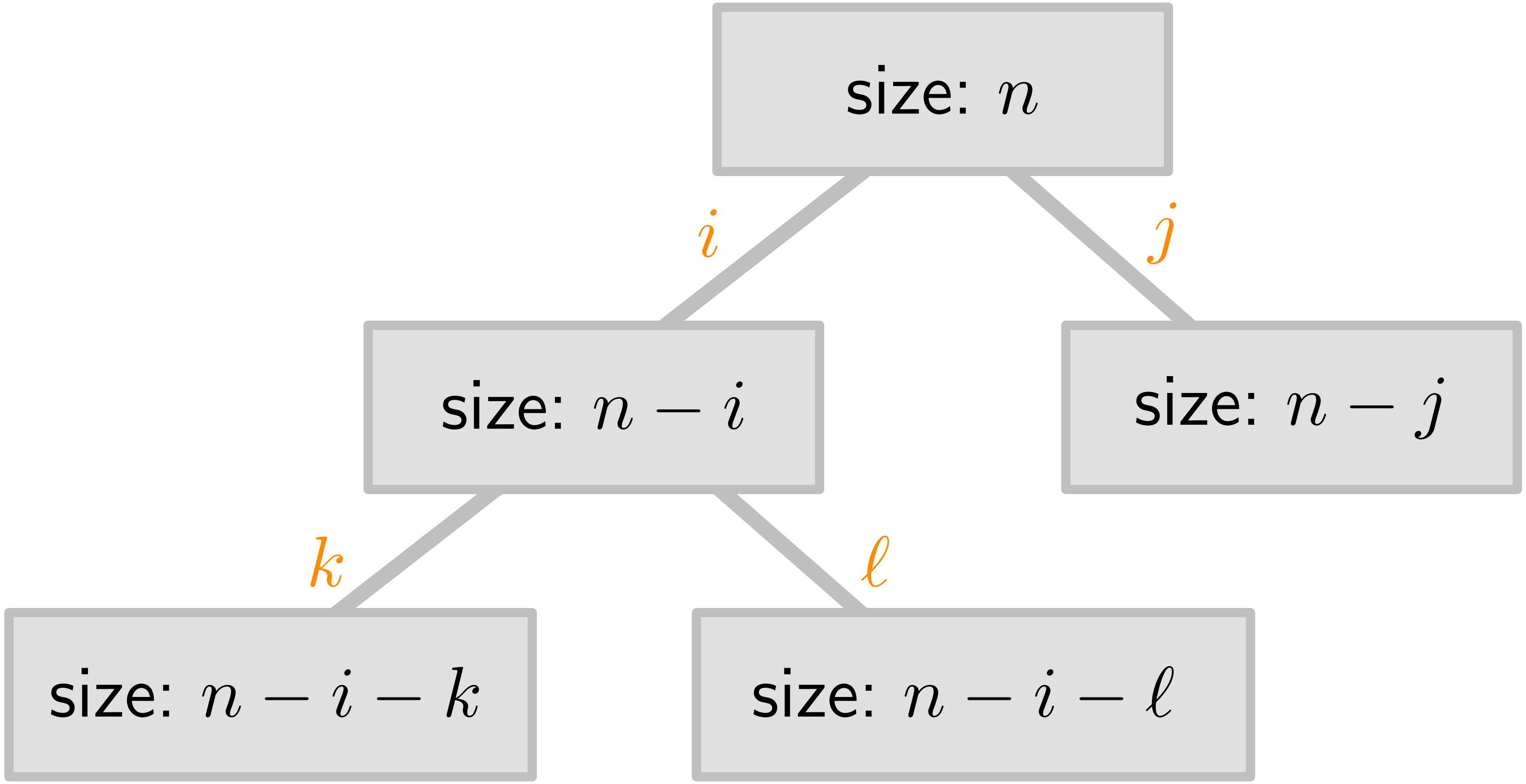

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size: $n-j$

k

ℓ

size: $n-i-k$

size: $n-i-\ell$

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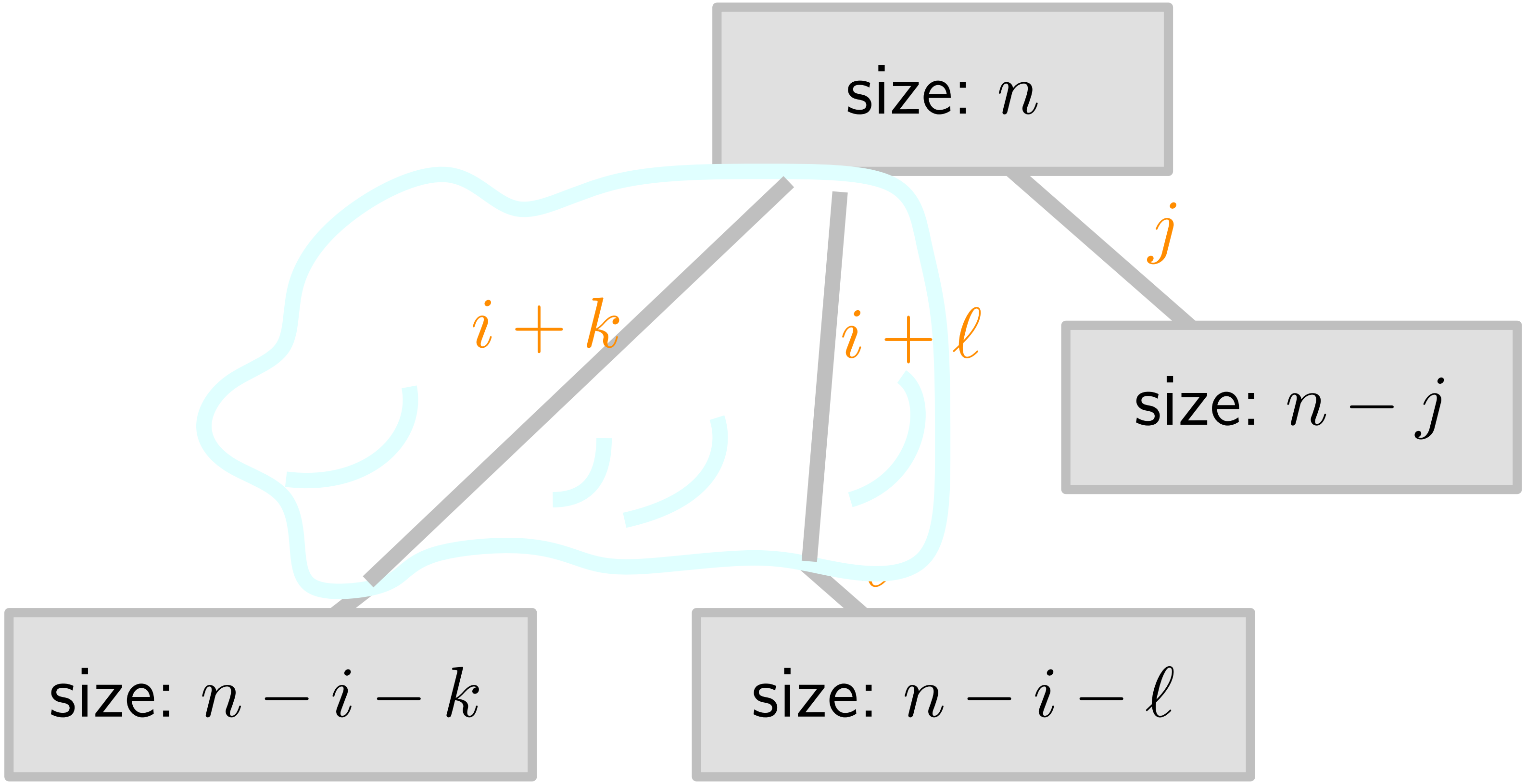

Branching-Vector: $(i+k,\ i+\ell,\ j)$

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Input: propositional logic formula F in conjunctive NF

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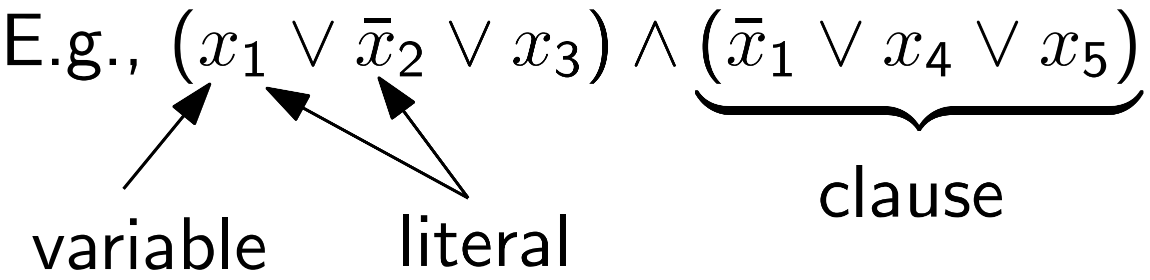

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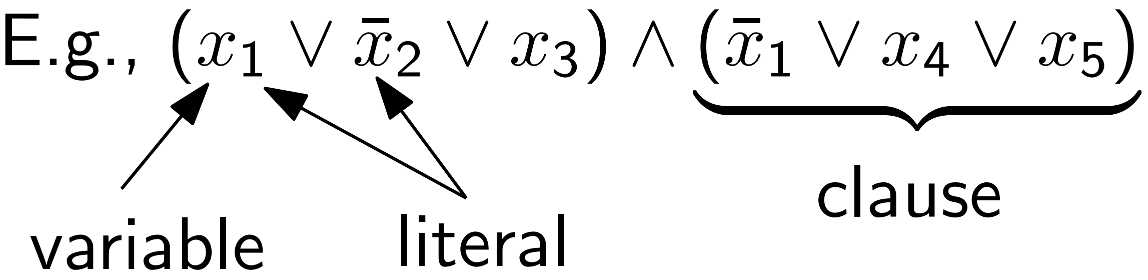

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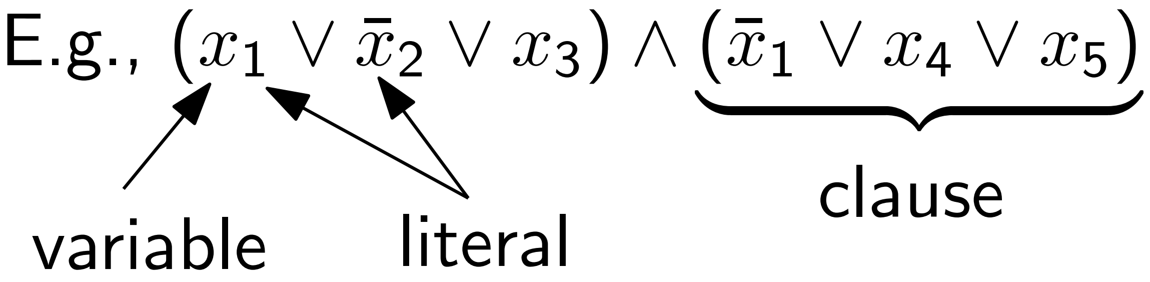

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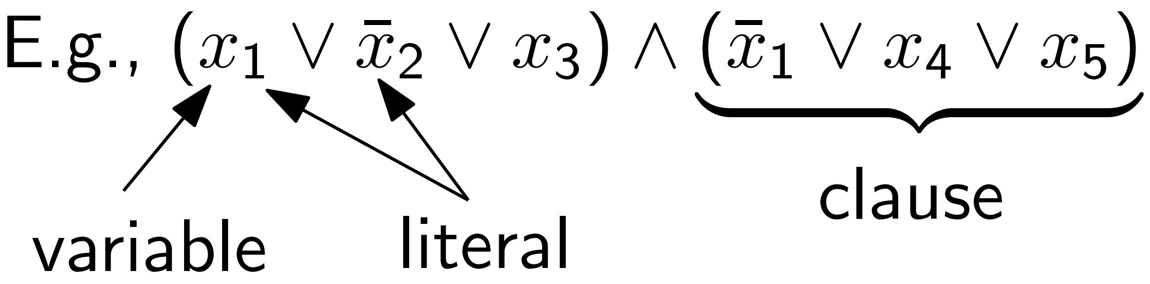

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For more on (S)ETH, see [Parameterized Algorithms 14.1].

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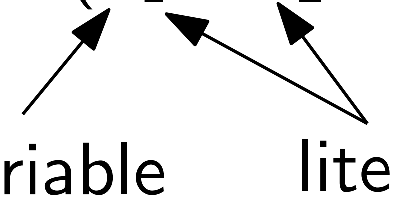

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$\nexists$ algorithm for SAT in $o(2^n)$ time, i.e., in $O^*((2-\varepsilon)^n), \varepsilon > 0$.

For more on (S)ETH, see [Parameterized Algorithms 14.1].

k-SAT: Each clause has $\leq k$ literals.

A Better Algorithm for k-SAT

- Goal:
  Solve k-SAT in time $O^*({\alpha_k}^n)$, where $\alpha_k < 2$ for every k.

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  - t: $x_1 =$ false
  - $F[t] = (\bar{x}_2 \vee x_3)$

Properties of Reduced Formulas

- $F[t]$ satisfiable $\Rightarrow$ F satisfiable

- $F = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_4 \vee x_5)$
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- $F[t]$ empty $\Rightarrow$ $F[t]$ satisfiable

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A First Algorithm for k-SAT

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Algorithm $k$-SAT-v1($F$)
  if $F$ is empty then
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  $t_1$: $\ell_1 =$ true
  $t_2$: $\ell_1 =$ false, $\ell_2 =$ true
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  $\vdots$
  $t_q$: $\ell_1 =$ false, $\ell_2 =$ false, $\ldots, \ell_{q-1} =$ false, $\ell_q =$ true
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- t_i: first true literal in c is ℓ_i

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- $\beta_1 = 1, \beta_2 < 1.6181, \beta_3 < 1.8393, \beta_4 < 1.9276, \beta_5 < 1.9660$

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t: $x_1 = x_4 =$ true

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Obs.
- If t is an autark, then: F satisfiable $\Leftrightarrow$ $F[t]$ satisfiable.

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- If t is an autark, then: F satisfiable $\Leftrightarrow$ $F[t]$ satisfiable.
- If t is *not* an autark,
  then $F[t]$ contains a clause of length $\leq k-1$.

An Improved k-SAT Algorithm

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Algorithm $k$-SAT-v2($F$)
    if $F$ is empty then
        return true
    if $F$ contains an empty clause then
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    if $t_i$ is an autark for some $i = 1, \ldots, q$ then
        return $k$-SAT-v2($F[t_i]$)
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        return $\bigvee_{i=1}^{q}$ $k$-SAT-v2($F[t_i]$)
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  if $t_i$ is an autark for some $i = 1, \ldots, q$ then
    return $k$-SAT-v2($F[t_i]$)          $\leftarrow$ "Reduce"
  else
    return $\bigvee_{i=1}^{q} k$-SAT-v2($F[t_i]$)   $\leftarrow$ "Branch"
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  Before:
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Now:	Before:
$\Rightarrow$ branching vector $(1, 2, \ldots, k-1)$	$(1, 2, \ldots, k-1, k)$
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$\Rightarrow \alpha_2 = 1, \alpha_3 < 1.6181, \alpha_4 < 1.8393, \alpha_5 < 1.9276, \alpha_6 < 1.9660$

$\square$